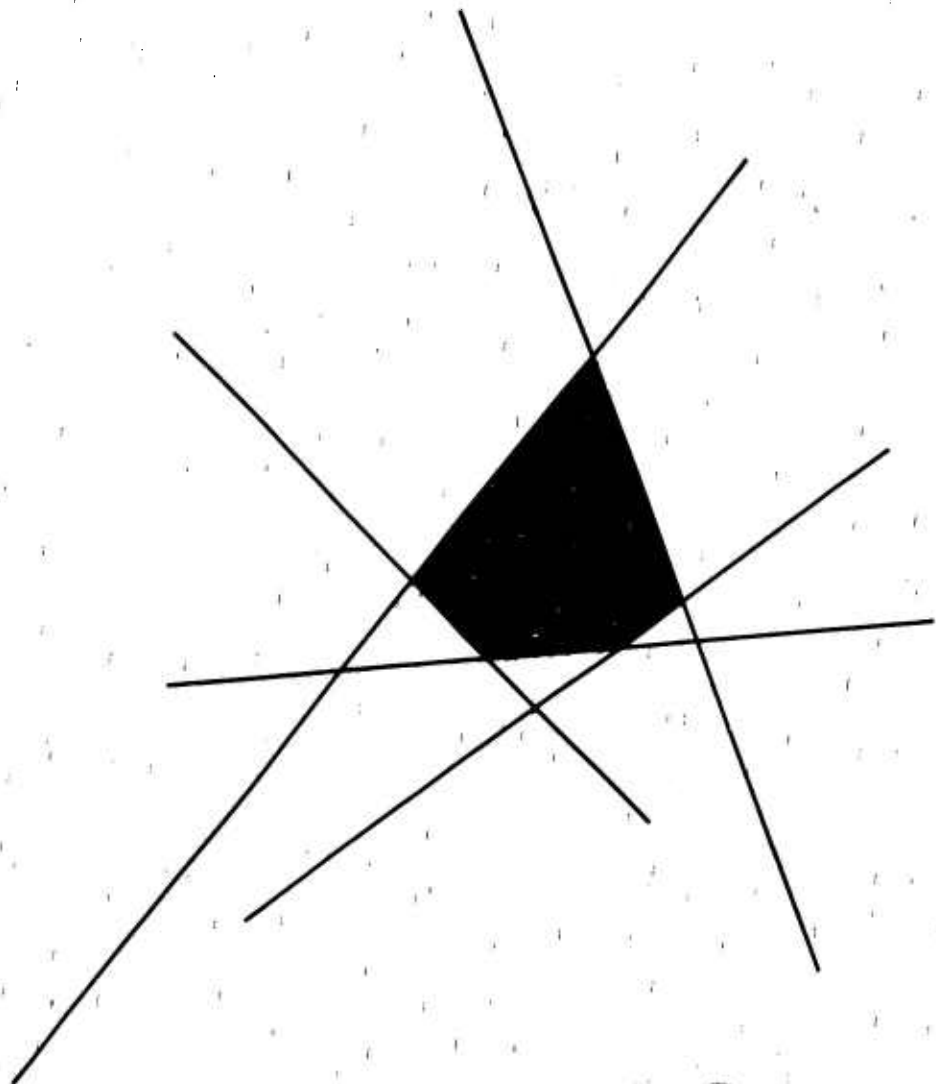


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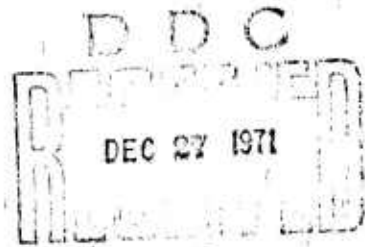
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by
SHELDON M. ROSS

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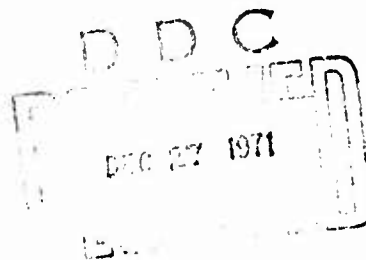
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ABSTRACT

A stationary independent increment process is the continuous time analogue of the discrete random walk, and, as such, has a wide variety of applications. In this paper we consider $M(t)$, the maximum value that such a process attains by time t . By using renewal theoretic methods we obtain results about $M(t)$. In particular we show that if μ , the mean drift of the process, is positive, then $\frac{M(t)}{t}$ converges to μ , and $E(M(t+h) - M(t)) \rightarrow h\mu$.

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Let $\{X(t), t \geq 0\}$ be a separable stationary independent increment process continuous in probability and with $X(0) = 0$. Suppose that $E[X(t)] = \mu t$ where $\mu > 0$, and let

$$M(t) = \sup_{0 \leq s \leq t} X(s).$$

Theorem 1:

(i) $M(t)/t \xrightarrow{a.s.} \mu$ as $t \rightarrow \infty$

(ii) $EM(t)/t \rightarrow \mu$ as $t \rightarrow \infty$.

Proof:

Fix some constant $b > 0$, and define $T_n, J_n, n \geq 1$, by

$$T_1 = \inf \{t : X(t) \geq b\}, \quad X(T_1) = b + J_1$$

$$T_n = \inf \{t : X(t) \geq b + X(T_{n-1})\}, \quad X(T_n) = b + X(T_{n-1}) + J_n, \quad n > 1.$$

Now, due to the stationary independent increment assumption, it follows that $\{T_n - T_{n-1}, n \geq 1\}$ is a renewal process. Letting $N(t)$ be the number of renewals by time t for this process, i.e., $N(t) = \max \{n : T_n \leq t\}$, we have

$$(1) \quad bN(t) + J_1 + \dots + J_{N(t)} \leq M(t) \leq bN(t) + J_1 + \dots + J_{N(t)} + b.$$

Now, from the strong law of large numbers and the renewal result $N(t)/t \xrightarrow{a.s.} 1/ET_1$, we have that

$$(2) \quad 1/t (bN(t) + J_1 + \dots + J_{N(t)}) \xrightarrow{a.s.} \frac{b + EJ_1}{ET_1}.$$

Now, from a similar theorem in discrete time (i.e., Stein's lemma), it follows that $ET_1 < \infty$ and thus from Martingale theory

$$(3) \quad E[b + J_1] = E[X(T_1)] = \mu ET_1,$$

and hence (i) follows from (1), (2) and (3). To prove (ii), we first note that

$$1/t E[bN(t) + J_1 + \dots + J_{N(t)}] = bEN(t)/t + \frac{EJ_1(EN(t) + 1)}{t} - \frac{E[J_{N(t)+1}]}{t}$$

by Wald's equation. However by regarding J_1 as a reward earned at the time of the i th renewal it follows, since $EJ_1 < \infty$, by standard arguments ([1], Page 53)

that $\frac{E[J_{N(t)+1}]}{t} \rightarrow 0$. Hence, by the elementary renewal theorem,

$$1/t E[bN(t) + J_1 + \dots + J_{N(t)}] \rightarrow \frac{b + EJ_1}{ET_1} = \mu$$

and (ii) is proven.

Q.E.D.

Theorem 2:

If $E[X^2(t)] < \infty$, then, for each $h > 0$,

$$E[M(t+h) - M(t)] \rightarrow h\mu.$$

Proof:

By (1) we have

$$b(N(t+h) - N(t)) + J_1 + \dots + J_{N(t+h)} - (J_1 + \dots + J_{N(t)}) - b \leq M(t+h) -$$

$$M(t) \leq b(N(t+h) - N(t)) + J_1 + \dots + J_{N(t+h)} - (J_1 + \dots + J_{N(t)}) + b.$$

Taking expectations, and using Wald's equation, yields

$$(b + EJ_1)E[N(t+h) - N(t)] + E[J_{N(t)+1}] - E[J_{N(t+h)+1}] - b \leq \\ E[M(t+h) - M(t)] \leq (b + EJ_1)E[N(t+h) - N(t)] + E[J_{N(t)+1}] - \\ E[J_{N(t+h)+1}] + b .$$

We shall show in the ensuing lemma that $\lim_{t \rightarrow \infty} E[J_{N(t)+1}]$ exists and is finite. Hence, the result follows from the above by first applying Blackwell's theorem and then letting $b \rightarrow 0$.

Q.E.D.

Lemma 1:

If $E[X^2(t)] < \infty$, then

$$\lim_{t \rightarrow \infty} E[J_{N(t)+1}] = \frac{E[J_1 T_1]}{E[T_1]} < \infty .$$

Proof:

Let $g(t) = E[J_{N(t)+1}]$ and let F denote the distribution of T_1 . Then

$$g(t) = \int_0^{\infty} E[J_{N(t)+1} \mid T_1 = x] dF(x) \\ = \int_0^t g(t-x) dF(x) + \int_t^{\infty} E[J_1 \mid T_1 = x] dF(x) .$$

This renewal type equation has the solution

$$g(t) = h(t) + \int_0^t h(t-x) dm(x)$$

where

$$h(t) = \int_t^{\infty} E[J_1 | T_1 = x] dF(x) .$$

From the key renewal theorem it follows that

$$\begin{aligned} \lim_{t \rightarrow \infty} g(t) &= \frac{\int_0^{\infty} h(t) dt}{E[T_1]} = \frac{\int_0^{\infty} \int_t^{\infty} E[J_1 | T_1 = x] dF(x) dt}{E[T_1]} = \\ &= \frac{\int_0^{\infty} x E[J_1 | T_1 = x] dF(x)}{E[T_1]} = \frac{E[J_1 T_1]}{E[T_1]} \end{aligned}$$

where the interchange is justified by the nonnegativity of rewards. That $E[J_1 T_1] < \infty$ follows from a similar result in discrete time. (The discrete time result states that for a random walk with positive drift whose steps have finite variance, the variance of the value of the process the first time it exceeds b is finite. The equivalent discrete time version of $E[J_1 T_1] < \infty$ then follows from Stein's lemma upon application of the Cauchy-Schwarz inequality.)

Q.E.D.

Remarks:

The technique employed in the above proof is similar to the ladder variable approach in fluctuation theory. That is, we say that a ladder point is reached not when the process attains a new high but when it passes the previous ladder height by at least some fixed value. It is to be hoped that this technique will prove useful in studying other properties of stationary independent increment processes. Also this approach should be contrasted with the approach, recently developed by M. Rubinovitch, in which a point t is said to be a ladder point if $M(t) = X(t)$. By applying the theory of regenerative events, as developed by Kingman, Rubinovitch

is then able to obtain many interesting results.

For a discrete time proof of (1), the interested reader should consult Heyde.

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