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METHODS OF COMPUTING PK FOR THE OFFSET CIRCLE PROBLEM

J. R. Williams

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AIR FORCE WEAPONS LABORATORY
Air Force Systems Command
Kirtland Air Force Base
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Four methods are surveyed briefly for computing the probability of kill (PK) for the offset circle problem which cannot be solved in closed form. ASA FORTRAN routines for these methods are presented. When ranked according to increasing accuracy, the order is the Wilson-Hilferty (W-H) approximation, the Pearson-Wilson-Hilferty (P-W-H) approximation, the Germond-Wegner (GW) approximation, and the series expansion (SE) approximation. When ranked according to increasing machine time requirements, the order is G-W, W-H, P-W-H, SE with the G-W approximation requiring 30 percent as much time as the SE. ( )

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FOREWORD

This research was performed under Project 5809, Task 8809-09.

Inclusive dates of research were April 1969 through April 1971. The report was submitted 8 June 1971 by the Air Force Weapons Laboratory Project Officer, J. R. Williams (SAB).

This technical report has been reviewed and is approved.

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Chief, Analysis Division

ABSTRACT

(Distribution Limitation Statement A)

Four methods are surveyed briefly for computing the probability of kill (PK) for the offset circle problem which cannot be solved in closed form. ASA FORTRAN routines for these methods are presented. When ranked according to increasing accuracy, the order is the Wilson-Hilferty (W-H) approximation, the Pearson-Wilson-Hilferty (P-W-H) approximation, the Germond-Wegner (GW) approximation, and the series expansion (SE) approximation. When ranked according to increasing machine time requirements, the order is G-W, W-H, P-W-H, SE with the G-W approximation requiring 30 percent as much time as the SE.

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INTRODUCTION

The problem at hand is to compute the probability of kill (PK) of a point target lying on a plane surface by a weapon having a circular "cookie-cutter" lethal area and delivered with a circular normal hit distribution centered around a point removed some distance from the point target. Figure 1 illustrates the problem. A point target is located at the origin O. An aim point is located at A. The impact point for some specific trial occurs at I. I is circular normally distributed about A; that is, the coordinates of I relative to A are independently and normally distributed with means zero and standard deviations $\sigma x = \sigma y = \sigma$. Surrounding the point target O is a vulnerable circle C of radius R. The vulnerable radius, R, is a deterministic variable. We desire to know the probability that, in a trial yet to occur, the impact point I will fall within circle C. Alternatively, (but mathematically the same) we could have described a circle of lethal radius R about the impact point and computed its probability of enveloping the point target O.

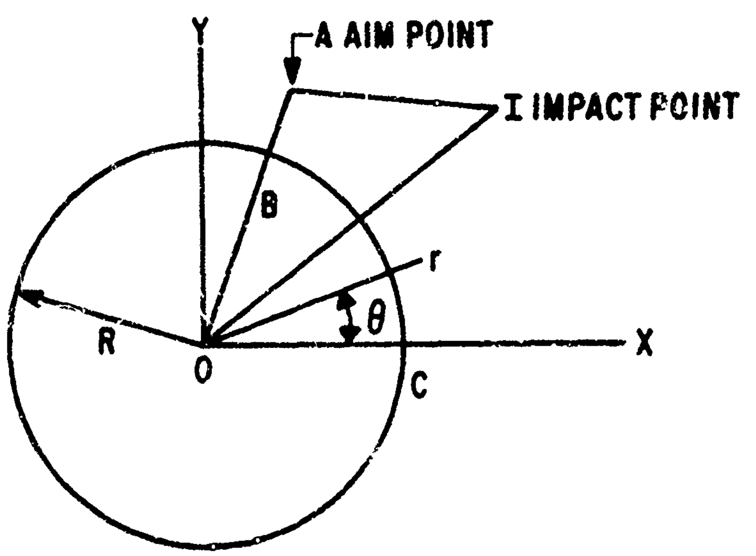

Figure 1. The Geometry of the Problem

The desired probability is

$$PK = P\ (I\epsilon C;\ B,\ R,\ \sigma)$$

$$= \int_C \frac{\partial P}{\partial r}\, dr\, d\theta$$

$$= \frac{1}{2\pi\sigma^2} \int_C r\, \exp(-r^2/2\sigma^2) dr\, d\theta \qquad (1)$$

Equation (1) reduces to

$$PK = 1/2 \left[1 - e^{-\left(B/\sigma\right)^2}\ I_o \left(\frac{B}{\sigma}\right)^2\right] \text{ when } B = R \qquad (1a)$$

$$PK = 1 - \exp(-R^2/2\sigma^2) \text{ when } B = 0 \qquad (1b)$$

In the past when hand methods of calculation were more common, published tables of PK = F(R/σ, B/σ) were frequently used. References 1, 2, and 3 are typical of such tables. Others are described in reference 4. Alternatively, these tables (comprising 185, 15, and 301 pages, respectively) can be transformed into a single graph such as figure 2, which was first drawn in the late 1950s by T. L. Franklin*. Note that it is convenient to make the plot with respect to the dimensionless coordinates σ/R and B/R with the parameter B/σ appearing as a series of radials. With this form of plot, a sink is seen to appear at B=R for high values of R/σ (or low values of the dispersion relative to the lethal radius). To help explain figure 2, the data have been replotted pictorially on figure 3.

Now that machine computation has replaced hand computation for most system analysis studies, it has become necessary to push these tables and graphs into the background and replace them with machine routines. This report presents a few useful routines.

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* The data for figure 2 were computed by PROGRAM MOLLY that used FUNCTION PKAREA (inf. series expansion) to compute PK and FUNCTIONS ROSA and ROSB to interpolate to desired values of PK. These latter functions are interesting because of their accuracy and simplicity. They are described in an AFWL technical report to be published as <u>The Effect of Input Dispersions and Uncertainties on the Kill Probability of a Point Target</u>.

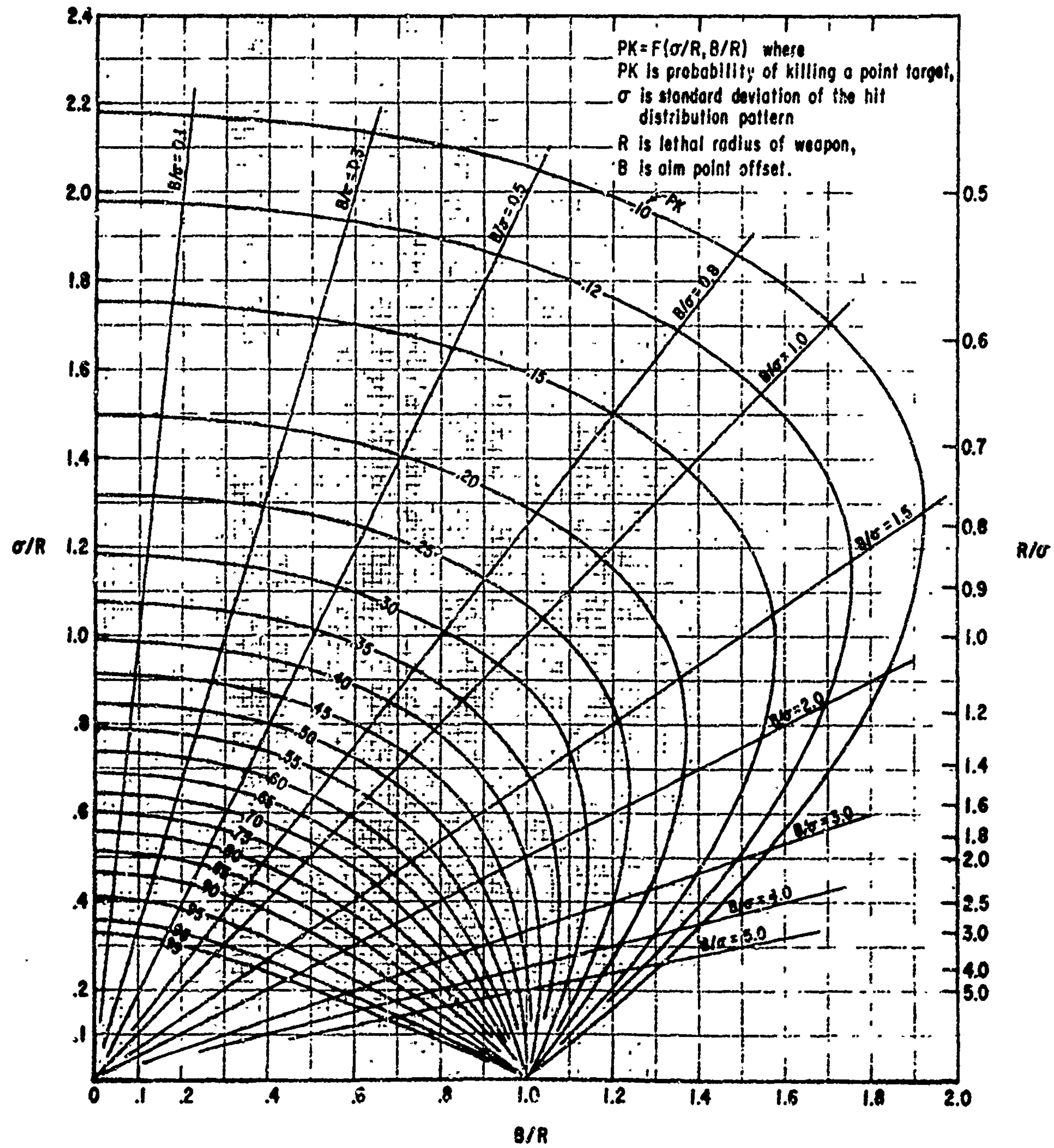

Figure 2. PK for the Offset Circle Problem

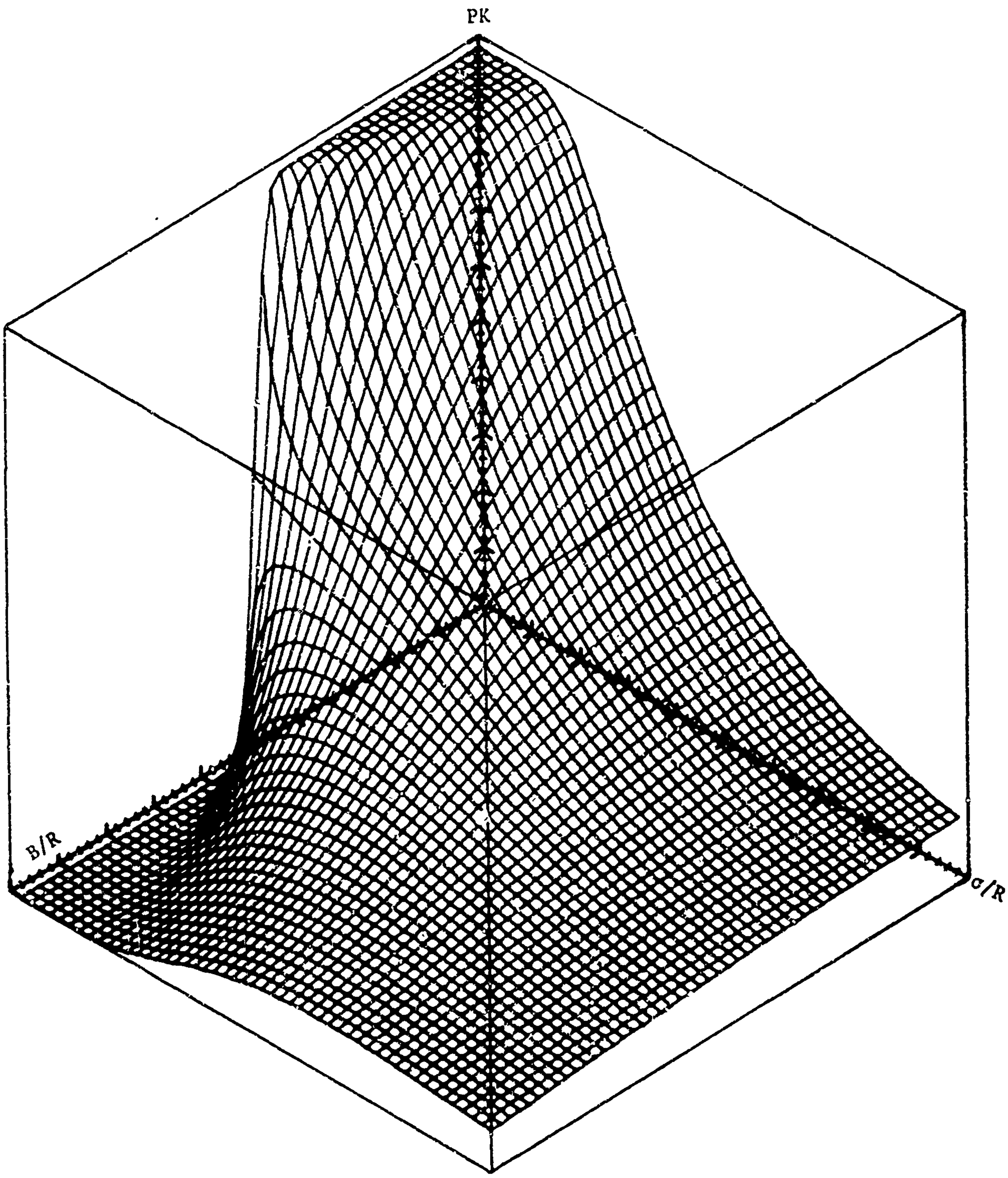

Figure 3. Pictorial Representation of PK = F(σ/R, B/R)

Since a closed solution* to this problem is not available, we will have to resort to approximations or a numerical integration of the appropriate function (as was done in preparing the referenced tables). Four approximations of varying accuracy and speed are presented in the following sections.

* Such a closed solution exists for the spherical case. See Guenther (1961) SIAM Review, Vol. 3, No. 3, Pages 247-250.

SECTION II

WILSON-HILFERTY APPROXIMATION

The distributions of the components (referred to the coordinate axes) of the distance from impact point to target point are distributed as noncentral χ^2 (Fisher 1928) with one-degree-of-freedom and noncentrality parameter a_i. Following Patnaik (1949) we can approximate a weighted sum of noncentral χ^2 by fitting its first two moments to an ordinary central χ^2. This central χ^2 then may be transformed to approximate normality by using the Wilson-Hilferty (W-H) transformation.

The mean and variance of the central χ^2 which approximates the given noncentral χ^2 are

$$m = 1.0 + 0.5\ (B/\sigma_1)^2 \qquad (2)^*$$

$$v = 1.0 + (B/\sigma_1)^2 \qquad (3)^*$$

where $\sigma_1{}^2 = \sigma_2^2 = 0.5\sigma^2$; σ_1 and σ_2 being the component standard deviations of the circular normal delivery pattern.

The equivalent number of degrees of freedom of the replacement central χ^2 distribution is

$$n' = 2m^2/v$$

The Wilson-Hilferty transformation is then used to transform the central χ^2 to approximate normality. This gives us the variate t which is approximately normally distributed with zero mean and unit variance.

---

* Equations (2) and (3) are equations (10) and (11) of Grubbs (Ref. 5) as simplified for the circular case.

$$t = \frac{\sqrt[3]{(R/\sigma_1)^2/2m} - \left(1 - \frac{v}{9m^2}\right)}{\sqrt{\frac{v}{9m^2}}} \qquad (4)^*$$

The technique, as we have briefly described it above, refers to an offset circle problem, but it is equally applicable (with some added complications, of course) to the offset sphere problem as well as to elliptical and ellipsoidal normal hit distributions. Grubbs (Ref. 5) fully describes the technique.

A suitable translation of the above into ASA FORTRAN takes the following form:

```
1  RSQ = ROS * ROS     (ROS = R/σ) where now σ = σ1 = σ2
   If (RSQ) 1, 2, 3
2  PKAREA = 0.0
   RETURN
3  BSQ = BOS * BOS     (BOS = B/σ)
   W = 1.0 + 0.5 * BSQ
   Y = (1.0 + BSQ)/(9.0 * W * W)
   T = ((0.5 * RSQ/W) ** 0.333333 + Y - 1.0)/SQRT(Y)
   PKAREA = PHI(T)
   RETURN
   END
```

The normal distribution function PHI(T) is detailed in the appendix.

To check out the W-H approximation for the offset circle problem, a small program was run to prepare a table of PK as a function of R/σ and B/R. This information is presented in table I with corresponding values for the Pearson-Wilson-Hilferty (P-W-H) approximation and for more accurate approximations as presented in the previously mentioned tables. In each block the uppermost value

---

* Equation (4) is derived from equation (20) of Grubbs (Ref. 5).

Table I

KILL PROBABILITY TABLE

(Comparison of Three Methods of Computing PK--The Series Expansion, the Pearson-Wilson-Hilferty, and the Wilson-Hilferty Methods)

ROS	BOR										
	0.0	.20	.40	.60	.80	1.00	1.20	1.40	1.60	1.80	2.00
.45	.096293 .102325 .102325	.09592 .101993 .101987	.09482 .101053 .100955	.09301 .099648 .099155	.09054 .097956 .096626	.08746 .096122 .093240	.08383 .094207 .089026	.07974 .092185 .084030	.07527 .089972 .078349	.07050 .087466 .072121	.06553 .084591 .063517
.50	.117503 .121673 .121673	.116953 .121174 .121165	.115317 .119746 .119620	.112642 .117568 .116977	.109001 .114850 .113176	.104491 .111756 .108195	.099232 .108355 .102078	.093356 .104629 .094951	.087007 .100522 .087011	.080331 .095988 .078509	.073473 .091024 .069724
.60	.164730 .165326 .165326	.16365 .164324 .164312	.16046 .161410 .161232	.15527 .156828 .156010	.14829 .150879 .148615	.139768 .143837 .139145	.13002 .135908 .127869	.11936 .127255 .115210	.10815 .118040 .101700	.09670 .108444 .087907	.085331 .098667 .074378
.70	.217296 .214919 .214919	.21542 .213136 .213123	.209910 .207897 .207699	.20102 .199522 .198610	.18920 .188517 .185982	.175023 .175487 .170232	.15912 .161059 .152076	.14216 .145829 .132450	.12482 .130336 .112372	.10770 .115046 .092812	.091320 .100339 .074585
.75	.245160 .241617 .241617	.24278 .239326 .239314	.23579 .232564 .232379	.22457 .221715 .220836	.20976 .207463 .204965	.19212 .190710 .185433	.17256 .172413 .163277	.15197 .153462 .139761	.13123 .134612 .116191	.11111 .116456 .093743	.09223 .099428 .073347
.80	.273851 .269378 .269378	.27089 .266494 .266485	.26221 .257959 .257805	.24834 .244240 .243462	.23013 .226279 .223946	.208657 .205369 .200247	.18509 .182876 .173778	.16062 .160017 .146177	.13634 .137760 .119054	.11320 .116806 .093792	.091918 .097614 .071407
.90	.333023 .327410 .327410	.32868 .323088 .323090	.31596 .310254 .310230	.29584 .289658 .289298	.26978 .263023 .261405	.239563 .232718 .228375	.20713 .201106 .192543	.17434 .170114 .156392	.14283 .141102 .122185	.11386 .114910 .091677	.088316 .091957 .065960
1.00	.393469 .387606 .387606	.387449 .381528 .381548	.369922 .363469 .363673	.342411 .334662 .335011	.307220 .297991 .297544	.267120 .257231 .254162	.225004 .215907 .208294	.183550 .176676 .163385	.144954 .141226 .122371	.110778 .110423 .087310	.081892 .084529 .059206

Table I (cont'd)

ROS	ROR 0.0	.20	.40	.60	.80	1.00	1.20	1.40	1.60	1.80	2.00
1.10	.453926 .448537 .448537	.44600 .440439 .440488	.42300 .416414 .416939	.38718 .378399 .379704	.34190 .330763 .331833	.291126 .278916 .277436	.23890 .227638 .221179	.18881 .180314 .167567	.14362 .138902 .120232	.10507 .104202 .081432	.073880 .076193 .051895
1.25	.542167 .538382 .538382	.53109 .526956 .527062	.49912 .493190 .494326	.44977 .440402 .443437	.38829 .375508 .479205	.32078 .306571 .307714	.25327 .240386 .235716	.19084 .181494 .169532	.13706 .132196 .113872	.09368 .093015 .071068	.06087 .063279 .041016
1.50	.675348 .674891 .674891	.659073 .658073 .658280	.612162 .608634 .610843	.540145 .532026 .538028	.451485 .438956 .447027	.356284 .341978 .347255	.264486 .251877 .249706	.184045 .175645 .164564	.119655 .116140 .098455	.072468 .072911 .053007	.040780 .043505 .025481
1.75	.783735 .785575 .785575	.76360 .765021 .765286	.70503 .704329 .707241	.61396 .608845 .617043	.50088 .490920 .502463	.37992 .367597 .376093	.26593 .255101 .254543	.17061 .164043 .153305	.09975 .097839 .080960	.052885 .054196 .036997	.025325 .027923 .014463
2.00	.864665 .867166 .867166	.843019 .845500 .845740	.778505 .780129 .783116	.674469 .672708 .681908	.541109 .534330 .548123	.396499 .386567 .397287	.262001 .253080 .253358	.154439 .149577 .138867	.080496 .079826 .064051	.036840 .038528 .024407	.014723 .016852 .007564
2.50	.956063 .957226 .957226	.938319 .940135 .940148	.879146 .882571 .884202	.767874 .770622 .778726	.605896 .604095 .619282	.418439 .412250 .425180	.246102 .240093 .240565	.120639 .118243 .107430	.048500 .049155 .036479	.015802 .017287 .009129	.004137 .005163 .001641
3.00	.988891 .988879 .988879	.978888 .979336 .979147	.937375 .939768 .939569	.836533 .840308 .845223	.658916 .659678 .673698	.432520 .428563 .441982	.225286 .221125 .220852	.089764 .088820 .078345	.026667 .027638 .018609	.005804 .006684 .002824	.000914 .001265 .000264
4.00	.999665 .999571 .999571	.998340 .998102 .997941	.986252 .986170 .984589	.924731 .926581 .925711	.744954 .747062 .756190	.449728 .447889 .460406	.180098 .177986 .175704	.044006 .044267 .036149	.006234 .006792 .003572	.000497 .000648 .000156	.000022 .000039 .000003
5.00	.999996 .999991 .999991	.999926 .999870 .999830	.997779 .997456 .996535	.969322 .969533 .966676	.812595 .814371 .818722	.459902 .458918 .470001	.137485 .136417 .132641	.018556 .018977 .014047	.001040 .001203 .000461	.000023 .000035 .000004	.000010 .000009 .000000

Table I (cont'd)

NOTES:

1. In each block, the three values shown are

    top - highly accurate values from reference 1 or 3 or NBS Applied Math Series 23 or the series expansion routine.

    middle - from the Pearson Wilson-Hilferty approximation

    bottom - from the Wilson-Hilferty approximation

2. The heavy lines delimit the area within which $PK \geq 0.10$.

is the accurate value; the middle value is given by the W-H approximation. Supplementing table I is table II which shows, as the middle and lower values in each block, the errors in probability resulting from use of the P-W-H and W-H approximations. An inspection of the lowermost values on table II (and within the heavy lines which delimit the problems for PK > 10 percent) shows us that the W-H approximation is generally rather good for engineering applications except in the region near $R/\sigma = 1.0$ to 1.5, $B/R \gtrsim 1.4$ where the errors are 0.020 to 0.024 in probability* or -11 to -22 percent of the correct probability. This is a bit too much of an error to accept willingly in a system analysis study.

---

* Mathur (Ref. 7) indicates that the W-H transformation from central χ^2 to approximate normality gives maximum errors in probability of .0344, .0122, and .0069 for 1, 2, or 3 degrees-of-freedom. The equivalent number of degrees-of-freedom in the intermediate central χ^2 is

$$n = 2m^2/v = 2(1 + 1/4\,[B/\sigma]^4/(1 + [B/\sigma]^2))$$

for the offset circle problem. This equation gives $n = 2$ when $B = 0$ and $n \approx 2.8$ in the region of poor performance mentioned in the text above. Consequently, as Mathur states, the maximum error in probability for the two areas mentioned will be .012 (for $n = 2$) and .007 (for $n = 2.8$) with respect to the W-H transformation from central χ^2 to approximate normality. In the larger view (the 2-step transformation from noncentral χ^2 to approximate normality) the errors in probability will be larger and as shown by table II.

Table II

KILL PROBABILITY TABLE

(Comparison of Several Methods of Computing PK--Errors in the PWH and WH Approximations)

ROS	BOR										
	0.0	.20	.40	.60	.80	1.00	1.20	1.40	1.60	1.80	2.00
.45	+6.3 +.006032 +.006032	+6.3 +.00607 +.00607	+6.6 +.00623 +.00613	+7.1 +.00664 +.00617	+8.2 +.00742 +.00609	+9.9 +.00866 +.00578	+12.4 +.01038 +.00519	+15.6 +.01244 +.00429	+19.5 +.01470 +.00308	+19.5 +.01697 +.00162	+29.0 +.01906 -.00001
.50	+3.5 +.004170 +.004170	+3.6 +.004221 +.004212	+3.8 +.004429 +.004303	+4.4 +.004926 +.004335	+5.4 +.005849 +.004175	+6.9 +.007265 +.003704	+9.2 +.009123 +.002846	+12.1 +.011273 +.001595	+15.5 +.013515 +.090004	+19.5 +.015657 -.001822	+23.9 +.017551 -.003749
.60	+0.4 +.000596 +.000596	+0.4 +.00067 +.00066	+0.6 +.00095 +.00077	+1.0 +.00166 +.00074	+1.7 +.00259 +.00032	+2.9 +.004069 -.000623	+4.5 +.00589 -.00215	+6.6 +.00789 -.00415	+9.1 +.00989 -.00645	+12.1 +.01174 -.00879	+15.6 +.013336 -.010953
.70	-1.1 -.002377 -.002377	-1.1 -.00228 -.00230	-1.0 -.00201 -.00221	-0.8 -.00150 -.00241	-0.4 -.00068 -.00421	+0.3 +.000464 -.005000	+1.2 +.00194 -.00704	+2.6 +.00367 -.00971	+4.4 +.00552 -.01245	+6.8 .00735 -.00489	+9.8 +.009019 -.016735
.75	-1.5 -.003543 -.003543	-1.4 -.00345 -.00347	-1.4 [illegible] -.00341	-1.3 -.00281 -.00373	-1.1 -.00230 -.00480	-0.7 -.00141 -.00669	-0.1 -.00015 -.00928	+1.0 +.00149 -.01221	+2.6 +.00338 -.01514	+4.8 +.00534 -.01737	+7.8 +.00720 -.01888
.80	-1.6 -.004473 -.004473	-1.6 -.00440 -.00441	-1.6 -.00425 -.00441	-1.7 -.00410 -.00488	-1.7 -.00385 -.00618	-1.6 -.003288 -.008410	-1.2 -.00221 -.01131	-0.4 -.00060 -.01444	+1.0 +.00142 -.01729	+3.2 +.00361 -.01941	+6.2 +.005696 -.020511
.90	-1.7 -.005613 -.005613	-1.7 -.00559 -.00559	-1.8 -.00571 -.00573	-2.1 -.00618 [illegible]	-2.5 -.00676 -.00838	-2.9 -.006845 -.011188	-2.9 -.00603 -.01459	-2.4 -.00423 -.01795	-1.2 -.00173 -.02065	+0.9 +.00105 -.02218	+4.1 +.003641 -.022356
1.00	-1.5 -.005863 -.005863	-1.5 -.005921 -.005901	-1.8 -.006453 -.006249	-2.3 -.007749 -.007400	-3.0 -.009229 -.009676	-3.7 -.009889 -.012958	-4.0 -.009097 -.016710	-3.7 -.006874 -.020165	-2.6 -.003728 -.022583	-0.3 -.000355 -.023468	+3.2 +.002637 -.022686

Table II (cont'd)

ROS	BOR 0.0	.20	.40	.60	.80	1.00	1.20	1.40	1.60	1.80	2.00
1.10	-1.2 -.005389 -.005389	-1.3 -.00556 -.00551	-1.6 -.00659 -.00606	-2.3 -.00878 -.00748	-3.3 -.01114 -.01007	-4.2 -.012210 -.013690	-4.7 -.01126 -.01772	-4.5 -.00850 -.02124	-3.3 -.00472 -.02339	-0.8 -.00087 -.02364	+3.1 +.002313 -.021985
1.25	-0.7 -.003785 -.003785	-0.8 -.00403 -.00403	-1.2 -.00593 -.00479	-2.1 -.00937 -.00633	-3.3 -.01278 -.00909	-4.4 -.01421 -.01307	-5.1 -.01288 -.01755	-4.9 -.00935 -.02131	-3.6 -.00486 -.02319	-0.7 -.00067 -.02261	+4.0 +.00241 -.01985
1.50	-0.1 -.000457 -.000457	-0.2 -.001000 .000793	-0.6 -.003528 -.001319	-1.5 -.008119 -.002117	-2.8 -.012529 -.004458	-4.0 -.014306 -.009025	-4.8 -.012609 -.014780	-4.6 -.008400 -.019481	-3.0 -.003515 -.021200	+0.6 +.000443 -.019461	+6.7 +.002725 -.015299
1.75	+0.2 +.001840 +.001840	+0.2 +.00142 +.00129	-0.1 -.00070 +.00221	-0.8 -.00512 +.00308	-2.0 -.00996 +.00158	-3.3 -.01233 -.00383	-4.1 -.01083 -.01139	-3.9 -.00657 -.01731	-1.9 -.00191 -.01879	+2.2 +.0012 -.0160	--- +.0026 -.0108
2.00	+0.3 +.001501 +.002501	+0.3 +.002481 +.002721	+0.2 +.001624 +.004611	-0.3 -.001761 +.007439	-1.3 -.006779 +.007014	-2.5 -.009932 +.000788	-3.4 -.008921 -.008643	-3.2 -.004862 -.015572	-0.8 -.000670 -.016445	--- +.001686 -.012433	--- +.002129 -.007159
2.50	+0.1 +.001163 +.001163	+0.2 +.001816 +.001829	+0.4 +.003425 +.005056	-0.3 -.002748 +.010852	-0.3 -.001801 +.013386	-1.0 -.006189 +.006741	-2.4 -.005009 -.005537	-2.0 -.002396 -.013209	+1.3 +.000655 -.012021	--- +.001485 -.006673	--- +.001026 -.002496
3.00	0.0 -.000012 -.000012	+0.0 +.000448 +.000259	+0.2 +.002393 +.002194	+0.4 +.003775 +.008690	+0.1 +.000762 +.014782	-1.5 -.003957 +.009462	-1.9 -.004161 -.004434	-1.1 -.000944 -.011419	--- +.000971 -.008058	--- +.000880 -.002980	--- +.000621 -.000380
4.00	0.0 -.000094 -.000094	0.0 -.000238 -.000399	0.0 -.000082 -.001663	+0.2 +.001850 +.000980	+0.3 +.002108 +.011236	-0.4 -.001839 +.010678	-1.2 -.002112 -.004394	+0.6 +.000261 -.007857	--- +.000558 -.002662	--- +.000151 -.000341	--- +.000017 -.000036
5.00	0.0 -.000005 -.000005	0.0 -.000056 -.000096	0.0 -.000323 -.001244	0.0 +.000211 -.002646	+0.2 +.001776 +.006137	-0.2 -.001084 +.010099	-0.8 -.001068 -.004844	--- +.000421 -.004509	--- +.000173 -.000579	--- +.000012 -.000019	--- --- ---

Table II (cont'd)

NOTES:

1. In each block, the three values shown are

   top - percentage error in the Pearson-Wilson-Hilferty approximation

   middle - error in the Pearson-Wilson-Hilferty approximation

   bottom - error in the Wilson-Hilferty approximation

2. The heavy lines delimit the area within which PK $\geq$ 0.10.

3. Percentage errors are not shown when PK < 0.04.

SECTION III

PEARSON-WILSON-HILFERTY APPROXIMATION

Something better than the W-H approximation is required for many probability problems. Pearson (Ref. 8) points out that a three-moment central chi-square approximation for the distribution of a noncentral chi-square may be quite accurate. The resulting central χ^2 distribution is then transformed to approximate normality by the W-H transformation. Grubbs (Ref. 5) describes this P-W-H transformation*. Applying his equations to the offset circle problem we see that

$$v_i = \sigma_i^2/\sigma^2 = 1/2 \tag{5}**$$

The third central moment is

$$\mu_3 = 8v_i^3 \sum_{i=1}^{i=2} \left(1 + 3\,a_i^2\right) = 8v_1^3 \left(2 + 3a_1^2 + 3a_2^2\right) \tag{6}**$$

but the noncentrality parameter is

$$a_i^2 = \frac{(\mu_i - \alpha_i)^2}{\sigma_i^2} \tag{7}**$$

so that $\mu_3 = 2 + 3 \left(a_1^2 + a_2^2\right) = 2 + 3\left(\frac{B}{\sigma_1}\right)^2$

Pearson's β_1 coefficient is

$$\beta_1 = \mu_3^2/v^3 \tag{8}**$$

where the variance, v, is $v = 1 + (B/\sigma_1)^2$ as shown earlier.

---

* Also known as the Pearson-Merrington approximation.

** Equations (5), (6), (7), and (8) appear in Grubbs (Ref. 5) as equations (4), (21), (5), and (22).

The equivalent number of degrees of freedom of the central χ^2 approximation is

$$n' = 8/\beta_1 \qquad (9)*$$

Let us take $\Psi^2 = \left(\frac{R}{\sigma}\right)^2 = 1/2\left(\frac{R}{\sigma_1}\right)^2$ as did Grubbs. Then the central χ^2 approximation is

$$\chi^2_{n'} = (\Psi^2 - m)\sqrt{\frac{2n'}{v}} + n \qquad (10)*$$

Next, we apply the W-H transformation to approximate normality to this central χ^2 distribution and find the variate t which is approximately normally distributed with zero mean and unit variance.

$$t = \frac{\sqrt[3]{\chi^2_{n'}/n'} + \frac{2}{9n'} - 1}{\sqrt{\frac{2}{9n'}}} \qquad (11)*$$

It will now be expedient to summarize these formulas, remove the subscripts, and redefine σ as $\sigma = \sigma_1 = \sigma_2$

$$v = 1 + \left(\frac{B}{\sigma}\right)^2$$

$$\mu = 2 + 3\left(\frac{B}{\sigma}\right)^2$$

Let $Q = \chi^2_{n'}/n'$

Then $Q = \frac{\mu}{2v^2}\left[1/2\left(\frac{R}{\sigma}\right)^2 - 1/2\left(\frac{B}{\sigma}\right)^2 - 1\right] + 1$

Let $S = 2/9n'$

* Equations (9), (10), and (11) appear in Grubbs (Ref. 5) as equations (24), (25), and (26).

$$\text{Then } S = \frac{\mu^2}{36v^3}$$

$$\text{and } t = \frac{Q^{1/3} + S - 1}{\sqrt{S}}$$

The quantity t is referred to a table of cumulative normal integrals to find the desired probability or evaluated by a computer routine.

A suitable translation of the above into ASA FORTRAN takes the following form:

```
1 RSQ = ROS * ROS     (ROS = R/σ)
  IF (RSQ) 1, 2, 3
2 PKAREA = 0.0
  RETURN
3 BSQ = BOS * BOS     (BOS = B/σ)
  V = 1.0 + BSQ
  U = 2.0 + 3.0 * BSQ
  Q = U * (RSQ - BSQ - 2.0)/(4.0 * V * V) + 1.0
  S = U * U/(36.0 * V * V * V)
  T = (Q**0.33333333 + S - 1.0)/SQRT(S)
  PKAREA = PHI(T)
  RETURN
  END
```

The normal distribution function PHI(T) is detailed in the appendix.

As mentioned earlier, tables I and II present a comparison of the P-W-H approximation to accurate computations. In table II, the center entry in each block gives the amount by which the P-W-H approximation errs in computing PK. We see that the greatest error is 0.0143 in probability and occurs at $R/\sigma = 1.5$ and $B/R = 1.0$*. Also in table II, the upper entry in each block gives the

* Johnson (Ref. 9) showed in 1959 that Pearson's three-moment approximation is remarkably accurate in both tails of the distribution, i.e., the upper and lower 5 percent points.

percentage error in computing PK by use of the P-W-H approximation. We see that the greatest percentage error is - 5.1 percent and occurs at $R/\sigma = 1.25$ and $B/R = 1.2$ when $PK = 0.25$.

We may conclude then that the P-W-H approximation is sufficiently accurate and easy for most system studies.

SECTION IV

GERMOND-WEGNER APPROXIMATION

The Germond-Wegner (G-W) three-region approximation* generally has a smaller percentage error (Table III) than the P-W-H approximation and a comparable run time. The equations take the following form:

$$PKAREA = 1 - \phi \left(BOS - \sqrt{ROS^2 - 1}\right) \qquad (12)$$

where $\phi(X)$ is the cumulative normal integral with zero mean and unit variance. See appendix.

$$PKAREA = ROS^2/A*EXP(-BOS^2/A) \qquad (13)$$

where

$$A = 2 + ROS^2/2 \qquad (13a)$$

$$PKAREA = 1-EXP\left[-ROS^2/\left[1.416+(.397-.0159\ ROS^2)BOS^2\right]^2\right] \qquad (14)$$

Use equation (12) if

$ROS > 3$

or

$ROS \geq 1.8$ and $BOS > 1.5$

Use equation (13) if

$ROS < 1.8$ and $BOS \geq 1.5$

or

$ROS \leq 1.0$ and $BOS \leq 1.5$

Use equation (14) otherwise.

---

* Prepared and submitted by George Schroeter of the US Army Aberdeen Research and Development Center, Aberdeen Proving Ground, Maryland 21005.

Table III

KILL PROBABILITY TABLE

(The Germond-Wegner Approximation and its Accuracy)

ROS	BOR										
	0.0	.20	.40	.60	.80	1.00	1.20	1.40	1.60	1.80	2.00
.45	+0.1 .096371 .000018	+0.1 .096000 .000017	+0.1 .094897 .000015	+0.1 .093085 .000070	+0.1 .090607 .000004	+0.1 .087517 .000057	+0.1 .083884 .000049	+0.1 .079784 .000041	+0.0 .075301 .000032	+0.0 .070525 .000024	+0.0 .065544 .000016
.50	+0.1 .117647 .000144	+0.1 .117095 .000142	+0.1 .115453 .000136	+0.1 .112768 .000126	+0.1 .109114 .000113	+0.1 .104589 .000098	+0.1 .099313 .000081	+0.1 .093420 .000063	+0.1 .087055 .000046	+0.0 .080360 .000029	+0.0 .073486 .000014
.60	+0.2 .165138 .000408	+0.2 .164050 .000400	+0.2 .160831 .000375	+0.2 .160831 .000375	+0.2 .148575 .000289	+0.2 .140000 .000232	+0.1 .130188 .000173	+0.1 .119475 .000114	+0.1 .108205 .000059	+0.0 .096712 .000012	-0.0 .085305 -.000026
.70	+0.5 .218263 .000967	+0.4 .216366 .000941	+0.4 .210772 .000866	+0.4 .201769 .000749	+0.3 .189808 .000603	+0.3 .175465 .000442	+0.2 .159398 .000280	+0.1 .142295 .000131	+0.0 .124829 .000005	-0.1 .107612 -.000092	-0.2 .091163 -.000157
.75	+0.6 .246575 .001415	+0.6 .244155 .001372	+0.5 .237037 .001247	+0.5 .225631 .001057	+0.4 .210579 .000822	+0.3 .192692 .000568	+0.2 .172880 .000321	+0.1 .152076 .000102	-0.1 .131162 -.000073	-0.2 .110915 -.000196	-0.3 .091961 -.000267
.80	+0.7 .275862 .002011	+0.7 .272835 .001942	+0.7 .263951 .001745	+0.6 .249783 .001445	+0.5 .231215 .001092	+0.3 .209357 .000699	+0.2 .185427 .000338	+0.0 .160647 .000032	-0.1 .136141 -.000197	-0.3 .112855 -.000342	-0.4 .091510 -.000408
.90	+1.1 .336798 .003775	+1.1 .332291 .003616	+1.0 .319129 .003166	+0.8 .298341 .002497	+0.6 .271492 .001714	+0.4 .240492 .000928	+0.1 .207368 .000239	-0.2 .174053 -.000287	-0.4 .142207 -.000620	-0.7 .113098 -.000766	-0.9 .087557 -.000759
1.00	+1.7 .400000 .006531	+1.6 .393651 .006202	+1.4 .375202 .005280	+1.2 .346355 .003944	+0.8 .309657 .002437	+0.4 .268128 .001008	-0.1 .224857 -.000147	-0.5 .182630 -.000919	-0.9 .143662 -.001292	-1.2 .109450 -.001328	-1.4 .080759 -.001134

Table III (cont'd)

ROS	BOR 0.0	.20	.40	.60	.80	1.00	1.20	1.40	1.60	1.80	2.00
1.10	-0.2 .453092 -.000834	-0.3 .444667 -.001331	-0.6 .420537 -.002465	-0.9 .383908 -.003276	-0.8 .339337 -.002564	+0.2 .291694 .000569	+2.6 .245154 .006255	-1.0 .186894 -.001915	-1.5 .141436 -.002183	-1.9 .103130 -.001940	-1.9 .072456 -.001424
1.25	-0.2 .541264 -.000903	-0.3 .529638 -.001437	-0.5 .496489 -.002633	-0.7 .446487 -.003288	-0.5 .386554 -.001736	+1.0 .324029 .003252	-1.2 .250173 -.003093	-2.1 .186796 -.004048	-2.7 .133345 -.003713	-2.9 .091005 -.002678	-2.5 .059380 -.001489
1.50	-0.1 .674425 -.000922	-0.2 .657799 -.001274	-0.4 .609890 -.002272	-0.5 .537328 -.002817	-0.0 .451300 -.000186	-1.6 .350462 -.005822	-3.5 .255303 -.009183	-4.6 .175572 -.008473	-4.7 .113983 -.005672	-3.6 .069857 -.002611	-0.9 .040417 -.000363
1.75	-0.1 .782898 -.000837	-0.1 .762916 -.000683	-0.2 .703727 -.001305	-0.4 .611401 -.002561	+0.0 .501084 .000205	-4.1 .364337 -.015580	-6.5 .248759 -.017169	-7.1 .158462 -.012153	-5.6 .094175 -.005575	-1.3 .052218 -.000668	--- .027013 .001688
2.00	-0.1 .863980 -.000684	+0.0 .843192 .000173	-0.0 .778353 -.000152	-0.5 .671297 -.003172	+2.1 .552528 .011419	-0.5 .394369 -.002130	-3.8 .252083 -.009918	-7.6 .142772 -.011667	-11.7 .071059 -.009437	--- .030885 -.005955	--- .011666 -.003057
2.50	-0.0 .955715 -.000348	+0.2 .940015 .001696	+0.2 .880970 .001824	-0.7 .762311 -.005563	+1.4 .614584 .008688	-0.3 .417337 -.001102	-2.8 .239252 -.006850	-6.0 .113387 -.007252	-9.8 .043752 -.004747	--- .013597 -.002205	--- .003377 -.000760
3.00	-0.0 .988764 -.000127	+0.2 .980952 .002064	+0.4 .941305 .003930	+1.4 .848125 .011593	+1.1 .665830 .006914	-0.2 .431887 -.000633	-2.3 .220184 -.005102	-5.2 .085098 -.004665	--- .024329 -.002337	--- .005062 -.000742	--- .000758 .000156
4.00	+0.0 .999946 .000282	+0.1 .998940 .000601	+0.2 .988486 .002234	+0.5 .929622 .004891	+0.6 .749521 .004567	-0.1 .449464 -.000264	-1.7 .176959 -.003139	-4.4 .042082 -.001923	--- .005752 -.000482	--- .000439 -.000058	--- .000018 -.000003
5.00	+0.0 .999999 .000003	+0.0 .999952 .000026	+0.0 .998128 .000349	+0.2 .971216 .002894	+0.4 .815668 .003073	-0.0 .459767 -.000134	-1.5 .135444 -.002041	--- .017820 -.000736	--- .000964 -.000075	--- .000021 -.000002	--- .000000 .000000

Table III (cont'd)

NOTES:

1. In each block, the three values shown are

   top - percentage error in the approximation from Germond and Wegner

   middle - value from the Germond and Wegner approximation

   bottom - error in the Germond and Wegner approximation

2. The heavy lines delimit the area within which $PK \geq 0.10$.

3. Percentage errors are not shown when $PK < 0.04$.

Equations (12) and (13) are from Germond (Ref. 10), Equation (14) is from Wegner (Ref. 11), and $\phi(x)$ is the cumulative normal distribution function.

A suitable translation of the above into ASA FORTRAN may take the following form:

```
    1  IF (ROS - 1.0) 2, 2, 3
       A = 2.0 + ROS * ROS/2.0
    2  PKAREA = ROS * ROS/A * EXP(-ROS*ROS/A)
       GO TO 9
    3  IF (ROS - 1.8) 4, 5, 6
    4  IF (ROS - 1.5) 7, 2, 2
    5  IF (ROS - 1.5) 7, 7, 8
    6  IF (ROS - 3.0) 5, 5, 8
    7  PKAREA = 1.0 - EXP (-ROS * ROS/(1.416+
                 (0.397 - 0.0159 * ROS * ROS) * ROS * ROS ** 2)
       GO TO 9
    8  Y = ROS - SQRT (ROS * ROS - 1.0)
       PKAREA = 1.0 - PHI(Y)
    9  CONTINUE
       RETURN
       END
```

SECTION V

A SERIES EXPANSION APPROXIMATION OF HIGH ACCURACY

Although the approximations previously described are sufficiently accurate for system analyses and are sufficiently easy to apply, a more accurate approximation to the value of the probability integral may at times be desired, even at the cost of more computer time. A series expansion approximation is available and may be used to compute PK to any desired level of accuracy.

Vitalis (Ref. 3) presents a few simple equations that require evaluation to compute PK. They are (using the present notation)

$$PK = e^{-y} \sum_{n=0}^{\infty} \frac{y^n}{n!} \left[1 - e^{-x} \sum_{m=0}^{n} \frac{x^m}{m!}\right]$$

$$x = \frac{1}{2}\left(\frac{R}{\sigma}\right)^2$$

$$y = \frac{1}{2}\left(\frac{B}{\sigma}\right)^2$$

To obtain his equations, Vitalis derived a conventional integral (which cannot be integrated in closed form), expanded it into an infinite series, integrated it term by term, and thus arrived at the infinite series given above. The series is convergent for all finite and positive values of x and y. The question naturally arises as to how many terms of this infinite converging series need to be evaluated in order to obtain a reasonable degree of accuracy. Vitalis shows that the error involved in the computations will be less than 6×10^{-7} if the number of terms is $T = 5.96 * B/\sigma + 3.96$ for $B/\sigma \leq 3.0$. In order to go up to $B/\sigma = 16$ and to ensure that the number of iterations is T rounded up to the next larger interger, we redefine T as

$$T = 5.0 + 6.0 * BOS + 0.31 * BOS^2$$

Computational difficulties arise if the equation for PK is evaluated in the form in which Vitalis has given it. It is easy to show that e^{-y} or e^{-x} may be too small to be registered by the computer or that y^n, x^n, or n! may be too large. For instance if $B/\sigma = 16$ then T = 180 and n! = 180!. This problem is avoided by a simple use of logarithms as follows:

$$PK = \sum_{n=0}^{T} \left\{ \frac{y^n}{e^y * n!} \left[1 - \sum_{m=0}^{n} \frac{x^m}{e^x * m!} \right] \right\}$$

$$\text{Let } Q_n = \log\left[\frac{y^n}{e^y * n!}\right] = -y + n * \log(y) - \sum_{j=0}^{n} \log(j)$$

$$QQ_m = \log\left[\frac{x^n}{e^x * m!}\right] = -x + m * \log(x) - \sum_{k=0}^{m} \log(k)$$

Note that the bracketed expression of the form $x^n/(e^x n!)$ represents a truncated series expansion of e^x divided by an infinite series expansion.

An elaboration of these equations into a FORTRAN routine follows. This routine is effective unless N is large as a result of B/σ being large. In this event, the time requirement becomes too large. We handle this problem by transferring to a different function when BOS > 16.

```
      FUNCTION PKAREA (ROS,BOS)
C
C     FUNCTION TO COMPUTE THE PROBABILITY OF KILL FOR A WEAPON OF KILL
C     RADIUS R, OFFSET AIM DISTANCE B, AND CIRCULAR NORMAL DELIVERY
C     DISPERSION SIGMA (R/SIGMA=ROS, B/SIGMA=BOS)
C
C     ROUTINE BY J. R. WILLIAMS, AFWL, KAFB, N. M. 87117
C
C     SEE VITALIS,J.A., ET AL *TABLE OF CIRCULAR NORMAL PROBABILITIES*
C     RPT NO 02-949-106, JUNE 1956, BELL AEROSYSTEMS CO, BUFFALO, N. Y.
C
C
1     X=0.5*ROS*ROS
      XX=ALOG(X)
      IF(BOS.GT.16.0) GO TO 3
      IF(BOS.LT.0.02) GO TO 4
      Y=0.5*BOS*BOS
      YY=ALOG(Y)
      T=5.0+6.0*BOS+0.31*BOS*BOS
      N=0
      Q=-Y
      QQ=-X
      SUMTT=EXP(QQ)
      TM=EXP(Q)*(1.0-SUMTT)
      SUMT=TM
2     N=N+1
      FNN=ALOG(FLOAT(N))
       Q=Q+YY-FNN
      QQ=QQ+XX-FNN
      SUMTT=SUMTT+EXP(QQ)
      TM=EXP(Q)*(1.0-SUMTT)
      SUMT=SUMT+TM
      IF(N.LT.T) GO TO 2
      PKAREA=SUMT
      RETURN
3     PKAREA=PKAREA2(ROS,BOS)
C     FUNCTION PKAREA2 IS THE PEARSON-WILSON-HILFERTY APPROXIMATION.
      RETURN
4     PKAREA=1.0-EXP(-X)
      RETURN
      END
```

SECTION VI

COMPUTER TIME REQUIREMENTS FOR THE FOUR METHODS

A computer time-requirement study was made using the CDC-6600 computer and a program whose major work was the computation of 17,340 PK values. Less compiling time, the central processor spent the following times on each PK value:

Wilson-Hilferty	0.25 millisecond
Pearson-Wilson-Hilferty	0.26 millisecond
Germond-Wegner	0.12 millisecond
Series Expansion	0.41 millisecond, average

For the first three approximations, the time requirements are essentially fixed. For the SE approximation, the time requirement is a very strong function of BOS (BOS = B/σ = $B/R \times R/\sigma$) and, consequently, increases rapidly as B/R and/or R/σ increase. Thus, the time requirement may be either more or less than the average value of 0.41 millisecond indicated above. If run on an IBM-7090, the time requirements would be about 12 times greater than indicated above.

Thus, we see that the series-expansion method is the best and also the most expensive. The G-W method is seen to be the next best and the cheapest. The other two methods may be rejected as being both poorer and more expensive.

APPENDIX

FUNCTION PHI (X)

```
C
C     FUNCTION TO RETURN THE CUMULATIVE NORMAL DISTRIBUTION FOR A VALUE
C     OF X (IN UNITS OF SIGMA), WHERE THE MEAN IS ZERO.
C
C     ROUTINE BY HARRY M. MURPHY, JR., 3 JULY 1969.
C     AIR FORCE WEAPONS LABORATORY, KIRTLAND AFB, NEW MEXICO.
C
C     ALGORITHM TAKEN FROM *HANDBOOK OF MATHEMATICAL FUNCTIONS*, (NBS
C     APPLIED MATH SERIES NO. 55, 1964), SECTIONS 26.2.1 and 26.2.17,
C     PAGES 931 AND 932.  (HM)
C
      REAL B1,B2,B3,B4,B5,C,D
C
      LOGICAL PLUS
C
      DATA B1,B2,B3,B4,B5,C,D/0.319381530,-0.356563782,1.781477937,
     1-1.821255978,1.330274431,0.3989422804,0.2316419/
C
C
1     AX=X
      PLUS=.TRUE.
      IF (AX) 3,2,4
C
2     PHI=0.5
      RETURN
C
3     AX=-AX
      PLUS=.FALSE.
C
4     T=1.0/(1.0+D*AX)
      Y=C*(T*(B1+T*(B2+T*(B3+T*(B4+T*B5)))))*EXP(-0.5*AX*AX)
      IF (PLUS) GO TO 5
C
      PHI=Y
      RETURN
C
5     PHI=1.0-Y
      RETURN
      END
```

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