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A SURVEY OF PLASTIC BUCKLING

M. J. Sewell

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ABSTRACT

An elementary up-to-date account is given of the basic features of buckling in perfect and imperfect plastic columns. A self-contained general theory of bifurcation and stability in arbitrarily shaped elastic/plastic and rigid/plastic bodies is described, with comments on applications. A bibliography of over 600 papers pertaining to plastic buckling is given, most of them describing work done during the past fifteen years.

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A SURVEY OF PLASTIC BUCKLING

M. J. Sewell

1. Introduction

In this article we do three things.

(a) In Section 2 we give a connected and up-to-date account in elementary style of the basic detailed features of one specific plastic buckling problem, namely the column problem. This includes the incipient buckling of the perfect column, seen as a bifurcation problem arising at a stable point on an equilibrium path under quasi-static loading. It also involves consideration of the finite deflection of perfect and imperfect columns, since the load maximum on the equilibrium path of the imperfect column is the most important single feature of interest. The actual mechanism of column buckling and the development of the unloading region within the column as buckling proceeds are also discussed.

(b) We give in Section 3 a self-contained formulation of the general boundary value problem for the continuing equilibrium of an arbitrarily shaped elastic or plastic body. The problem is posed in a configuration which is supposed known, but nothing need be said about the various possible loading histories which may have lead to that given state. The incremental or incipient quasi-static behaviour which can then ensue is sought, allowing for the possibility of geometry changes - so that material spin may be large compared with

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distortion, and buckling can be discussed. A general theory of bifurcation is then developed in Section 4 and its application to elastic/plastic and rigid/plastic solids discussed. This is essentially the theory developed by Hill [13-24], except that it is extended here in a way which offers the prospect that bifurcations from stress distributions at <u>corners</u> of yield surfaces may be treated. This hinges on the new inequality (3.60) or (3.65). An account of appropriate quasi-static constitutive equations of incremental plasticity is included in subsections 3(iii)-(viii). Some discussion of stability is given in Section 5.

The level of this account of general theory is necessarily less elementary than the discussion of the column problem, but its purpose is different. The literature shows that it can provide the starting point for a wide range of investigations. So that although some attention to detail is required in reading it, the benefit gained is proportional to the effort put in. As we said, the account given here is selfcontained.

(c) We give a Bibliography of literature on plastic buckling problems which includes nearly 650 items, and for which purpose a literature search concentrated over the past fifteen years has been conducted. The Bibliography centres on time-independent work-hardening elastic/plastic and rigid/plastic solids, with buckling interpreted as the appearance of bifurcation or instability points on equilibrium paths. Typically the buckling failures may appear as ordinary flexural or torsional buckling, bulging or barreling, necking and certain kinds of surface disturbances, in either thin or thick bodies. We have also included a few references to work in the contiguous areas of elastic buckling, collapse loads and plastic limit analysis, buckling under impulse loading, and creep buckling, since an overall view of the subject of plastic buckling will need to carry these areas in mind. But no systematic coverage of these contiguous areas has been attempted (for example, during the compilation we noted in passing - but do not cite - approximately 500 papers from the past ten years on the buckling of elastic cylinders alone, and evidently [28B] more than three times this number of papers on cylindrical shell buckling can be quoted [523]).

Broadly speaking these three main topics in Section 2, Sections 3-5, and Section 6 can be taken independently, even though there is obviously some degree of interconnection.

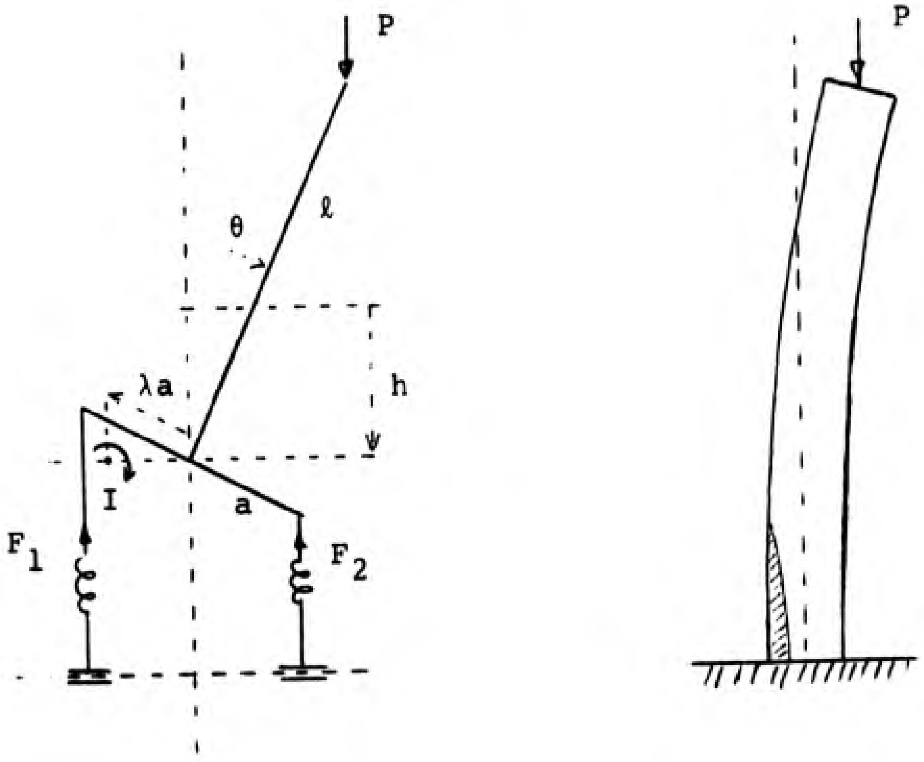
2. Plastic Column Buckling

There has never been an adequate survey of what has been achieved in plastic column buckling theory. The object here is to try to cover some of the more basic aspects of quasi-static flexural bifurcations and associated instability modes. We omit some of the more applied topics such as the effect of residual stresses and local buckling in built-up members (summarized in design manuals such as [203]), time-dependent loading and time-dependent material properties (e.g. creep and strain-rate effects). However, the Bibliography contains some references to these topics.

(i) Introduction

To be definite we shall phrase our remarks in the contexts of the two columns whose end conditions and associated finitely deflected equilibrium positions are indicated in Fig. 2.1. The applied axial load is P. The two-spring model in Fig 2.1(a) represents many of the features of the built-in 'real' column in Fig. 2.1(b). The model consists of a rigid T-piece (length l, with arms each of length a) whose join is constrained to move on a vertical datum axis (h denotes its vertical distance moved, measured positive downwards). The T can also rotate sideways through an angle θ, and the ends of its arms are always acted upon by vertical spring forces F_1 and F_2 arranged as shown via smooth slides at a fixed level. The shaded region in the 'real' prismatic built-in column of arbitrary cross section in Fig. 2.1(b) is the region of elastic unloading behaviour which can be associated with a combination of bending and overall compression; the unshaded region is that of plastic loading. In the spring model the interface between loading and unloading regions is located at the instantaneous center I of rotation of the rigid T, which will be at a horizontal distance λ a cos θ (say) from the root of T as shown.

There are four basically important values of the direct compressive stress σ in the fibres of such a cylindrical column compressed by an axial load P (= σ A in the straight position, where A is section area), namely



(a) Spring model (b) Continuum column

Fig. 2.1 Columns in buckled configuration

Euler stress $\sigma_e = E(\frac{\pi k}{2\ell})^2$, (2.1)

tangent modulus stress $\sigma_t = E_t(\frac{\pi k}{2\ell})^2$, (2.2)

initial yield stress in compression σ_y ,

reduced modulus stress $\sigma_r = E_r(\frac{\pi k}{2\ell})^2$. (2.3)

Here k is the radius of gyration of the cross-section about its buckling axis, E is Young's modulus and E_t is the tangent modulus or slope of the compressive stress-strain curve. For continuing loading beyond σ_y we have $E_t < E$. The reduced modulus E_r lies between them and will be defined later (equation (2.20)).

We shall be considering the two kinds of stress/strain curve shown in Fig. 2.2 (the two-segment idealization) and in Fig. 2.3 (the more realistic smooth-knee case). In both cases

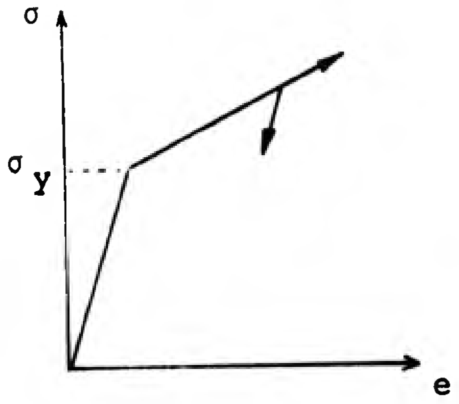


Fig. 2.2 Two-segment stress/strain curve

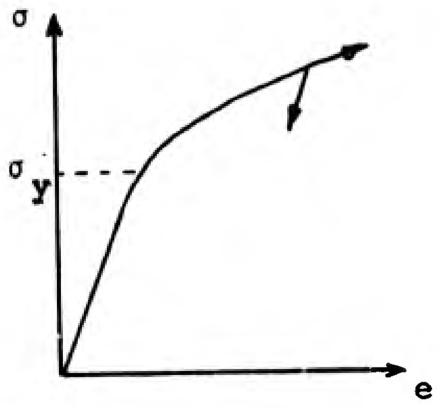


Fig. 2.3 Smooth knee stress/strai curve

the compressive strain e in the fibre increases from zero with the compressive stress, but unloading from any stress higher than σ_y proceeds with modulus E parallel to the initial elastic portion, as the arrows indicate. The tangent modulus is defined as

$$E_t = \frac{d\sigma}{de} \qquad (2.4)$$

by differentiating the function

$$\sigma = \sigma(e) \qquad (2.5)$$

given by the stress/strain curve, taking due account of whether loading or unloading is taking place in the fibre. For monotonic loading $d\sigma/de$ drops discontinuously at σ_y in Fig. 2.2 from E to another constant value E_t, whereas in Fig. 2.3 it falls off continuously (and sometimes quickly) to a slowly decreasing value. We apply Figs. 2.2 or 2.3 to the fibres of the model (namely the springs) by regarding them as force/compression relations instead of stress/strain relations.

For a given column whose fibres obey (2.5) we therefore have

$$\sigma_t < \sigma_r < \sigma_e , \tag{2.6}$$

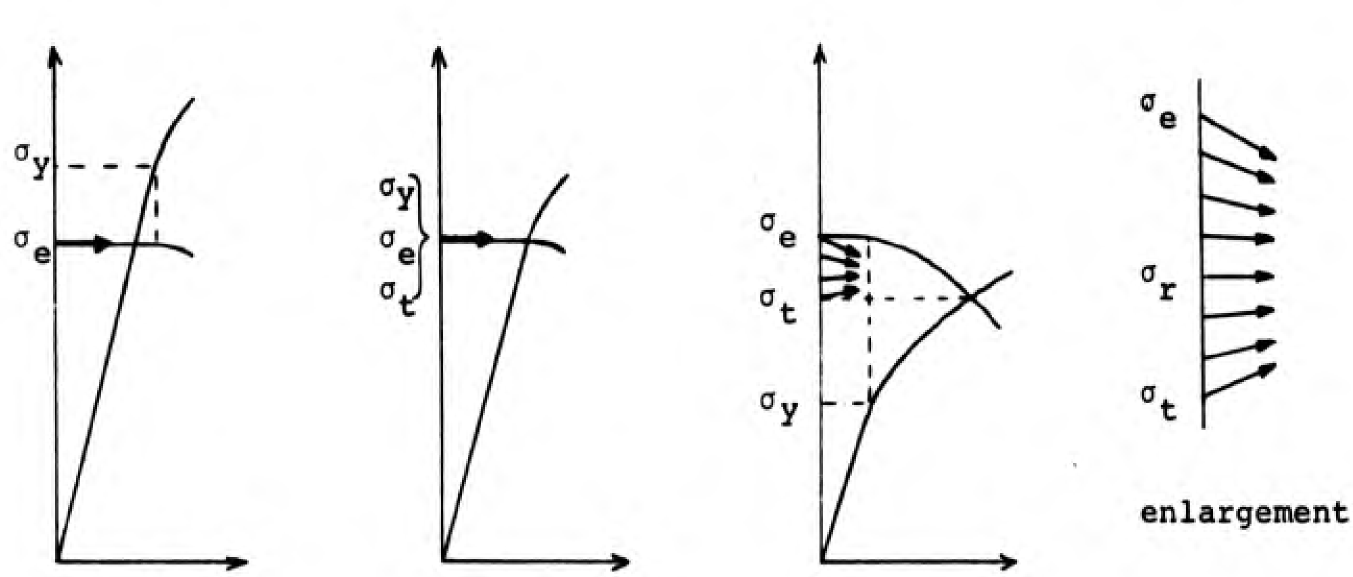
unless $\sigma \leq \sigma_y$ in which case we can have $\sigma_t = \sigma_e$. There are four different stress ranges defined by (2.6), and the buckling mechanism will depend in the first place upon which of these ranges contains the yield stress σ_y.

(ii) Incipient buckling of the perfect column.

We try to give here a complete description of what is known of the mechanism of incipient buckling. By this we mean quasi-static bifurcation from the straight equilibrium position possible under uniform compressive stress in the perfect column. This description only requires linearized theory. We describe the mechanism in an elementary style without proofs, but with references. Enough will be implied later about proofs, when we discuss the nonlinear finite deflection analysis of the imperfect column from the direct 'beam bending' viewpoint, and the general three-dimensional bifurcation theory from the variational viewpoint (in particular see Section 4(v)).

The restriction to incipient behaviour means that all equations refer to the straight configuration, and incremental variables may be regarded as quasi-static rates calculated then - with respect to any quantity (such as lateral deflection) providing a regular local parameterization of the expected new equilibrium paths. Equivalently, the needed linearized equations may be seen as just the first order equations resulting from a 'static perturbation procedure' [46] performed on the exact large deflection equations.

The arrows in Fig. 2.4 represent the initial directions of possible equilibrium paths in a load/lateral deflection space when the smooth stress/strain curve of Fig. 2.3 applies. They are superimposed on plots showing how the stress/strain curve intersects the function $E_t(\pi k/2\ell)^2$ to define the bifurcation stress σ_e or σ_t. By considering how this intersection is arrived at it is clear that σ_y cannot lie within a range $\sigma_t < \sigma_y < \sigma_e$ when the stress/strain curve is smooth. When the two-segment idealization of Fig. 2.2 is used in place of Fig. 2.3, the fact that $E_t(\pi k/2\ell)^2$ is now a step-function of strain



(a) $\sigma_e \leq \sigma_y$ (b) $\sigma_e = \sigma_y = \sigma_t$ (c) $\sigma_y < \sigma_e$

Fig. 2.4 Incipient buckling directions for smooth stress/strain curve

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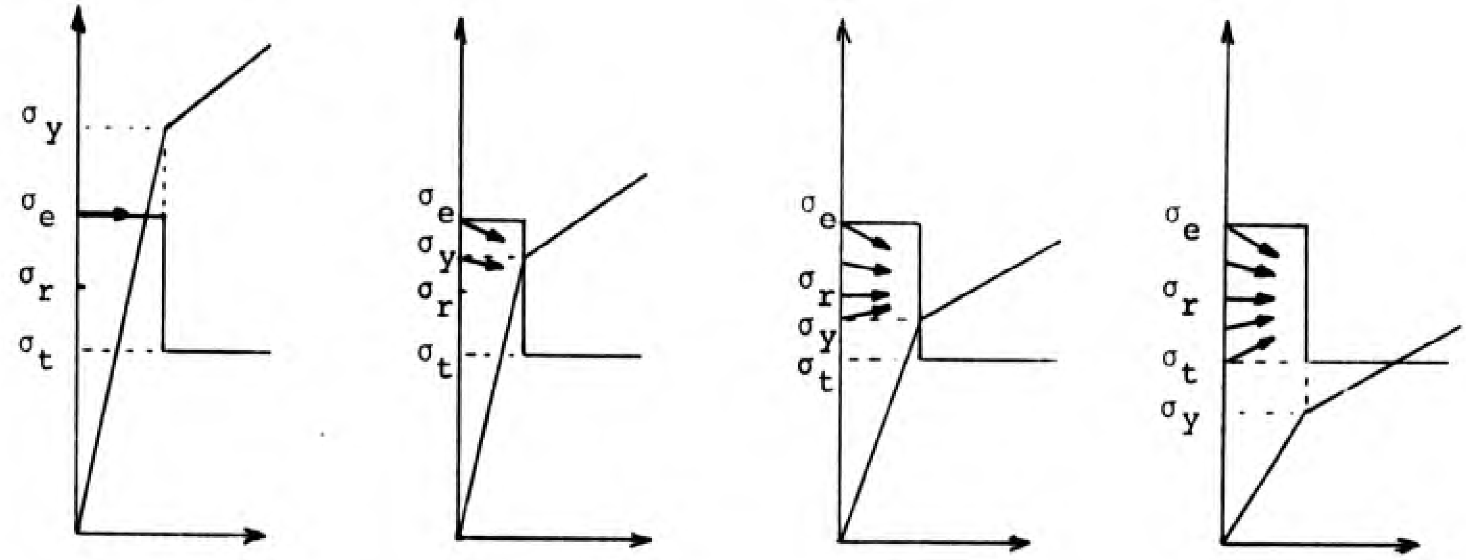


(a) $\sigma_e < \sigma_y$ (b) $\sigma_r \leq \sigma_y \leq \sigma_e$ (c) $\sigma_t \leq \sigma_y \leq \sigma_r$ (d) $\sigma_y \leq \sigma_t$

Fig. 2.5 Incipient buckling directions for two-segment stress/strain curve

modifies Fig. 2.4 to the forms shown in Fig. 2.5. These two sets of diagrams, and their correlation with the extent of incipient unloading regions in the columns which are given below, contain all that can be deduced from a <u>linearized</u> theory (apart from the mode <u>shape</u>, which is assumed sinusoidal for the present as long as shear stiffening is neglected).

Fig. 2.4(a) expresses the classical result that an elastic column can first begin to deflect in equilibrium when the Euler stress σ_e is reached, and that it can then take up 'adjacent equilibrium positions' which will require no change of load to first order. Fig. 2.4(b) shows that linearized theory will predict the same result in the transitional situation when the Euler stress coincides with the yield stress if the stress/strain curve is smooth. Fig. 2.4(c) says that when the straight column yields before the Euler stress is reached, the first plastic buckling stress is the tangent modulus value σ_t. Moreover, there is now a continuous range of stresses between σ_t and σ_e, at each of which the column may begin to deflect quasi-statically from the straight position, but in one definite determinable direction which is different for each of these starting stresses. At σ_t this direction requires the column load P to begin increasing, at σ_e to begin decreasing, and in between to range monotonically through the continuous spectrum of arrows shown in the enlargement in Fig. 2.4(c). That intermediate stress σ_r which admits the horizontal arrow (adjacent equilibrium position) is called the reduced modulus stress.

There is a direct connection between the individual members of this spectrum of arrows and buckling loads, and the extent of the unloading region in the column at bifurcation. At σ_t, the instantaneous center I in Fig. 2.1(a) is on the 'convex' spring F_1 whose compression-rate is therefore zero while the 'concave' spring F_2 continues to load. Higher loads have a one-to-one correspondence with the location of I, which is between the springs so that F_1 unloads and F_2 loads, until at σ_e I is at F_2 which then has stationary load. Similarly, for the real column of Fig. 2.1(b), the unloading region when buckling begins at σ_t is confined to the extreme point on the convex side in the bottom section (of greatest curvature), while at σ_e the loading region is confined to the extreme point on the concave side in that section. At σ_r the loading/unloading interface divides the whole length of the column into two parts. At intermediate loads in the range (2.6) the extent of the incipient loading and unloading regions are indicated in Fig. 2.8, and are calculable for a cylindrical column of any cross-sectional shape [191], as we explain below.

Figs. 2.5(b) and (c) show how Fig. 2.4(b) is modified when a pronounced knee on the stress-strain curve is idealized by a sharp corner at the yield stress σ_y. Plastic buckling can begin <u>at</u> the yield stress σ_y when $\sigma_t < \sigma_y < \sigma_e$ (σ_y may be regarded as a tangent modulus value associated with any one of the fan of possible slopes of the stress/strain curve at the corner). But the initial direction of the equilibrium path at starting loads between σ_y and σ_e will have to be chosen in Figs. 2.5(b) and (c) by <u>omitting</u> the steepest rising directions between σ_t and σ_y (and possible in Fig. 2.5(d)). In particular, if $\sigma_r < \sigma_y < \sigma_e$, bifurcation cannot begin until σ_y is reached, but the associated equilibrium path then requires <u>falling</u> load, suggesting that the straight configuration will become unstable as soon as yielding takes place. Even though this conclusion is based on idealizations not attainable in practice, experimental evidence has been produced [152] (cf. [275, p.287]) indicating such instability for certain columns which yield only just before their Euler stress. A finite part of the column will immediately begin unloading when the bifurcation buckling begins at yield in the range $\sigma_t < \sigma_y < \sigma_e$, and a progressively greater part the nearer σ_y is to σ_e, as we indicated above.

If we consider a sequence of columns having different slenderness ratios but the same stress-strain curve, say with a smooth but pronounced knee, the buckling stresses σ_e, σ_y or σ_t so far discussed can be represented in the familiar pseudo-hyperbolic diagram of Fig. 2.6. One can also combine the information contained in Figs. 2.4 and 2.6 in a single three-dimensional diagram (omitting the buckling stresses above σ_t).

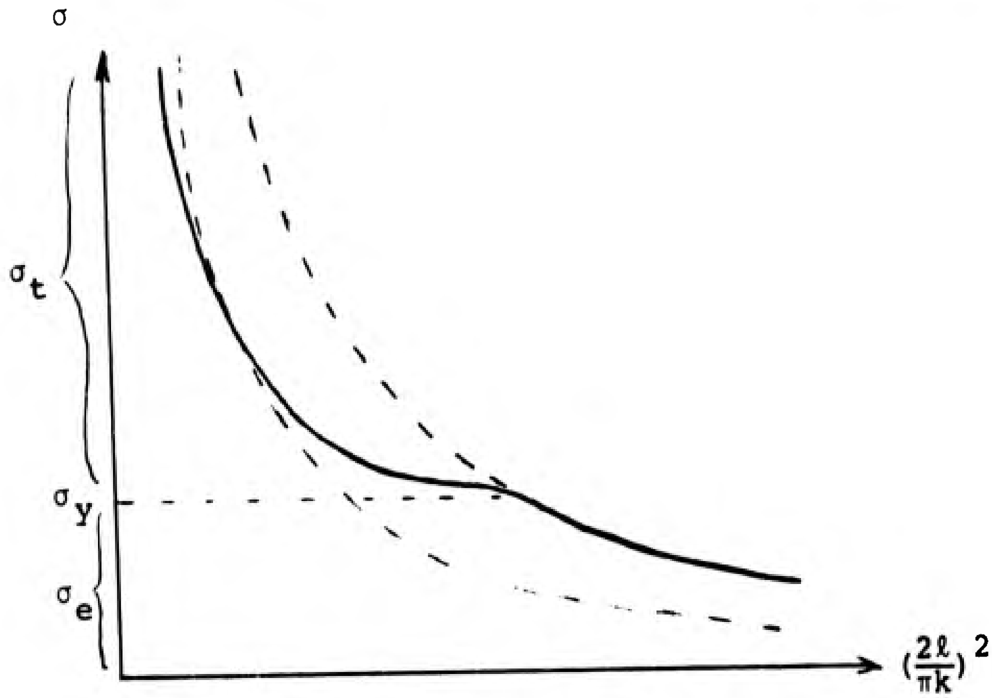


Fig. 2.6 Buckling stress/slenderness ratio pseudo-hyperbola

This is shown in Fig. 2.7, which can be regarded as an incipient <u>equilibrium surface</u> [48] in the space of the parameters indicated. The downward slope at σ_y in Fig. 2.5(b) does not emerge until the sharp corner of Fig. 2.2 is actually introduced as a limiting case, when Fig. 2.6 has a straight segment at σ_y joining the two hyperbolae.

The information summarized up to now has been put together

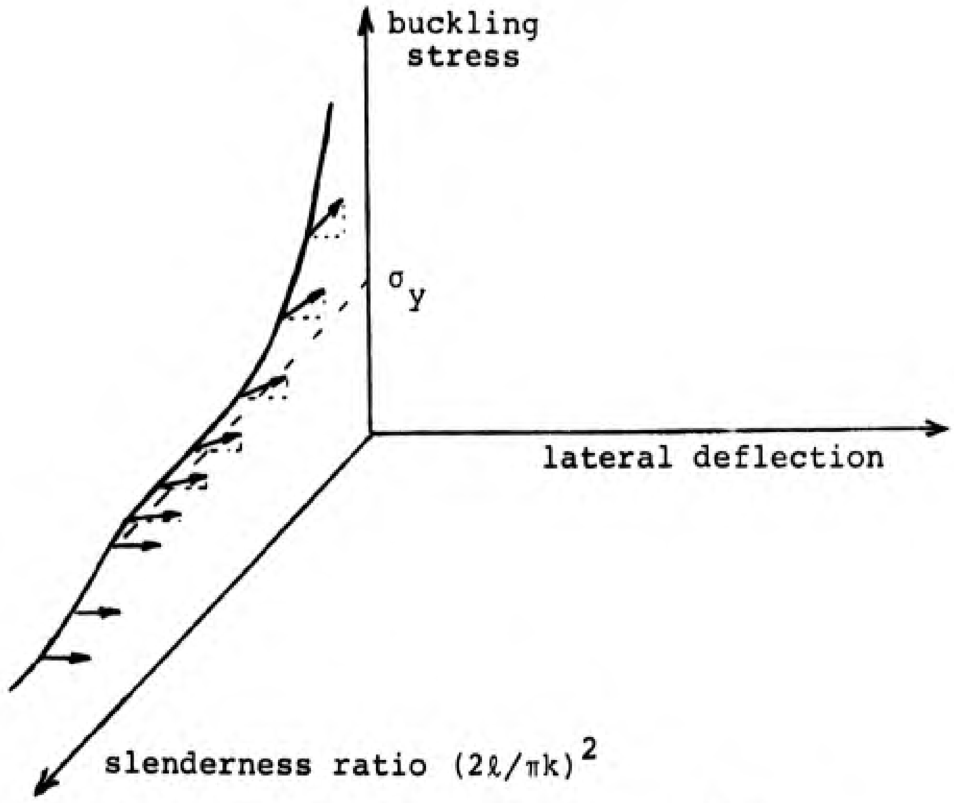


Fig. 2.7 Incipient equilibrium surface

from a variety of sources. In 1889 Engesser [169] first proposed the tangent modulus load to replace the Euler load in the plastic stress-range, but on the erroneous basis that it permitted adjacent equilibrium positions. Considère [156] in 1895 first established the reduced (or double) modulus load as the lowest load permitting deflection under no change of load, and consequently involving unloading over part of the column. Engesser [170-1] acknowledged his error, and the reduced modulus load became firmly established especially after von Karman's [205] use of it in 1910, until 1947. Then Shanley [261-2] pointed out that quasi-static deflection could indeed begin at the tangent modulus value, provided the load were simultaneously increased sufficiently to permit continued plastic loading over the whole cross-section. Von Karman agreed with this and clearly expressed the idea of bifurcation [206]. This curious and interesting history of the theory is summed up in Table 2.1.

	Hypothesis	Argument	Conclusion
Engesser 1889	Wrong (adjacent equilibrium)	Wrong (no reversal)	Right (tangent modulus)
Considère 1895	Wrong (adjacent equilibrium)	Right (strain reversal)	Wrong (reduced modulus)
Shanley 1947	Right (unrestricted bifurcation)	Right (no reversal)	Right (tangent modulus)

Table 2.1 History of plastic column buckling theory

Experimental results had always tended to indicate buckling only slightly above the tangent modulus load, and this was widely used in practice even before 1947. Actually there is now another theoretical refinement, namely the correction for shear stiffening contained in equation (2.7) below, which has still to be assessed experimentally.

Shanley worked with an idealized column equivalent to that in Fig. 2.1(a), and gave (in effect) only the linearized theory, but his papers rejuvenated the subject. Duberg and Wilder [165-6] gave a clear account (for the spring model, and the idealized H-section column) of the spectrum of directions shown in Fig. 2.5(d). They showed in particular that the slope of the arrow emanating from σ_t is proportional to σ_t itself (lateral deflection here means sideways translational - not angular - displacement of the end of the column). This is why the arrows become steeper with buckling stress in Fig. 2.7. Cicala [153] gave a method for calculating the extent of the incipient unloading region in an H-section column, at each starting load between the tangent modulus and reduced modulus values. A number of other authors commented upon some of the features of the incipient behaviour described above, sometimes in the course of nonlinear investigations of the imperfect column (which we discuss later), often in the context of the spring model or the idealized H-section, and always getting their equilibrium equations directly from elementary 'beam bending' arguments - for example see [140-1, 164, 226, 246].

The foregoing results are all obtainable under the tacit assumption that the longitudinal fibres of the column can slide freely along each other during the bending and compression which takes place during buckling - i.e. that any shear stiffening effect which will inhibit such sliding in a real column is negligible. These transverse shear terms can be accounted for convincingly by setting up and exploiting (in the form of equation (4.36) below) a variational characterization of the full three-dimensional equilibrium equations. This general approach to the incipient buckling mechanism in an inelastic column was worked out in detail in 1960 by Hill and Sewell [189-91] and Sewell [260]. It is wide enough to be valid in particular for orthotropic cylindrical columns of arbitrary cross-section and of much wider material properties than were considered hitherto.

For example, it was shown [189, equation (12)] that the correct bifurcation load σ_b is

$$\sigma_b = E_t (\frac{\pi k}{2\ell})^2 \left[\frac{1 + 3\alpha}{1 + \frac{\pi^2}{4}\alpha}\right] \quad (2.7)$$

where

$$\alpha = \frac{1}{(12/\pi^2 - 1)} (\frac{j}{2k})^4 (\frac{k}{\ell})^2 (\frac{4\nu^2 G}{E_t}) , \quad (2.8)$$

$$Ak^2 = \int x_1^2 \, dA \quad , \quad Aj^4 = \int (x_1^2 + x_2^2)^2 \, dA , \quad (2.9)$$

ν is the lateral contraction ratio in a transversely isotropic column, A is cross-sectional area, and x_1 and x_2 are measured in the section from the centroid, with x_1 in the buckling direction. For a circular section $(j/2k)^4 = 1/3$, so that for many shapes of cross-section $\alpha \sim (\frac{k}{\ell})^2 \frac{G}{E_t}$. Since $G \sim E$, we see that for a slender column α is negligible for purely elastic buckling ($E_t = E$), and therefore $\sigma_b = \sigma_e$ from (2.7). Shear stiffening is therefore genuinely negligible in the purely elastic case. When the uniform axial stress in the column goes beyond the yield point, the current shear modulus G can retain its elastic value (but see Fig. 2 of [28A]) while E_t begins to fall. The tangent modulus stress σ_t then underestimates the correct bifurcation load σ_b in (2.7) by a factor which can rise eventually to $12/\pi^2$, i.e. by an error of about 20%. The discrepancy becomes appreciable when G/E_t is of order $(\ell/k)^2$, which may happen in a structural metal with a rate of hardening small compared with the shear modulus. A significant shear stiffening effect is therefore possible in those circumstances, and is greatest for the limiting case of the rigid/plastic solid obtained as $G/E_t \to \infty$. The bifurcation load resulting from (2.7) for this limit is

$$\sigma_b = \frac{12}{\pi^2} E_t (\frac{\pi k}{2\ell})^2 = 3 E_t (\frac{k}{\ell})^2 . \tag{2.10}$$

This has also been obtained [15] by treating the rigid/plastic solid per se, with a Mises-type yield surface normal. In so far as this 20% increase in bifurcation stress has never been commented upon by an experimenter, the reason may lie in a lowering of the shear modulus G associated with the development of a corner on the plastic yield surface (again see Hutchinson's Fig. 2 in [28A]).

The fundamental buckling mode when α is not negligible is no longer sinusoidal, but for the built-in axially loaded column of Fig. 2.1(b) it is adequately represented [189, equation (11)] by

$$f(z) = f(1) [1 - \cos \frac{\pi z}{2} + \frac{3\pi}{4} \alpha z^2] , \tag{2.11}$$

where $z = x_3/\ell$ is the non-dimensional axial coordinate. The quadratic limiting form of this mode for large α is consistent with the known restricted set of displacement fields (4.20) possible in a uniaxially stressed rigid/plastic solid [45, equations (15)] at a smooth yield surface point.

The new deflecting equilibrium path which can begin from the modified critical load (2.7) will involve a combination of the lateral deflection (2.11) together with a continuing uniaxial compression. The net effect will be that unloading is confined to the extreme point on the convex side of the bottom end if the ratio of downward/lateral deflection of the top end is

$$\frac{\text{compressive axial displacement}}{\text{lateral deflection}} = \frac{\pi^2}{4} (1 + \frac{6\alpha}{\pi}) \frac{a}{\ell} \tag{2.12}$$

where a is here the greatest distance of the perimeter from the weaker section (x_2-) axis [189, equation (15)]. When shear-stiffening is negligible ($\alpha \approx 0$), the direction given by (2.12) corresponds to the arrow emanating from σ_t in the load/lateral-deflection plots of Figs. 2.4(c) and 2.5(d). Actually the linearized theory alone will permit a whole fan of possible directions (not shown here) to emanate from the single load (2.7), all involving no unloading within the column, the limiting member of which is (2.12) - this has the smallest component of axial compression consistent with the requirement that no unloading shall begin within the column at that load. But it can be shown [46, p.258] that only this limiting member is compatible with a finite deflection theory, the other members of the fan being spurious effects introduced by the linearization - which is why we do not show them.

The three-dimensional theory of the long column with arbitrary uniform shape of cross-section has also been refined by Hill and Sewell [191] to the point where the location of the interface between the regions of incipient loading and unloading within the column can be determined at the start of buckling from the straight position, at each load in the range displayed in Fig. 2.4(c) (or their values adjusted to allow for sheer stiffening). Even though these situations can never be precisely realized in a 'test', it is of definite methodological interest to see how the interface is found in this relatively simple case of a straight configuration. For the only 'buckling load' which an engineer will recognize is the maximum load occurring at nonzero deflection during the test of an imperfect column - and we shall see that the interface in that problem is expected to lie within the column when such a maximum load is reached. Therefore, one hopes that the eventual general theory which is still needed for the latter problem may be compared by the rather complete knowledge now available about the interface location for incipient departure from the <u>straight</u> configuration at loads above the tangent modulus value. The actual buckling mode from this straight position is sought from among the following class [191, equation (3.1)] of quasi-static velocity fields:

$$v_1 = f + \nu\left[\frac{x_1^2 - x_2^2}{2\ell^2} f'' + \frac{x_1}{\ell} h'\right] \tag{2.13}$$

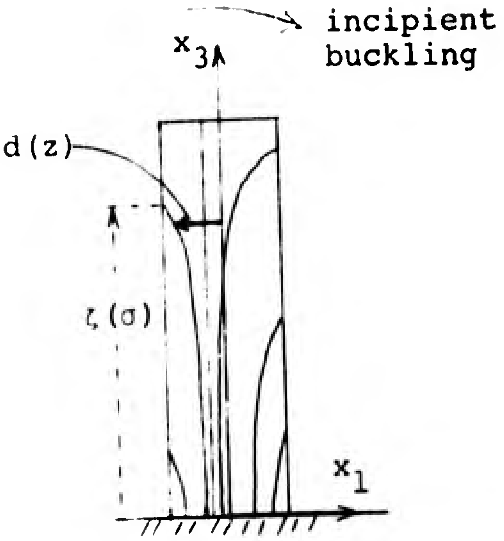
$$v_2 = \nu \frac{x_2}{\ell}\left[\frac{x_1}{\ell} f'' + h'\right] \tag{2.14}$$

$$v_3 = -\left[\frac{x_1}{\ell} f' + h\right] \tag{2.15}$$

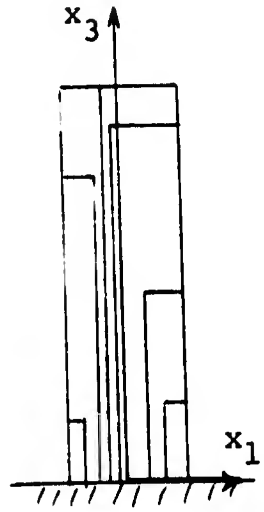
where $f(z)$ and $h(z)$ are functions of $z = x_3/\ell$ to be determined subject to the conditions of end-fixity $f(0) = \dot{f}'(0) = h(0) = 0$ implied by Fig. 2.1(b), and $f' \equiv df/dz$, etc. The incipient deformation in any such field is a combination of axial compression and lateral expansion (non-uniform if $h'' \neq 0$) with bending in which plane sections remain plane and perpendicular to the deflecting line of centroids. This undergoes a sideways motion $f(z)$ in the x_1 x_3 plane of symmetry, so that the local rate of curvature is f''/ℓ^2.

The loading/unloading interface at the start of buckling is where the axial strain rate $\partial v_3/\partial x_3$ is zero, i.e. at

$$x_1 = -d(z) \text{ where } d(z) = \ell \frac{h'(z)}{f''(z)} . \tag{2.16}$$



(a) Exact



(b) Approximate - ref. [191]

Fig. 2.8 Schematic profiles of incipient loading/unloading interface

Let $z = \zeta(\sigma)$ be the height at which this interface profile passes outside the column. Several profile locations for different bifurcation stresses σ are shown in Fig. 2.8(a), where buckling is assumed to be beginning to the right, and the unloading region is to the left of each profile. The point where the profile leaves the column passes monotonically up its left side from $\zeta(\sigma_t) = 0$ to $\zeta(\sigma_r) = \ell$, and then down the right side to $\zeta(\sigma_e) = 0$, as the bifurcation stress is imagined to increase through the range displayed in Fig. 2.4(c) or 2.5(d). Equations have been formulated [191, (3.7) and (3.8)], by an orthogonalizing overall approximate method having some affinity with Galerkin/ Kantorovich/weighted residual type methods, which should describe these properties even in the presence of shear stiffening, but no attempt has been made to solve them in that general case. When shear is negligible (for G/E_t small compared with ℓ^2/k^2) the two equilibrium equations have been cast into a form suitable for computer solution in terms of the two unknown functions f(z) and d(z). They were not solved in that way, because an analytic approximate solution was devised [191, §5] which has the effect of replacing the exact loading/unloading interface profiles sketched in Fig. 2.8(a) by the approximate interfaces shown in Fig. 2.8(b). These involve the assumption of constant d over a determinable length of the column, which is certainly reasonable when σ is near σ_r (and is exact at $\sigma = \sigma_r$) because the actual interface then only curves appreciably just before it leaves the column; and near σ_t (or σ_e) the contribution from the unloading (loading) region is small and the profile is of little consequence. Detailed calculations of this type were carried out for a column of rectangular section and for the idealized H-section. A closed form solution was obtained for the unapproximated equations for any shape of section in the case when the unloading elastic modulus is infinite. These remarks show that the method is capable of explicit application. The main features of incipient bifurcation behaviour in a perfect column can evidently by regarded as established with reasonable rigour and generality.

Of particular interest is the stress at which the column can first deflect quasi-statically under dead load, i.e. such that the incipient direction displayed in Figs. 2.4(c) or 2.5(d) necessarily be precisely horizontal. This has interest in situations where the applied load has no means of changing its value as deflection begins, as with gravity loading or an infinitely flexible testing machine. (The circumstances are different for an elastic testing machine, which might permit the equilibrium value of applied load on the column to fall with lateral deflection, or for a rigid testing device which prescribes the axial deflection of the end of the column). The condition for zero change of load across an arbitrary cross-section is, to first order,

$$\int \frac{d\sigma}{de} \frac{\partial v_3}{\partial x_3} dA = 0 \; . \tag{2.17}$$

Using (2.15), (2.16) and the appropriate tangent modulus from (2.4) reduces this to a condition determining a unique value d_0 of d:

$$E_t \int_{x_1+d_0>0} (x_1 + d_0) \, dA + E \int_{x_1+d_0<0} (x_1 + d_0) \, dA = 0 \; . \tag{2.18}$$

This verifies that the location $x_1 = -d_0$ of the loading/unloading interface in this case is the same <u>all</u> along the column (as Fig. 2.8 depicts). If we define a 'weighted modulus' $\bar{E}(d)$ by

$$Ak^2\bar{E} = E_t \int_{x_1+d>0} (x_1 + d)^2 \, dA + E \int_{x_1+d<0} (x_1 + d)^2 \, dA \; , \tag{2.19}$$

it may be shown [190, §4] that the 'reduced modulus' E_r is the absolute minimum of $\bar{E}$ with respect to values of d within the section, and occurs at d_0. That is,

$$E_r = \bar{E}(d_0) = \operatorname*{Min}_{d} \bar{E}(d) \tag{2.20}$$

$$= \frac{1}{Ak^2}[E_t \int_{x_1>-d_0} x_1(x_1 + d_0) \, dA + E \int_{x_1<-d_0} x_1(x_1 + d_0) \, dA] .$$

From the moment equilibrium equation the stress which permits such a deflection to begin under dead load is the 'reduced modulus stress' (2.3). This does not take account of shear stiffening, but a modified formula to allow for this effect along the lines of (2.7) has been suggested [190, equation (22)], which is in line with what is known about the rigid/plastic limit when treated on its merits.

That completes our summary of what is known about the incipient quasi-static deflections from the straight configuration of a perfect column whose stress is in the plastic range.

(iii) Column stability.

We have deferred general comments on stability until Section 5. Here it suffices to make the point that stability of equilibrium is essentially a dynamical subject, involving a discussion of all the motions which can ensue from any small disturbance (suitably defined) of the considered equilibrium. The equilibrium may be regarded as stable if the maximum amplitude of every such motion tends to zero when the disturbance itself does.

This is conceptually distinct from the purely static concept of the bifurcation of equilibrium paths, and the dynamical criterion is hard to apply in plastic bodies because the sequence of the different loading and unloading responses in each material element will vary with each considered motion, and be unknown in advance. Nevertheless a viable criterion can be given (equation (5.6)) according to which the equilibrium may reasonably be described as at least quasi-stable.

This has been applied [190, 260] to show that the perfect cylindrical column of arbitrary cross-section under dead loading is (quasi-)stable below the reduced modulus load when shear stiffening is negligible. When shear stiffening is included the stress σ_s terminating the range of (quasi-)stability becomes [190, equation (17)]

$$\sigma_s = E_r \left(\frac{\pi k}{2\ell}\right)^2 \left[\frac{1 + 3\,\alpha_0}{1 + \frac{\pi^2}{4}\,\alpha_0}\right], \tag{2.21}$$

where

$$\alpha_0 = \frac{1}{(12/\pi^2-1)} \left(\frac{j_0}{2k}\right)^4 \left(\frac{k}{\ell}\right)^2 \left(\frac{4\,\nu^2\,G}{E_r}\right), \tag{2.22}$$

$$A j_0^4 = \int (x_1^2 + x_2^2)\,[(x_1 + 2\,d_0)^2 + x_2^2]\,dA\,. \tag{2.23}$$

The limit of (2.21) as $G/E_t \to \infty$ can be shown [190] to coincide with the value obtained by treating the rigid/plastic solid on its merits [18], namely

$$\sigma_s = 3\,E_t\,\frac{(d_0^2 + k^2)}{\ell^2}, \tag{2.24}$$

d_0 here being the distance between the x_2-axis and the parallel tangent at the convex side (cf. (2.24) with (2.10)). Instability of the straight equilibrium position under dead load is implied above these values σ_s.

We observe that some authors [126, 128-9, 220, 276] have studied the stability of inelastic columns by directly exploring solutions of the equations of motion near the equilibrium point, in varying degrees of generality. In particular Lee [220] addresses himself to the path-dependence difficulty.

It will be noticed that the critical stress σ_s for stability in the dynamical sense of the inelastic column under dead load turns out to have the same value as the stress which admits adjacent equilibrium positions - at least when shear is negligible, the common value then being σ_r. Put otherwise, the stability or instability in the dynamical sense of the straight equilibrium configurations can be inferred with reasonable certainty from the purely static information provided by the tangents to the incipient load/lateral deflection equilibrium paths shown in Figs. 2.4 and 2.5. Stable states are those for which the tangents slope upwards, and unstable states are those for which the tangents slope downwards. In practice such a stability criterion is widely used intuitively - witness the idea than stability will fail when maximum load is reached on the equilibrium path of an imperfect system, whether the loaded body is elastic or inelastic. But it is well to be aware that dynamical conclusions are being inferred from purely statical data and such a procedure needs to be backed by concrete arguments such as those indicated in Section 5.

If the applied load is not a dead load, but is applied via an elastic or rigid loading device whose 'load/deflection characteristic' is inclined (downwards to the right in Figs. 2.4 and 2.5) instead of being horizontal (as for a dead load), it would be interesting to know in what generality one can prove that stability (instability) of the system will occur when the (algebraic) slope of the equilibrium path for the body is locally greater (less) than that of the characteristic of the loading device.

Some of the analytic ideas are outlined by Sewell [48, §7] in an elementary context of elastic structures. A prerequisite is to check that changing the elasticity of the loading device will not change the form of the equilibrium path for the body. Introducing stiffness into a loading device can extend the range of stability on a given equilibrium path.

(iv) Finite deflection of perfect and imperfect column models.

In subsection 2(ii) we described what is known about the starting tangents to load/deflection equilibrium paths for a perfect column in the straight configuration. Now we wish to consider how such paths actually evolve as the lateral deflection increases, at least as far as the point where the load reaches its maximum (when this exists). We must consider both perfect and imperfect columns, since it is the load maximum occurring for the latter which will be of more practical interest.

There is a certain rather patchy accumulation of knowledge in the literature, which is sometimes hard to follow, and has not reached anything like the same degree of generality or completeness as the incipient buckling theory. We pick out some principal ideas.

In this subsection we consider the spring model of Fig. 2.1, and in the next subsection the real column. The equilibrium of a finitely deflected position of the model is expressed by

$$\left.\begin{aligned} F_2 + F_1 &= P \quad , \\ F_2 - F_1 &= P \frac{\ell}{a} \tan \theta \quad . \end{aligned}\right\} \qquad (2.25)$$

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The continuing of this equilibrium for incrementally different values of load and displacement is expressed by

$$\dot{F}_2 + \dot{F}_1 = \dot{P} \tag{2.26}$$

$$\dot{F}_2 - \dot{F}_1 = \dot{P}\,\frac{\ell}{a}\tan\theta + P\,\frac{\ell}{a}\sec^2\theta\cdot\dot{\theta} \quad .$$

The dots signify small increments, or more precisely 'quasi-static rates' with respect to any parameter ϵ (say) which can provide a regular parametrization of the sought-for equilibrium paths. (The actual choice of this parameter may best be left until later [46] - the choice $\epsilon = \theta$ itself is used in the paths described by (2.28) and (2.29) below). The downward displacements, from a symmetrical configuration, of the points of application of the spring forces are

$$\left.\begin{matrix} e_2 \\ e_1 \end{matrix}\right\} = h \pm a \sin\theta \ . \tag{2.27}$$

These equations (2.25-7) are valid for genuinely <u>finite</u> deflection, and for any spring laws. In particular these equations apply to the <u>imperfect</u> as well as the perfect system - because the subsequently specified spring laws $F = F(e)$ could have a datum such that $F_1(0) \neq F_2(0)$ in the imperfect case.

For the perfect column whose spring laws are the two-segment idealization of Fig. 2.2, Sewell [46] obtained the equilibrium paths shown in Fig. 2.9. They are described by the exact formula

$$P = \frac{P_0(E-E_t) + 4\,E\,E_t\,a\sin\theta}{(E-E_t) + (E+E_t)\,\frac{\ell}{a}\tan\theta} \tag{2.28}$$

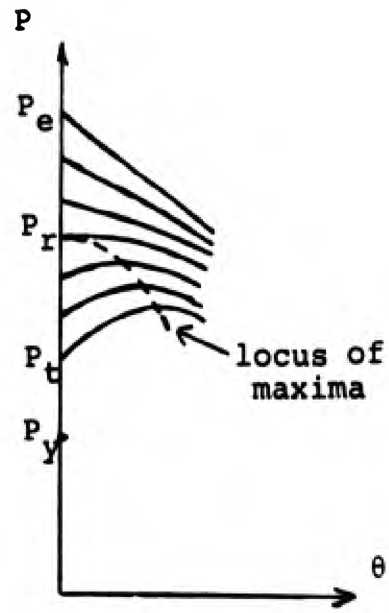

Fig. 2.9 Load/finite deflection equilibrium paths for perfect column model

valid beyond the maxima shown, and involving continuing loading in the concave spring F_2 and continuing unloading in the convex spring F_1. The starting load P_0 to reach a deflected position necessarily lies somewhere between the tangent modulus value $P_t = 2\ E_t\ a^2/\ell$ and the Euler value $P_e = 2\ E\ a^2/\ell$. The axial displacement beyond the starting value h_0 is also found to be given by an exact formula

$$h-h_0 = \frac{a \sin\theta\ [(E+E_t) + (E-E_t)\ \frac{\ell}{a}\tan\theta] - P_0\ \frac{\ell}{a}\tan\theta}{(E-E_t) + (E+E_t)\ \frac{\ell}{a}\tan\theta} \ . \tag{2.29}$$

The extent of the unloading region is specified by the horizontal distance $\lambda(\theta)$ a cos θ to the instantaneous centre (see Fig. 2.1(a)) defined by

$$\lambda(\theta) = \frac{1}{a\cos\theta}\ \frac{dh}{d\theta} \ . \tag{2.30}$$

In terms of this the slope of the paths shown in Fig. 2.9 are

$$\frac{dP}{d\theta} = a(E+E_t)\cos\theta\left[\lambda(\theta) - \frac{E-E_t}{E+E_t}\right] . \qquad (2.31)$$

The advantage of these exact formulae (2.28)-(2.31), valid for large deflection and any ℓ/a ratio, is that they provide a quick tangible way of seeing the small deflection theory in perspective. For example, the spectrum of directions displayed in Fig. 2.5(d) is consistent with Fig. 2.9, and may be obtained for this model from (2.31) evaluated for $\theta = 0$. In fact, it follows that one equilibrium path of type (2.28) with (2.29), and for which the location $\lambda(\theta)$ of the instantaneous centre is in the range

$$-1 \le \lambda(\theta) \le 1 \qquad (2.32)$$

between the springs, can start at each load

$$P_0 = \frac{1}{2}\left[(P_e+P_t) - (P_e-P_t)\,\lambda(0)\right] \qquad (2.33)$$

between P_t and P_e, corresponding to $\lambda(0)$ given the values in (2.32) in turn.

The load/deflection curves are stationary where $dP/d\theta = 0$, and these points can be shown [46] to exist and be maxima for paths starting in $P_t \le P_0 \le P_r$, where $P_r = 2\,P_e\,P_t/(P_e+P_t)$ is the reduced modulus load for this model. These maximum values decrease below P_r as P_0 drops below P_r, but are always above P_t. Notice, however, that when the maximum load occurs the location within the column of the loading/unloading interface or instantaneous centre is (from (2.31)) at a position relative to the column given by

$$\lambda(\theta) = \frac{E-E_t}{E+E_t} \; . \tag{2.34}$$

(see Fig. 2.1(a)). This is actually independent of θ and the same for all paths starting between P_t and P_r - in particular it is the same location relative to the column as in the case of incipient deflection of a straight column to an adjacent equilibrium position at reduced modulus load.

The solutions described here cease to apply when the value of $\lambda(\theta)$ given by (2.30) passes below -1, which it ultimately does. ($\lambda(\theta)$ actually decreases monotonically when $P_0 < P_r$). This corresponds to the interface passing out of the column on the concave side, so that elastic unloading then applies everywhere.

Naturally this information for the spring model can only be a qualitative indication of what happens in a real column. However, it is worth noting that the decrease of tangent modulus in the loading fibres of the real column seems to contribute significantly to the observed occurrence of maximum load at very small deflection values, and in the model we achieve a weakening effect and maximum load differently by allowing the distance between the springs to decrease with cos θ. This means that with linear springs the model could not describe the rising load/deflection curve of the elastica, but nevertheless it may be a more reliable indicator of the plastic postbuckling behaviour associated with a real falling tangent modulus. If linearization of the θ terms is introduced too early in the analysis while

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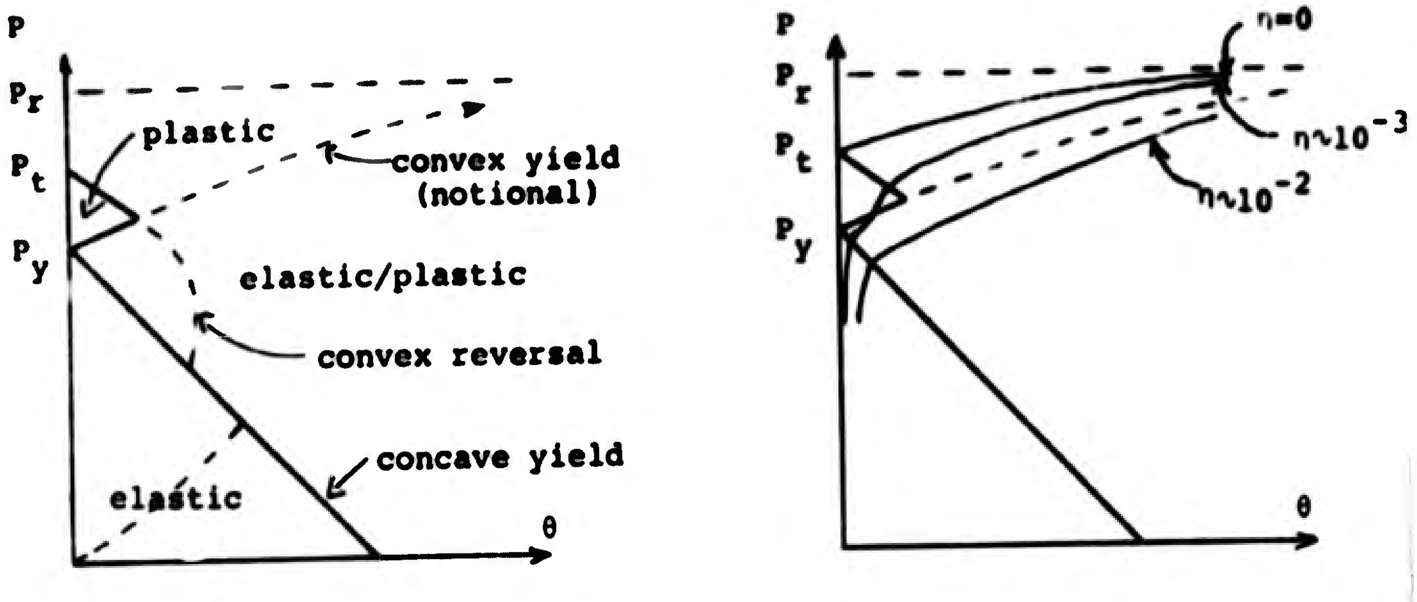
retaining a constant tangent modulus, the genuine maximum will be lost - in the sense that the load/deflection curves will all appear to approach the reduced modulus load at infinite deflection.

Continuity suggests that when an imperfection is present the equilibrium path will be contained within the region $\theta > 0$ in Fig. 2.9 and below the path branching out from P_t, with a maximum load which may either be above or below the tangent modulus load P_t, depending on the size of the imperfection. Detailed investigation reveals a number of different loading/unloading sequences by which equilibrium paths for imperfect columns may evolve. It would be of interest to know how much general local information about such paths can be obtained from (2.26) without precise specification of the imperfection.

Direct analysis of the spring model for prescribed initial deflection by Duberg and Wilder [165-6, Fig. 4] revealed equilibrium paths of the type shown in Fig. 2.10. These paths all tend to the reduced modulus load because the early linearization mentioned above was used, with constant tangent modulus. The significant feature of these results is the existence of the three regimes: (a) elastic, in which neither spring has yet yielded; (b) elastic/plastic, in which the spring on the concave side has yielded and continues to load, while the convex spring has either not yielded at all, or (if the imperfection is small enough) has yielded and then reversed; (c) plastic, a relatively small regime bounded by a closed curve joining P_t to P_y (the yield load for the perfect column) in which for small enough



(a) Elastic, elastic-plastic and plastic regimes separated by solid lines

(b) Equilibrium paths with $\frac{\text{imperfection}}{\text{column width}} \equiv \eta = 0, 10^{-3}, 10^{-}$

Fig. 2.10 Duberg and Wilder spring model results

imperfections both springs have yielded and are continuing to load. It would be of interest to know the results of an imperfect spring model analysis with a stress-strain curve having <u>decreasing</u> tangent modulus.

The equilibrium paths of Fig. 2.10(b) have the defects that the load maxima are not reached at a determinable finite deflection value (for the reasons already indicated). This defect has been incorporated in design manuals [203, Fig. 2.3] via the concept of 'ultimate load'. The consequent eventual unloading of <u>both</u> springs predicted by the analysis associated with Fig. 2.9 cannot emerge either. As a basis for discussion of the mechanism of plastic column buckling in the presence of imperfections we put forward the conjectured regimes and equilibrium paths shown in Fig. 2.11. At <u>very</u> small deflection

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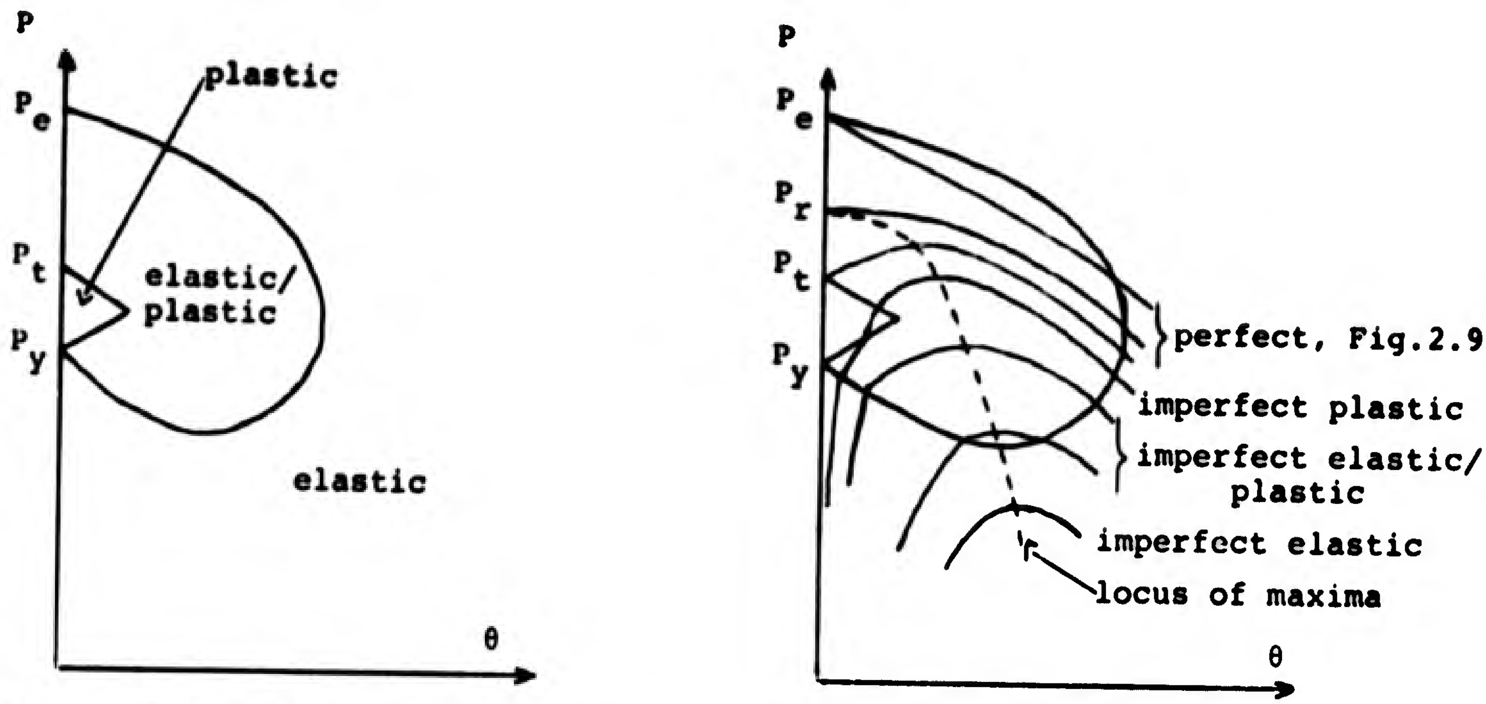

(a) Conjectured regimes

(b) Conjectured imperfect equilibrium paths

Fig. 2.11 Schematic basis for discussion of column buckling mechanism

this agrees with Fig. 2.10, but at larger deflections all the paths eventually involve unloading in both springs (as in Fig. 2.9). Load maxima are reached for imperfect columns, as experiment requires, and if the imperfection is large enough the load maximum is reached before either spring yields (even though the corresponding perfect column would yield below the Euler load).

This last feature is present in a (different) column model analyzed by Hutchinson [537]. His model contains an extra and lateral spring exerting a force $\beta\theta^2$ on the top end perpendicular to P in Fig. 2.1(a). This induces asymmetry and consequently a stronger imperfection-sensitivity than is present in our case (Hutchinson's main purpose is to use this as an introduction to the study of the equilibrium paths and plastic buckling of an imperfect spherical shell under external pressure). If the modulus β is large enough this nonlinearity alone is enough to induce load maxima. The relation between load maximum P^{max} (normalized by the maximum P_0^{max} on the 'perfect path' starting from P_t) and imperfection $\bar{\theta}$ is found [537, Fig. 4] to be of the general type shown in Fig. 2.12. This diagram loosely corresponds to the locus of maxima shown in Fig. 2.11(b). In particular it contains the region of elastic buckling under large imperfections which we mentioned above. It also has the significant feature of a half-parabolic cusp with vertical

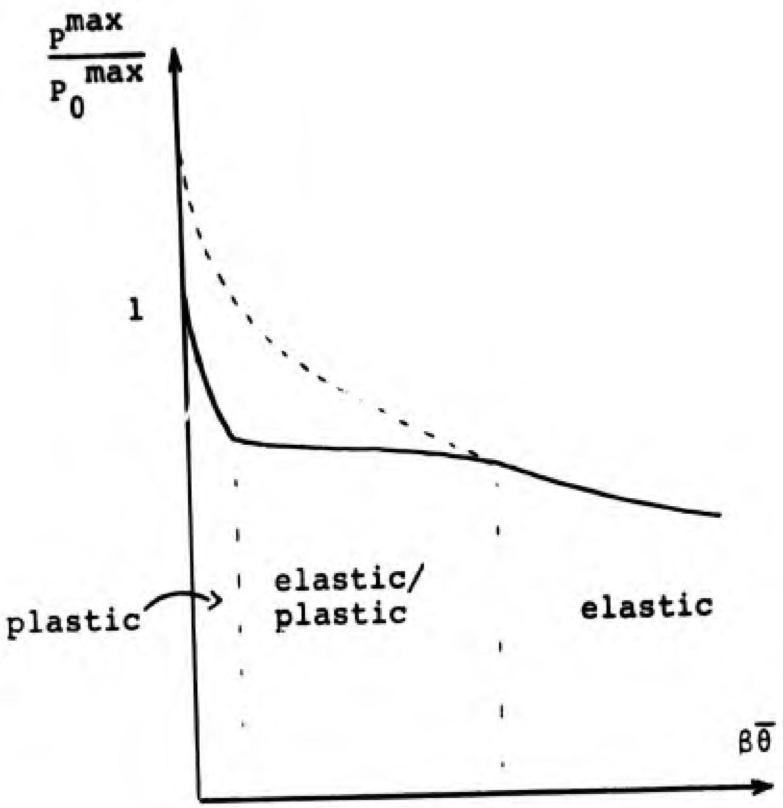

Fig. 2.12 Imperfection sensitivity of load maximum - after Hutchinson [537]

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tangent as $\beta\bar{\delta} \to 0$. This quantifies the idea of high imperfection sensitivity, somewhat in the spirit of Koiter's treatment of the effect of imperfections on elastic bifurcation points [28B; 29, Fig. 6], except that here it is a load maximum or 'limit point' and not a bifurcation point which is associated with the point of the cusp.

However, there is nothing approaching an adequate general theory of the effect of imperfections in plastic buckling. Even for the spring model there are no adequate precise details of the shapes of the regimes conjectured in Fig. 2.11. For example, it is no more than speculation (informed as described) to suggest that there is a single continuous curve joining P_y to P_e (instead of separate curves leaving P_y and P_e which, perhaps, might not join up). It would be of interest to know, from genuine nonlinear theory, to what extent such regimes do exist and have determinable shapes independent of the size (and if possible the type) of the imperfections. Such regimes of the load/deflection space are defined here in the context of those specific loading histories which generate monotonically evolving equilibrium paths, and not in the context of arbitrary loading histories.

(v) Deflection of imperfect columns with distributed flexibility.

The approximate equations usually used to express the equilibrium of a slightly deflected equilibrium position of the 'real' column in Fig. 2.1(b) are

$$\left.\begin{aligned} &\int \sigma(e)\, dA = P \quad, \\ &\int x_1\, \sigma(e)\, dA = P(y(1) - y(z)) \;. \end{aligned}\right\} \tag{2.35}$$

We suppose all cross-sections to be of the same shape, which is arbitrary. Buckling is supposed to be in the direction of the x_1-principal axis. The deflection of the line of centroids is $y(z)$ $(z = x_3/\ell)$ and x_1 is measured from this line. The inclination of the compressive fibre force $\sigma\, dA$ is assumed negligible in evaluating its moment in $(2.35)_2$. In expressing (2.35) nothing need be said about how the deflected state was reached, and in particular nothing need be said about whether the column is perfect or imperfect.

The same generality applies to the equations of continuing equilibrium (cf.(2.26))

$$\int E_t\, \dot{e}\, dA = \dot{P} \quad, \tag{2.36}$$

$$\int x_1\, E_t\, \dot{e}\, dA = \dot{P}(y(1) - y(z)) + P(\dot{y}(1) - \dot{y}(z)) \;.$$

As before the dots signify small increments $\dot{P}\delta\epsilon$, $\dot{e}\delta\epsilon$, $\dot{y}\delta\epsilon$ associated with 'quasi-static rates' with respect to any parameter ϵ which serves to order the deformation. The tangent modulus in (2.36) is given by (2.4) and the subsequent remarks, i.e.,

$$E_t = \begin{cases} E_t(\sigma) = \dfrac{d\sigma}{de} & \text{if } \dot{e} > 0 \\ E & \text{if } \dot{e} < 0 \end{cases} \tag{2.37}$$

where the stress-strain curve (2.5) may be that of either

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Fig. 2.2 or Fig. 2.3. In the latter case $E_t(\sigma)$ may vary across the loading part of the section. By an extension of (2.16) to the slightly bent configuration for which (2.36) applies we assume the compressive fibre strain-rate to be

$$\dot{e} = (x_1 + d(z)) \frac{\dot{y}''}{\ell^2} \tag{2.38}$$

so that the loading/unloading interface is again given by $x_1 = -d(z)$. The incremental equations of equilibrium now become

$$\left[\int x_1 E_t \, dA + d(z) \int E_t \, dA\right] \frac{\dot{y}''}{\ell^2} = \dot{P} \ , \tag{2.39}$$

$$\left[\int x_1^2 E_t \, dA + d(z) \int x_1 E_t \, dA\right] \frac{\dot{y}''}{\ell^2} = \dot{P}(y(1) - y(z)) + P(\dot{y}(1) - \dot{y}(z)) \ . \tag{2.40}$$

These would furnish the incipient behaviour from the straight position, described in subsection 2(ii), by putting $y(z) \equiv 0$, $\dot{y}(z) = f(z)$, and treating E_t as constant over the loading part $x_1 \geq -d(z)$ of the cross-section.

They can also be used as the basis for numerical determination of load/deflection paths in columns with initial imperfections of assumed form. In view of the small deflection assumptions, due allowance should be made for the variation of E_t along the length of each fibre, and over the loading part of the cross-section, if we are to avoid 'losing' the maximum load. Lin carried out such a calculation [33, 226] for a rectangular steel column having slenderness ratio $\ell/k = 75$, linearly decreasing E_t after yield, and imperfection magnitude $\sim 10^{-3}k$. He found an equilibrium path agreeing with an experimental curve and arriving at a maximum load when the sideways deflection reached $\sim 10^{-2}k$. We reproduce these results in Fig. 2.13. Lin's calculation used the device of the 'modified section', according to which the origin of coordinates in a cross-section is chosen not at the centroid as we have done, but at a point such that the variation of E_t over the section always makes $\int x_1 E_t \, dA = 0$. This point is therefore different from section to section (because E_t is so). The advantage is the formal simplication achieved in (2.39) and (2.40). The calculation was not carried beyond the maximum.

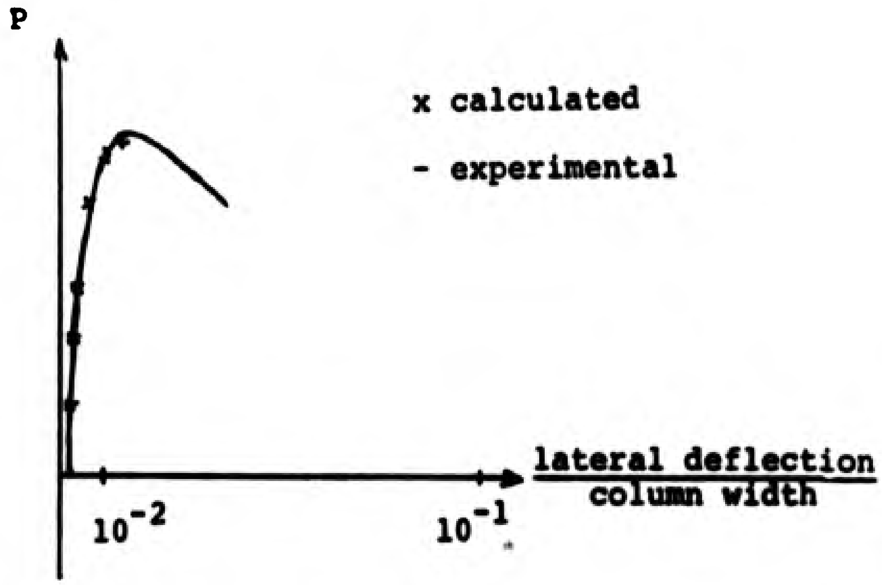


Fig. 2.13 Lin's equilibrium path for an imperfect steel column

Another direct numerical investigation of the equilibrium paths for an imperfect (and perfect) column of rectangular section was carried out by Malvick and Lee [228], this time for an aluminimum alloy column with $\ell/k \sim 55$. They introduced an imperfection consisting of an assumed initial lateral deflection of the centre line having two terms

$$y_0 = a_1 \sin \frac{\pi x_3}{\ell} + a_3 \sin \frac{3\pi\, x_3}{\ell} \qquad (2.41)$$

(for the pin-ended column of length ℓ and origin of x_3 at one end). This contains not only the fundamental eigenfunction (with amplitude a_1) of the eigenvalue problem associated with the bifurcation problem for the perfect column, but also the next higher mode with coefficient a_3 (excluded by Lin). The equilibrium load/midpoint deflection paths calculated by Malvick and Lee are shown in Fig. 2.14 (where values of a_1 and a_3 are normalized by the column width). The bifurcation point found by the computer for the perfect column $(a_1, a_3) = (0, 0)$ 'agrees exactly with that given by the tangent modulus formula'. The maximum load for columns $(a_1, 0)$ with only the fundamental imperfection was found to be sensitive to the amplitude a_1 of that imperfection. It decreased as a_1 increased and for the

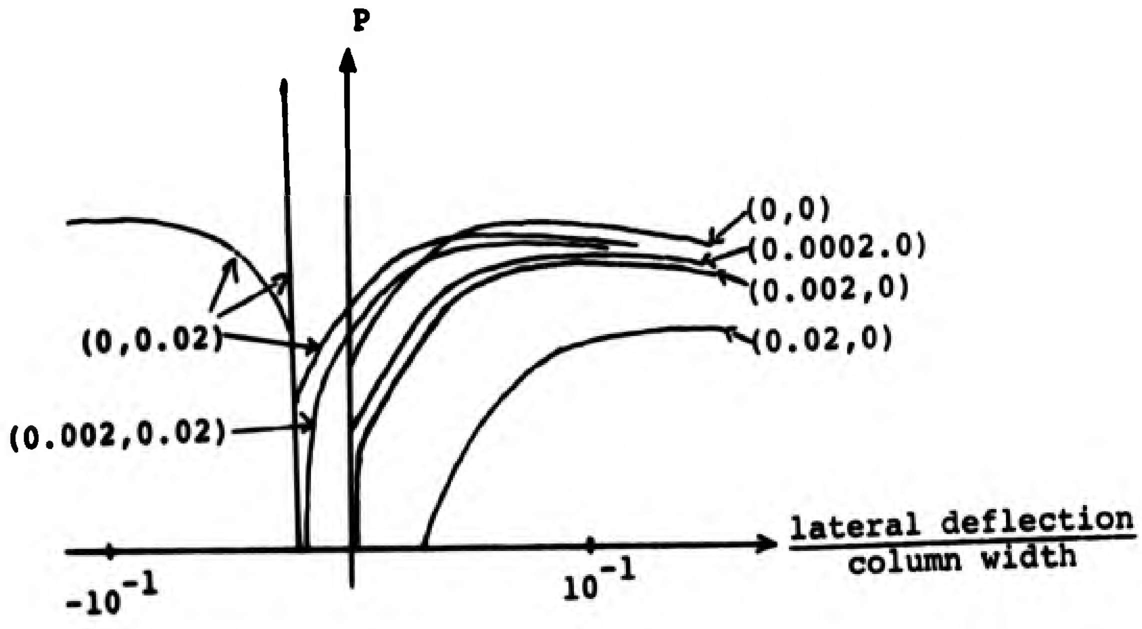
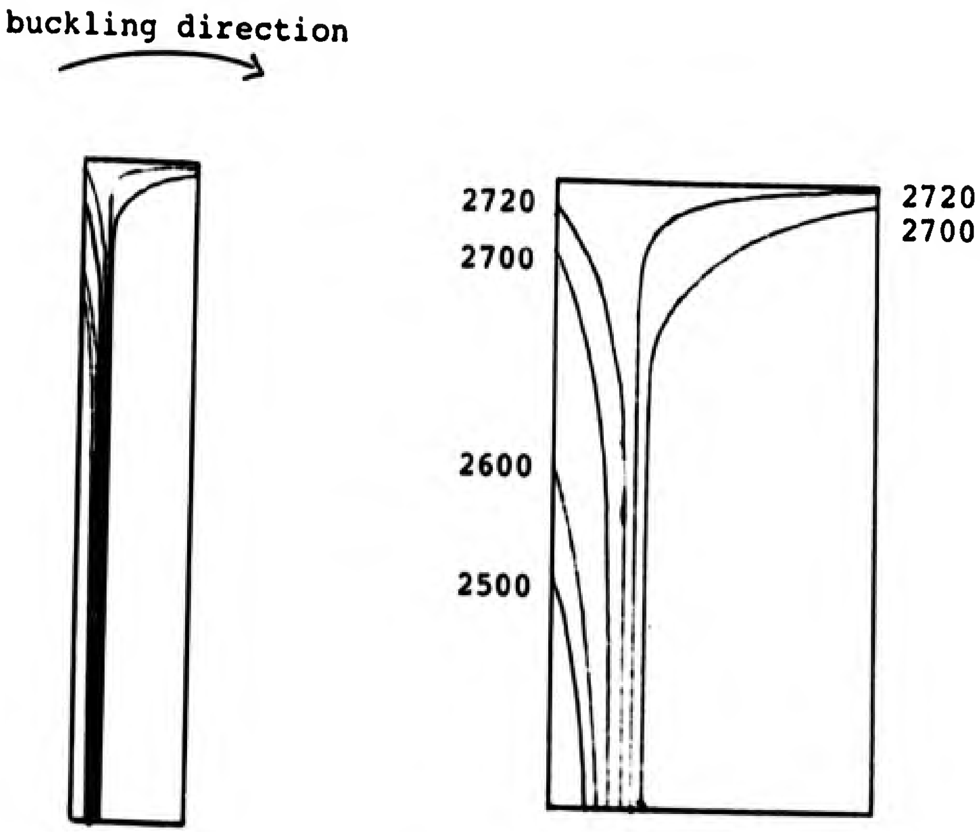


Fig. 2.14 Load/deflection equilibrium paths of Malvick and Lee [228]

perfect column (0, 0) it was below the reduced modulus load. These features are in general agreement with what was predicted by the spring model analyses described in the previous subsection. (Malvick and Lee comment that for deflections beyond those shown in Fig. 2.14, the Bauschinger effect and a nonlinear stress/strain relation in tension should be considered.)

Less expected is the indicated effect of the higher a_3 mode, which even when the fundamental imperfection is absent ($a_1 = 0$), can induce a load maximum slightly lower than that for the perfect column. The wide separation between the first and second Euler loads in the elastic column problem (as distinct from shell problems) indicates that interaction between the associated modes can be safely ignored, at least for small deflections. Perhaps this assumption should be viewed more critically in the plastic column problem. In fact Malvick and Lee attribute the similarity of their load/deflection curves to the development of strain reversal dominating the rate of lateral deflection.

They calculate the loading/unloading interface as the deflection proceeds, for the perfect column (0, 0) and also for the column (0, 0.02) with an imperfection in the higher but not the fundamental mode. Their results for the perfect column are displayed in Fig. 2.15. (The separations are a little wider for the imperfect column). These calculations of Malvick and Lee seem to provide good confirmation of the approximate method proposed by Hill and Sewell [191] and summed up in Fig. 2.8(b) - which is strikingly similar to Fig. 2.15. Malvick and Lee do



(a) Whole column (b) Enlargement of top third, load values in lbs.

Fig. 2.15 Motion of loading/unloading interface with deflection as calculated by Malvick and Lee [228] for perfect column

not refer to this paper [191].

The column with idealized H-section has its flexibility concentrated in two flanges $x_1 = \pm a$ which run along the whole length of the column, and is often used as an intermediate simplification of the real column which can permit longitudinal variations ruled out by the spring model. In the present problem it permits the lateral variation of E_t to be omitted (since the integrals in (2.35) and (2.36) reduce to a sum of two terms from $x_1 = \pm a$), but the effect of the longitudinal variation of E_t, and in particular the longitudinal spread of the loading/unloading interface, can be examined. Wilder, Brooks and Mathauser [277] calculated load/lateral deflection curves for such a column with assumed imperfections in the fundamental mode form. They used the Ramberg-Osgood stress-strain curve, with various values of curvature at the knee in Fig. 2.3, and enforced the equilibrium equations (2.35) only at the midpoint of their pin-ended column (equivalent to the bottom section in Fig. 2.1(b)). They found load maxima at finite deflection values, the latter becoming large when $P_t \to P_e$ (other parameters fixed). The maximum load was always less than that found for the perfect column (computed by Duberg and Wilder [166], and found to be lower for a sharper knee), and it could be greater or smaller than the tangent modulus load, depending upon the column proportions, the magnitude of the initial curvatures, and the shape of the stress-strain curve.

Evidently any apparent agreement found between the maximum load obtained from an experimental test, and the tangent modulus load which is the bifurcation load of a theoretical perfect column, should be regarded as fortuitous from a strict viewpoint.

We comment now on the kind of deductions which may be made from (2.39) and (2.40) about the local properties of equilibrium paths, which will be valid (as we said) without needing to specify anything about the imperfection nor how the considered equilibrium state was arrived at. (This is also the underlying viewpoint of the general theory of Sections 3 and 4).

Hoff [194] has made some such deductions for the idealized H-section by applying (2.39) and (2.40) only to the section of greatest curvature, and with an assumed deflection form. Here this corresponds to using $y(z) = y(1)[1 - \cos \pi z/2]$ and $\dot{y}(z) = \dot{y}(1)[1 - \cos \pi z/2]$ at $z = 0$. For an <u>arbitrary</u> shape of section this gives

$$\frac{\pi^2}{4\ell^2} \int E_t(x_1 + d(0))\, dA = \frac{dP}{dy} \quad , \tag{2.42}$$

$$\frac{\pi^2}{4\ell^2} \int E_t(x_1 + d(0))^2\, dA - P = \frac{dP}{dy}\,(y(1) + d(0)) \quad , \tag{2.43}$$

where $dP/dy = \dot{P}/\dot{y}(1)$ is the local slope of the equilibrium path.

This slope is zero, implying at least stationary load, when $d(0)$ takes a value d_0^* (say) which makes the left side of (2.42) zero, making due allowance for the possible variation of E_t across the section. This result, for a slightly bent configuration, generalizes (2.18) which applies to the straight configuration. The actual stationary load value is given by (2.43), and is a corresponding generalization P_r^* of the reduced modulus load (2.3) with (2.20), namely

$$P_r^* = E_r^* A \left(\frac{\pi k}{2\ell}\right)^2 \quad , \tag{2.44}$$

where

$$E_r^* = \frac{1}{Ak^2} \int E_t\, x_1(x_1 + d_0^*)\, dA \quad . \tag{2.45}$$

(A corresponding formula in our <u>general</u> theory is (5.8) with $\dot{\underline{F}} = 0$).

For purely elastic incremental behaviour over the whole of the bottom section, whether before yield or after unloading, so that we are in the elastic regime (cf. Fig. 2.11), E_t has the value E over the whole section. In this case $dP/dy = 0 \Rightarrow d_0^* = 0 \Rightarrow P_r^* = P_e$, even if $y(1) \neq 0$. The small deflection assumption built into (2.35) therefore precludes a stationary load except at the Euler value (perhaps with $y(1) \neq 0$ then, in the imperfect case). In particular this means that the conjectured maximum load in the elastic regime of Fig. 2.11(b) can only occur, if it exists at all, at larger values of deflection than are admitted by (2.35), so that a more exact large deflection theory is required to predict any such maximum load. This ought to show whether the boundary between the elastic and elastic/plastic regimes shown in Fig. 2.10(b) really does extend down to the zero load line, and whether or not complete unloading across the bottom section does occur or is precluded by a plastic hinge effect for large enough imperfection.

After the entire bottom section has yielded, in order that loading shall be continuing in every fibre we must have $d(0) \geq a$. Then we shall be in the plastic regime (cf. Fig. 2.11) and (2.42) shows that the slope dP/dy cannot be zero but must exceed a definite positive value

$$\frac{dP}{dy} \geq \frac{\pi^2}{4\ell^2} \int E_t(x_1 + a)\, dA \,, \tag{2.46}$$

which it can assume on the upper boundary of the plastic regime (emanating from P_t) at loads having the property (from (2.43) with $y(1) \geq 0$) that

$$P \leq \frac{\pi^2}{4\ell^2} \int E_t \, x_1(x_1 + a) \, dA \, . \qquad (2.47)$$

The bounds in (2.46) and (2.47) simplify if the variation of E_t across the section is ignored (as in Hoff's analysis of the H-section). When the unspecified strain history and imperfections permit this assumption, we get

$$\frac{dP}{dy} \geq \frac{a}{k^2} P_t \quad , \quad P \leq P_t \quad . \qquad (2.48)$$

A load maximum calculated on the basis of equations (2.35) can therefore only occur in the elastic/plastic regime, at the value P_r^* given by (2.44). Hoff [194] gives an argument indicating that

$$P_r^* \leq P_r \qquad (2.49)$$

where P_r is the reduced modulus load $P_r = \sigma_r A$ given by (2.3) and (2.20), corresponding to the straight configuration.

The equilibrium path of a perfect column in which the Euler load is reached before plastic yielding ($\sigma_e < \sigma_y$) will begin by following the rising curve for the elastica from the bifurcation point. Unless the material has unlimited elasticity, however, yielding will ultimately set in on the concave side, followed not long afterwards by the occurrence of a maximum load as the equilibrium path falls away from the elastica solution. Leites [224] has computed this maximum for both perfect and eccentrically loaded bars of rectangular section, and his results are illustrated in Fig. 2.16.

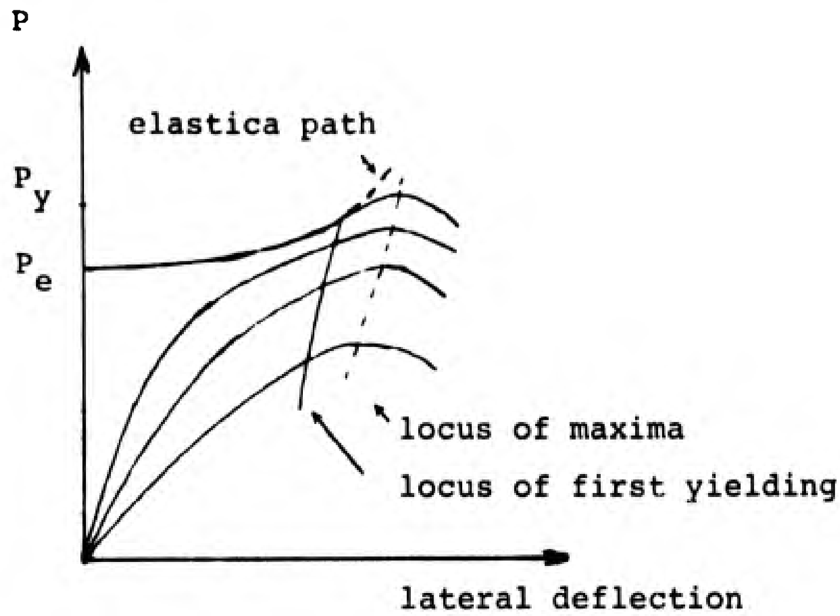


Fig. 2.16 Leites [224] equilibrium paths ($\sigma_e < \sigma_y$)

3. General Boundary Value Problem and Constitutive Equations

(i) Formulation of the quasi-static boundary value problem in arbitrary bodies.

Consider a body of arbitrary shape made of any material. Assign material coordinates θ^i ($i = 1, 2, 3$) to the particles of the body, by 'scratching' a curvilinear coordinate net onto the body when in some particular 'reference configuration'. Throughout any subsequent motion a typical particle will then carry some constant triad of values ($\theta^1, \theta^2, \theta^3$) as a permanent label. Let its position vector from a fixed origin 0 be $\underline{r}$ in the reference configuration and $\underline{R}$ in some other 'current configuration' (Fig. 3.1). The displacement $\underline{u} = \underline{R} - \underline{r}$ between the configurations may be large - these configurations need not be adjacent.

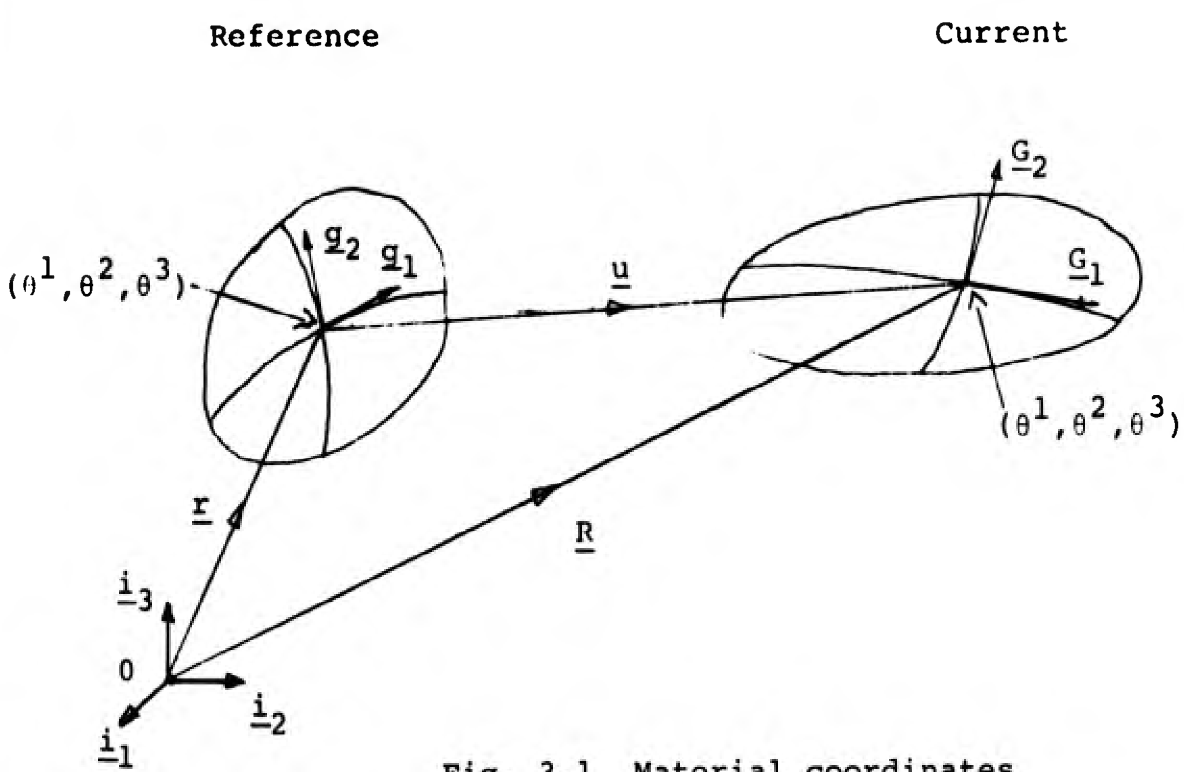


Fig. 3.1 Material coordinates

Introduce a 'background frame' of fixed rectangular cartesian coordinate axes with origin at 0, and axes parallel to the unit triad ($\underline{i}_1$, $\underline{i}_2$, $\underline{i}_3$). The cartesian components of $\underline{r} = (\underline{r} \cdot \underline{i}_i)\, \underline{i}_i$ (sum on i) will depend on the θ^i, and the partial derivatives $\partial\underline{r}/\partial\theta^i$ of such a function $\underline{r}(\theta^1, \theta^2, \theta^3)$ will be tangential to the curvilinear axes in the reference configuration. We write

$$\underline{r} = \underline{r}(\theta^i) \quad , \quad \underline{g}_i \equiv \frac{\partial\underline{r}}{\partial\theta^i} \quad . \qquad (3.1)$$

Suppose the current configuration can be reached as the end result of a sequence of configurations generated by the monotonic increase of a single scalar parameter ϵ, from the value $\epsilon = 0$ defining the reference configuration. In other words, let the cartesian components of $\underline{R}$ and $\partial\underline{R}/\theta^i$ depend on ϵ, writing

$$\underline{R} = \underline{R}(\theta^i, \epsilon) \quad , \quad \underline{G}_i \equiv \frac{\partial \underline{R}}{\partial \theta^i} \quad , \tag{3.2}$$

in a way such that

$$\underline{R}(\theta^i, 0) = \underline{r}(\theta^i) \quad , \quad \underline{G}_i(\theta^i, 0) = \underline{g}_i \ . \tag{3.3}$$

In a dynamic problem ϵ would be the real time t. In quasi-static distortion ϵ can be any quantity providing a regular parametrization of the equilibrium paths to be examined (as illustrated in Section 2).

The displacement referred to the cartesian axes will be a function of the form $\underline{u}(\theta^i, \epsilon) = \underline{R}(\theta^i, \epsilon) - \underline{r}(\theta^i)$. It may also be referred either to the $\underline{g}_i$ or the $\underline{G}_i$, which are called the 'covariant base vectors' of the θ^i-system in the two configurations. The resulting contravariant components of $\underline{u}$ will be corresponding functions of the θ^i and ϵ (but different from the cartesian components). For example, we could write

$$\underline{u} = u^j \underline{g}_j \quad , \quad u^j = u^j(\theta^i, \epsilon) \quad , \quad \underline{g}_j = \underline{g}_j(\theta^i) \ . \tag{3.4}$$

Any other vector associated with the typical particle θ^i at the current instant ϵ can likewise be expressed in terms of components referred to either (i) the cartesian fixed triad $(\underline{i}_1, \underline{i}_2, \underline{i}_3)$, (ii) the curvilinear fixed triad $(\underline{g}_1, \underline{g}_2, \underline{g}_3)$, or (iii) the curvilinear moving triad $(\underline{G}_1, \underline{G}_2, \underline{G}_3)$.

Consider an area element $\underline{n}$ dS in the reference configuration, having unit normal $\underline{n}$ and area dS then. In the current configuration its direction will have changed to $\underline{N}$ and its magnitude to $d\sigma$ (Fig. 3.2), neither by small amounts in

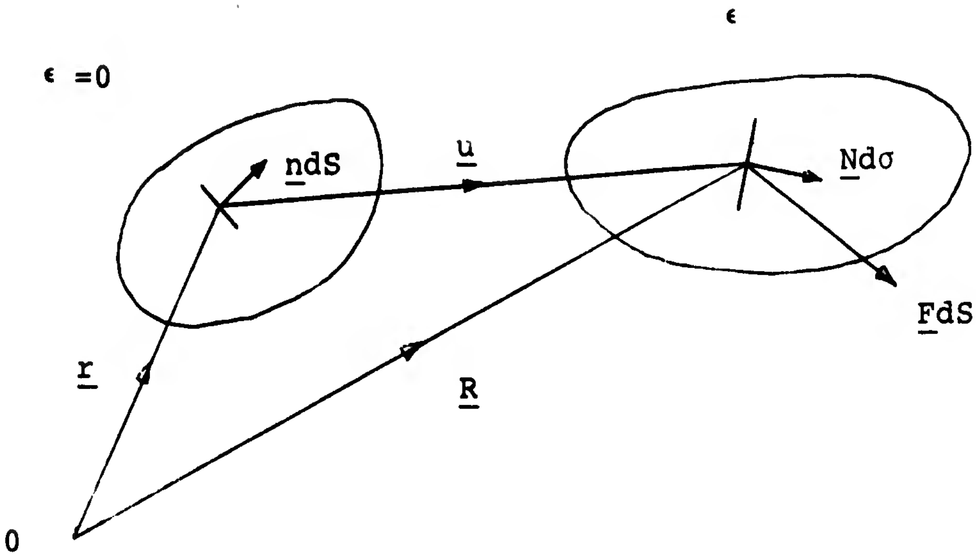


Fig. 3.2 Nominal traction on current area

general. Denote the load vector acting on this area in the current configuration by $\underline{F}$ dS, even if the area itself has changed to $d\sigma \neq dS$. Then $\underline{F}$ is the nominal traction, i.e. the current load per unit reference area. Refer $\underline{F}$ to the fixed triad ($\underline{g}_1$, $\underline{g}_2$, $\underline{g}_3$) at the reference position of the area:

$$\underline{F} = F^j \underline{g}_j \ . \tag{3.5}$$

Cauchy's law says that the dependence of $\underline{F}$ on $\underline{n}$ is linear, and we write

$$F^j = n_i s^{ij} \tag{3.6}$$

where $n_i = \underline{g}_i \cdot \underline{n}$. The coefficients s^{ij} so defined are called the (contravariant) components of the nominal stress (Hill [17]).

If the curvilinear net scratched on the body was momentarily coincident with the cartesian coordinates in the reference configuration, so that $\underline{g}_1 = \underline{i}_1$, $\underline{g}_2 = \underline{i}_2$, $\underline{g}_3 = \underline{i}_3$, the individual s^{ij} are written s_{ij} and take a simple interpretation: the quantity s_{ij} dS is the component of current load in the j^{th} cartesian direction on an area element which in its reference position had faced in the i^{th} cartesian direction and had area dS then.

The nominal stress is well suited to the formulation of boundary value problems for the moving current configuration, because it makes the kinematics tangible by expressing spatial integrations over the fixed reference region, without introducing any assumptions about small strain or displacement. For example, the current state is in (translational) equilibrium under body forces $\underline{b} = b^i \underline{g}_i$ per unit mass if

$$\int \underline{F}\, dS + \int \rho\, \underline{b}\, dV = 0 \tag{3.7}$$

where V is the volume at $\epsilon = 0$ and $\rho = \rho(\theta^i)$ is the density then. From Green's theorem applied to the reference volume it follows that

$$s^{ij}{}_{,i} + \rho\, b^j = 0 \tag{3.8}$$

are the differential equations of current translational equilibrium. Subscripts following a comma will denote covariant derivatives (here calculable from $\partial(s^{ij}\, \underline{g}_i\, \underline{g}_j)/\partial\theta^k \equiv s^{ij}{}_{,k}\, \underline{g}_i\, \underline{g}_j$ by differentiating in turn the three factors in the invariant 'dyadic' product, and equivalent to cartesian partial derivatives if the θ^i are cartesian in the reference configuration).

The nominal stress is not symmetric, but if we write the current load vector in the alternative forms

$$\underline{F}\ dS = n_i\ s^{ij}\ \underline{g}_j\ dS = n_i\ \tau^{ij}\ \underline{G}_j\ dS$$

$$= N_i\ \sigma^{ij}\ \underline{G}_j\ d\sigma \tag{3.9}$$

(where $N_i = \underline{G}_i \cdot \underline{N}$) it may be shown that local rotational equilibrium of the current state requires

$$\tau^{ij} = \tau^{ji} . \tag{3.10}$$

This 'Kirchoff stress tensor' is related to the Cauchy true stress σ^{ij} by

$$\sigma^{ij} = \frac{\rho(\epsilon)}{\rho(0)}\ \tau^{ij} \tag{3.11}$$

(Hill [20]), i.e. simply by the density quotient as a proportionality factor; and it is related to the nominal stress by

$$s^{ij} = \tau^{ij} + \tau^{ik}\ u^j{}_{,k} \tag{3.12}$$

from $(3.2)_2$, $(3.4)_1$ and (3.9) (with $\partial(u^j\ \underline{g}_j)/\partial\theta^k \equiv u^j{}_{,k}\ \underline{g}_j$).

Common boundary conditions on different parts of S in equilibrium problems include

(i) assigned displacement $\underline{u} = \underline{h}(\epsilon)$, (3.13)

where $\underline{h}(\epsilon)$ is given function of 'quasi-static time' ϵ . (for rigid constraints $\underline{h}(\epsilon) \equiv 0$);

(ii) assigned traction $\underline{F} = \underline{c}(\epsilon)$ (3.14) where $\underline{c}(\epsilon)$ is a given function of ϵ, constant for dead loading; and

(iii) hydrostatic pressure loading $\underline{F}\, dS = -p(\epsilon)\, \underline{N}\, d\sigma$ (3.15)

always perpendicular to the current area, where the pressure magnitude $p(\epsilon)$ is a given scalar function of ϵ. It has been shown by Sewell [47, 49] that this $\underline{F}$ is derivable from a potential function of displacement and surface gradient calculated in the reference configuration. This requires some differential geometry to express $\underline{N}\, d\sigma/dS$ in terms of the <u>fixed</u> reference vectors $\underline{g}_i$. The explicit dependence of $\underline{F}$ on the deformation turns out to be

$$\underline{F} = -p(\epsilon)\Big[n_k + (n_k\, u^j{}_{,j} - n_j\, u^j{}_{,k}) + \tfrac{1}{2}\,\epsilon_{ijk}\,\epsilon^{pqr}\, n_r\, u^i{}_{,p}\, u^j{}_{,q}\Big]\underline{g}^k \qquad (3.16)$$

however large the displacement. Here we have used the alternating tensor components ϵ_{ijk} and ϵ^{pqr}, and contravariant base vectors $\underline{g}^1$, $\underline{g}^2$, $\underline{g}^3$ which are orthogonal to the θ^i = constant surfaces at $\epsilon = 0$. This loading is 'configuration-dependent' in that it senses the motion of the surface, and its variation depends explicitly on that motion, unlike (3.14) which is prescribed in advance regardless of how the surface moves. The 'linear elastic foundation' is another type of configuration-dependent loading which is derivable from a potential.

The equations of equilibrium for the large displacement of an arbitrary body from a reference configuration are expressible in terms of that fixed reference configuration by inserting (3.12) (with (3.10)) into (3.8), and by expressing boundary conditions on current load in terms of nominal traction as in (3.14) or (3.16).

When the material coordinates θ^i are chosen to be cartesian at $\epsilon = 0$ (say a_1, a_2, a_3) the basic equations may be written

$$F_j = n_i \, s_{ij} \tag{3.17}$$

$$\frac{\partial s_{ij}}{\partial a_i} + \rho \, b_j = 0 \quad , \quad \tau_{ij} = \tau_{ji} \tag{3.18}$$

$$s_{ij} = \tau_{ij} + \tau_{ik} \frac{\partial u_j}{\partial a_k} \tag{3.19}$$

and (3.16) becomes

$$F_k = -p(\epsilon) \left[n_k + \left(n_k \frac{\partial u_j}{\partial a_j} - n_j \frac{\partial u_j}{\partial a_k}\right) + \frac{1}{2} \epsilon_{ijk} \epsilon_{pqr} \, n_r \frac{\partial u_i}{\partial a_p} \frac{\partial u_j}{\partial a_q} \right] \tag{3.20}$$

where now ϵ_{ijk} is +1, -1 or 0 if the sequence ijk is cyclic, acyclic or has two equal members.

Incipient quasi-static behaviour from the current state is governed by equations obtained from the above by differentiating with respect to ϵ. This procedure is valid because ϵ and the θ^i (or the a_i) are independent variables, so that a small change in ϵ generates an adjacent configuration for the particle with labels $(\theta^1, \theta^2, \theta^3)$. We denote the corresponding ϵ-derivatives by superposed dots - they are called convected derivatives. For example, the first convected derivative of (3.19) evaluated in the current configuration is

$$\dot{s}_{ij} = \dot{\tau}_{ij} + \dot{\tau}_{ik} \frac{\partial u_j}{\partial a_k} + \tau_{ik} \frac{\partial \dot{u}_j}{\partial a_k} \quad . \tag{3.21}$$

When the parameters of stress, shape and incremental material response can be regarded as known in the current state, the boundary value problem for the incipient behaviour from it can be simplified by choosing that configuration <u>itself</u> as the reference configuration. For then we can put $\underline{u} \equiv 0$ in expressions like (3.21) <u>after</u> the ϵ-differentiation, but not before.

The general equations of continuing equilibrium from a known reference configuration $\epsilon = 0$ are therefore

$$\dot{s}^{ij}{}_{,i} + \rho\, \dot{b}^j = 0 \ , \quad \dot{\tau}^{ij} = \dot{\tau}^{ji} \tag{3.22}$$

$$\dot{s}^{ij} = \dot{\tau}^{ij} + \sigma^{ik} v^j{}_{,k} \tag{3.23}$$

where we have written $\underline{v} \equiv \underline{\dot{u}}|_{\epsilon = 0}$, and $\tau^{ik} = \sigma^{ik}$ at $\epsilon = 0$. More precisely, these are the <u>first</u> order equations in a 'static perturbation' scheme. The first order change in load on an area element would be $\underline{\dot{F}}\, dS\, \delta\epsilon$ corresponding to a small change $\delta\epsilon$ in ϵ, where from (3.9)

$$\dot{F}^j = n_i\, \dot{s}^{ij} \quad . \tag{3.24}$$

For example, pressure loading would prescribe $\dot{\underline{F}}$ to be of the form

$$\dot{\underline{F}} = - \dot{p}\underline{n} - p(n_k \, v^j{}_{,j} - n_j \, v^j{}_{,k}) \, \underline{g}^k \qquad (3.25)$$

by differentiating (3.16) and putting $\epsilon = 0$. Higher order equations in the static perturbation scheme correspond to higher order derivatives with respect to ϵ, evaluated at $\epsilon = 0$. These would be needed to discuss smooth bifurcations on equilibrium paths (as in Sewell [52, §5(ii)]). The more common abrupt bifurcations will involve bifurcation of the quasi-static velocity field $\underline{v}$ appearing in the above first order equations, however.

The equations developed here are those employed by Hill in a sequence of investigations of the incremental boundary value problem in elastic, elastic/plastic and rigid/plastic bodies [12-24]. They have the advantages of being simple in structure and of having direct physical interpretation. Many other notations, terminology and stress measures have been used by different authors, more particularly for elastic materials, but we shall not digress to review these.

(ii) Constitutive equations for elastic solids.

Introduce the large strain tensor

$$e_{ij} \equiv \frac{1}{2} (\underline{G}_i \cdot \underline{G}_j - \underline{g}_i \cdot \underline{g}_j) \; . \qquad (3.26)$$

This measures the distortion which has taken place between the reference configuration and the current configuration. A compressible elastic solid whose strain energy is $E(e_{ij})$ per unit reference volume will have Kirchoff stress given by

$$\tau^{ij} = \frac{\partial E}{\partial e_{ij}} \, . \tag{3.27}$$

Since $E(e_{ij})$ is symmetrized, (3.10) is satisfied by (3.27). The reference state itself need not be unstressed. If there are workless internal kinematical constraints, like incompressibility, (3.27) should be augmented by the most general term which will do no work in all virtual motions compatible with those constraints (e.g. as in [51, equation (5.10)]). From (3.2), (3.4) and (3.26) it follows that

$$e_{ij} = \frac{1}{2} \left(\underline{g}_i \cdot \frac{\partial \underline{u}}{\partial \theta^j} + \underline{g}_j \cdot \frac{\partial \underline{u}}{\partial \theta^i} + \frac{\partial \underline{u}}{\partial \theta^i} \cdot \frac{\partial \underline{u}}{\partial \theta^j} \right) \tag{3.28}$$

$$= \frac{1}{2} \left(u_{i,j} + u_{j,i} + u^k{}_{,i}\, u_{k,j} \right) , \tag{3.29}$$

and then from (3.12) and (3.27) that the nominal stress in the elastic solid is given by

$$s^{ij} = \frac{\partial E}{\partial u_{j,i}} \, , \tag{3.30}$$

as shown by Hill [17, equation (3)].

The rate of change of e_{ij}, calculated in the current configuration from (3.26) or (3.28), is (Hill [20, equation (36)]

$$\dot{e}_{ij} = \frac{1}{2} \left(\underline{G}_i \cdot \frac{\partial \dot{\underline{u}}}{\partial \theta^j} + \underline{G}_j \cdot \frac{\partial \dot{\underline{u}}}{\partial \theta^i} \right) . \tag{3.31}$$

The incremental response of an elastic solid from the <u>current</u> configuration may therefore be written

$$\dot{\tau}^{ij} = \frac{\partial^2 E}{\partial e_{ij}\, \partial e_{k\ell}}\, \dot{e}_{k\ell} \tag{3.32}$$

or

$$\dot{s}^{ij} = \frac{\partial^2 E}{\partial u_{j,i}\, \partial u_{\ell,k}}\, \dot{u}_{\ell,k} \quad . \tag{3.33}$$

The Hessians in these equations are matrices of instantaneous or 'tangent' moduli. In structural metals the total possible elastic strain from a stress-free state is usually so small (in the absence of heavy overstraining) that the coefficients in (3.32) may be approximated by their values in that stress-free state. And for elastic isotropy $\partial^2 E/\partial e_{ij}\, \partial e_{k\ell}$ will then contain only Young's modulus and Poisson's ratio, or the two Lamé constants λ and μ thus:

$$\frac{\partial^2 E}{\partial e_{ij}\, \partial e_{k\ell}} = \lambda\, g^{ij}\, g^{k\ell} + \mu (g^{ik}\, g^{j\ell} + g^{i\ell}\, g^{jk}) \tag{3.34}$$

where $g^{ij} \equiv \underline{g}^i \cdot \underline{g}^j$ is the contravariant metric tensor.

The incremental response of an elastic solid from the reference configuration is obtained by evaluating (3.32) at $\epsilon = 0$. Let us write (3.32) at $\epsilon = 0$ as

$$\dot{\tau}^{ij} = K^{ijk\ell}\, \epsilon_{k\ell} \quad , \tag{3.35}$$

where

$$\epsilon_{k\ell} \equiv \dot{e}_{k\ell}\big|_{\epsilon = 0} = \frac{1}{2}\left(\underline{g}_i \cdot \frac{\partial \underline{v}}{\partial \theta^j} + \underline{g}_j \cdot \frac{\partial \underline{v}}{\partial \theta^i}\right)$$

$$= \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \tag{3.36}$$

is the strain-rate from the reference configuration. The nominal stress-rate at $\epsilon = 0$ is obtained by inserting (3.35) and (3.36) into (3.23).

It will be convenient to use a matrix notation of capital letters for fourth order tensors, and bold-face symbols for symmetric second order tensors (as well as for vectors). Thus (3.35) will be written

$$\dot{\underline{\tau}} = K\underline{\epsilon} \quad . \tag{3.37}$$

The context will make it clear whether a bold face symbol means a second order tensor or an ordinary vector in future.

<u>(iii) Constitutive equations for plastic solids</u>.

We confine attention here to incremental response from the <u>reference</u> configuration $\epsilon = 0$, in which the shape of the body and its stress distribution are supposed known. The strain history may have been arbitrary and is unspecified except for the instantaneous values of the 'tangent moduli' which appear as given coefficients in the piecewise linear stress-rate/strain-rate equations now to be laid down.

In an elastic element we take these 'rate equations' to be just (3.37) for all $\underline{\epsilon}$, with given non-singular K having the same symmetries as the Hessian in (3.32). Write the inverse of (3.37) as

$$\underline{\epsilon} = M\dot{\underline{\tau}} \quad , \qquad M = K^{-1} \quad . \tag{3.38}$$

In a plastic element we take the strain-rate to be given by

$$\underline{\epsilon} = M\dot{\underline{\tau}} + \gamma_\alpha \underline{\nu}_\alpha \tag{3.39}$$

(summation on $\alpha = 1, \ldots, N$) where the $\underline{\nu}_\alpha$ are N given symmetric second order tensors, and the γ_α are N scalar parameters such that for each α

$$\dot{\underline{\tau}} \cdot \underline{\nu}_\alpha - h_{\alpha\beta}\, \gamma_\beta \leq 0 \quad , \tag{3.40}$$

$$\gamma_\alpha \geq 0 \quad , \tag{3.41}$$

$$\gamma_\alpha \, (\dot{\underline{\tau}} \cdot \underline{\nu}_\alpha - h_{\alpha\beta}\, \gamma_\beta) = 0 \quad . \tag{3.42}$$

In these equations $\dot{\underline{\tau}} \cdot \underline{\nu}_\alpha$ means the double sum $\dot{\tau}_{ij}\, \nu_{ij\alpha}$, and $h_{\alpha\beta}$ are the elements of a given symmetric N × N 'hardening' matrix which need not be diagonal. Equations (3.39)-(3.42) are obtained by ascribing point-wise validity over the continuum to new equations proposed recently by Hill [26] for the incremental behaviour of single metal crystals by multislip. In the single crystal context each γ_α is interpretable as a shear rate in a vector direction $\underline{t}_\alpha$ on a glide plane with vector normal $\underline{n}_\alpha$, with $\nu_{ij\alpha} = \frac{1}{2}\,(t_i\, n_j + t_j\, n_i)_\alpha$, so that each $\underline{\nu}_\alpha^2 = 1/2$. The plastic part of the strain-rate is $\gamma_\alpha\, \underline{\nu}_\alpha$. The connexion with classical plasticity can associate the N $\underline{\nu}_\alpha$ with the normals to the facets of a singular yield surface at the given stress point (but see [26, §2]). A smooth yield surface corresponds to the case N = 1. More detailed comments on the constitutive equations for polycrystalline aggregates can be found in [25] and [28A], and on the circumstances in which yield surface corners exist in

[28].

We have written equations (3.40)-(3.42) in a form which is a slight but significant variant of that which is usual in the plasticity literature, in that we introduce an equation (3.42) to replace a verbal or parenthetic qualification. This form now shows better how general manipulations of the equations (such as proofs of uniqueness theorems and extremum principles) are of the same kind as occur in nonlinear programming and other branches of applied mathematics (see Noble and Sewell [37]). Each of the N terms in the sum in (3.42) must be separately zero when (3.40) and (3.41) hold, and only one of the factors γ_α and $\dot{\underline{\tau}} \cdot \underline{\nu}_\alpha - h_{\alpha\beta}\, \gamma_\beta$ in each such term can be non-zero. This is illustrated by the schematic representation of (3.40)-(3.42) in Fig. 3.3.

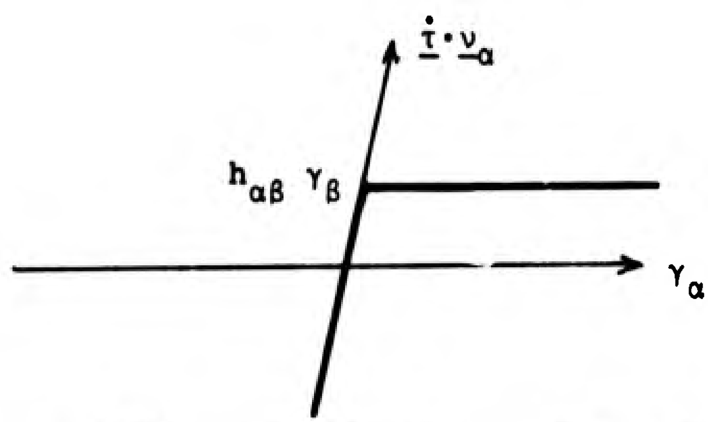


Fig. 3.3 Schematic representation of multislip equations

A more familiar version of (3.39)-(3.42) is easily deduced in the case of only one glide system N = 1. The unit normal to a smooth yield surface is now $\sqrt{2}\ \underline{\nu}$. The hardening matrix $h_{\alpha\beta}$

reduces to a single term $\frac{1}{2}h$ (say), and we must distinguish the three cases of $h > 0$ (hardening), $h = 0$ (non-hardening) and $h < 0$ (softening) in order to express the plastic part $\underset{\sim}{\epsilon}_p = \gamma\underset{\sim}{\nu}$ of the strain-rate explicitly in terms of the stress-rate $\dot{\underset{\sim}{\tau}}$. Purely elastic behaviour $\gamma = 0$ requires no loading ($\underset{\sim}{\nu} \cdot \dot{\underset{\sim}{\tau}} \leq 0$) whatever h may be, i.e.

$$\underset{\sim}{\epsilon}_p = 0 \quad \Rightarrow \quad \gamma = 0 \quad \Rightarrow \quad \dot{\underset{\sim}{\tau}} \cdot \underset{\sim}{\nu} \leq 0 \; . \qquad (3.43)$$

Plastic straining $\gamma > 0$ requires loading (unloading) in the hardening (softening) case, i.e.

$$\underset{\sim}{\epsilon}_p = \gamma\underset{\sim}{\nu} \neq 0 \quad \Rightarrow \quad \gamma > 0 \quad \Rightarrow \quad \dot{\underset{\sim}{\tau}} \cdot \underset{\sim}{\nu} = \tfrac{1}{2}h\gamma \; , \qquad (3.44)$$

which implies

$$\underset{\sim}{\epsilon}_p = \begin{cases} \dfrac{\dot{\underset{\sim}{\tau}}\cdot\underset{\sim}{\nu}}{\frac{1}{2}h}\,\underset{\sim}{\nu} & \text{and } \dot{\underset{\sim}{\tau}} \cdot \underset{\sim}{\nu} > 0 \quad \text{if } h > 0 \; , \\ \gamma\underset{\sim}{\nu} & \text{and } \dot{\underset{\sim}{\tau}} \cdot \underset{\sim}{\nu} = 0 \quad \text{if } h = 0 \; , \\ \dfrac{\dot{\underset{\sim}{\tau}}\cdot\underset{\sim}{\nu}}{\frac{1}{2}h}\,\underset{\sim}{\nu} & \text{and } \dot{\underset{\sim}{\tau}} \cdot \underset{\sim}{\nu} < 0 \quad \text{if } h < 0 \; . \end{cases} \qquad (3.45)$$

It should be mentioned that the terms loading and unloading, hardening and softening, are used here specifically in relation to the particular definition of stress-rate $\dot{\underset{\sim}{\tau}}$, i.e. to the convected derivative of Kirchoff stress evaluated in the reference configuration. Use of a different admissible stress-rate in the constitutive equations might involve hardening and loading parameters whose signs differ from the present h and

$\dot{\underline{\tau}} \cdot \underline{\nu}$ (see [23], [45]).

Returning to the case of general N, since $K = M^{-1}$ we can write (3.39) as

$$\dot{\underline{\tau}} = K\underline{\epsilon} - \gamma_\alpha K \underline{\nu}_\alpha \quad . \tag{3.46}$$

Inserting this into (3.40)-(3.42), and writing

$$g_{\alpha\beta} \equiv h_{\alpha\beta} + \underline{\nu}_\alpha K \underline{\nu}_\beta \tag{3.47}$$

permits (3.40)-(3.42) to be written as

$$\underline{\nu}_\alpha K\underline{\epsilon} - g_{\alpha\beta} \gamma_\beta \leq 0 \quad , \tag{3.48}$$

$$\gamma_\alpha \geq 0 \quad , \tag{3.49}$$

$$\gamma_\alpha (\underline{\nu}_\alpha K\underline{\epsilon} - g_{\alpha\beta} \gamma_\beta) = 0 \quad . \tag{3.50}$$

Purely elastic behaviour (all $\gamma_\alpha = 0$) now requires no loading on any glide system (all $\dot{\underline{\tau}} \cdot \underline{\nu}_\alpha \leq 0$) whatever $h_{\alpha\beta}$ might be; but activation of a particular glide system (<u>some</u> $\gamma_\alpha > 0$) requires for that value of α that

$$\dot{\underline{\tau}} \cdot \underline{\nu}_\alpha = h_{\alpha\beta} \gamma_\beta = \underline{\nu}_\alpha K\underline{\epsilon} - \gamma_\beta \underline{\nu}_\alpha K \underline{\nu}_\beta \tag{3.51}$$

and therefore

$$\underline{\nu}_\alpha K\underline{\epsilon} = g_{\alpha\beta} \gamma_\beta = \bar{g}_{\alpha\beta} \gamma_\beta \qquad \text{(active } \alpha\text{)} . \tag{3.52}$$

Here $\bar{g}_{\alpha\beta}$ is that (symmetric) submatrix of $g_{\alpha\beta}$ which corresponds only to the active systems, so that the activation conditions in terms of the strain-rates are

$$\gamma_\alpha = \bar{g}^{-1}_{\alpha\beta} \underline{\nu}_\beta K\underline{\epsilon} > 0 \tag{3.53}$$

for each active α-value, together with linear inequalities in $\underline{\epsilon}$ derived by inserting (3.53) into (3.48) applied to the non-active α-values.

It follows from (3.39) and (3.42) that

$$\dot{\tau}^{ij}\,\epsilon_{ij} \equiv \dot{\underline{\tau}} \cdot \underline{\epsilon} = \dot{\underline{\tau}}\, M\, \dot{\underline{\tau}} + h_{\alpha\beta}\,\gamma_\alpha\,\gamma_\beta \tag{3.54}$$

$$= \dot{\underline{\tau}}\, M\, \dot{\underline{\tau}} + \bar{h}^{-1}_{\alpha\beta}\,(\underline{\nu}_\alpha \cdot \dot{\underline{\tau}})(\underline{\nu}_\beta \cdot \dot{\underline{\tau}}) \tag{3.55}$$

since only the active

$$\gamma_\alpha = \bar{h}^{-1}_{\alpha\beta}\,(\underline{\nu}_\beta \cdot \dot{\underline{\tau}}) > 0 \tag{3.56}$$

will contribute to (3.54), $\bar{h}_{\alpha\beta}$ being the associated 'active submatrix' of $h_{\alpha\beta}$. Of course (3.53) and (3.56) require the inverses to be assumed to exist. Expressions (3.54) and (3.55) are valid in the rigid/plastic limit $M = 0$.

To express $\dot{\underline{\tau}} \cdot \underline{\epsilon}$ in terms of the strain-rates alone we can, in the elastic/plastic case, use (3.46) and (3.53) to get

$$\dot{\underline{\tau}} \cdot \underline{\epsilon} = \underline{\epsilon}\, K\underline{\epsilon} - \bar{g}^{-1}_{\alpha\beta}\,(\underline{\nu}_\alpha\, K\underline{\epsilon})(\underline{\nu}_\beta\, K\underline{\epsilon}) \quad . \tag{3.57}$$

It remains to be seen, when $N > 1$, whether $\dot{\underline{\tau}} \cdot \underline{\epsilon}$ can be expressed as a quadratic in $\underline{\epsilon}$ in the rigid/plastic limit as $K \to \infty$ (cf. the limiting procedure in [395, equation (29)]). When $N = 1$ we can argue directly just by inserting the square of $\underline{\epsilon} = \gamma\underline{\nu}$ into (3.54), giving

$$\dot{\underline{\tau}} \cdot \underline{\epsilon} = h\,\underline{\epsilon}^2. \tag{3.58}$$

(iv) Elastic/plastic comparison theorem.

Let $\underline{\epsilon}_+$ and $\underline{\epsilon}_-$ by any two strain-rates, with difference $\underline{\epsilon}_+ - \underline{\epsilon}_- \equiv \Delta\underline{\epsilon}$ (say). Associated via (3.39)-(3.42) will be corresponding values $\dot{\underline{\tau}}_+$, $\gamma_{\alpha+}$ and $\dot{\underline{\tau}}_-$, $\gamma_{\alpha-}$, with similarly ordered differences $\Delta\dot{\underline{\tau}}$ and $\Delta\gamma_\alpha$. We assume that

$$g_{\alpha\beta} \text{ is positive definite} \tag{3.59}$$

with positive definite inverse $g^{-1}_{\alpha\beta}$, and thence prove that

$$\Delta\dot{\underline{\tau}} \cdot \Delta\underline{\epsilon} \geq \Delta\underline{\epsilon}\ K\ \Delta\underline{\epsilon} - g^{-1}_{\alpha\beta}\ (\underline{\nu}_\alpha\ K\ \Delta\underline{\epsilon})(\underline{\nu}_\beta\ K\ \Delta\underline{\epsilon}) \ . \tag{3.60}$$

The proof begins by taking the difference of (3.46) applied to both 'plus' and 'minus' sets:

$$\begin{aligned}
&\Delta\dot{\underline{\tau}} \cdot \Delta\underline{\epsilon} - \Delta\underline{\epsilon}\ K\ \Delta\underline{\epsilon} + g^{-1}_{\alpha\beta}\ (\underline{\nu}_\alpha\ K\ \Delta\underline{\epsilon})(\underline{\nu}_\beta\ K\ \Delta\underline{\epsilon}) \\
&= g^{-1}_{\alpha\beta}\ (\underline{\nu}_\alpha\ K\ \Delta\underline{\epsilon})(\underline{\nu}_\beta\ K\ \Delta\underline{\epsilon}) - \Delta\gamma_\alpha\ (\underline{\nu}_\alpha\ K\ \Delta\underline{\epsilon}) \\
&= (\underline{\nu}_\alpha\ K\ \Delta\underline{\epsilon})\ g^{-1}_{\alpha\beta}\ (\underline{\nu}_\beta\ K\ \Delta\underline{\epsilon} - g_{\beta\delta}\ \Delta\gamma_\delta) \\
&= g^{-1}_{\alpha\beta}\ (\underline{\nu}_\alpha\ K\ \Delta\underline{\epsilon} - g_{\alpha\lambda}\ \Delta\gamma_\lambda)(\underline{\nu}_\beta\ K\ \Delta\underline{\epsilon} - g_{\beta\delta}\ \Delta\gamma_\delta) \\
&\qquad + \Delta\gamma_\beta\ (\underline{\nu}_\beta\ K\ \Delta\underline{\epsilon} - g_{\beta\delta}\ \Delta\gamma_\delta) \geq 0 \ .
\end{aligned} \tag{3.61}$$

The quadratic first term is non-negative because $g^{-1}_{\alpha\beta}$ will be positive definite, whereas the second term is equal to

$$\begin{aligned}
&-\gamma_{\beta+}\ (\underline{\nu}_\beta\ K\ \underline{\epsilon}_- - g_{\beta\delta}\ \gamma_{\delta-}) \\
&-\gamma_{\beta-}\ (\underline{\nu}_\beta\ K\ \underline{\epsilon}_+ - g_{\beta\delta}\ \gamma_{\delta+}) \geq 0
\end{aligned} \tag{3.62}$$

because the pairs of quantities $\underline{\epsilon}_+$, $\gamma_{\beta+}$ and $\underline{\epsilon}_-$, $\gamma_{\beta-}$ separately satisfy (3.48)-(3.50). Q.E.D.

The theorem generalizes a result of Hill [19, Lemma I] for $N = 1$ to values of $N > 1$. See also [20, §4(ii)] and [395, §3]. It may be shown [26, 37] that when M (and therefore K) and $h_{\alpha\beta}$ are positive definite, equations (3.39)-(3.42) can have at most one solution for $\dot{\underline{\tau}}$ and for the N γ_α when $\underline{\epsilon}$ is given. This theorem (3.60) requires the slightly weaker postulate (3.59).

(v) Linear comparison solid.

It is convenient to express the foregoing theorem in a different way by defining a hypothetical 'linear comparison solid' satisfying (3.38) in elastic elements, but in plastic elements having rate equations

$$\left.\begin{aligned} \underline{\epsilon} &= M\,\dot{\underline{\tau}} + \gamma_\alpha\,\underline{\nu}_\alpha\,, \\ 0 &= \dot{\underline{\tau}}\cdot\underline{\nu}_\alpha - h_{\alpha\beta}\,\gamma_\beta\,, \end{aligned}\right\} \qquad (3.63)$$

in place of (3.39)-(3.42). For this solid the schematic representation in Fig. 3.3 is converted into that of Fig. 3.4. Property (3.54) still holds as it stands. In fact properties (3.53)-(3.57) all hold for the comparison solid provided that the inequalities > 0 in (3.53) and (3.56) are replaced by $\neq 0$, with corresponding extended meanings of the 'active submatrices' $\bar{g}_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$, since (3.63) do not restrict the sign of the γ_α.

If now $\underline{\epsilon}_+$ and $\underline{\epsilon}_-$ are the same two strain-rates that we considered in the comparison theorem (3.60), and if $\dot{\underline{\tau}}^L_+$, $\gamma^L_{\alpha+}$ and $\dot{\underline{\tau}}^L_-$, $\gamma^L_{\alpha-}$ are the corresponding stress-rates and γ_α-values in the

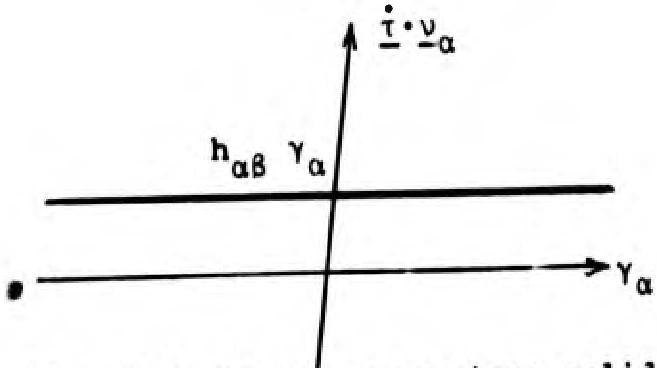


Fig. 3.4 Linear comparison solid

linear solid (3.63), it follows that

$$\Delta\dot{\underline{\tau}}^L \cdot \Delta\underline{\epsilon} = \Delta\underline{\epsilon}\ K\ \Delta\underline{\epsilon} - g^{-1}_{\alpha\beta}\ (\underline{\nu}_\alpha\ K\ \Delta\underline{\epsilon})(\underline{\nu}_\beta\ K\ \Delta\underline{\epsilon})\ . \quad (3.64)$$

Hence the elastic/plastic comparison theorem (3.60) may be written

$$\Delta\dot{\underline{\tau}} \cdot \Delta\underline{\epsilon} \geq \Delta\dot{\underline{\tau}}^L \cdot \Delta\underline{\epsilon}\ . \quad (3.65)$$

We shall see that this inequality is the germ of a far-reaching generalization of the column property that bifurcation cannot begin with unloading. This is pertinent to any problem in which the boundary conditions make the elastic/plastic solid respond like its linear comparison solid over the whole volume of the body.

(vi) Rigid/plastic comparison theorem.

It is easy to prove a similar comparison theorem for the rigid/plastic solid ($M = 0$), although this is somewhat less explicit for $N > 1$ than we can get for $N = 1$.

If $\underline{\epsilon}_+$ and $\underline{\epsilon}_-$ are any two strain-rates in a plastic element,

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with associated stress-rates and shear-rates satisfying (3.39)-(3.42), we get the inequality [26, equation (9)]

$$\Delta\dot{\underline{\tau}} \cdot \Delta\underline{\epsilon} - h_{\alpha\beta}\,\Delta\gamma_\alpha\,\Delta\gamma_\beta$$

$$= \Delta\gamma_\alpha\,\Delta(\underline{\nu}_\alpha \cdot \dot{\underline{\tau}} - h_{\alpha\beta}\,\gamma_\beta) \geq 0 \quad . \tag{3.66}$$

In terms of the comparison solid (3.63) with M = 0 we have

$$\Delta\dot{\underline{\tau}}^L \cdot \Delta\underline{\epsilon} = h_{\alpha\beta}\,\Delta\gamma_\alpha\,\Delta\gamma_\beta \tag{3.67}$$

for the same $\underline{\epsilon}_+$ and $\underline{\epsilon}_-$ as in (3.66). Therefore (3.65) also holds in the rigid/plastic limit. The difficulty mentioned after (3.57) again arises when we wish to express (3.67) directly in terms of the given $\Delta\underline{\epsilon}$. If N = 1, however, we have

$$\Delta\dot{\underline{\tau}}^L \cdot \Delta\underline{\epsilon} = \frac{1}{2}\,h(\Delta\gamma)^2 = h(\Delta\underline{\epsilon})^2 \quad . \tag{3.68}$$

(vii) Strain-rate potentials.

Consider constitutive rate equations of the form

$$\dot{\underline{\tau}} = \frac{\partial W}{\partial \underline{\epsilon}} \quad , \quad \text{i.e.} \quad \dot{\tau}^{ij} = \frac{\partial W}{\partial \epsilon_{ij}} \quad , \tag{3.69}$$

where $W(\epsilon_{ij})$ is a given symmetrized homogeneous function of degree two in the ϵ_{ij}, with continuous first derivatives. By Euler's theorem 2W will have the value of $\dot{\tau}^{ij}\,\epsilon_{ij}$. This function may also depend parametrically on the quantities characterizing the given reference state [20, 24]. Such rate equations include the elastic solid (3.35), with quadratic

$$W = \frac{1}{2}\,\underline{\epsilon}\,K\,\underline{\epsilon} = \frac{1}{2}\,K^{ijk\ell}\,\epsilon_{ij}\,\epsilon_{k\ell} \tag{3.70}$$

which has continuous first and second derivatives with respect to strain-rate.

The elastic/plastic solid described in subsection 3(iii) above is also included in (3.69), the required W now having second derivatives which are only piecewise continuous in strain-rate space. This W is a different quadratic in each of a number of pyramidal domains of strain-rate space with a common vertex, the coefficients of the quadratic jumping discontinuously across the hyperplane boundaries of these domains [27, §2], while the first derivatives of W and therefore the stress-rates remain continuous. For example, if the submatrices of $g_{\alpha\beta}$ corresponding to some particular combination of active ($\gamma_\alpha > 0$) and latent ($\gamma_\alpha = 0$) systems are written as

$$[g_{\alpha\beta}] = \left[\begin{array}{c:c} \bar{g}_{\alpha\beta} & \overset{o}{\bar{g}}_{\alpha\beta} \\ \hdashline \overset{o}{\bar{g}}_{\alpha\beta} & \overset{o}{g}_{\alpha\beta} \end{array}\right] \begin{array}{l} \} \text{ active} \\ \\ \} \text{ latent} \end{array} , \qquad (3.71)$$

then for this combination of 'glide systems' the stress-rate is

$$\dot{\underline{\tau}} = K\underline{\epsilon} - (\bar{g}^{-1}_{\alpha\beta}\, \underline{\nu}_\beta\, K\underline{\epsilon})\, K\, \underline{\nu}_\alpha \qquad (3.72)$$

in the strain-rate domain

$$\bar{g}^{-1}_{\alpha\beta}\, \underline{\nu}_\beta\, K\underline{\epsilon} > 0 \qquad \text{(active } \alpha\text{)} , \qquad (3.73)$$

$$\underline{\nu}_\alpha\, K\underline{\epsilon} - \overset{o}{\bar{g}}_{\alpha\beta}\, \bar{g}^{-1}_{\beta\delta}\, \underline{\nu}_\delta\, K\underline{\epsilon} \leq 0 \qquad \text{(latent } \alpha\text{)} . \qquad (3.74)$$

These results follow for (3.46)-(3.53). The strain-rate potential W which expresses (3.72) as an example of (3.69) is

$$W = \frac{1}{2} \underline{\epsilon} K \underline{\epsilon} - \frac{1}{2} \overline{g}^{-1}_{\alpha\beta} (\underline{\nu}_\alpha K\underline{\epsilon})(\underline{\nu}_\beta K\underline{\epsilon}) , \tag{3.75}$$

which is half of (3.57). Thoroughly nonlinear constitutive laws involving a potential which may be convex, but need not be quadratic as (3.75) is, have also been envisaged by Hill [28] to represent more exactly the behaviour of polycrystals at yield vertices, and associated detailed calculations have been performed by Hutchinson [28A].

When N = 1 the strain-rate potential has only two branches

$$W = \frac{1}{2} \underline{\epsilon} K \underline{\epsilon} \begin{cases} - \dfrac{(\underline{\nu} K \underline{\epsilon})^2}{h + 2 (\underline{\nu} K \underline{\nu})^2} & \text{if } \underline{\nu} K \underline{\epsilon} > 0 , \\ 0 & \text{if } \underline{\nu} K \underline{\epsilon} \le 0 , \end{cases} \tag{3.76}$$

separated by the hyperplane $\underline{\nu} K \underline{\epsilon} = 0$ through the origin. Continuity of W and $\partial W/\partial \underline{\epsilon}$ across this hyperplane are immediately obvious in this case, as are the jumps in second derivatives.

The rigid/plastic solid cannot be expressed in the form (3.69).

The linear comparison solid (3.63) has the property (3.46) whenever $K = M^{-1}$ exists, whence all $\underline{\nu}_\alpha K \underline{\epsilon} = g_{\alpha\beta} \gamma_\beta$ (in place of (3.52)). Hence the linear comparison solid has a strain-rate potential W_L (say) given by

$$W_L = \frac{1}{2} \underline{\epsilon} K \underline{\epsilon} - \frac{1}{2} g^{-1}_{\alpha\beta} (\underline{\nu}_\alpha K\underline{\epsilon})(\underline{\nu}_\beta K\underline{\epsilon}) . \tag{3.77}$$

This potential is the same quadratic <u>throughout</u> strain-rate space, and has continuous second derivatives. It is exploited, for example, in [21] and [189] when N = 1. Terms associated with any γ_α which happen to be zero will drop out from the double sum in W_L.

(viii) Kirchhoff stress-rate potentials.

Although we shall not exploit them in this article, for completeness we mention rate equations of the form

$$\underline{\epsilon} = \frac{\partial W_c}{\partial \dot{\underline{\tau}}}, \quad \text{i.e.} \quad \epsilon_{ij} = \frac{\partial W_c}{\partial \dot{\tau}^{ij}} \tag{3.78}$$

where $W_c(\dot{\tau}_{ij})$ is a given symmetrized homogeneous function of degree two in the $\dot{\tau}_{ij}$, with continuous first derivatives. $2W_c$ will also have the value of $\dot{\tau}_{ij}\epsilon_{ij}$, and when both (3.69) and (3.78) exist W_c will be related to W by a Legendre dual transformation, so that numerically

$$W + W_c = \dot{\tau}^{ij}\epsilon_{ij} \quad . \tag{3.79}$$

The elastic solid has $W_c = \frac{1}{2}\dot{\underline{\tau}}\, M\, \dot{\underline{\tau}}$, and the elastic/plastic solid has branches of W_c given by expressions of the type (3.55) in suitably defined regions of stress-rate space (via arguments parallel to those resulting in (3.73)-(3.75)). In the rigid/plastic limit M = 0, $2W_c(\dot{\tau}_{ij})$ still exists and is given by (3.55), but (3.78) are not invertible and $W(\epsilon_{ij})$ does not exist (even in the case N = 1 when the values of $2W_c$ are those of (3.58) in the loading domain).

For the comparison solid (3.63)

$$W_{cL} = \frac{1}{2}\dot{\underline{\tau}}\, M\, \dot{\underline{\tau}} + \frac{1}{2} h^{-1}_{\alpha\beta}\, (\underline{\nu}_\alpha \cdot \dot{\underline{\tau}})(\underline{\nu}_\beta \cdot \dot{\underline{\tau}}) \tag{3.80}$$

over all of stress-rate space.

(ix) Velocity-gradient potentials.

We have shown that the boundary value problem for the incremental quasi-static response of an arbitrary body from the reference configuration involves equations such as (3.22)-(3.25). For elastic and plastic bodies the system of equations is completed by constitutive equations of type (3.69) or (3.78), as we have now seen by detailed consideration of examples.

Whenever (3.69) apply they may be inserted with (3.36) directly into (3.23) to express the nominal stress-rate in terms of a velocity gradient potential [17, 19-21]:

$$\dot{s}^{ij} = \frac{\partial U}{\partial v_{j,i}} \tag{3.81}$$

where

$$U(v_{j,i}) = W(\epsilon_{ij}) + \frac{1}{2}\sigma^{ik} v_{j,i} v^{j}{}_{,k} \quad . \tag{3.82}$$

For the elastic solid (3.70) and the linear comparison solid (3.77) we see that $U(v_{j,i})$ is a single quadratic function of velocity-gradient (but not of strain-rate only), and (3.81) are linear equations. For the elastic/plastic solid the composite nature of W is reflected in U. We recall that the stresses σ^{ik} are regarded as given. We observe from (3.23) and (3.82) that the function $U(v_{j,i})$ will have the numerical values of

$$U = \frac{1}{2}\dot{s}^{ij} v_{j,i} \quad . \tag{3.83}$$

We can regard (3.81) as a derived version of the constitutive equations, suitable for insertion into (3.22) and (3.24) to obtain a concise statement of the boundary value problem, valid whether the spin be large or small compared with the distortion-rate.

(x) Nominal stress-rate potentials.

It may happen [24] that there exists a second degree function $U_c(\dot{s}^{ij})$ such that

$$v_{j,i} = \frac{\partial U_c}{\partial \dot{s}^{ij}} \tag{3.84}$$

and taking the numerical value of

$$U_c = \frac{1}{2}\,\dot{s}^{ij}\,v_{j,i} = \frac{1}{2}\,\dot{\tau}^{ij}\,\epsilon_{ij} + \frac{1}{2}\,\sigma^{ik}\,v_{j,i}\,v^{j}{}_{,k}$$

$$= W_c + \frac{1}{2}\,\sigma^{ik}\,v_{j,i}\,v^{j}{}_{,k}\,. \tag{3.85}$$

When both $U(\dot{s}_{ij})$ and $U_c(v_{j,i})$ exist they will be the Legendre duals of each other, so that numerically

$$U + U_c = \dot{s}^{ij}\,v_{j,i}\,. \tag{3.86}$$

4. General Bifurcation Theory

(i) Résumé of problem.

The body is in equilibrium with known shape and stress distribution. The first order equations of continuing equilibrium are

$$\dot{s}^{ij}{}_{,i} + \rho\,\dot{b}^{j} = 0 \tag{4.1}$$

(from (3.22)), to be solved within the region V occupied by the body, where $\rho\,\dot{b}^j$ are given body force-rate components. To be definite suppose that the conditions on complementary parts S_F and S_v of the boundary S of V are

(i) assigned nominal traction-rate components (cf. (3.14))

$$n_i\,\dot{s}^{ij} = \dot{c}^j \qquad \text{on } S_F \quad , \tag{4.2}$$

(ii) rigid constraints (cf. (3.13))

$$v_j = 0 \qquad \text{on } S_v \; . \tag{4.3}$$

The constitutive equations describing the quasi-static (isothermal) response in the material may be, for the most part, any one of those sets described for elastic, elastic/plastic and rigid/plastic bodies from equations (3.35) onwards. It must be assumed that all the parameters or 'tangent moduli' in these constitutive equations are known, such as the matrix K of elastic moduli, the hardening matrix $h_{\alpha\beta}$ and the N matrices $\underline{\nu}_\alpha$. These parameters can vary in piecewise continuous fashion through the region V occupied by the extended body, which need not therefore be homogeneous or isotropic. Further, the previous strain history which produces the given parameter values does not need to be specified, and may have been arbitrarily complicated. The Kirchhoff stress-rate $\dot{\tau}^{ij}$ actually experienced by a material 'coupon' in the reference configuration is related to the nominal stress-rate $\dot{s}^{ij}$ observed from a fixed frame by (3.23), i.e.

$$\dot{s}^{ij} = \dot{\tau}^{ij} + \sigma^{ik}\,v^j{}_{,k} \; . \tag{4.4}$$

The existence of two or more different velocity field solutions to these equations can be expected to imply a sharp (as distinct from smooth) bifurcation on equilibrium paths generated as ϵ passes through the value $\epsilon = 0$ defining the considered state, since the problem posed is a first order one obtained from a supposedly non-singular perturbation procedure.

(ii) Uniqueness criteria.

Consider any two velocity fields v_{j+} and v_{j-}, and any two nominal stress-rate fields $\dot{s}_{+}^{ij}$ and $\dot{s}_{-}^{ij}$. Use the prefix Δ as an ordered finite difference operator applied to the two members of such pairs thus: write $\Delta v_j \equiv v_{j+} - v_{j-}$ and $\Delta \dot{s}^{ij} \equiv \dot{s}_{+}^{ij} - \dot{s}_{-}^{ij}$, and read as '$\Delta \Rightarrow$ plus member less minus member'.

If the two stress-rate fields satisfy (4.1) and (4.2), and the two velocity fields satisfy (4.3), it follows from the divergence theorem that

$$\int \Delta \dot{s}^{ij} \, \Delta v_{j,i} \, dV = 0 \ . \tag{4.5}$$

This is true regardless of what the constitutive equations may be, because constitutive equations are not involved in (4.5).

Now suppose, instead of what is postulated to get (4.5), that v_{j+} and $\dot{s}_{+}^{ij}$ are related by the constitutive equations (either elastic, elastic/plastic or rigid/plastic as the case may be), and also that v_{j-} and $\dot{s}_{-}^{ij}$ are also related by those constitutive equations. The argument of reductio ad absurdum leads to the following two sufficient criteria for uniqueness.

(i) If

$$\int \Delta \dot{s}^{ij} \Delta v_{j,i} \, dV > 0 \tag{4.6}$$

holds for all such v_{j+} and v_{j-} which also satisfy the rigid constraints (4.3), then there cannot be more than one velocity solution to the entire boundary value problem. For the difference of two solutions would have to satisfy both (4.5) and (4.6). In the elastic and elastic/plastic cases it follows (e.g. from (3.69)) that the stress-rate is also unique (but not necessarily in the rigid/plastic case).

(ii) If (4.6) holds for all such $\dot{s}^{ij}_{+}$ and $\dot{s}^{ij}_{-}$ (i.e. satisfying the constitutive equations) which also satisfy (4.1) and (4.2), there cannot be more than one solution for the nominal stress-rate in the actual problem, again because of a contradiction which ensues from (4.5) and (4.6).

These criteria can be made more explicit [24] by inserting (when appropriate) (3.81) into (4.6) for (i), or (3.84) into (4.6) for (ii). The relative strengths of the two criteria depend on the details of the particular problem. Comparison has been made for the classical elastic solid [22], but the criterion (ii) in terms of stress-rates seems not to have been explored for plastic solids. We concentrate attention subsequently here upon what follows from the first criterion (i).

In terms of the stress-rates $\dot{s}^{ijL}$ and $\dot{\tau}^{ijL}$ appropriate to v_j in the linear comparison solid, (4.6i) becomes

$$\int \Delta \dot{s}^{ijL} \Delta v_{j,i} \, dV + \int \Delta(\dot{\tau}^{ij} - \dot{\tau}^{ijL}) \Delta \epsilon_{ij} \, dV > 0 \tag{4.7}$$

for the differences associated with all pairs of distinct velocity fields admissible in (4.6i). We have used $\dot{s}^{ij} - \dot{s}^{ijL} = \dot{\tau}^{ij} - \dot{\tau}^{ijL}$, which follows from (4.4) for the same v_j and given σ^{ik}.

At this point we exploit the comparison theorem (3.65) or (3.66-7), whichever is appropriate. This theorem applied to the two strain-rates derived from v_{j+} and v_{j-} tells us that the second integral in (4.7) is always non-negative. Uniqueness is therefore assured in the actual solid whenever the first integral in (4.7) is positive for every pair of velocity fields compatible with the rigid constraints (4.3) and the constitutive equations (3.63) of the comparison solid (the last condition on the admissible fields is only a real restriction in the rigid/plastic case M = 0). In other words, uniqueness of the velocity field cannot fail in the actual nonlinear solid whenever the sufficient criterion (4.6i) applied to its linear comparison solid is satisfied. This applies specifically to the elastic solid (trivially since it is its own comparison solid), to the elastic/plastic solid, and to the rigid/plastic solid (at least for N = 1 when (3.68) holds, and for general N if it proves possible to generate the admissible velocity fields from assumed distributions of the γ_α subject to (4.3) as a constraint).

The central problem thus becomes the investigation of states for which the first integral in (4.7) is non-negative for all

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pairs of velocity fields admissible in the comparison solid. Since for rigid constraints the difference between any two admissible velocities is also admissible, and since (4.4) and the constitutive equations (3.63) are linear, the central problem becomes the investigation of states for which

$$\int \dot{s}^{ijL} v_{j,i} \, dV \geq 0 \qquad (4.8)$$

for all admissible velocity fields. 'Admissible' now means satisfying (4.3) and, in the rigid/plastic case, $\underline{\epsilon} = \gamma_\alpha \underline{\nu}_\alpha$ without a sign restriction on the γ_α. If we use (4.4) to bring out explicitly the role of the stresses, (4.8) becomes

$$\int \dot{\tau}^{ijL} \epsilon_{ij} \, dV + \int \sigma^{ik} v_{j,i} v^{j}{}_{,k} \, dV \geq 0 \quad . \qquad (4.9)$$

The first term can be expressed via (3.63) entirely in terms of the admissible velocity fields (or at least in terms of the shear rates when N > 1 in the rigid/plastic case), and its coefficients will then be the 'tangent moduli'.

For example, when the entire body is elastic we get

$$\int K^{ijk\ell} \epsilon_{ij} \epsilon_{k\ell} \, dV + \int \sigma^{ik} v_{j,i} v^{j}{}_{,k} \, dV \geq 0 \qquad (4.10)$$

for all velocity fields satisfying the rigid boundary constraints. This is formally the same as a criterion known in elasticity theory (e.g. see Pearson [41] or Koiter [29,30] for a modern treatment). However there is no restriction here on the magnitude of pre-strain or the strain history. For example (4.10) would apply even if the overstraining were such that the body could not get back to the zero stress-state without some plastic yielding (i.e. the yield surface need not enclose the origin).

For the elastic/plastic case it follows from (3.63) that

$$\dot{\underline{\tau}}^L = K\underline{\epsilon} - g^{-1}_{\alpha\beta}\,(\underline{\nu}_\beta\, K\underline{\epsilon})\, K\, \underline{\nu}_\alpha\ ,$$
$$\equiv C\underline{\epsilon} \quad \text{(say)}\ . \tag{4.11}$$

(The only $g_{\alpha\beta}$ actually needed here are those associated with the non-zero γ_α, as we indicated after (3.63)). Consequently we have only to replace the matrix K in (4.10) by C. In other words, (4.9) with (4.11) gives

$$\int (K^{ijk\ell} - g^{-1}_{\alpha\beta}\, \nu_{pq\alpha}\, \nu_{rs\beta}\, K^{pqij}\, K^{rsk\ell})\, \epsilon_{ij}\, \epsilon_{k\ell}\, dV$$
$$+ \int \sigma^{ik}\, v_{j,i}\, v^j{}_{,k}\, dV \geq 0 \tag{4.12}$$

for all velocity fields satisfying the rigid boundary constraints.

For the rigid/plastic case it follows from (3.63) that

$$\dot{\underline{\tau}}^L \cdot \underline{\epsilon} = h_{\alpha\beta}\, \gamma_\alpha\, \gamma_\beta \quad \text{when } N \geq 1$$
$$= h\, \underline{\epsilon}^2 \qquad \text{when } N = 1\ . \tag{4.13}$$

Then when the deformable region extends over the whole body (4.9) requires

$$\int h_{\alpha\beta}\, \gamma_\alpha\, \gamma_\beta\, dV + \int \sigma^{ik}\, v_{j,i}\, v^j{}_{,k}\, dV \geq 0 \tag{4.14}$$

for shear-rate distributions γ_α compatible via $\underline{\epsilon} = \gamma_\alpha\, \underline{\nu}_\alpha$ with velocity fields satisfying the rigid constraints (4.3). The more explicit version of (4.14) applies when N = 1 [15], and if the hardening rate h is then uniform (4.14) can be expressed as an inequality on h itself, so guaranteeing uniqueness for large enough h.

In the elastic and elastic/plastic cases there exists a velocity-gradient potential $U_L(v_{j,i})$ for the linear comparison solid. It has continuous first and second derivatives and has properties like (3.81-3). In fact $2U_L$ is just the sum of the integrands of (4.10) or (4.12), so that they may be written

$$\int U_L(v_{j,i})\, dV \geq 0 \quad , \tag{4.15}$$

for all $v_j = 0$ on S_v.

<u>(iii) First bifurcation state</u>.

We indicate now how the 'first bifurcation state' in an elastic/plastic solid can often be found in the following two stages:

(a) application of variational methods to find from (4.12) the first bifurcation state in the linear comparison solid. This is the state in which the zero minimum in (4.12) is actually achieved with certain v_j fields, say w_j, and is positive for all others. Usually only <u>one</u> of the parameters defining the state is imagined to vary, such as a load magnitude or modulus, so generating mathematically a sequence of non-bifurcation states in which the strict inequality applies in (4.12), and terminating with (4.12) - hence the adjective 'first';

(b) determination of those combinations of these w_j with the often known and fairly trivial 'fundamental solution' (such as continuing uniform compression in a strut or externally pressurized body) of the original problem which are such that the actual elastic/plastic solid behaves like its linear comparison solid everywhere. Comparison of (3.63) with (3.39)-(3.42) shows that this will happen for combinations whose strain-rates satisfy

$$\text{all} \quad \gamma_\alpha = g^{-1}_{\alpha\beta} \, \underline{\nu}_\beta \, K\underline{\epsilon} > 0 \quad \text{everywhere.} \qquad (4.16)$$

The range of values $\dot{c}^j$ of assigned traction-rate which are compatible with this loading everywhere can be found a posteriori. (Even if some of the γ_α should be zero, (3.42) does not necessarily require the inequality in (3.40), so that in place of (4.16) we might admit (3.73) and (with equality) (3.74).)

This procedure provides the first bifurcation state in the elastic/plastic solid because it shows that some of the non-unique modes possible in the comparison solid (fundamental solution plus arbitrary linear combination of the w_j) are also possible in the actual elastic/plastic solid (fundamental solution plus restricted linear combination of the w_j).

To justify the method described it is convenient to argue from (4.15). An eigenfunction expansion of $\int U_L(v_{j,i})\, dV$ shows [20, equation (25)] that the w_j provide equality in (4.15) if and only if they are eigensolutions for the corresponding homogeneous problem in the comparison solid (i.e. with $\dot{c}^j = 0$ in (4.2) and $\dot{b}^j = 0$ in (4.1)). It is then clear that an arbitrary additive multiple of the w_j does represent the extent of non-uniqueness possible in the original inhomogeneous problem for the comparison solid. Variational methods can be used to find the w_j because the zero value in (4.15) is stationary with respect to arbitrary variations from the eigensolutions, i.e.

$$\begin{aligned} \delta \int U_L \, dV &= \int \dot{s}^{ijL} \, \delta w_{j,i} \, dV \\ &= \int \dot{c}^j \, \delta w_j \, dS_F + \int \rho \, \dot{b}^j \, \delta w_j \, dV = 0 \end{aligned} \tag{4.17}$$

when $\dot{c}^j = \dot{b}^j = 0$. This reduces to a conventional eigenvalue problem and associated Rayleigh principle.

In short, then, the first bifurcation state in an elastic/plastic solid can often be investigated as if it were an elastic solid, but whose given tangent moduli are the coefficients in the first integral of (4.12) instead of (4.10). The plastic contributions to this matrix of coefficients can make it an orthotropic matrix, even for a body which is both elastically and plastically isotropic (e.g. see [395, §4]). Therefore, existing computations of eigensolutions and bifurcation states of orthotropic elastic bodies might be immediately available in the first stage of the elastic/plastic analysis, with their parameters suitably re-interpreted.

A similar two stage procedure can be applied in the rigid/plastic case. There is not quite the same assurance that the w_j which provide the zero value in (4.14) are actually eigensolutions for the homogeneous problem in the comparison solid, because an eigenfunction expansion of $\int \dot{s}^{ijL} v_{j,i} \, dV$ has not been given for that case. But uniqueness in the actual rigid/plastic solid is certainly assured whenever the strict inequality holds in (4.14) for all admissible non-zero v_j, which in itself is significant information.

The admissible fields in the rigid/plastic criterion (4.14) have to be compatible with $\underline{\epsilon} = \gamma_\alpha \underline{\nu}_\alpha$, as we explained after (4.8). In the case N = 1 let us write

$$\underline{\epsilon} = \lambda \, \underline{m} \quad , \qquad \underline{m}^2 = 1 \tag{4.18}$$

so that $\underline{m} = \sqrt{2} \, \underline{\nu}$, $\lambda\sqrt{2} = \gamma$. Thus $\underline{m}$ is now interpretable as the unique unit normal to a smooth yield surface. It may be shown that when $\underline{m}$ is constant over an extended region (as would happen when the stress is uniform, or at least varies only over a plane facet of the yield surface), the full set of velocity fields compatible with (4.18) can be obtained explicitly. Referring $\underline{m}$ to its principal axes, we need to integrate

$$\left.\begin{aligned} &\epsilon_{11} = \lambda \, \ell, \quad \epsilon_{22} = \lambda \, m \;, \quad \epsilon_{33} = \lambda \, n \\ &\epsilon_{23} = \epsilon_{31} = \epsilon_{12} = 0 \;, \quad \ell^2 + m^2 + n^2 = 1 \end{aligned}\right\} \tag{4.19}$$

where ℓ, m and n are constants. Continuous fields so obtained by Sewell [45, 395] are

(i) when $\ell mn \neq 0$,

$$\left.\begin{aligned} v_1 &= \tfrac{1}{2}a(\ell x_1^2-mx_2^2-nx_3^2)+\ell bx_2x_1+\ell cx_3x_1+\ell dx_1 , \\ v_2 &= max_1x_2+\tfrac{1}{2}b(mx_2^2-nx_2^2-\ell x_1^2)+mcx_3x_2+mdx_2 , \\ v_3 &= nax_1x_3+nbx_2x_3+\tfrac{1}{2}c(nx_3^2-\ell x_1^2-mx_2^2)+ndx_3 , \end{aligned}\right\} \quad (4.20)$$

$$\lambda = ax_1 + bx_2 + cx_3 + d ; \quad (4.21)$$

(ii) when $m = 0,\ \ell|n| = -n|\ell| \neq 0$,

$$\left.\begin{aligned} v_1 &= \ell|\ell|^{-\frac{1}{2}} (\phi - \psi) + \ell bx_1x_2 , \\ v_2 &= -\tfrac{1}{2} b(\ell x_1^2 + nx_3^2) , \\ v_3 &= n|n| (\phi + \psi) + nbx_3x_2 , \end{aligned}\right\} \quad (4.22)$$

$$\begin{aligned} \lambda = {} & \phi'(|n|^{\frac{1}{2}} x_3 + |\ell|^{\frac{1}{2}} x_1) + \\ & \psi'(|n|^{\frac{1}{2}} x_3 - |\ell|^{\frac{1}{2}} x_1) + bx_2 . \end{aligned} \quad (4.23)$$

Here a, b, c, d are arbitrary constants, ϕ and ψ are arbitrary functions of the arguments indicated in (4.23), and a prime denotes differentiation with respect to the argument in each case. Cases (i) and (ii) refer to the physically most realistic $\underline{m}$. They include the Mises cylinder and the Coulomb pyramid, so that they are applicable to soil mechanics as well as the plasticity of metals. Incompressibility ($\ell + m + n = 0$) is not assumed. Prager [43] obtained corresponding results in the incompressible case, and commented upon discontinuities. Miles [81] has solved (4.18) to get the velocity fields compatible with a spherically symmetric and therefore non-uniform stress distribution. When $N > 1$ linear combinations of such results would generate at least some solutions of $\underline{\epsilon} = \gamma_\alpha \underline{\nu}_\alpha$.

(iv) Other boundary conditions.

The same basic approach contained in (4.5) and (4.6) can be extended to deal with boundary conditions other than (4.2) and (4.3). For example if pressure loading is applied to S_F instead of assigned traction-rate, (4.2) is replaced by (3.25) with assigned p and $\dot{p}$. That is, the traction-rate vector is an assigned function not only of position on the boundary, but also of the velocity $\underline{v}$, so that stress-rate and velocity are 'coupled' on the boundary S_F. It follows that in place of (4.5), any two pairs of stress-rate and velocity fields which separately satisfy (4.1) and (4.3), and for which each pair is coupled by (3.25) on S_F,

$$\int \Delta\dot{s}^{ij} \, \Delta v_{j,i} \, dV$$

$$+ \int p(n_k \, \Delta v^j{}_{,j} - n_j \, \Delta v^j{}_{,k}) \, \Delta v^k \, dS_F = 0 \, . \qquad (4.24)$$

Therefore a sufficient condition for uniqueness of the velocity field is that, for all pairs of velocity fields satisfying (4.3) and the constitutive equations,

$$\int \Delta \dot{s}^{ij} \Delta v_{j,i} \, dV$$

$$+ \int p(n_k \Delta v^j{}_{,j} - n_j \Delta v^j{}_{,k}) \Delta v^k \, dS_F > 0 \ . \qquad (4.25)$$

The argument now proceeds much as before [15] and has been extended by Hill [24] to more general configuration-dependent loading.

A different kind of example is provided by passive constraints. Suppose that (4.2) is retained, but (4.3) is replaced by a smooth and rigid but passive bounding surface in contact with the body on S_v. That is, the bounding surface exerts zero tangential traction-rate, but it can only push and not pull in the normal direction and it cannot follow the body if contact is lost. The conditions on the normal components of velocity and traction-rate are therefore

$$\underline{n} \cdot \underline{\dot{F}} \leq 0 \ , \qquad (4.26)$$

$$\underline{n} \cdot \underline{v} \leq 0 \ , \qquad (4.27)$$

$$(\underline{n} \cdot \underline{\dot{F}})(\underline{n} \cdot \underline{v}) = 0 \ , \qquad (4.28)$$

on S_v. Equation (4.28) represents another kind of coupling between stress-rate and velocity (but weaker than (3.25)). For any two stress-rate/velocity pairs satisfying (4.1) and all the boundary conditions, it follows that [27, 51]

$$\int \Delta \dot{s}^{ij} \Delta v_{j,i} \, dV \leq 0 \ , \qquad (4.29)$$

which replaces (4.5). Therefore a sufficient condition for uniqueness of the velocity field is that, for all pairs of velocity fields satisfying (4.27) and the constitutive equations,

$$\int \Delta\dot{s}^{ij}\, \Delta v_{j,i}\, dV > 0 \quad . \tag{4.30}$$

It is possible that application of such criteria might need to be informed by the search techniques of nonlinear programming [37, 51]. A bifurcation or other instability under conditions (4.26)-(4.28) might correspond to the so-called 'one-way buckling'. Some experimental work on this topic (for elastic bodies) has been reported by Allan [121, 122].

(v) Applications to columns, plates and shells.

The only applications which have been worked out so far have been for the case N = 1. As in (4.18), we write $\underline{\nu} = \frac{1}{\sqrt{2}}\underline{m}$ in the general theory and interpret $\underline{m}$ as the unit normal to a smooth yield surface at the given stress point (the shape of the yield surface elsewhere need not be specified). With $h_{11} = \frac{1}{2}h > 0$, the rate equations for the linearized solid (3.63) become

$$\underline{\epsilon} = M\,\underline{\dot{\tau}}^L + \frac{\underline{m}\cdot\underline{\dot{\tau}}^L}{h}\,\underline{m} \quad . \tag{4.31}$$

For example, when the material is elastically isotropic (with Young's modulus E and Poisson's ratio ν) and when

$$m_{ij} = \begin{bmatrix} m_1 & m_4 & 0 \\ m_4 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{4.32}$$

referred to cartesian axes of reference, (4.31) is (see [395])

$$\begin{bmatrix} \dot\epsilon_{11} \\ \dot\epsilon_{22} \\ \dot\epsilon_{33} \\ \dot\epsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} + \frac{m_1^2}{h} & -\frac{\nu}{E} + \frac{m_1 m_2}{h} & -\frac{\nu}{E} + \frac{m_1 m_3}{h} & \frac{2m_1 m_4}{h} \\ -\frac{\nu}{E} + \frac{m_2 m_1}{h} & \frac{1}{E} + \frac{m_2^2}{h} & -\frac{\nu}{E} + \frac{m_2 m_3}{h} & \frac{2m_2 m_4}{h} \\ -\frac{\nu}{E} + \frac{m_3 m_1}{h} & -\frac{\nu}{E} + \frac{m_3 m_2}{h} & \frac{1}{E} + \frac{m_3^2}{h} & \frac{2m_3 m_4}{h} \\ \frac{2m_1 m_4}{h} & \frac{2m_2 m_4}{h} & \frac{2m_3 m_4}{h} & \frac{2(1+\nu)}{E} + \frac{4m_4^2}{h} \end{bmatrix} \begin{bmatrix} \dot\tau^L_{11} \\ \dot\tau^L_{22} \\ \dot\tau^L_{33} \\ \dot\tau^L_{12} \end{bmatrix} \tag{4.33}$$

together with $\widehat{23}$ and $\widehat{31}$ purely elastic shear relations. It is the inverse of this matrix of coefficients which then appears in the first integral of the bifurcation criterion (4.12). When the body is plastically orthotropic with principal axes coinciding with these axes of reference, and the stress is plane, the rate equations can be of this form with $m_4 \propto \sigma_{12}$. In the absence of shear stress we can put $m_4 = 0$ and regard the simplified equations as referred to the principal axes of stress.

When the stress is axial compression in the 3-direction, and the yield criterion coincides locally with that of von Mises, so that

$$\underline{m} = (m_1, m_2, m_3) \equiv (\ell, m, n) = \sqrt{\tfrac{2}{3}}(\tfrac{1}{2}, \tfrac{1}{2}, -1) \ , \tag{4.34}$$

the matrix of coefficients in (4.33) becomes transversely isotropic. The tangent modulus in the 3-direction, with $\dot\tau^L_{11} = \dot\tau^L_{22} = 0$, is

$$E_t = \frac{\dot{\tau}^L_{33}}{\dot{\epsilon}_{33}} = \frac{1}{\frac{1}{E} + \frac{m_3^2}{h}} \rightarrow \frac{h}{m_3^2} \quad \text{when } E \rightarrow \infty \tag{4.35}$$

(this can be expressed in terms of the current slope of the true stress/strain curve, with true stress measured with respect to fixed axes, but the difference between the two definitions of tangent modulus is frequently negligible - see [189, §3]).

Hill and Sewell [189] and Sewell [260] obtained the bifurcation stress (2.7) for the elastic/plastic column under uniaxial compressive stress $\sigma_{ij} = -\sigma\, \delta_{i3}\, \delta_{j3}$ $(\sigma > 0)$ by applying the above rate equations to express (4.12) in the form

$$\int [E_t\, \epsilon_{33}^2 + 4\, G(\epsilon_{13}^2 + \epsilon_{23}^2)]\, dV - \int \sigma\, v_{k,3}^2\, dV \geq 0 \tag{4.36}$$

and then obtaining the stationary minimum among the class (2.13)-(2.15) of approximating fields satisfying the rigid constraint conditions $f(0) = f'(0) = h(0) = 0$. (In Section 2 we used ν to denote the overall lateral contraction ratio $(\frac{\nu}{E} - \frac{m_1 m_3}{h})/(\frac{1}{E} + \frac{m_3^2}{h})$, instead of its purely elastic part (as here).) Hill used the fields (4.20) with (4.34) to get the stress (2.10) limiting the uniqueness range for the (incompressible) rigid/plastic column directly ([15], where the tensile problem $\sigma < 0$ and the associated necking is discussed too - see also [27]). Note that when shear stiffening is negligible $(G \sim E_t)$ the yield surface normal does not enter the bifurcation stress formula $E_t(\pi k/2\ell)^2$ independently of E_t - therefore it is adequate to know the stress/strain curve in uniaxial compression to find the bifurcation

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stress.

The situation is quite different in the plate problem. The foregoing general theory has been applied by Sewell [395-6], and for a uniaxially compressed plate in its $\widehat{31}$ plane simply supported on all four edges the bifurcation stress is given by

$$\frac{\sigma 3\beta^2}{\pi^2 t^2}\,\frac{(1-\nu^2)}{E} = \frac{\left(\frac{m\beta}{\alpha}\right)^2\left(1+\frac{E}{h}m_1^2\right)+\left(\frac{\alpha}{m\beta}\right)^2\left(1+\frac{E}{h}m_3^2\right)+2\left[1+\left(m_3^2+m_1^2-(1-\nu)m_3m_1\right)\frac{E}{h(1+\nu)}\right]}{1+\left(m_3^2+m_1^2+2\nu m_3m_1\right)\frac{E}{h(1-\nu^2)}} \tag{4.37}$$

where the integer m is chosen to minimize the right hand side with all other parameters held fixed (see [396, equation (2.22) and Figs. 2.2-2.4]). Here α is the width of the plate perpendicular to the applied compression in the 3-direction, β is the length and 2t the thickness. The least right side of (4.37) is an explicit function of m_1, m_3 and E_t (given, e.g., by (4.35)), whereas the left hand side depends only on E_t through the compressive stress/strain curve. The solution of this equation clearly depends on m_1 and m_3 as well as E_t and has been presented graphically by Sewell [396].

It is generally agreed that the experimental buckling load for plastic plates is well below that predicted by inserting the Mises normal (4.34) into (4.37), and a number of authors have asserted that deformation theory gives much better agreement with experiment. A detailed discussion of the literature was given in [396, §2.6]. The present author observed that the available experimental data gave no attention to the yield surface shape, which was usually assumed a priori to be that of von Mises, without regard to effects such as plastic anisotropy or even residual stress which may have been induced by the forming process. (That residual stress can cause a significant reduction in the buckling strength of steel columns has been revealed in particular by the Lehigh investigations [9, 203 p.15].) The bifurcation stress given by (4.37) was computed in [396] for a range of 'incompressible normals' (i.e. such that $m_1 + m_2 + m_3 = 0$) covering the Tresca 'fan' and so including the Mises normal, but with elastic isotropy retained. A significant reduction in bifurcation stress was found as compared with the Mises value. Since the formula (4.37) was derived under the general restriction $N = 1$ consistent with a smooth yield surface, some degree of plastic anisotropy has to be invoked to permit a smooth yield surface with normal different from that of Mises. Plastic anisotropy in the plate buckling problem has also been examined both theoretically and experimentally by Reckling [391-2], who used aluminum alloy plates with anisotropy induced by rolling. Now that we have available the basic comparison theorem proved for $N > 1$ in (3.65), the way is open to the calculation of bifurcation stresses in situations more closely resembling corners on yield surfaces. For example it ought now to be possible to retain plastic isotropy and find to what extent (4.37) needs to be modified to include the case when the bifurcation stress is at a corner of the Tresca criterion itself. Onat and Drucker [383] suggested that imperfections could account for the difference between the experimental data and the Mises incremental theory (by contrast with purely elastic plate buckling, in which the bifurcation load is not imperfection sensitive to anything like the same extent that it is in curved panels and shells). A similar conclusion has been reached by Hutchinson [537, 28B] for the spherical shell under external pressure, in which it is also found that the discrepancy between the prediction of deformation theory and Mises flow theory largely disappears in the presence of imperfections. Probably the precise truth involves a combination of these effects (anisotropy, yield surface corners, imperfections) - to establish it will require painstaking calculations and unambiguous experimental data. Even if deformation theory provides reasonable empirical correlation, that is not the same as fundamental understanding.

Applications of this type of theory (for N = 1) to cylindrical shell problems have been made by Dubey [505] and Chakrabarty [493-4], although it is possible to question some of their remarks. The rigid/plastic spherical shell under internal pressure has been considered by Miles [81]. There are, of course, many other analyses of plastic buckling in shells [479-599] from other viewpoints, but we make no attempt to unify these in this article.

(vi) Direct method.

We now distinguish between bifurcation states found by the variational method applied to minimize expressions such as

(4.12), as discussed so far, and the direct search for solutions of the governing equations (4.1)-(4.4) with the constitutive equations. This direct method has been successful in exhibiting, for thick bodies under certain symmetrically distributed boundary conditions, both symmetric modes of bifurcation (necking or bulging) and antisymmetric modes (buckling), as well as certain kinds of surface instabilities. Such solutions in elastic bodies have been demonstrated by Biot [1] and others, and in view of what has been said about the linear comparison solid, we expect such work to inform the search for similar types of solution in elastic/plastic bodies, with the expectation that assumptions of continuing loading everywhere can be verified a posteriori. Of the works cited in the Bibliography, the most extensive discussion of elastic/plastic thick body bifurcations is by Dubey [104] and Ariaratnam and Dubey [99, 105]. In the rigid/plastic case such investigations were initiated by Cowper [101] and Cowper and Onat [103] for the incompressible body, and by Sewell [260, §5.6; 45] for the compressible body - so that applications to soils as well as metals may be envisaged. Goodier and Newman [107] discussed bifurcations in a thick rigid/plastic circular cylinder associated with a vertex of the Tresca yield surface.

5. Stability

(i) Derivation of stability criterion.

We regard stability as an intrinsically dynamical subject. A discussion of the stability of an equilibrium configuration of any body will involve an examination of all the motions which can pass near (in some precise sense) to that configuration. The question of stability is therefore conceptually quite distinct from the discussion of the bifurcation of a sequence of equilibrium configurations - for instance we need say nothing in the bifurcation analysis about the density distribution, which will certainly affect the motions of the considered body. However, the two questions - of stability, and of the evolution of equilibrium states - are related, and we need to explore this relation.

An exact treatment of the stability of inelastic bodies is notably more difficult than that for elastic bodies and conservative systems. Nevertheless some progress can be made if certain assumptions are adopted. For inelastic bodies not involving time effects such as creep some general deductions have been made by Hill [18-20] and Hill and Sewell [190]. First suppose that during any considered motion the time-rate of increase of internal energy per unit current volume is the purely mechanical amount $\sigma^{ij} \frac{de_{ij}}{dt}$ with no thermal contribution (this uses the notation of (3.9), and the time-differentiation procedure is formally the same as the ϵ-differentiation in (3.31)). This rate of energy increase is just $s^{ij} \frac{d}{dt}(u_{j,i})$ per unit reference volume, as (3.9), (3.11) and (3.31) show. Consider any particular motion between time t = 0 and t = T. However the body or its shape varies, we can employ nominal stress s^{ij}, nominal surface tractions $\underline{F}$ and nominal body forces $\rho\,\underline{b}$ per unit reference volume, to express the quantity

$$w \equiv \int_0^T \left[\int s^{ij} \frac{d}{dt}(u_{j,i})\, dV - \int \underline{F} \cdot \frac{d\underline{u}}{dt}\, dS - \int \rho\, \underline{b} \cdot \frac{d\underline{u}}{dt}\, dV \right] dt \qquad (5.1)$$

in terms of integrals over the fixed reference region V with surface S. This w is the excess of the total energy absorbed within the body over the total work done by the surface forces $\underline{F}$ and the body forces $\underline{b}$ during that motion. The total kinetic energy K of the body will change from K_0 to K_T during this motion, and from the general balance law of work and energy we find

$$K_0 = w + K_T \geq w \, . \qquad (5.2)$$

In a conservative system w would be just the gain in total potential energy; but to evaluate w in the circumstances which interest us here, we need to take account not only of whatever variation may be prescribed for the external forces $\underline{F}$ and $\underline{b}$ (they might still be conservative), but also of the multi-valued nature of the constitutive equations for inelastic bodies. This last feature makes exact evaluation of the absorbed energy $\int_0^T [\int s^{ij} \frac{d}{dt}(u_{j,i})\, dV]\, dt$ particularly difficult because it may be expected to depend on the actual path taken between two configurations. Paths might be direct, or arbitrarily circuitous and the sequences of loading and unloading on two such paths can clearly be quite different. The difficulty still remains even if it supposed that the isothermal (quasi-static) constitutive equations are presumed to hold during the actual motions (which will certainly not all be quasi-static).

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Plausible progress may be made by supposing all the motions from the considered equilibrium position to be such that $w \geq 0$, and also such that each motion must be contained in a region which tends to that equilibrium position as $w \to 0$ (i.e. as if w were representable by a potential energy surface having a local minimum at that equilibrium position). Then if the 'disturbance' K_0 is supposed to tend to zero, it follows from (5.2) and what has just been said that

$$K_0 \geq w \geq 0 \tag{5.3}$$

and therefore that the displacement tends to zero with the disturbance - in that sense its equilibrium position may be called stable.

Concrete calculations of w to apply this criterion require even more assumptions. Suppose the body is held by rigid constraints on part of its boundary, while everywhere else the applied loading is dead. For small enough times t a Taylor expansion of the absorbed energy integrand may be written

$$\begin{aligned} &s^{ij}(t)\,\frac{d}{dt}\,(u_{j,i}(t)) = \\ &[s^{ij}(0) + t\,\frac{d}{dt}\,s^{ij}\Big|_0]\,\frac{d}{dt}\,(u_{j,i}(t)) + o(t^2)\ . \end{aligned} \tag{5.4}$$

Equilibrium of the given state balances out the $s^{ij}(0)$ term here with the external force integrals in (5.1) (in the rigid/plastic solid an extra possibility needs to be accounted for - see [18]). Now confine attention to those special motions for which the actual velocity $\frac{d\underline{u}}{dt}$ has the value $\underline{v}$ which is constant in time, with associated initial stress-rate $\left.\frac{ds^{ij}}{dt}\right|_0$ calculated from the quasi-static constitutive equations, and therefore denoted by $\dot{s}^{ij}$ (recall that dots denote ϵ-derivatives). For such direct paths taking place in the now small time T, (5.1) gives

$$w \simeq \frac{1}{2} T^2 \int \dot{s}^{ij} v_{j,i}\, dV \quad . \tag{5.5}$$

We therefore arrive at the following criterion for stability (or at least 'quasi-stability' - in view of the assumptions which have been made): we require

$$\int \dot{s}^{ij} v_{j,i}\, dV > 0 \tag{5.6}$$

for all differentiable distributions of velocity which satisfy the rigid constraints and are compatible via the constitutive equations with stress-rates $\dot{s}^{ij}$. Another dynamical argument leading to sufficient conditions for instability can also be put forward and utilized for direct paths [190] - it involves (5.6) but with the inequality reversed in direction.

At present it is an open question whether the more recent refinements of stability theory (e.g. see [4, 182]) can produce a substantial improvement over both the arguments and conclusions indicated here for the inelastic body, making due allowance for the heavy path-dependence of the net absorbed energy w.

As in the transition from (4.8) to (4.9) we can use (4.4) to bring out more explicitly the role of the stresses. From the point of view of applications the central stability problem therefore becomes the investigation of states for which

$$\int \dot{\tau}^{ij} \epsilon_{ij} \, dV + \int \sigma^{ik} v_{j,i} v^{j}{}_{,k} \, dV \geq 0 \qquad (5.7)$$

for all the fields admissible in (5.6). In (5.7) it is the actual stress-rates $\dot{\tau}^{ij}$ corresponding to these fields in the nonlinear solid which appear, as distinct from the $\dot{\tau}^{ijL}$ calculated for the linear comparison solid which appear in (4.9). Due allowance must therefore be made in (5.7) for the difference between loading and unloading regions associated with any considered field. We can then insert expressions such as (3.57) for the elastic/plastic solid, and for the rigid/plastic solid (3.54) (with $M = 0$) or (3.58) (when $N = 1$).

(ii) Relation between stability and uniqueness.

This relation rests upon a comparison of the inequalities (4.7) and (5.6), and thence of (4.9) and (5.7) and the ranges of the moduli and stresses which satisfy them. Detailed discussions have been given by Hill for the rigid/plastic [18, §4] and elastic/plastic [19, p.247] solids. The broad conclusion is that uniqueness implies stability, but not conversely. Stable bifurcations can actually be exhibited, as in the column problem. In fact, in stable states guaranteed by (5.6) the inequality must hold in particular for any equilibrium distributions of velocity $\underline{v}$, so that the associated surface loading $\dot{\underline{F}}$ on S_F must satisfy

$$\int \dot{s}^{ij} v_{j,i} \, dV = \int \dot{\underline{F}} \cdot \underline{v} \, dS_F > 0 \qquad (5.8)$$

(regardless of whether $\underline{v}$ is unique or not). Therefore at such a stable bifurcation the loading cannot remain constant (to this order) but must change with any possible mode in a way which satisfies (5.8) - and therefore makes an 'acute angle' with the mode in the obvious overall sense. (By contrast in the elastic solid the higher order stability calculations of Koiter [29] (see also [50, §5]) show that stable bifurcations with associated adjacent equilibrium positions can exist with $\dot{\underline{F}} = 0$, as in the elastica and compressed plate problems, provided the loading increases in some higher order term. Unstable bifurcations in shell problems are associated with loading which decreases either in the first or some higher order term.)

(iii) Applications.

The stability criterion (5.7) has been applied by Hill and Sewell [190] and Sewell [260] to confirm that the perfect prismatic column under dead loading is stable below the reduced modulus load when shear stiffening is negligible, as explained in Section 2(iii). The cross-sectional shape is arbitrary, and it is found that (5.7) applied with the fields (2.13)-(2.15) reduces to an expression like (4.36) but with E_t replaced by E over that part of the cross-section where $\epsilon_{33} > 0$ (unloading). Minimization then produces the results described in Section 2(iii), including the correction for shear stiffening.

In the elastic and elastic/plastic cases the velocity gradient potential $U(v_{j,i})$ exists and (5.7) becomes

$$\int U(v_{j,i})\, dV \geq 0 \qquad (5.9)$$

for all $\underline{v}$ satisfying any rigid constraint which may be present on the boundary.

When the entire boundary is subjected to dead loading, particular interest attaches to states in which (5.9) holds for all non-uniform velocity fields, with equality for some deformation mode. The zero minimum is stationary with respect to arbitrary variations from the solution of the homogeneous problem - by an argument like that of (4.17) but applied here to the actual possibly nonlinear solid instead of to its linearization. Hill [27] defines an 'eigenstate' in the nonlinear solid to be a state in which the homogeneous problem does have non-trivial 'eigensolutions' - i.e. solutions of (4.1) with $\dot{b}^j = 0$, and (4.2) with $\dot{c}^j = 0$ over the whole of S. He conducts a detailed investigation of 'primary eigenstates', which have the property (5.9) and are met first on (quasi-) stable loading paths satisfying (5.6). The necking of a tension specimen at maximum load is related to the existence of primary eigenstates.

6. A Bibliography of Plastic Buckling

(i) Preface.

The papers listed in the Bibliography extend over a variety of viewpoints. It is hoped thereby to convey a sense of perspective of the whole subject of plastic buckling, not only for future investigators but also for the authors whose works are cited. No comment upon a paper is implied merely by its inclusion in this list. The time available for this article

(3 months) has not been sufficient both to compile the Bibliography and to provide the conventional commentary. Nevertheless we have deemed it useful to present the Bibliography as it stands, and to complement it with a personal account of certain particular but fundamental topics with which this author happens to be familiar. Readers will therefore appreciate that the Bibliography contains articles which are of fundamental interest even though they are not referred to in the text, as well as articles which are unduly special in various ways, or even questionable in certain respects. As explained in the Introduction, the net has been deliberately cast rather wide so that some peripheral papers are included, because it seemed that the subject of plastic buckling could still be presented as a bounded subject on an overall view. References to papers in languages other than English are followed by the volume and review number from Applied Mechanics Reviews when these numbers are known. Papers listed under one of the following headings sometimes also contain material appropriate to another heading.

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