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DIFFERENTIAL GAMES WITH INFORMATION TIME LAG: A COMPARISON OF STRATEGIES

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ABSTRACT

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I. INTRODUCTION

In discussing some possible extensions of his work on differential games (DG), Isaacs mentioned the need for a relaxation of the assumption that each player in a DG has perfect information [1]. In many practical situations of dynamic conflict, the assumption cannot be justified. A particular problem which Isaacs cited is that in which a player in a DG has a time lag associated with the availability of his measurements of the opponent's state vector. In practice, such delays can occur in a DG when data processing and/or transmission takes place prior to the generation of control signals. The importance of the problem stems from the practical need to know precisely how much information delay leads to intolerable performance degradation. At the same time, knowledge of the effects of time delay might permit relaxation of system design specifications to achieve an overall economy of design while maintaining acceptable performance.

The historical basis for the problem appears to be the classical "bomber-battleship duel", in which a bomber must take into account ordnance delivery time in deciding when to strike an evasive battleship. Since 1950, a number of papers have successfully treated the bomber-battleship duel as a multistage game with probabilistic strategies [2-9]. However, no treatment of the more general problem was presented, and little was known about the possible affects of information delay in dynamic conflict. Subsequently, Ciletti [10] treated finite games, multistage games and a class of differential games in which one player has an information time lag. Function space methods provided the solution to versions of the well known linear regulator differential game with information time lag (DGWITL) in [11, 12, 13]. The above-referenced works showed that a) the player with information lag can construct a payoff bound and an associated worst-case strategy by considering past as well as future behavior of his opponent, b) an optimal control problem must be solved to provide the boundary condition for the game with information time lag, and c) the opponent's strategy which induces the worst-case payoff - an ideal exploitation strategy - is not physically realizable because it requires knowledge of future states. These structural features were shown to be common to finite, multistage and differential games with information time lag. N-player DGWITL (linear plants, quadratic payoffs) also admit to analysis by function space methods [14]. Despite the availability of results for the linear problem, there was a lack of results for more general systems and payoffs. Then, an extension of the well-known Hamilton-Jacobi theory for optimal control and the familiar "main equation" analysis of Isaacs was developed to treat DGWITL [15]. The Hamilton-Jacobi equation was shown to have a dual-time-reference form in which the observation times (τ,t) of the state vectors $(x(\tau),y(t))$ as well as the spatial data appeared. It was pointed out that, in general, an optimal control problem must be solved to provide the boundary condition for the potential value function, $V^o(x,\tau,y,t)$ over the τ-t data reference plane. The generalized Hamilton-Jacobi equation for $V^o(x,\tau,y,t)$ over the τ-t plane can be converted to a partial differential equation in t along the line $\tau = t-\sigma$ for fixed σ to produce the equation satisfied by the parametric solution, $V^o(x,y,t;\sigma)$ [16]. In an independent work, Sokolov and Chernous'ko [17, 18] showed that, under certain conditions of separability, a class of DGWITL can be considered equivalent to a related DG without information time lag. Petrosjan [19] treated a discretization of a DGWITL within the context of mixed strategies.

This paper continues the developments of [15,16]. Section II briefly summarizes the main results of [15,16], and Section III demonstrates the application of the theory for DGWITL to the well-known linear regulator game. The problem will be solved in closed form over the data reference plane to provide feedback control laws which optimally utilize the delayed information. We also present the specialized solution along the locus of play for fixed σ. Structural properties which distinguish DGWITL control laws from suboptimal control laws are examined. Section IV presents results of practical significance. By means of a simple two-dimensional example, we examine the precise effects of information time lag on the outcome of a game. Then we consider alternate control schemes which may be suggested for use in DGWITL. Simulation results show the effect of using the DGWITL optimal strategy, a direct insertion feedback strategy (DIFBK), and a simple predictor feedback scheme (PRFBK).

II. SUMMARY OF DGWITL THEORY

As a point of reference for the ensuing work, we begin with the dynamic systems controlled respectively by the pursuer (P) and the evader (E):

$$\dot{x} = f(x,\tilde{u}(t),t) \tag{1a}$$

$$\dot{y} = g(y,\tilde{v}(t),t) \tag{1b}$$

when $x,y \in R^n, \tilde{u} \in R^q, \tilde{v} \in R^m$. At each time t the control inputs $\tilde{u}$ and $\tilde{v}$ are to be determined by control laws u and v which are members of sets K_u and K_v, where $K_u = \{u:R^n \times R^n \times R^1 \to A_u \subset R^q\}$, $K_v = \{v:R^n \times R^n \times R^1 \to A_v \subset R^m\}$, A_u and A_v are locally compact, and members of K_u and K_v are continuous in t and Lipschitzian in (x,y). Given regions G_u and G_v to which motions of (1a) and (1b) are restricted and a target set $S \subset G = G_u \times G_v$, we form $K_a \subset K_u \times K_v$ to include only those control law pairs which produce for any initial phase in G-S a solution which reaches S without leaving G. Accordingly, the terminal time, t_f (depending on x_o,y_o,t_o,u,v) is the first instant at which a solution to (1) penetrates S . With $(u,v) \in K_a$, $(x_o,t_o,y_o,t_o) \in G$, we associate a scalar payoff made by P to E and defined by:

$$J(x_o,y_o,t_o\, u,v) = \int_{t_o}^{t_f} L_1(x(\alpha),u(x(\alpha),y(\alpha),\alpha),\alpha)d\alpha$$

$$+ \int_{t_o}^{t_f} L_2(y(\alpha),v(x(\alpha),y(\alpha),\alpha),\alpha)d\alpha$$

$$+ \tilde{W}(x(t_f),y(t_f),t_f)$$

where $x(\cdot)$ and $y(\cdot)$ are understood to be the solution to (1) associated with the control law pair (u,v).

The methodology of game theory then centers about the task of finding a saddlepoint of $J(\cdot)$ w.r.t. u and v. The well-known results are that under appropriate smoothness assumptions the saddlepoint, or value, $V^\circ(\cdot)$, satisfies a Hamilton-Jacobi type partial differential equation. The analysis also produces an implicit definition of the optimal strategies in terms of the states and the components of the gradient of $V^\circ(\cdot)$. Solution of the HJE leads to closed form explicit definitions of the control laws u° and v° which induce the saddlepoint. Thus, implementation of the control laws requires that $x(t)$ and $y(t)$ be available to each player at every t.

The class of DGWITL which are studied here are those in which the evader E has access to $x(t-\sigma)$ instead of $x(t)$ at each time t. As shown in [15], this leads naturally to consideration of control laws $u(\cdot)$ and $v(\cdot)$ which are maps on $R^n \times R^1 \times R^n \times R^1$ so that temporal references as well as spatial data may be considered. This also leads to study of the following coupled dynamical systems:

$$\frac{d}{dt} x(t-\sigma) = f(x(t-\sigma),u(x(t-\sigma),t-\sigma,y(t),t),t-\sigma) \tag{2a}$$

$$\frac{d}{dt} y(t) = g(y(t),v(x(t-\sigma),t-\sigma,y(t),t),t) \tag{2b}$$

with $x(t_o - \sigma) = x_{o\sigma}$, $y(t_o) = y_o$. This system is one which the evader can think of as defining the evolution of his observations in time. The usual structural assumptions on $u(\cdot)$ and $v(\cdot)$ are made here to provide the existence of a unique solution to (2).

We then address the question of termination. Upon receipt of data (x,τ,y,t) E must first decide whether the game is over, i.e., could P have applied controls over $[\tau,t]$ in such a manner that the path emanating from x puts the present phase on S? This leads to an open loop boundary value control problem and the following definitions.

Def. The preliminary target set, S_p, is the set of all $(x,\tau,y,t) \in R^n \times R^1 \times R^n \times R^1$ having the property that there exists a piecewise continuous $\bar{u}(\cdot):[\tau,t] \to R^1$ and $(\bar{x},\tau,y,t) \in S$, where $\bar{x}$ is understood to be the solution of (1a) at time $t \geq \tau$ when driven by $\bar{u}$ from (x,τ). Members of S_p will be called potentially terminal phases.

Def. The potential terminal time, $t_p(x_{o\sigma},\tau_o,y_o,t_o,u,v)$ is the smallest t such that the motion of (1) is a member of S_p. Figure 1 illustrates a potentially terminal phase and a preliminary target set. Various $\bar{u}(\cdot)$ may be available for termination, and E has no way of knowing which, if any, have been used by P.

Def. For a potentially terminal admissible pair (u,v) we define the remaining payoff:

$$J_r(x_{o\sigma},\tau_o,y_o,t_o,u,\bar{u},v) = \int_{\tau_o}^{t_p-\sigma} L_1(x(\alpha),u(x(\alpha),\alpha,y(\alpha+\sigma),\alpha+\sigma),\alpha)d\alpha$$

$$+ \int_{t_o}^{t_p} L_1(y(\alpha),v(x(\alpha-\sigma),\alpha-\sigma,y(\alpha),\alpha),\alpha)d\alpha$$

$$+ \int_{t_p-\sigma}^{t_p} L_1(\bar{x}(\alpha),\bar{u}(\alpha),\alpha)d\alpha + \tilde{W}(\bar{x}(t_p),y(t_p),t_p)$$

Def. The potential terminal payoff on S_p is given by:

$$W(x,\tau,y,t) = \min_{\bar{u}\varepsilon\bar{K}_u} \tilde{W}(\bar{x}(t_p),y(t_p),t_p) +$$

$$\int_{\tau}^{t_p} L_1(x(\alpha),\bar{u}(\alpha),\alpha)d\alpha$$

Def. The potential payoff, $J_p(\cdot)$, is defined to be:

$$J_p(x_o,\tau_o,y_o,t_o) = W(x(t_p-\sigma),t_p-\sigma,y(t_p),t_p)$$

$$+ \int_{\tau_o}^{t_p-\sigma} L_1(x(\alpha),u(x(\alpha),\alpha,y(\alpha+\sigma),\alpha+\sigma),\alpha)d\alpha$$

$$+ \int_{t_o}^{t_p} L_2(y(\alpha),v(x(\alpha-\sigma),\alpha-\sigma,y(\alpha),\alpha)d\alpha$$

The problem of DGWITL is that of finding the potential value, $V^\circ(\cdot)$, or the saddlepoint of $J_p(\cdot)$ w.r.t. u and v (we assume that $V^\circ(\cdot)$ exists). With each (x,τ,y,t) we associate $V^\circ(x,\tau,y,t)$, and the results of [15] show that, under the assumption that $V^\circ(\cdot)$ is continuously differentiable in its argument in $G-S_p$, the potential value satisfies the following generalized Hamilton Jacobi equation:

$$\frac{\partial V^\circ}{\partial \tau} + \frac{\partial V^\circ}{\partial t} + H^\circ(x,\tau,y,t,V^\circ_x,V^\circ_y) = 0 \qquad (3)$$

where:

$$H^\circ(x,\tau,y,t,V^\circ_x,V^\circ_y) = H(x,\tau,y,t,V^\circ_x,V^\circ_y,k_1(x,\tau,y,t,V^\circ_x,V^\circ_y),k_2(x,\tau,y,t,V^\circ_x,V^\circ_y)),$$

$$\min_{\mu}\max_{\beta} H(x,\tau,y,t,\mu,\beta,\lambda_x,\lambda_y) = \max_{\beta}\min_{\mu} H(x,\tau,y,t,\mu,\beta,\lambda_x,\lambda_y) =$$

$$= H(x,\tau,y,t,k_1(x,\tau,y,t,\lambda_x,\lambda_y),k_2(x,\tau,y,t,\lambda_x,\lambda_y),\lambda_x,\lambda_y)$$

and

$$H(\cdot) = \langle\lambda_x, f(x,\mu,\tau)\rangle + \langle\lambda_y, g(y,\beta,t)\rangle + L_1(x,\mu,\tau) + L_2(y,\beta,t)$$

with

$$\mu \in A_u \subset R^q,\ \beta \in A_v \subset R^m, \lambda_x,\lambda_y \in R^n.$$

The generalized Hamilton-Jacobi equation is defined on the playing space for P and E and over the τ-t "data reference plane"; its boundary condition is given by the potential terminal payoff, $W(\cdot)$, on S_p. The novel feature of this equation is the presence of τ, the time corresponding to the delayed state variable x. In optimal control problems and in DG with perfect information, it is implicit that all spatial variables are associated with a single time, t.

The procedure for forming the generalized Hamilton-Jacobi equation specifies the control law for E which implements the data (x,τ,y,t) in an optimal fashion while accounting for the age of the data.

III LINEAR REGULATOR REVISITED

We now present the linear regulator as an example to illustrate application of the theory of DGWITL to a tractable problem. The systems are described by

$$\dot{x} = A_p(t)x(t) + B_p(t)\tilde{u}(t);\ x(\tau_o) = x_{o\sigma}$$

$$\dot{y} = A_e(t)y(t) + B_e(t)\tilde{v}(t);\ y(t_o) = y_o,$$

with $\tilde{u}(t) \in A_u \subset R^q,\ \tilde{v}(t) \in A_v \subset R^m.$

The target set is given by: $S = \{(x,t,y,t) : t = T\}$, therefore, the preliminary target set is simply $S_p = \{(x,\tau,y,t) : \tau = T-\sigma,\ t=T\}$, where

$\sigma \triangleq \tau_o - t_o$. The terminal time is $t_f = T$, and the potential terminal time is $t_p(x_{o\sigma}, \tau_o, y_o, t_o, u, v) = T$, since the game has specified duration. The remaining payoff function is

$$J_r(x_{o\sigma}, \tau_o, y_o, t_o, u, \bar{u}, v) = ||x(T) - y(T)||_F^2 +$$

$$\int_{\tau_o}^{T-\sigma} ||u||^2_{R_p(\alpha)} d\alpha + \int_{T-\sigma}^{T} ||\bar{u}||^2_{R_p(\alpha)} d\alpha - \int_{t_o}^{T} ||v||^2_{R_e(\alpha)} d\alpha$$

where $F \geq 0$, $R_p(\cdot) > 0$, $R_e(\cdot) \geq 0$, $||u||^2_{R_p(\alpha)}$ denotes $u'(\alpha)R_p(\alpha)u(\alpha)$ and all matrices are of consistent dimension. The potential terminal payoff on S_p is the solution to the following optimal control problem. Let $(x,\tau,y,t) \in S_p$, i.e., $\tau = T-\sigma$ and $t=T$. Then:

$$W(x, T-\sigma, y, T) = \min_{\bar{u} \in \bar{K}_u} ||\bar{x}(T) - y||_F^2 + \int_{T-\sigma}^{T} ||\bar{u}||^2_{R_p(\alpha)} d\alpha$$

and $\bar{K}_u$ is the set of piecewise continuous $\bar{u}$: $[T-\sigma, T] \rightarrow R^q$. Function space or other methods can be applied to this problem, and we simply provide here the result:

$$W(x, T-\sigma, y, T) = [\phi_p(T, T-\sigma)x - y]'[F^{-1} + SR_p^{-1}S^*]^{-1}[\phi_p(T, T-\sigma)x - y]$$

where

$$SR_p^{-1}S^* \triangleq \int_{T-\sigma}^{T} \phi_p(T,\alpha)B_p(\alpha)R_p^{-1}(\alpha)B_p'(\alpha)\phi_p'(T,\alpha)d\alpha$$

and $\phi_p(\cdot)$ is the transition matrix for $A_p(\cdot)$. The optimal terminating control, $\bar{u}$, is given by

$$\bar{u}(\alpha) = -R_p^{-1}(\alpha)B_p'(\alpha)\phi_p'(T,\alpha)[F^{-1} + SR_p^{-1}S^*]^{-1}[\phi_p(T, T-\sigma)x - y], \alpha \in [T-\sigma, T]$$

Thus, $\bar{u}(\cdot)$ is E's determination at time T of the best control P could have applied over $[T-\sigma,T]$ from $x(T-\sigma)$ with knowledge of $y(T)$. It follows that the potential payoff is simply

$$J_p(x_{o\sigma},\tau_o,y_o,t_o,u,v) = [\phi_p(T,T-\sigma)x-y]'[F^{-1} + SR_p^{-1}S^*]^{-1}[\phi_p(T,T-\sigma)x-y]$$

$$+ \int_{t_o-\sigma}^{T-\sigma} ||u||^2_{R_p(\alpha)} d\alpha - \int_{t_o}^{T} ||v||^2_{R_e(\alpha)} d\alpha$$

where $x = x(T-\sigma)$ and $y = y(T)$. To apply the theory for DGWITL to this problem we form the generalized pre-Hamiltonian:

$$H(x,\tau,y,t,\lambda_x,\lambda_y,\mu,\beta) = \langle A_p(\tau)x + B_p(\tau)\mu,\lambda_x\rangle +$$

$$\langle A_e(t)y + B_e(t)\beta,\lambda_y\rangle + \langle \mu,R_p(\tau)\mu\rangle - \langle \beta,R_e(t)\beta\rangle$$

The usual min max operation leads to:

$$k_1(x,\tau,y,t,\lambda_x,\lambda_y) = -\frac{1}{2} R_p^{-1}(\tau)B_p'(\tau)\lambda_x$$

$$k_2(x,\tau,y,t,\lambda_x,\lambda_y) = \frac{1}{2} R_e^{-1}(t)B_e'(t)\lambda_y$$

and the following generalized Hamiltonian

$$H^\circ(x,\tau,y,t,\lambda_x,\lambda_y) = x'A_p'(\tau)\lambda_x - \frac{1}{4}\lambda_x'B_p(\tau)R_p^{-1}(\tau)B_p'(\tau)\lambda_x$$

$$+ \frac{1}{4}\lambda_y'B_e(t)R_e^{-1}(t)B_e'(t)\lambda_y + y'A_e'(t)\lambda_y$$

The generalized Hamilton-Jacobi equation satisfied by the potential value is:

$$V^\circ_\tau + V^\circ_t + x'A'_p(\tau)V^\circ_x + y'A'_e(t)V^\circ_y$$

$$- \frac{1}{4}V^\circ_x{}'B_p(\tau)R_p^{-1}(\tau)B'_p(\tau)V^\circ_x + \frac{1}{4}V^\circ_y{}'B_e(t)R_e^{-1}(t)B'_e(t)V^\circ_y = 0$$

with boundary condition:

$$V^\circ(x,\tau,y,t)\Big|_{\substack{\tau=T-\sigma \\ t=T}} = [\phi_p(T,T-\sigma)x-y]'[F^{-1}+SR_p^{-1}S^*]^{-1}[\phi_p(T,T-\sigma)x-y]$$

$$= \begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} \phi'_p(T,T-\sigma) \\ \hline -I \end{bmatrix} [F^{-1} + SR_p^{-1}S^*]^{-1}[\phi_p(T,T-\sigma) \;\vdots\; -I] \begin{bmatrix} x \\ y \end{bmatrix}$$

where I denotes the identity matrix for R^n. We point out here that letting $\sigma \to 0$ and $\tau \to t$ produces the well know results for DG without time lag. Following the procedures for linear optimal control and DG we next assume that

$$V^\circ(x,\tau,y,t) = \begin{bmatrix} x \\ y \end{bmatrix}' P(\tau,t) \begin{bmatrix} x \\ y \end{bmatrix}$$

where $P(\cdot)$ is a 2n x 2n symmetric matrix. Then

$$V^\circ_x(x,\tau,y,t) = 2(P_{11}(\tau,t)x + P_{12}(\tau,t)y)$$

$$V^\circ_y(x,\tau,y,t) = 2(P_{12}(\tau,t)x + P_{22}(\tau,t)y),$$

and we are naturally led to the condition:

$$0 = \begin{bmatrix} x \\ y \end{bmatrix}' \{\frac{\partial P}{\partial \tau}(\tau,t) + \frac{\partial P}{\partial t}(\tau,t) + P(\tau,t)A(\tau,t) + A'(\tau,t)P(\tau,t)$$

$$+ P(\tau,t)B(\tau,t)P(\tau,t)\} \begin{bmatrix} x \\ y \end{bmatrix},$$

in $G - S_p$ with

$$A(\tau,t) = \left[\begin{array}{c|c} A_p(\tau) & 0 \\ \hline 0 & A_e(t) \end{array}\right] \qquad B(\tau,t) = \left[\begin{array}{c|c} -B_p(\tau)R_p^{-1}(\tau)B_p'(\tau) & 0 \\ \hline 0 & B_e(t)R_e^{-1}(t)B_e'(t) \end{array}\right]$$

$$\text{and } P(\tau,t) = \left[\begin{array}{cc} P_{11}(\tau,t) & P_{12}(\tau,t) \\ \hline P_{12}(\tau,t) & P_{22}(\tau,t) \end{array}\right]$$

Since the condition holds for arbitrary x and y, the following generalized matrix Riccati equation must be satisfied over the τ-t plane with $\tau \leq t \leq T$:

$$\frac{\partial P}{\partial \tau}(\tau,t) + \frac{\partial P}{\partial t}(\tau,t) + P(\tau,t)A(\tau,t) + A'(\tau,t)P(\tau,t)$$

$$+ P(\tau,t)B(\tau,t)P(\tau,t) = 0 \tag{4}$$

subject to:

$$P(\tau,t)\Bigg|_{\substack{\tau=T-\sigma \\ t=T}} = \left[\begin{array}{c} \phi_p'(T,T-\sigma) \\ \hline -I \end{array}\right] [F^{-1} + SR_p^{-1}S*]^{-1}[\phi_p(T,T-\sigma) \mid -I]. \tag{5}$$

From the sufficiency theorem for DGWITL [15] we conclude that the non-realizable optimal exploitation strategy for the pursuer is

$$u°(x,\tau,y,t) = -R_p^{-1}(\tau)B_p'(\tau)[P_{11}(\tau,t)x + P_{12}(\tau,t)y],$$

the DGWITL optimal strategy for the evader is:

$$v°(x,\tau,y,t) = R_e^{-1}(t)B_e'(t)[P_{12}(\tau,t)x + P_{22}(\tau,t)y],$$

and the potential value is

$$V^\circ(x,\tau,y,t) = \begin{bmatrix} x \\ y \end{bmatrix}' \; P(\tau,t) \begin{bmatrix} x \\ y \end{bmatrix}$$

in the neighborhood of $(T-\sigma,T)$ in which the solution to (4) is positive definite. Although the quadratic payoff-linear control structure of the solution has been preserved, it is important to note that the generalized Riccati equation is defined on two temporal parameters, while the potential value is defined on the spatial variables and their associated temporal parameters. This point is emphasized by Figure 2, which illustrates the so-called "data reference plane" for this problem. The shaded region of the plane below $\sigma=0(\tau=t)$ is that region over which (4) must be solved. The line $t=T$ is the locus on which the potential terminal payoff provides boundary data for $V^\circ(\cdot)$ over the shaded region. If the DG with zero time lag has already been solved, it may be used to provide boundary data along $\sigma=0$. The behavior of V° along a line of fixed t with $\tau \leq t$ is the "aging data problem," a problem of practical interest.

In general, the generalized HJE over the τ-t plane must be converted to a "delayed argument" partial differential equation in t along the line $\tau = t-\sigma$. So, we must convert the generalized Riccati equation in τ and t into a Riccati equation in t with parameter σ appearing in delayed arguments of the time varying matrices. That is, we wish to generate solutions in parametric form by solving the GHJE along a fixed-σ locus for a DGWITL in Figure 2. For a fixed σ, let

$$\hat{P}(t;\sigma) \triangleq P(\tau,t)\Big|_{\tau=t-\sigma}$$

Then:

$$\frac{d\hat{P}}{dt}(t;\sigma) = \frac{\partial P}{\partial \tau}(\tau,t)\Big|_{\tau=t-\sigma} \frac{d\tau}{dt} + \frac{\partial P}{\partial t}(\tau,t)\Big|_{\tau=t-\sigma}$$

It follows directly that the solution to (4) along the line $\tau = t-\sigma$ with boundary condition (5) must satisfy:

$$\frac{d}{dt}\hat{P}(t;\sigma) + P(t;\sigma)A(t-\sigma,t) + A'(t-\sigma,t)\hat{P}(t;\sigma)$$

$$+ \hat{P}(t;\sigma)B(t-\sigma,t)\hat{P}(t;\sigma) = 0 \qquad (6)$$

subject to:

$$\hat{P}(T;\sigma) = \begin{bmatrix} \phi_p'(T,T-\sigma) \\ - I \end{bmatrix} [F^{-1} + SR_p^{-1}S*]^{-1}[\phi_p(T,T-\sigma)\big| -I] \qquad (7)$$

To indicate the solution along $\tau = t-\sigma$ we write:

$$V^\circ(x,y,t;\sigma) = \begin{bmatrix} x \\ y \end{bmatrix}' \hat{P}(t;\sigma) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$u^\circ(x,y,t;\sigma) = -R_p^{-1}(t-\sigma)B_p'(t-\sigma)[\hat{P}_{11}(t;\sigma)x + \hat{P}_{12}(t;\sigma)y]$$

$$v^\circ(x,y,t;\sigma) = R_e^{-1}(t)B_e'(t)[\hat{P}_{12}(t;\sigma)x + \hat{P}_{12}(t;\sigma)y]$$

Figure 3 illustrates the block diagram for the evader's DGWITL feedback control law. For additional emphasis, we write $x_p(\cdot) = x(\cdot)$ and $x_e(\cdot) = y(\cdot)$. While the linear feedback structure has been preserved, the DGWITL control law is distinct from the control law for the game with zero time lag. The important difference is that the control law uses the pursuer's delayed state vector and

Riccati gain function obtained from the solution to a Riccati equation with delayed arguments.

As an alternate scheme, one can define

$$\tilde{P}(t;\sigma) = \left[\begin{array}{c|c} \phi_p'(t-\sigma,t) & 0 \\ \hline 0 & -I \end{array}\right] \hat{P}(t;\sigma) \left[\begin{array}{c|c} \phi_p(t-\sigma,t) & 0 \\ \hline 0 & -I \end{array}\right]$$

Then:

$$\begin{bmatrix} u^\circ(\cdot) \\ v^\circ(\cdot) \end{bmatrix} = \left[\begin{array}{c|c} -R_p^{-1}(t-\sigma)B_p'(t-\sigma)\phi_p'(t,t-\sigma) & 0 \\ \hline 0 & R_e^{-1}(t)B_e'(t) \end{array}\right] \tilde{P}(t;\sigma) \left[\begin{array}{c} \phi_p(t,t-\sigma)x(t-\sigma) \\ \hline y(t) \end{array}\right]$$

where $\tilde{P}(\cdot;\sigma)$ satisfies the Riccati equation in the control law block diagram of Figure 4. This representation of $v^\circ(\cdot)$ exposes the "simple predictor" portion of the control law, i.e., $\tilde{x}_p(t) = \phi_p(t,t-\sigma)x_p(t-\sigma)$. The evader can form $\tilde{x}_p(t)$ with certainty, since it is determined solely by P's dynamics. This control law also requires the solution of a Riccati equation with delay arguments. It is clear from this representation that the effect of introducing the time delay is to create both a predictor and a σ-dependent Riccati gain in E's optimal strategy.

IV. CONTROL LAW COMPARISON

In practice, an information delay may not be detected by E. E would then be led to synthesize the DGWFI control law in ignorance of the fact that $x(t-\sigma)$ was being observed instead of $x(t)$. Or, an aware E may simply conjecture that the delay is of no consequence. We call the control law which uses the delayed state in the DGWFI law a "direct insertion feedback law"(DIFL). Another possibility is that E can simply predict - in the manner already mentioned - P's present state and continue to use the Riccati gain of the DGWFI law. These two suboptimal control laws may be appealing because they don't require the use of delayed arguments in the Riccati equation solutions for their feedback gain functions. One might even expect reasonable performance from the predictor law since it synthesizes a portion of the DGWITL law. Both of the alternate control laws can be formed in Figure 4 by setting $\sigma=0$ in both the predictor and gain generator and in only the gain generator, respectively. In applying game theory we would like to know whether it is important to take into account information time delays in a competitive situation, or whether it is sufficient to use one of the suboptimal schemes described here. Although the answer to these questions depends on the individual DG, we provide, by means of an example, a quantitative illustration of the effect of information delays, with the intention of giving both an insight about the approach that can be taken for the study of other examples and a demonstration of the importance of such consideration in practical problems. We demonstrate that even small information lags (relative to the time constants) can lead to serious performance degradation if the delay is ignored and the DIFBK law is used. At the same time, a simple predictor may likewise be inadequate. We also show that the DGWITL law effectively compensates for the information time lag and results in a marked improvement in performance. In addition to comparing these control laws, we examine the effect of the time lag on the existence of the solution to a DG example, i.e., the effect of time lag on the occurrence of a conjugate point in the delayed argument Riccati equation solution.

To illustrate the various points in the preceding discussion we consider the scalar version of the linear regulator results. The equations of motion are

$$\dot{x}_p = a_p x_p + b_p \tilde{u} \tag{8}$$

$$\dot{x}_e = a_e x_e + b_e \tilde{v}, \tag{9}$$

and all parameters are scalar constants. The remaining payoff is taken to be:

$$J_r(\cdot) = |x_p(T) - x_e(T)|^2 + \frac{1}{c_p} \int_\tau^{T-\sigma} u^2(\alpha)d\alpha$$

$$- \frac{1}{c_e} \int_t^T v^2(\alpha)d\alpha + \frac{1}{c_p} \int_{T-\sigma}^T u^2(\alpha)d\alpha, \tag{10}$$

with $\sigma = \tau - t$, $c_p > 0$, $c_e > 0$ and T fixed. The potential terminal payoff is the solution to the optimal control problem given by

$$W(x_p(T-\sigma),T-\sigma,x_e(T),T) = \min_{\bar{u}\varepsilon\bar{K}_u}\{|\bar{x}_p(T) - x_e(T)|^2 + \frac{1}{c_p} \int_{T-\sigma}^T \bar{u}^2(\alpha)d\alpha\}$$

subject to $\dot{x}_p = a_p x_p + b_p \bar{u}$, $T-\sigma \leq t \leq T$, $x_p(T-\sigma)$ given.

The solution can easily be shown to be:

$$W(x_p(T-\sigma),T-\sigma,x_e(T),T) = \frac{[e^{a_p} x_p(T-\sigma) - x_e(T)]^2}{1 + \frac{c_p b_p^2}{2a_p}[e^{2a_p\sigma}-1]} \quad ,$$

The potential payoff is:

$$J_p = \frac{[e^{a_p\sigma} x_p(T-\sigma) - x_e(T)]^2}{1 + \frac{c_p b_p^2}{2a_p}[e^{2a_p\sigma}-1]} + \frac{1}{c_p} \int_\tau^{T-\sigma} u^2(\cdot)d\alpha - \frac{1}{c_e} \int_t^T v^2(\cdot)d\alpha.$$

By performing the analysis which was described in the preceding sections we are able to show that the potential value and the optimal strategies have the following solution:

$$V^\circ(x_p,\tau,x_e t) = \frac{[e^{a_p(T-\tau)}x_p - e^{a_e(T-t)}x_e]^2}{G(\tau,t)} \tag{11}$$

$$u^\circ(x_p,\tau,x_e,t) = \frac{-c_e b_e[e^{a_p(T-\tau)}x_p - e^{a_e(T-t)}x_e]\, e^{a_p(T-\tau)}}{G(\tau,t)} \tag{12}$$

$$v^\circ(x_p,\tau,x_e,t) = \frac{-c_e b_e[e^{a_p(T-\tau)}x_p - e^{a_e(T-t)}x_e]\, e^{a_e(T-t)}}{G(\tau,t)} \tag{13}$$

where:

$$G(\tau,t) = 1 + \frac{c_p b_p^2}{2a_p}[\, e^{2a_p(T-\tau)} - 1] - \frac{c_e b_e^2}{2a_e}[e^{2a_e(T-t)} - 1] \tag{14}$$

Alternately:

$$V^\circ(x_p,x_e,t;\sigma) = \frac{[e^{a_p(T-t+\sigma)}x_p(t-\sigma) - e^{a_e(T-t)}x_e(t)]^2}{\hat{G}(t;\sigma)} \tag{15}$$

$$u^\circ(x_p,x_e,t;\sigma) = \frac{-c_p b_p[e^{a_p(T-t+\sigma)}x_p(t-\sigma) - e^{a_e(T-t)}x_e(t)]\, e^{a_p(T-t+\sigma)}}{\hat{G}(t;\sigma)} \tag{16}$$

$$v^{\circ}(x_p,x_e,t;\sigma) = \frac{-c_e b_e [e^{a_p(T-t+\sigma)} x_p(t-\sigma) - e^{a_e(T-t)} x_e(t)]\, e^{a_e(T-t)}}{\hat{G}(t;\sigma)} \tag{17}$$

and

$$\hat{G}(t;\sigma) = 1 + \frac{c_p b_p^2}{2a_p}[e^{2a_p(T-t+\sigma)} - 1] - \frac{c_e b_e^2}{2a_e}[e^{2a_e(T-t)} - 1] \tag{18}$$

Figure 5 contains the block diagram for (15), with the "simple predictor" portion highlighted.

It is easily verified that the conjugate point of the Riccati gain matrix corresponding to (15) is solely determined by the zero crossing of $\hat{G}(t;\sigma)$ in the neighborhood of T, and ultimately on the relative value of the parameters in (8), (9) and (10).

To explore the effect of information delay on the existence of solutions to the linear DGWITL, we examine (14) and seek the locus of conjugate points in the data reference plane. Letting $\tau_{CR}(t)$ denote the value of $\tau \leq T$ at which (14) has a zero crossing we are able to locate the conjugate points for fixed system parameters. For our purposes it is enough to present the results in Figure 6, which show $\tau_{CR}(t)$ vs t for various system configurations, with $b_p = 0.8$, $c_p = c_e = 1.0$, $a_p = -0.2$, $a_e = -0.4$, and $T = 5.0$. The area under the curve defined by $\tau_{CR}(\cdot)$ is the region of the plane for which $\hat{G}(\cdot)$ is positive. Lines drawn parallel to and below the line $\tau = t$ correspond to DGWITL. For $b_e = 1.35$ the line $\tau = t$ does not intersect τ_{CR}, so neither the DGWFI or the DGWITL have conjugate points. For $b_e = 1.4$ the DGWFI has a conjugate point at $t = 3.5$ secs. A DGWITL with $0 \leq \sigma \leq 0.7$ also has a conjugate point, and the time at which it occurs is farther from T than the time associated with the DGWFI conjugate point. Hence, the information delay extends the half-interval about T for which (18) is valid. For $\sigma > 0.7$ the DGWITL does not have a conjugate point. Thus, if sufficient information delay is present the conjugate point vanishes. For $b_e = 1.45$ the DGWFI has a conjugate point at $t = 3.8$. The DGWITL also has a finite-time conjugate point at $\hat{t}(\sigma)$. For $\sigma = 3$, $\hat{t} = -1.25$. These three choices of b_e demonstrate that the information delay effectively lengthens the solution interval about T and may extend it to $-\infty$, depending on the problem parameters. Similar results can be obtained for formulations in which the pursuer has an information delay.

To demonstrate the importance of properly compensating for information time lag in a differential game, we now consider three control laws which may be implemented in a DGWITL. The first control law is the DGWITL law of Figure 5. The "direct insertion feedback" law (DIFBK) is synthesized by using the perfect information control law in spite of the presence of delayed data. In Figure 5 this corresponds to setting $\sigma=0$ in the Riccati gain and in the predictor. By ignoring the time delay, E synthesizes this law. The third law we synthesize is the "predictor feedback" law (PRFBK). This law approximates the DGWITL law by setting $\sigma=0$ in the Riccati gain while maintaining the correct σ in the predictor. In general, E might be motivated to do this by the requirement of solving the generalized Riccati equation when the various system matrices are time varying. Certainly this might be a reasonable scheme when $\tilde{P}(\cdot;\sigma)$ is not sensitive to σ over the interval of interest.

To compare these control laws we simulated (8) and (9) and evaluated (10) on a digital computer, using Runge-Kutta integration. Because the pursuer law given by (16) is not physically realizable (see [15]) we simulated a pursuer that used the following DGWFI control law:

$$u^o(x_p,x_e,t) = \frac{-c_p b_p [e^{a_p(T-t)} x_p(t) - e^{a_e(T-t)} x_e(t)] e^{a_p(T-t)}}{\hat{G}(t;0)} \tag{19}$$

For the purpose of simulating this control law, E's path history over $[t_o-\sigma, t_o]$ was taken to be the autonomous motion of (9) with terminal boundary condition given by $y(t_o)$. This corresponds more closely to a physical situation in which E is inactive prior to being alerted that the conflict has commenced. For the purposes of simulation, the following parameters remained: $a_p = -.2$, $a_e = -.4$, $b_p = .8$, $b_e = 1.2$, $c_p = 1.0$, $c_e = 1.0$, $T = 5.0$. For a fixed $x_p(t_o-\sigma)$ and $x_e(t_o)$ we then chose t_o, $t_o \leq 5.0$ and simulated the game for various σ, with $0 \leq \sigma \leq 5.0$. Thus, each choice of the pair $(t_o-\sigma)$ corresponds to a different game. Letting $t_o \to T$ results in simulations having a shorter interval of play for E. Since the physical data, $x_p(t_o-\sigma)$ and $x_e(t_o)$, remain fixed for all simulations, increasing σ effectively changes the time corresponding to P's location at the fixed x_p. This corresponds to the practical problem of determining how the outcome varies as a function of the observation time associated with the fixed spatial data. In general, control over starting times and initial data may not be allowable, so the important comparison to be made is that which relates the payoffs associated with the three control laws for a given t_o, σ, $x_p(t_o-\sigma)$, and $x_e(t_o)$.

By letting $x_p(t_o-\sigma) = 10.0$ and $x_e(t_o) = 20.0$ we obtain the potential value, V^o, directly from (15). The behavior of V^o is displayed in figure 7. Figures 8-10 illustrate the simulated J_{TL}, J_{DI}, and J_{PR}, the remaining payoffs when the time lag control law, the direct insertion feedback law, and the predictor feedback law are used, respectively, by E. The graphs show the behavior of these payoffs for games beginning at various times and having various information lags. Examination of the curves for a given t_o and σ shows that the DGWITL control law bounds the payoff from below by the potential value. Since P can't realize the optimal exploitation control law, J_{TL} is actually better than V^o. At the same time, the DIFBK and PRFBK laws permit large negative payoffs. It is also of interest to note that serious performance degradation can occur when the information time lag is ignored (DIFBK), even for data lags that are small relative to the time constants of the systems, e.g., for $t_o = 0.0$ and $\sigma = 1.0$ (the system time constants are 2.5 secs. and 5.0 secs.). Figure 11 illustrates the relative performance of the control laws for $t_o = 0.0$ by showing the loss in payoff incurred by DIFBK (η_{DI}) and PRFBK (η_{PR}) normalized by the maximum loss incurred by DIFBK for $0 \leq \sigma \leq 5.0$.

To interpret the numerical data in more detail it should be noted that, because the plants are stable and the initial coordinates were both taken to be positive, the natural motion of (8) over $[t_o-\sigma, t_o]$ works to P's disadvantage - in the sense that if P was inactive over $[t_o-\sigma, t_o]$ the actual separation at time t_o would be greater than $|x_p(t_o-\sigma) - x_e(t_o)|$. In Fig. 7 the potential value is always bounded from below by zero, since E can choose to do nothing in (9). For $t_o = 5.0$ the solution corresponds to the potential terminal payoff.

Figure 12 shows the behavior of J_{TL}, J_{DI} and J_{PR} for the same spatial data but with $t_o = -5.0$. For this game of longer duration (10 secs.) non-negligible payoff degradation accompanies DIFBK and PRFBK control even for relatively small information delays. Fig. 13 shows the P and E trajectories resulting from DGWITL and DIFBK control by E. Although the terminal miss for DIFBK is better, E's net performance is worse because the DIFBK law erroneously uses excessive energy in the terminal phase of play. Figure 14 shows the remaining payoffs for a game of 5.0 secs. duration with different spatial coordinates. Here the detrimental use of the DIFBK law is very apparent. Figure 15 shows the payoffs for the same spatial data but for play of 10 secs. duration. Again, it is clear that ignoring the information delay can lead to serious performance degradation. At the same time, even the predictor control law leads to what might be considered to be unacceptable performance. On the other hand, the DGWITL control law appears to lead to performance that is much less sensitive to the information delay.

To display the effect of the relative location of P and E w.r.t. the origin, we let $x_p(t_o-\sigma) = 20.0$ and $x_e(t_o) = 10.0$. Figure 16 shows the plot of the potential value. For ;iven t_o, $V°(\sigma)$ decreases w.r.t. σ for the range of σ shown, except for $t_o = 5.0$, where a slight increase occurs near $\sigma = 5.0$. The main effect causing this decrease of potential value as the data ages is the natural component of P's motion over $[t_o-\sigma,t_o]$. This tends to place the actual $x_p(t_o)$ closer to $x_e(t_o)$ with no effort on behalf of P. For σ beyond 3.5 secs., however, this effect works to E's advantage.

Figures 17, 18 and 19 show J_{TL}, J_{DI} and J_{PR} for the various t_o and σ. The superiority of the DGWITL control law is evident. Figure 20 shows these payoffs for $t_o = -5.0$ (i.e., a 10 sec. game). ' ;ain, the payoff degradation due to failure to compensate properly for the information delay is evident.

As an alternate scheme, we simulated the situation in which the pursuer was inactive over $[t_o-\sigma,t_o]$, thus the motion of both P and E was autonomous over this interval. Figures 21, 22 and 23 contain plots of $x_p(t_o-\sigma) = 10.0$ and $x_e(t_o) = 20.0$. Here we see that for large σ and a short interval of play (e.g., $t_o = 4.5$ secs.) the DIFBK law does slightly better than the DGWITL law. Since P is not synthesizing the exploitation control law, this is possible. However, the DIFBK law does not provide E with the payoff-bounding property.

Also, for small σ the superiority of the DGWITL law is retained. In comparing Figs. 21-23 with Figs. 8-10, we expect the greatest difference in performance to occur for larger σ, where the effect of the different initialization scheme is more pronounced. Fig. 24 illustrates the performance due to the three control laws for $t_o = 0.0$ and $x_p(t_o-\sigma) = 10.0$, $x_e(t_o) = 20.0$. Figs. 25-27 are for spatial data of $x_p(t_o-\sigma) = 20.0$ and $x_e(t_o) = 10.0$. Fig. 28 contains $J_{TL}(\sigma)$, $J_{DI}(\sigma)$ and $J_{PR}(\sigma)$ for the same data and $t_o = 0.0$.

V. CONCLUSIONS.

It is clear from the few basic examples presented here that failure to include a consideration of information delay in the synthesis of a control law can result in serious performance degradation, even for relatively small amounts of time delay. We expect that similar phenomena can occur in more complex and realistic problems. Our work in this paper serves to demonstrate, for the first time, that detection of information delays and determination of their significance by means of the DGWITL theory should be part of the overall synthesis procedure in situations of dynamic conflict.

From an engineering viewpoint, the DGWITL theory provides the framework within which the system designer can assess the payoff degradation due to information delay and establish acceptable amounts of delay. The control law synthesized by the DGWITL theory can also be used as a means for comparing the performance of suboptimal control laws, such as the predictor law used in the examples. In general it is necessary to perform the assessment over the regime of spatial data for the problem. Some spatial regimes may admit the use of a suboptimal law, while others may require the DGWITL law in order to maintain system performance. Not only does the DGWITL theory allow the system designer to assess the payoff degradation; it also provides the designer with an important engineering option. If the DGWITL law can maintain acceptable performance over a spatial regime, then the designer has the option of deliberately introducing information delay so that a single slower computer - data acquisition system may be used, or a sufficiently fast computer - data acquisition system may be used to simultaneously control several systems, each of which employs a DGWITL controller*. These engineering tradeoffs underscore the relevance and importance of the theory of differential games with information delay.

* For a hypothetical example, from Fig. 20 it is clear that the DGWITL law would enable use of a computer - data acquisition system which generated a one second information delay; alternately, more than one system may possibly be controllable by a faster computer which presented each system with the same one second delay.

References.

[1] R. Isaacs, Differential Games, J. Wiley and Sons, New York, 1965.

[2] L.E. Dubins, "A Discrete Evasion Game," Institute of Air Weapons Research Technical Note No. 2, University of Chicago.

[3] R. Isaacs and S. Karlin, "A Game of Aiming and Evasion," Rand Corporation Report RM-1316, 1954.

[4] H. Scarf and L. Shapley, "Games with Information Lag," Rand Corporation Report RM-1320, 1954.

[5] R. Isaacs, "A Game of Aiming and Evasion: General Discussion and the Marksman's Strategies," Rand Corporation Report RM-1385, 1954.

[6] H. Scarf and L. Shapley, "Games with Partial Information," Contributions to the Theory of Games, Vol. III, Annals of Mathematics Study No. 39, Princeton University Press, 1957.

[7] S. Karlin, "An Infinite Move Game with a Lag," Contributions to the Theory of Games, Vol. III, Annals of Mathematics Study No. 39, Princeton University Press, 1957.

[8] L.E. Dubins, "A Discrete Evasion Game," Contributions to the Theory of Games, Vol. III, Annals of Mathematics Study No 39, Princeton University Press, 1957.

[9] R. Isaacs, "The Problem of Aiming and Evasion," Naval Res. Logistics Quarterly, Vol. 2, Nos 1 and 2, June 1955.

[10] M.D. Ciletti, "On a Class of Deterministic Differential Games with Imperfect Information," Dept. of Elect. Engineering T.R. EE679, Univ. of Notre Dame, Notre Dame, Ind., Dec. 1967.

[11] M.D. Ciletti, "Functional Analysis and an Imperfect Information Differential Game," Proc. of Eleventh Midwest Symposium on Circuit Theory,Univ. of Notre Dame, Notre Dame, Ind., 1968.

[12] M.D. Ciletti, "A Differential Game with Information Time Lag," Proc. of the First International Conference on the Theory and Applications of Differential Games, Univ. of Massachusetts, Amherst, Mass., 1969.

[13] M.D. Ciletti, "Results in the Theory of Linear Differential Games with Information Time Lag," Journal of Optimization Theory and Applications, Vol. 5, No. 5, 1970, pp. 347-362.

[14] M.D. Ciletti, "Open-Loop Nash Equilibrium Strategies for an N-Person, Nonzero-Sum Differential Game with Information Time Lag," in Differential Games and Related Topics, ed. by H.W. Kuhn and G.P. Szego, North Holland Publishing Company, Amsterdam, 1971.

[15] M.D. Ciletti, "New Results in the Theory of Differential Games with Information Time Lag," Journal of Optimization Theory and Applications, Vol. 8, No. 4, 1971, pp. 287-315.

[16] M.D. Ciletti, "A Note on the Generalized Hamilton-Jacobi Equation for Differential Games with Information Time Lag," Proc. of Ninth Allerton Conference on Circuits and System Theory, University of Illinois, Urbana, Ill., 1971.

[17] B.N. Sokolov and F.L. Chernous'ko, "Differential Games with Information Lag," Journal of Applied Mathematics and Mechanics, Vol. 34, No. 5, 1970, pp. 779-785.

[18] F.L. Chernous'ko, "Differential Games with Information Delay," Soviet Physics-Doklady, Vol. 14, No. 10, 1970, pp. 952-954.

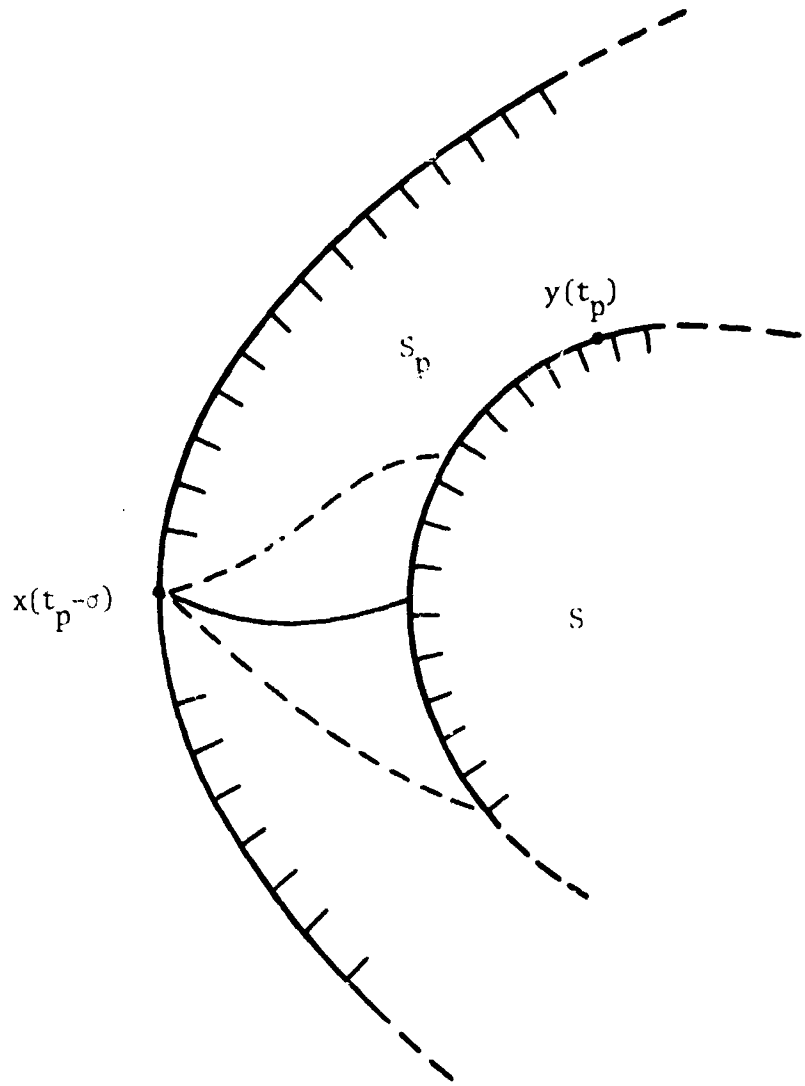

Fig. 1. Preliminary Target Set.

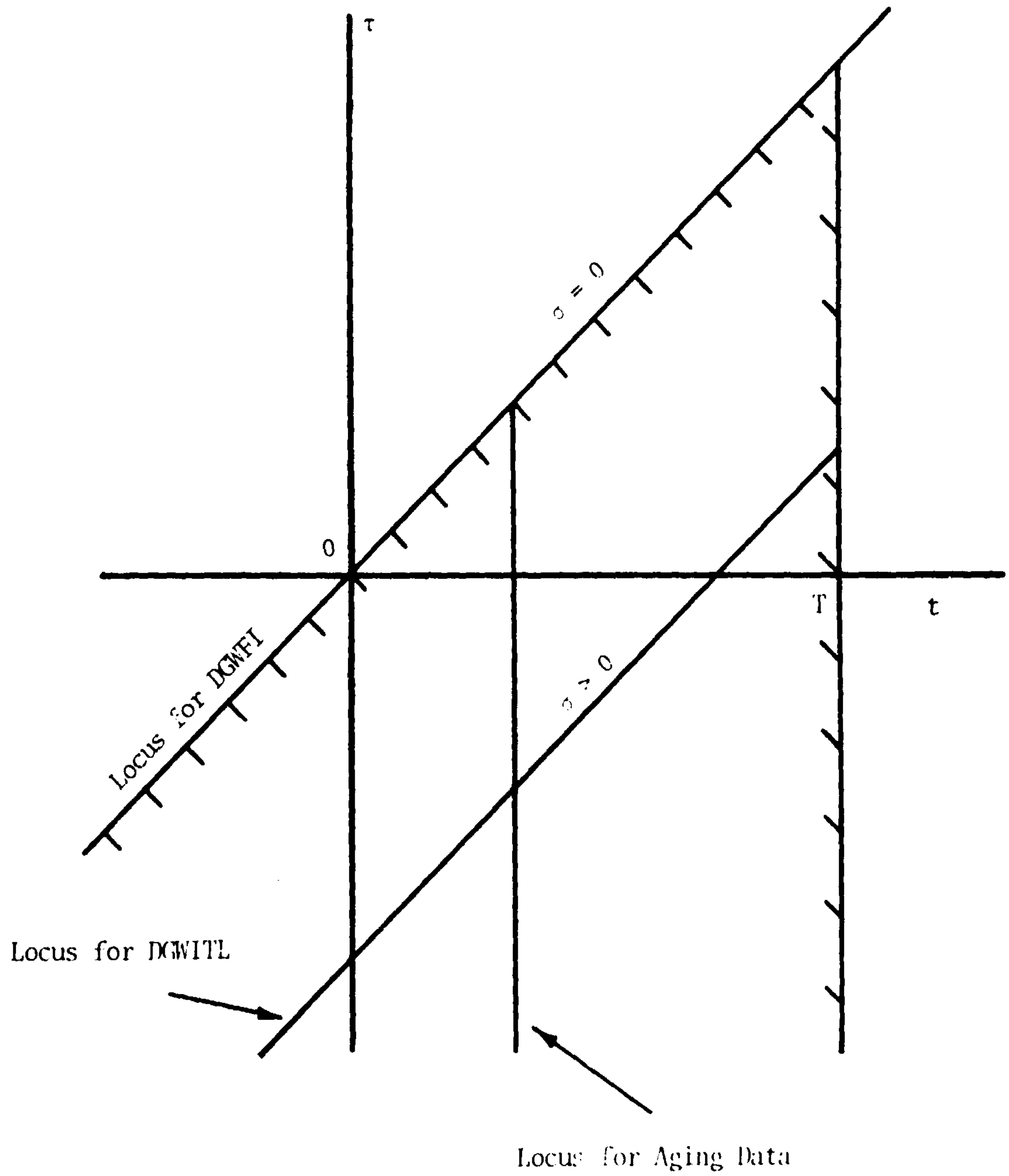

Fig. 2. Data Reference Plane.

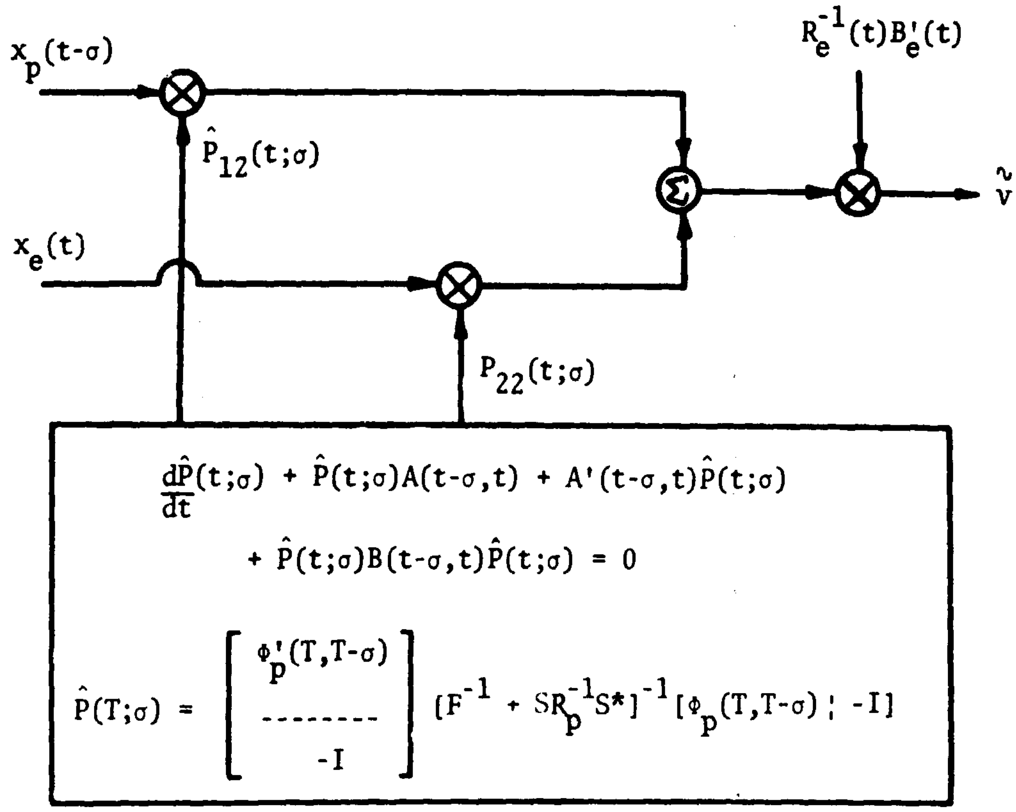

Fig. 3. Evader DGWITL Control Law.

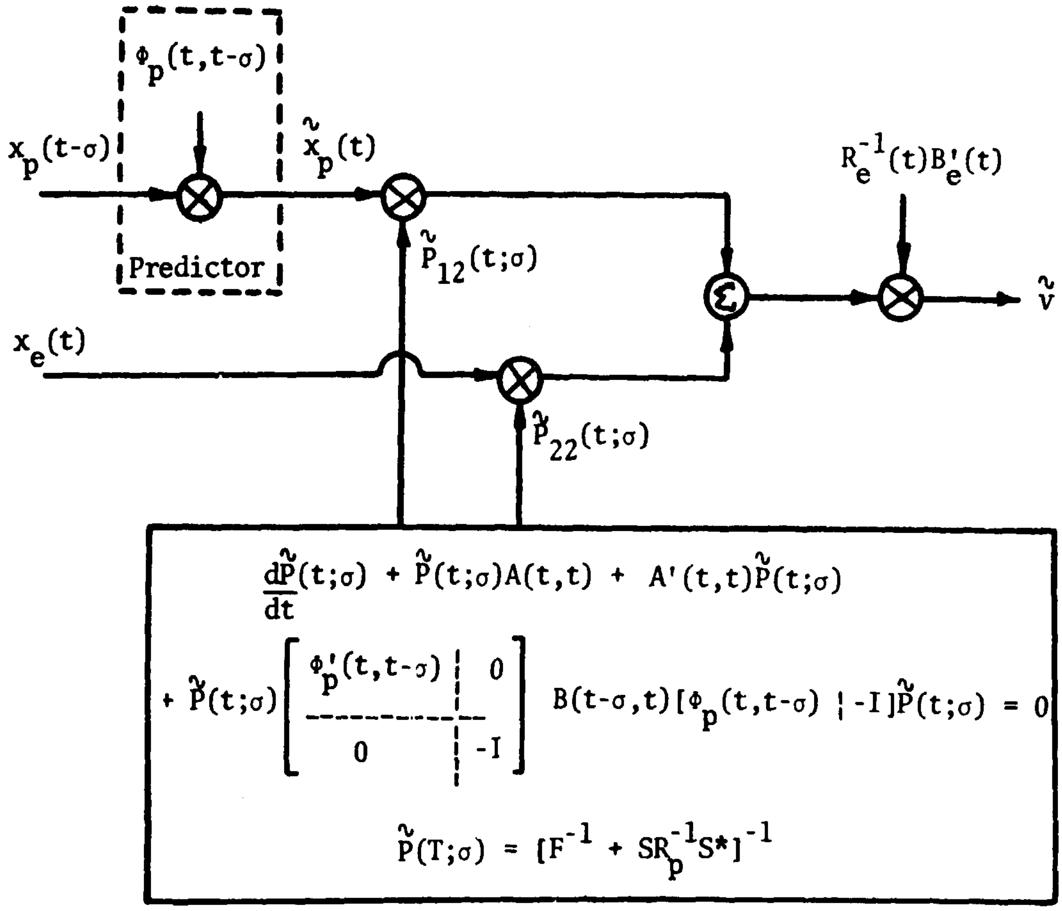

Fig. 4. Predictor Decomposition of DCMITL Control Law.

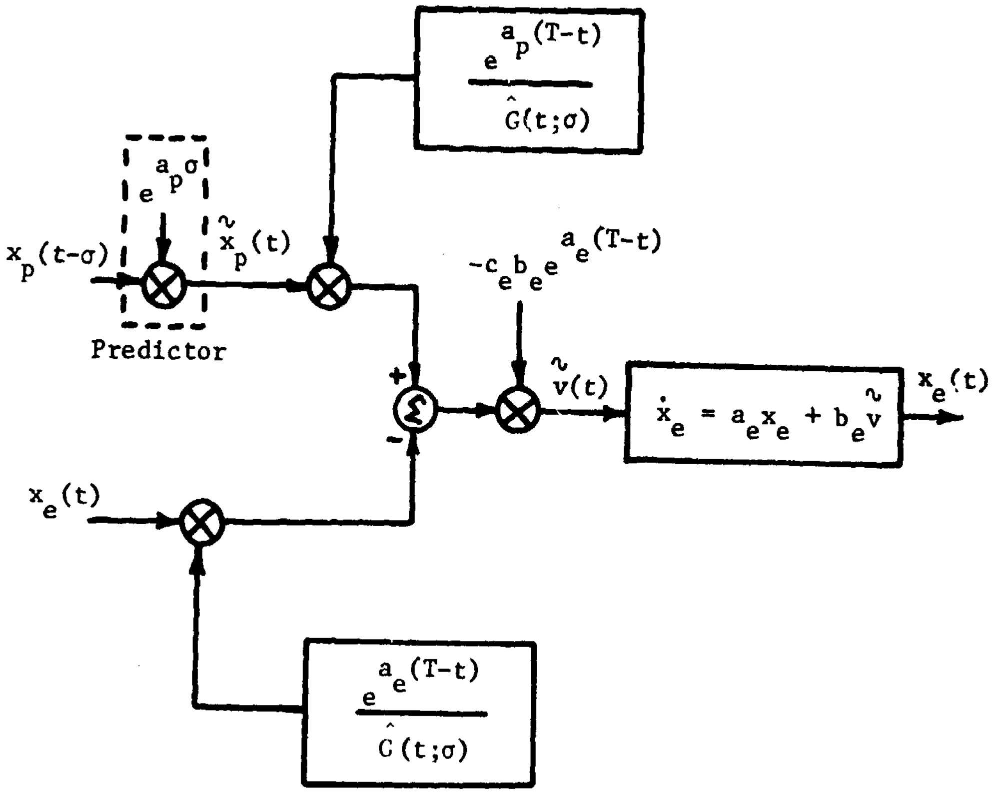

Fig. 5. Evader Control Law.

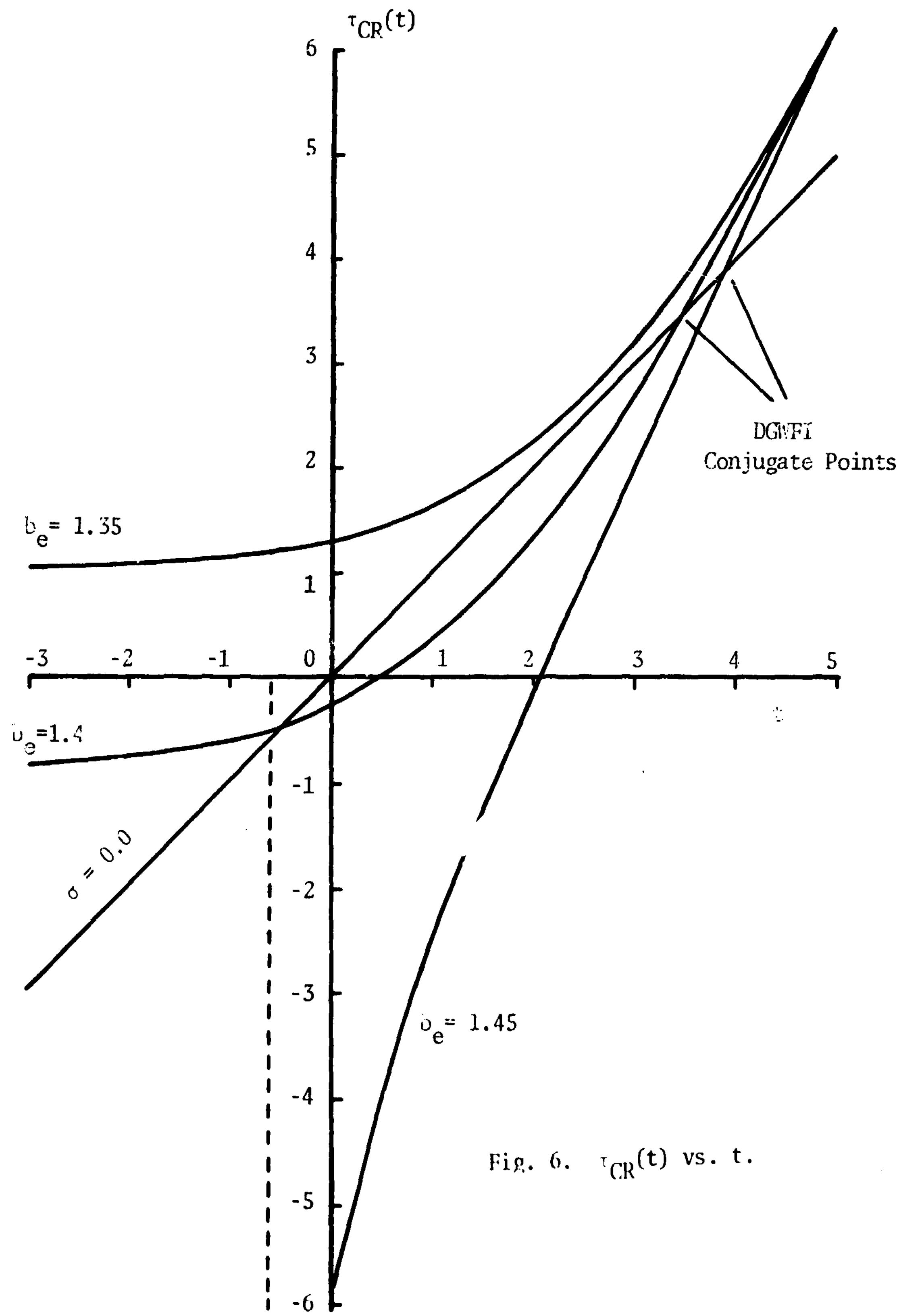

Fig. 6. $\tau_{CR}(t)$ vs. t.

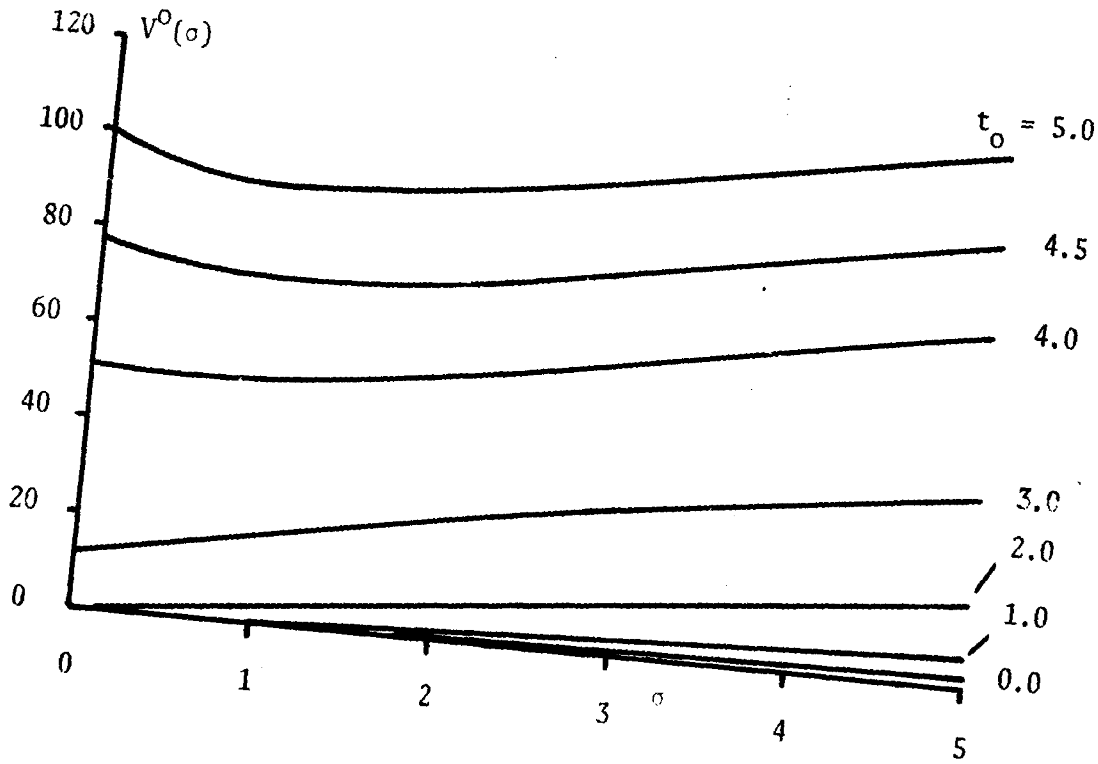

Fig. 7. Potential Value vs. Data Delay.

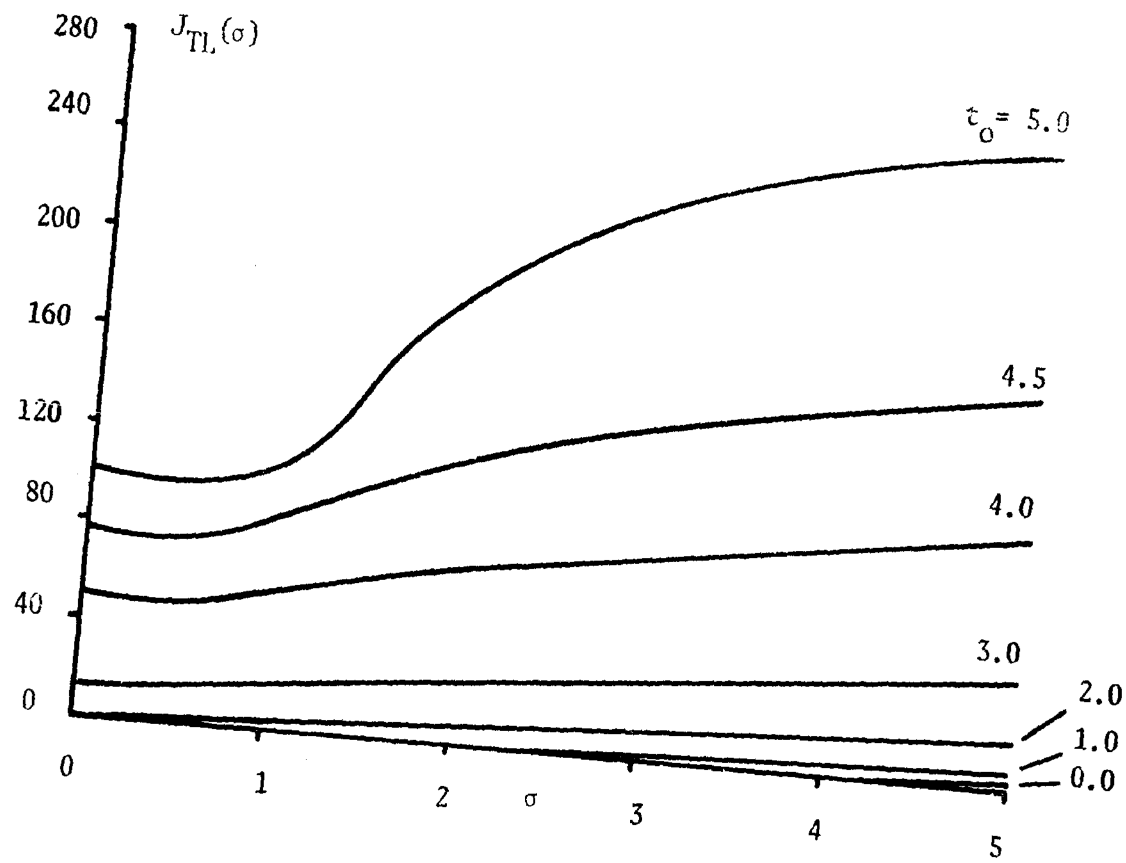

Fig. 8. $J_{TL}(\sigma)$ vs. σ.

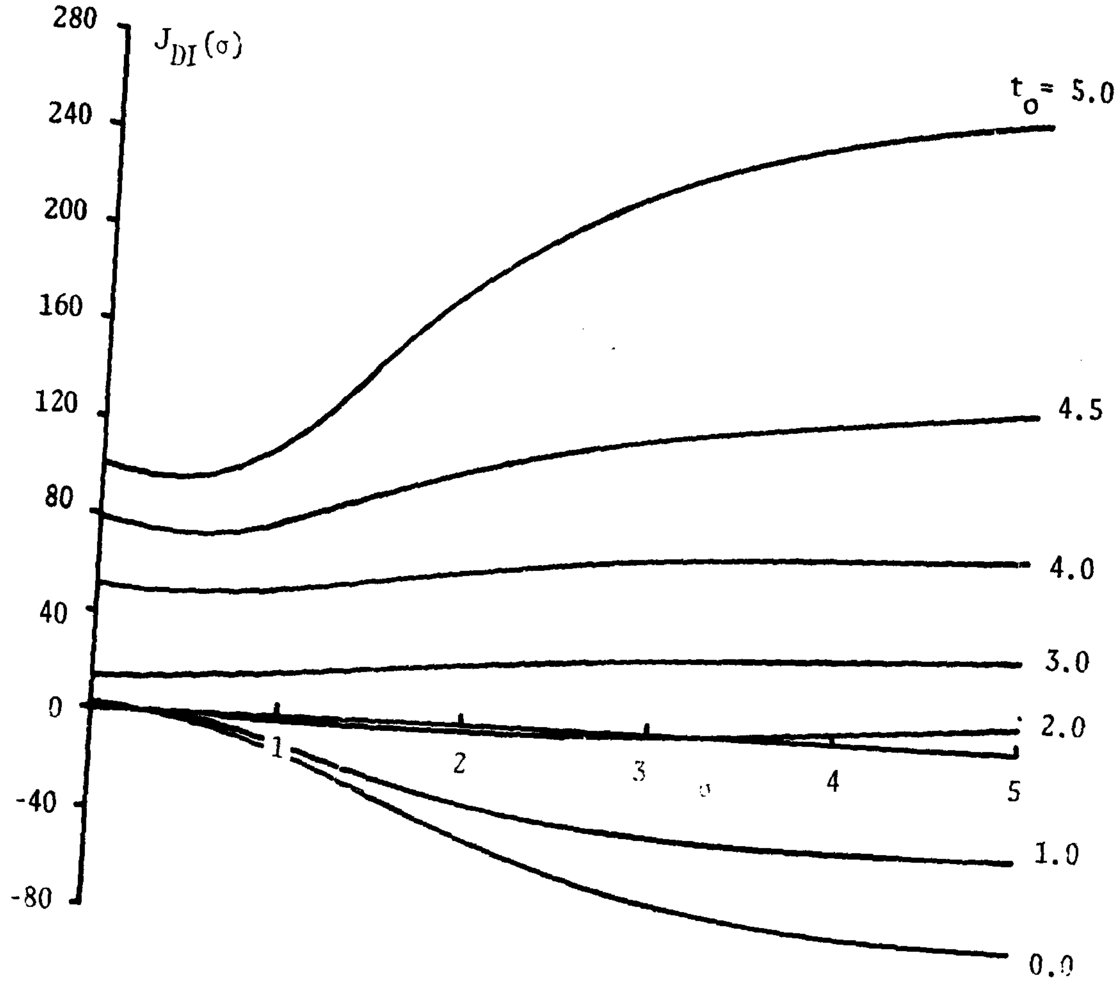

Fig. 9. $J_{DI}(\sigma)$ vs. σ.

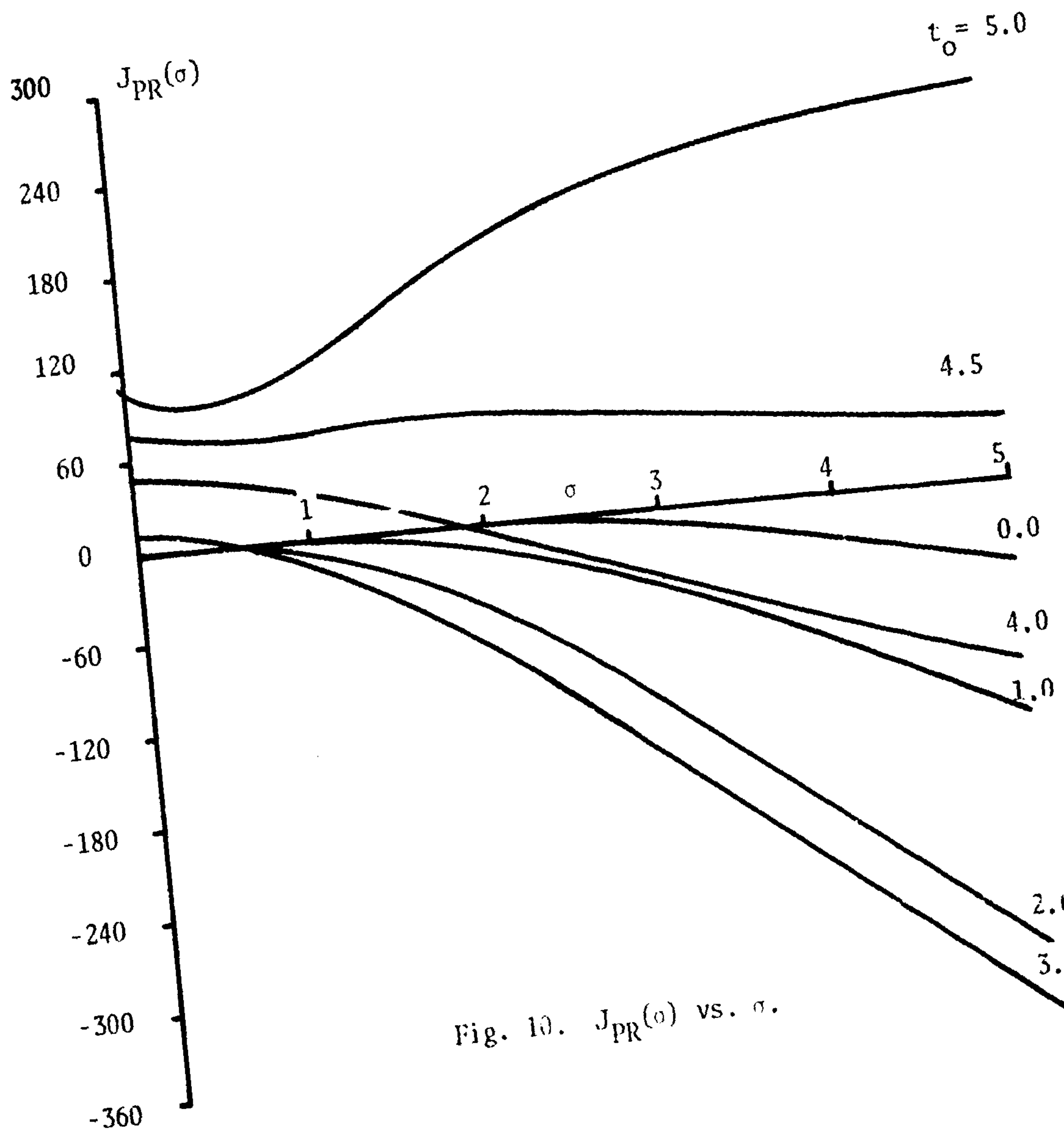

Fig. 10. $J_{PR}(\sigma)$ vs. σ.

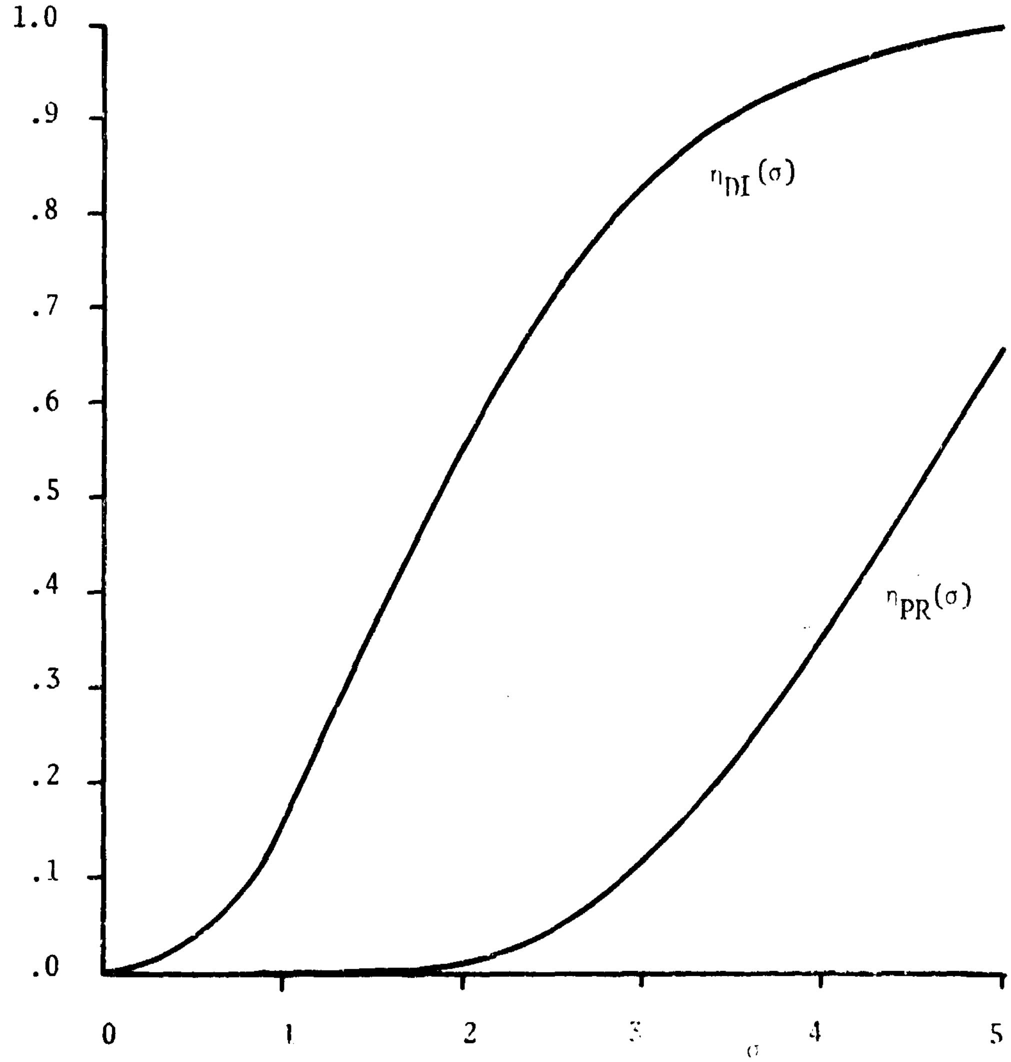

Fig. 11. η_{DI} and η_{PR} vs. σ.

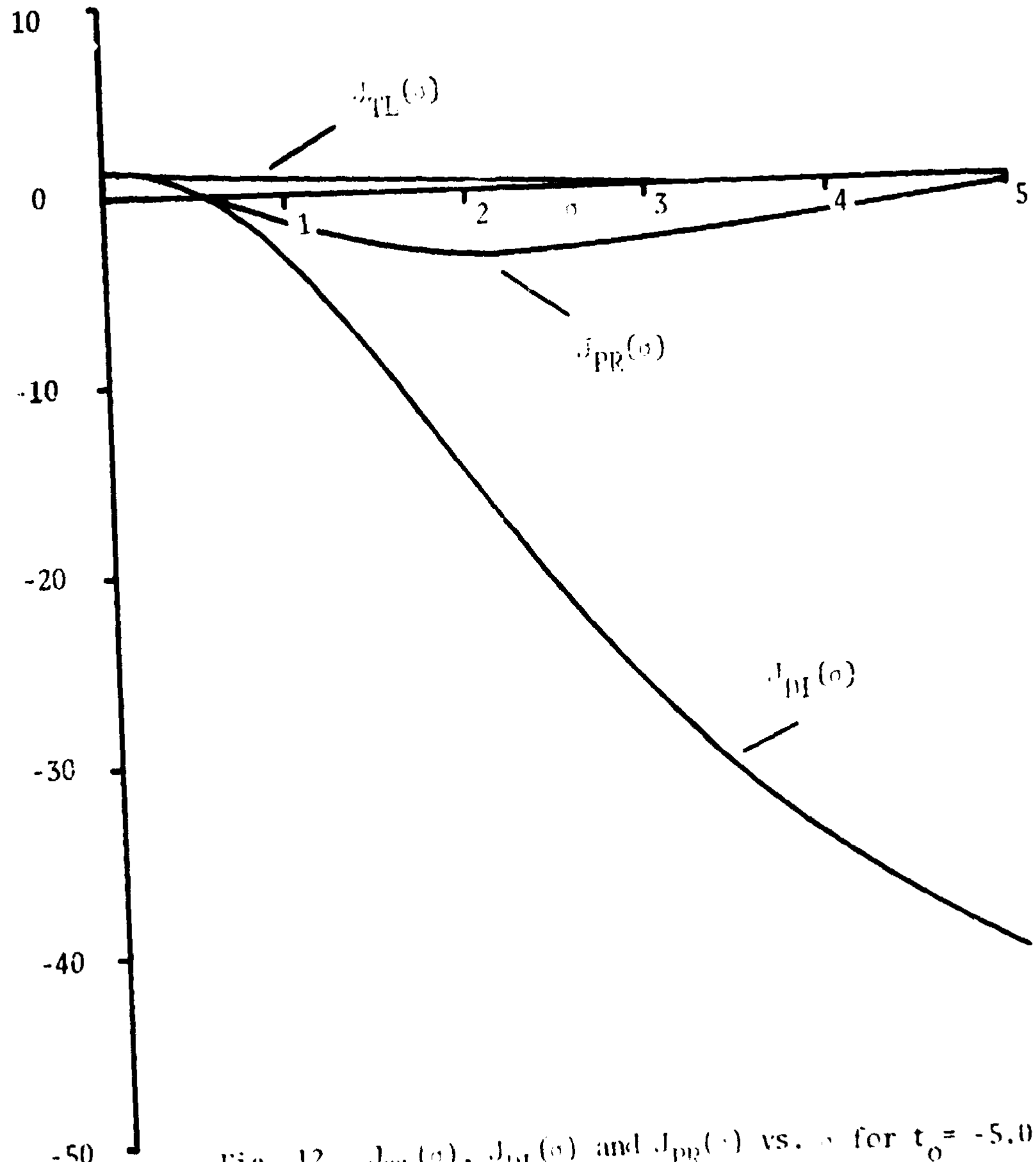

Fig. 12. $J_{TL}(\sigma)$, $J_{DI}(\sigma)$ and $J_{PR}(\sigma)$ vs. σ for t_o= -5.0.

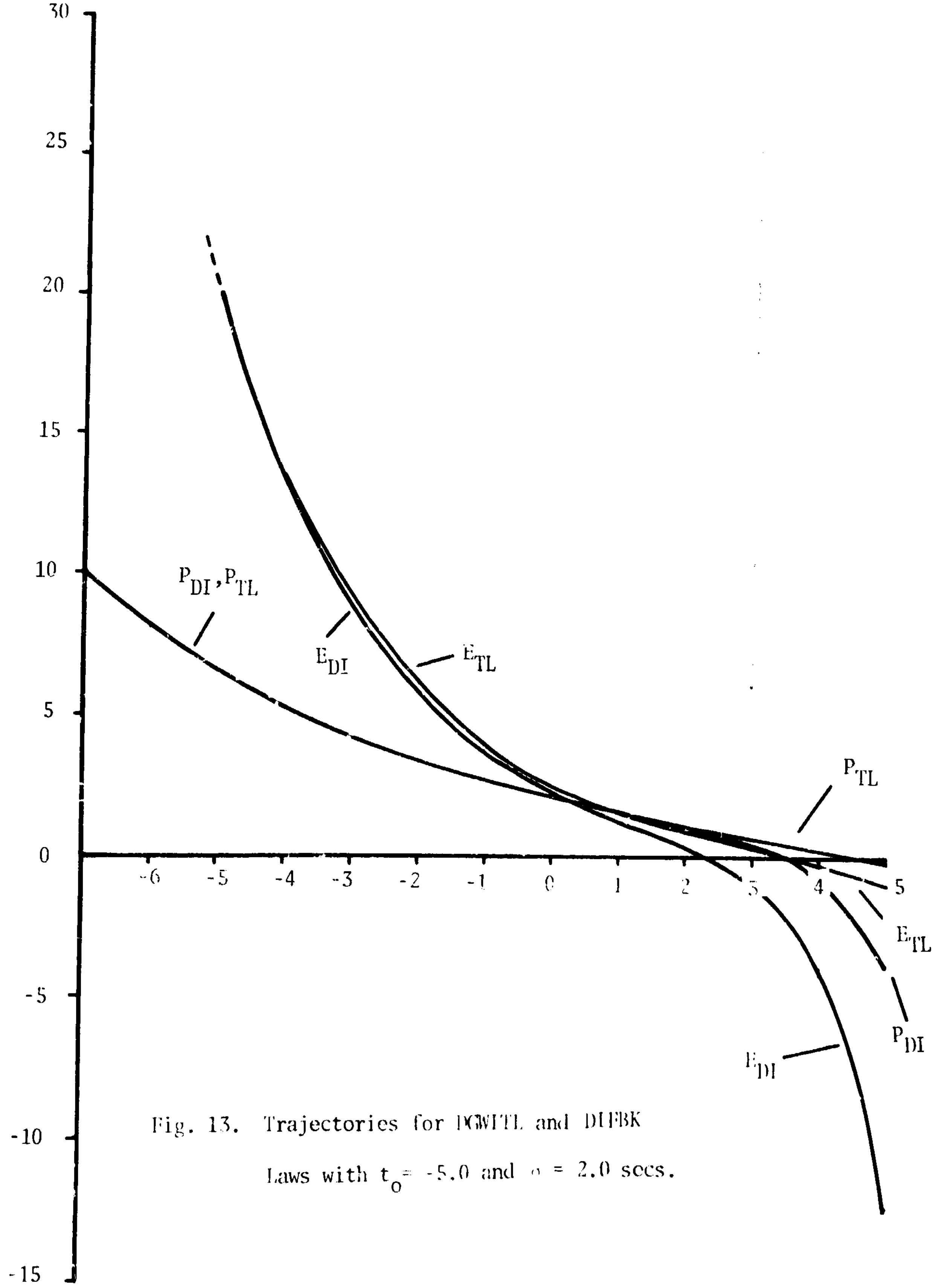

Fig. 13. Trajectories for DGWITL and DIFBK
Laws with $t_o = -5.0$ and $\sigma = 2.0$ secs.

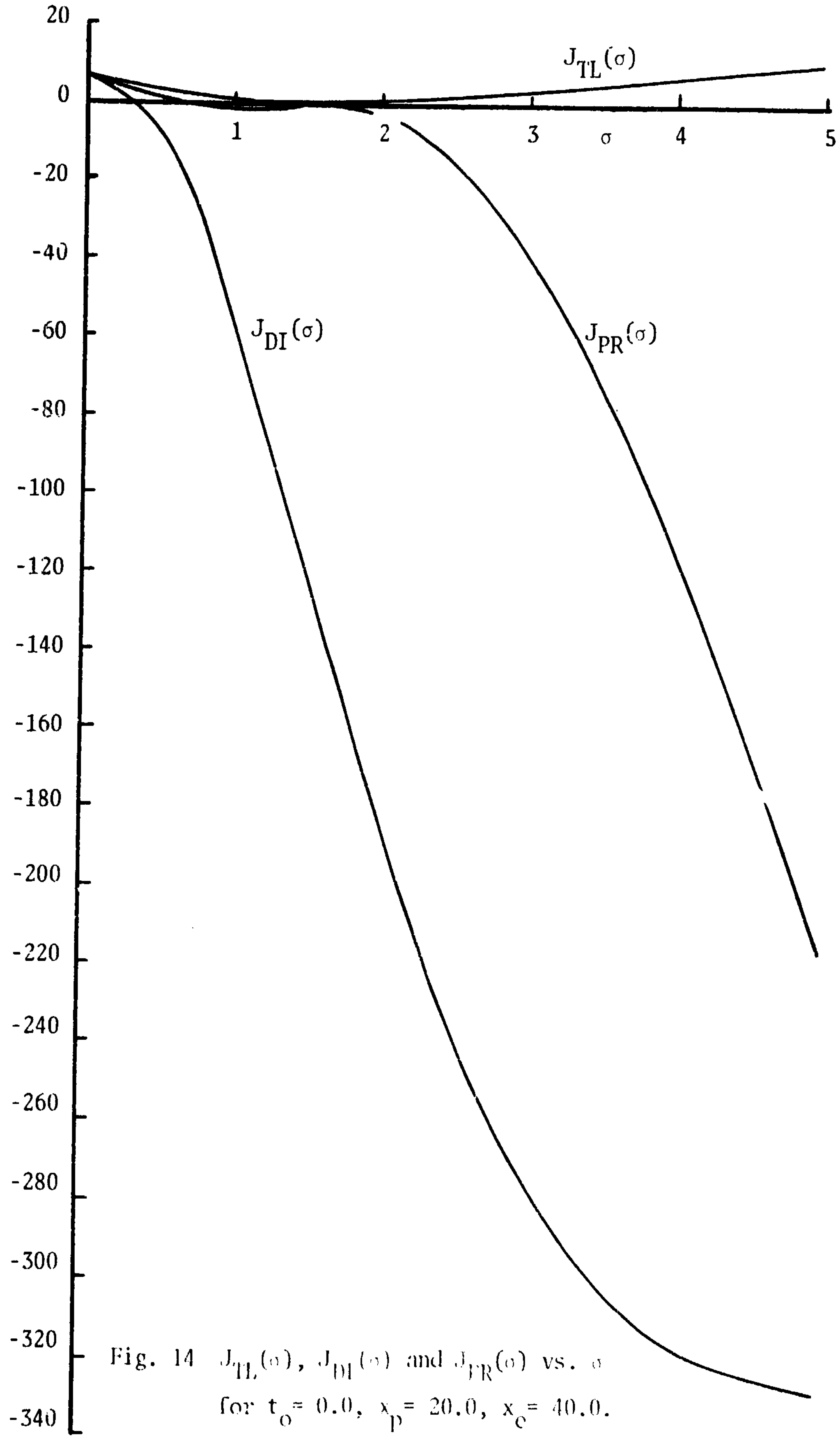

Fig. 14 $J_{TL}(\sigma)$, $J_{DI}(\sigma)$ and $J_{PR}(\sigma)$ vs. σ for $t_o = 0.0$, $x_p = 20.0$, $x_c = 40.0$.

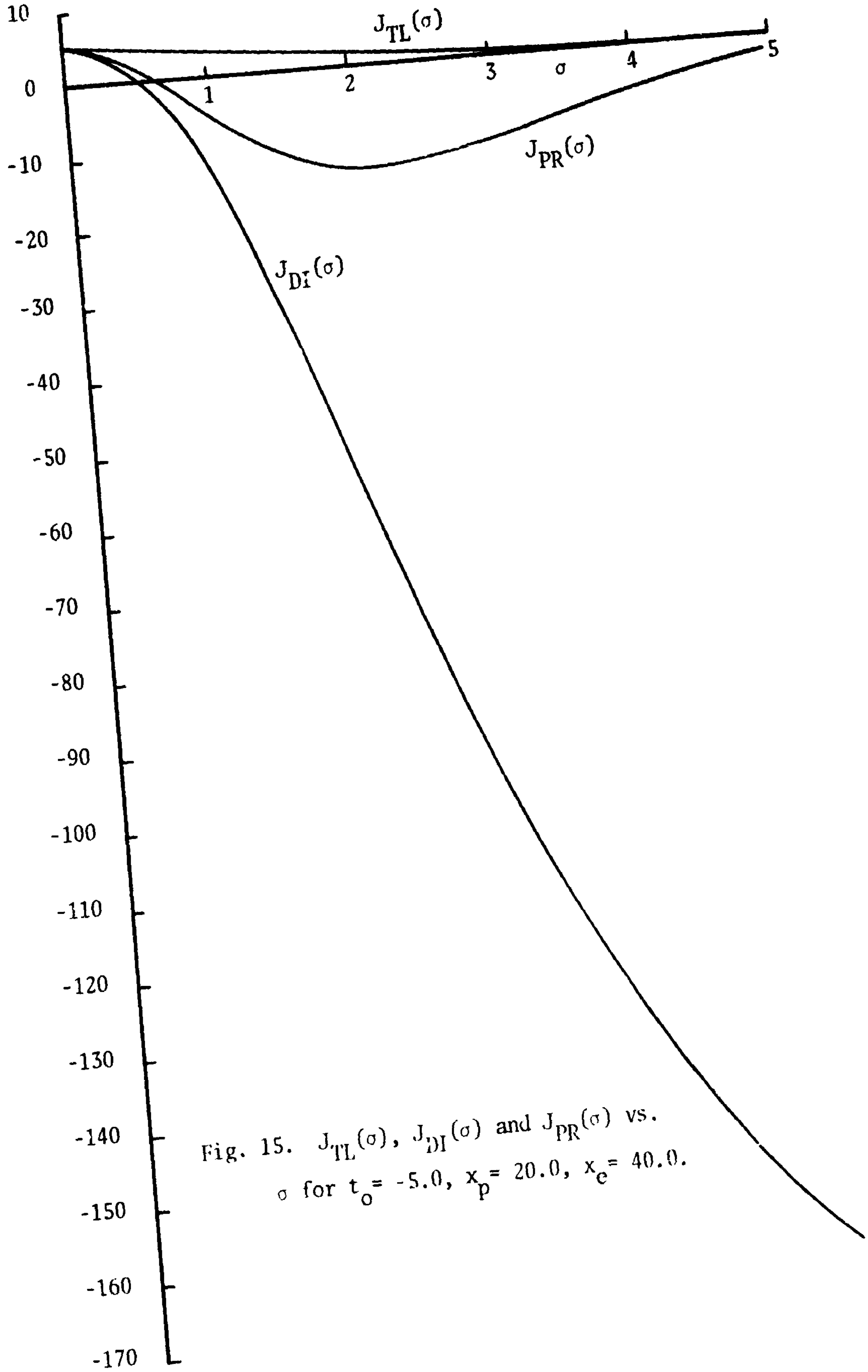

Fig. 15. $J_{TL}(\sigma)$, $J_{DI}(\sigma)$ and $J_{PR}(\sigma)$ vs. σ for $t_o = -5.0$, $x_p = 20.0$, $x_e = 40.0$.

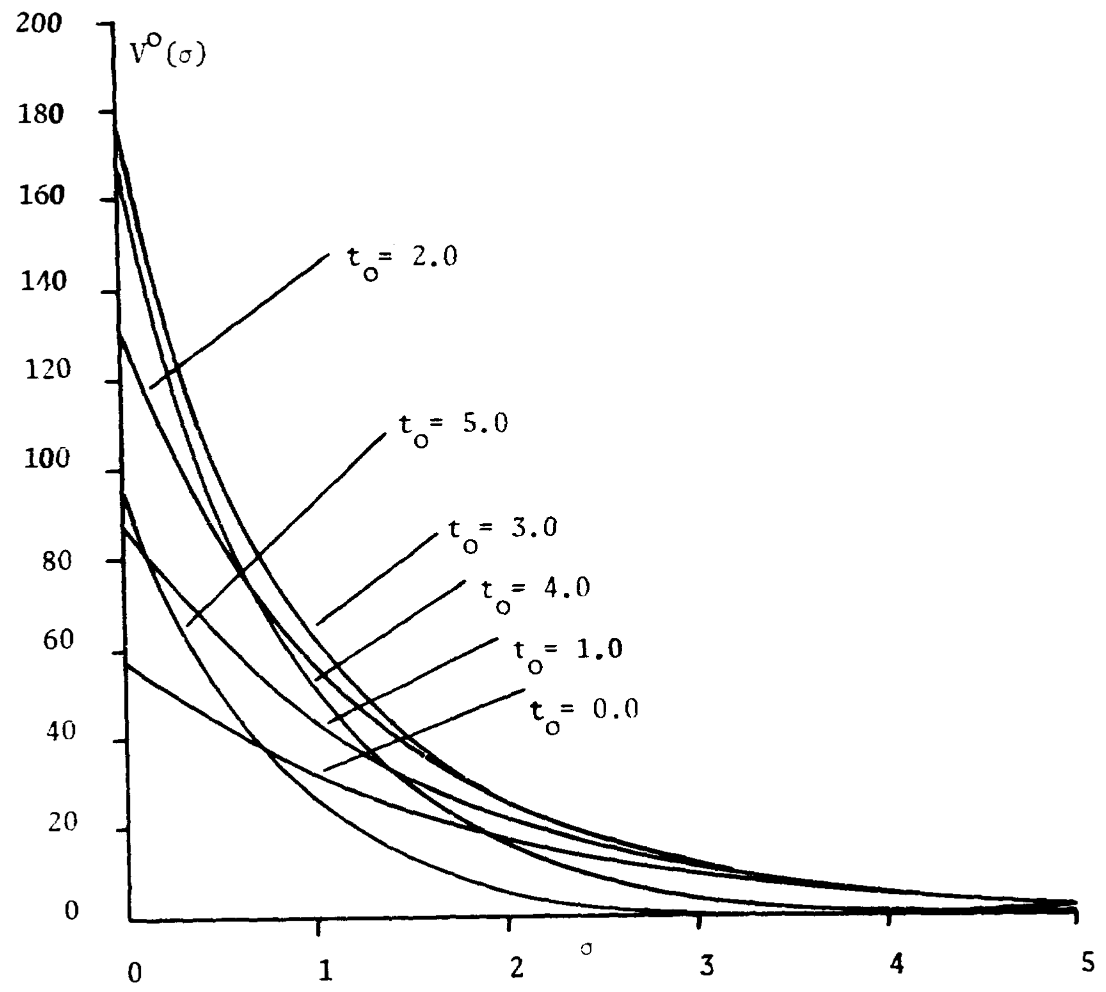

Fig. 16. Potential Value vs. Data Delay.

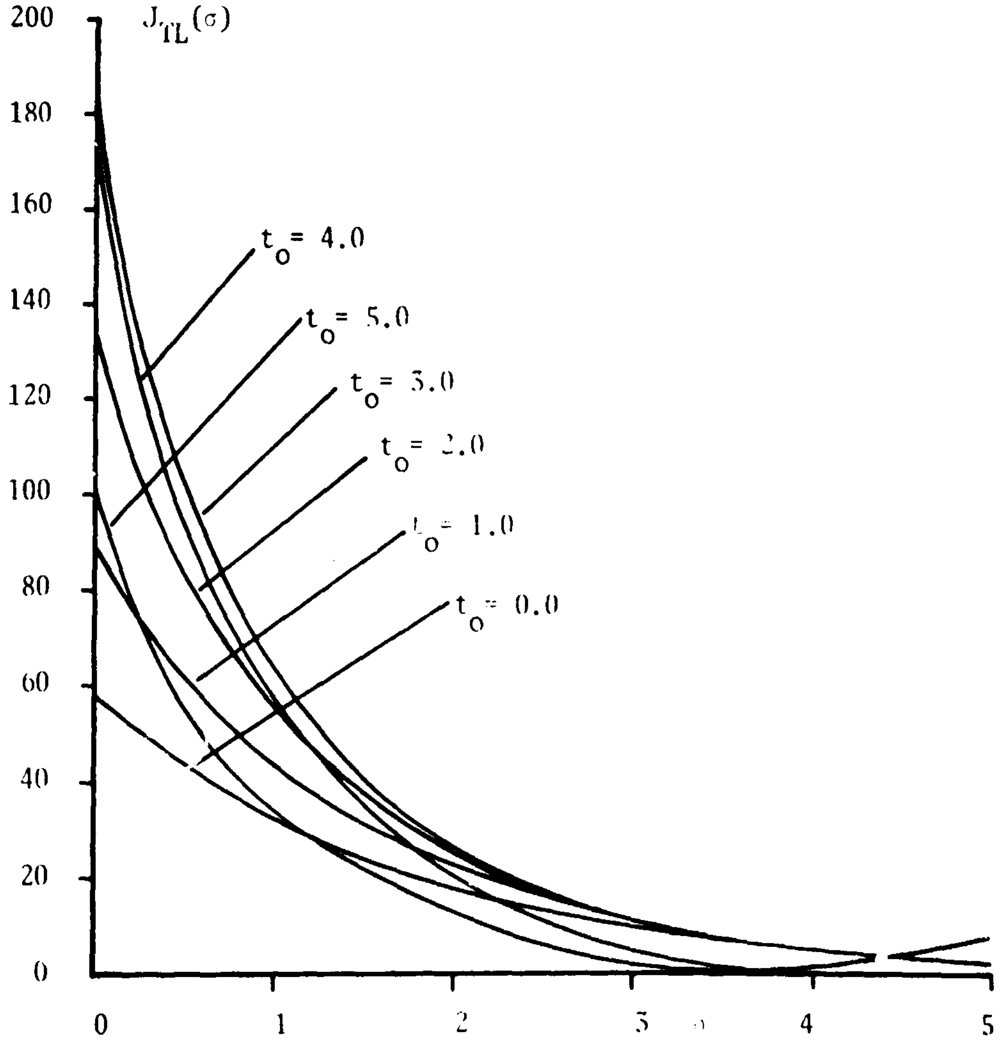

Fig. 17. $J_{TL}(\sigma)$ vs. σ.

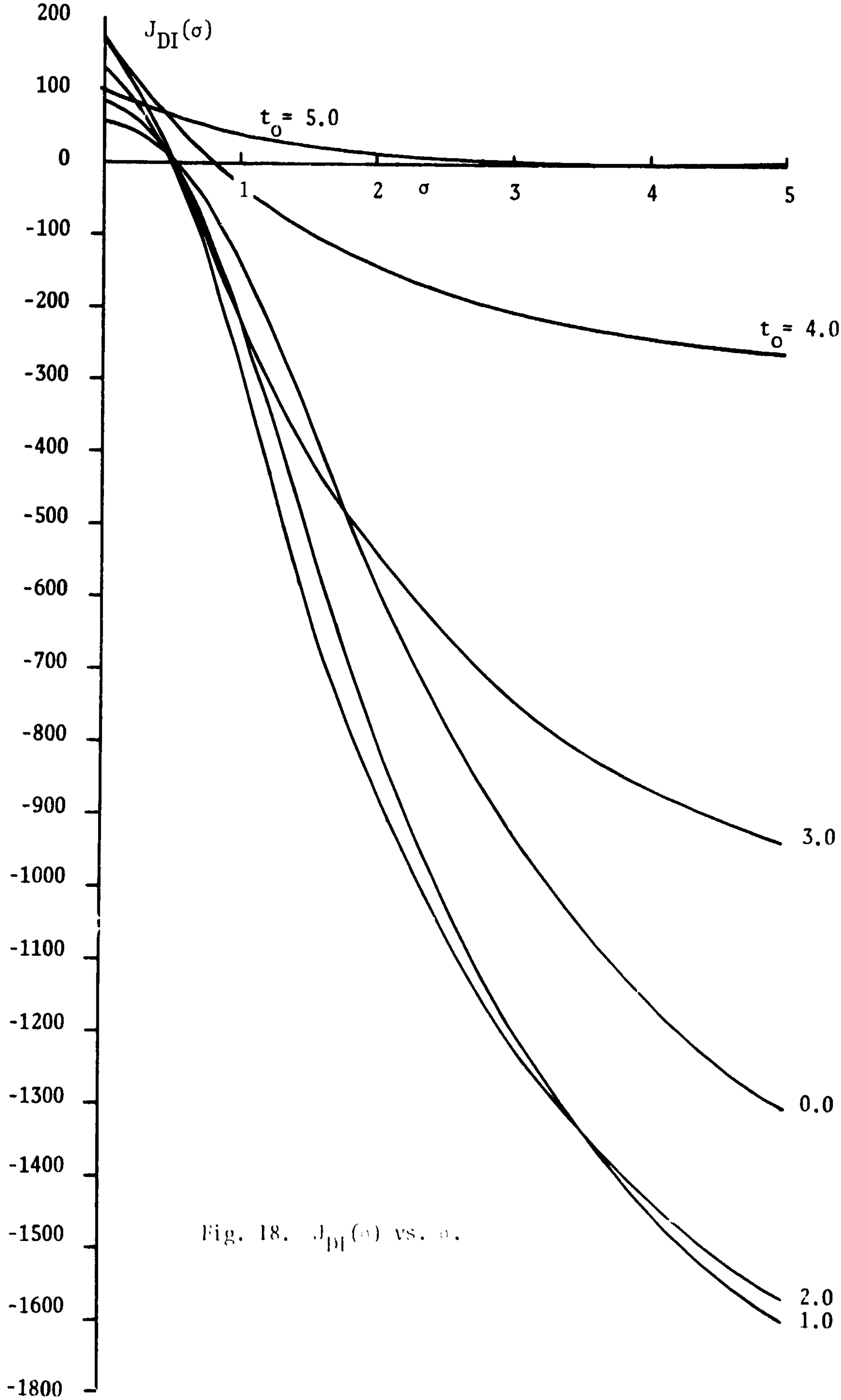

Fig. 18. $J_{DI}(\sigma)$ vs. σ.

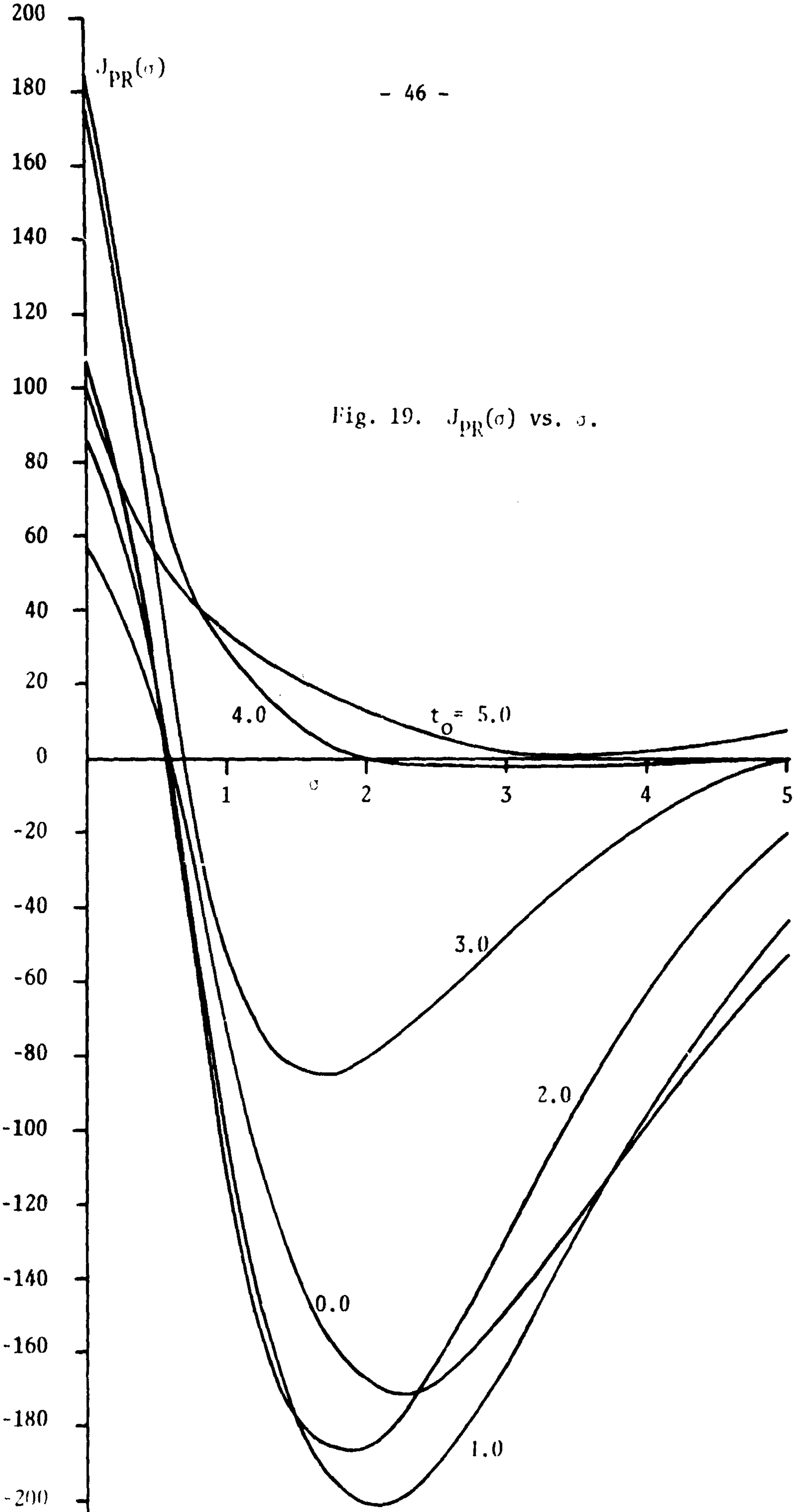

Fig. 19. $J_{PR}(\sigma)$ vs. σ.

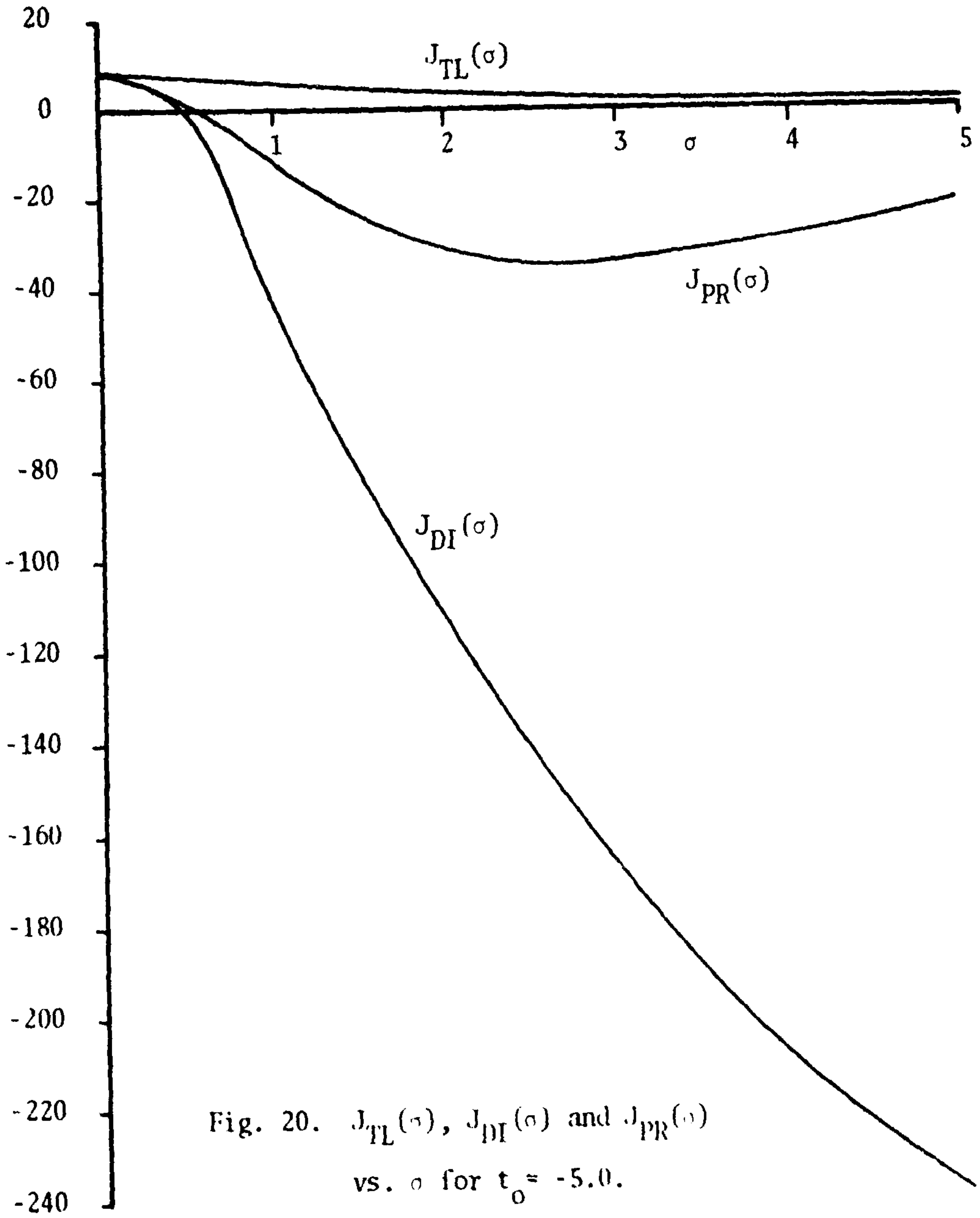

Fig. 20. $J_{TL}(\sigma)$, $J_{DI}(\sigma)$ and $J_{PR}(\sigma)$ vs. σ for $t_o = -5.0$.

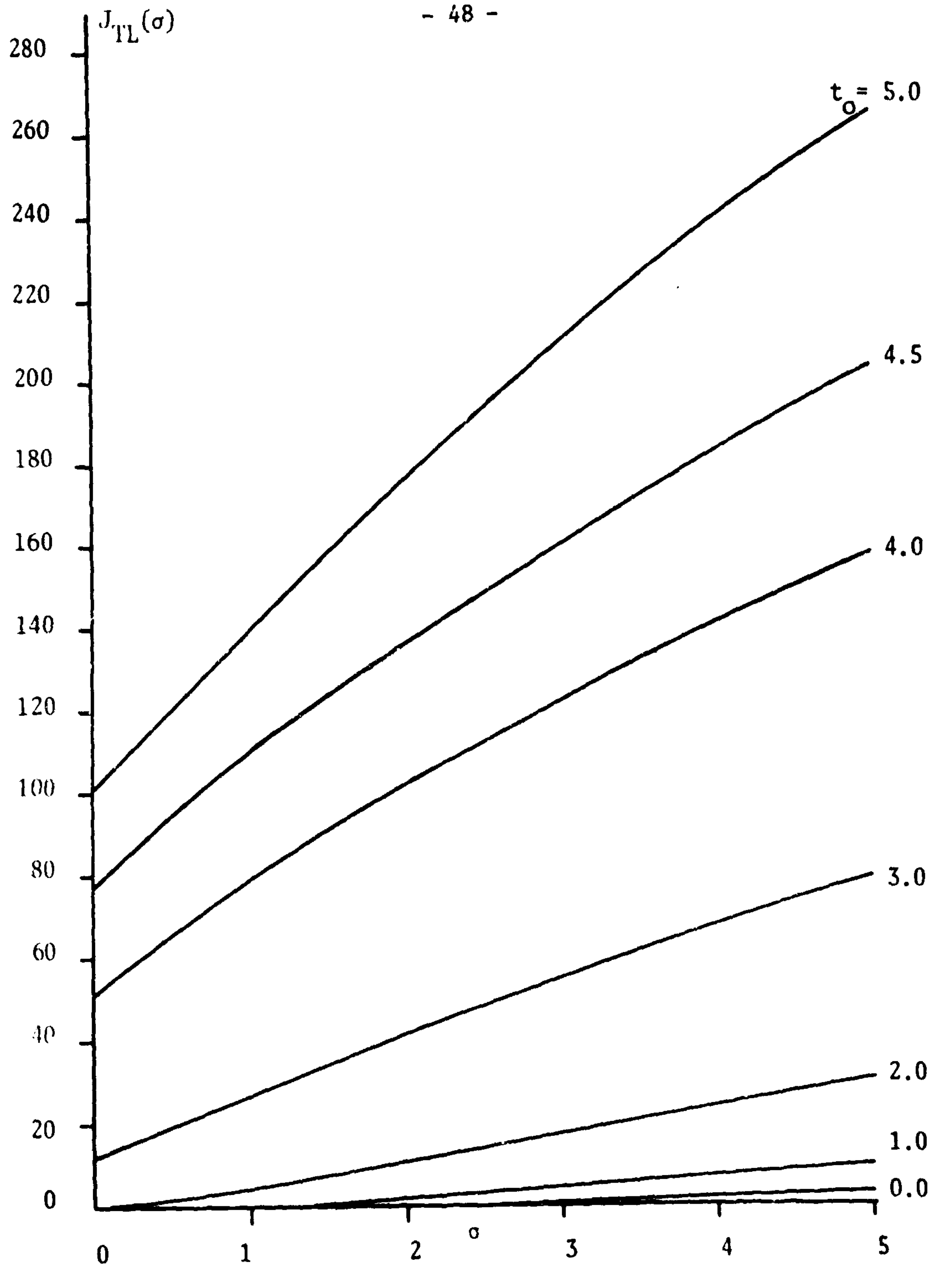

Fig. 21. $J_{TL}(\sigma)$ vs. σ.

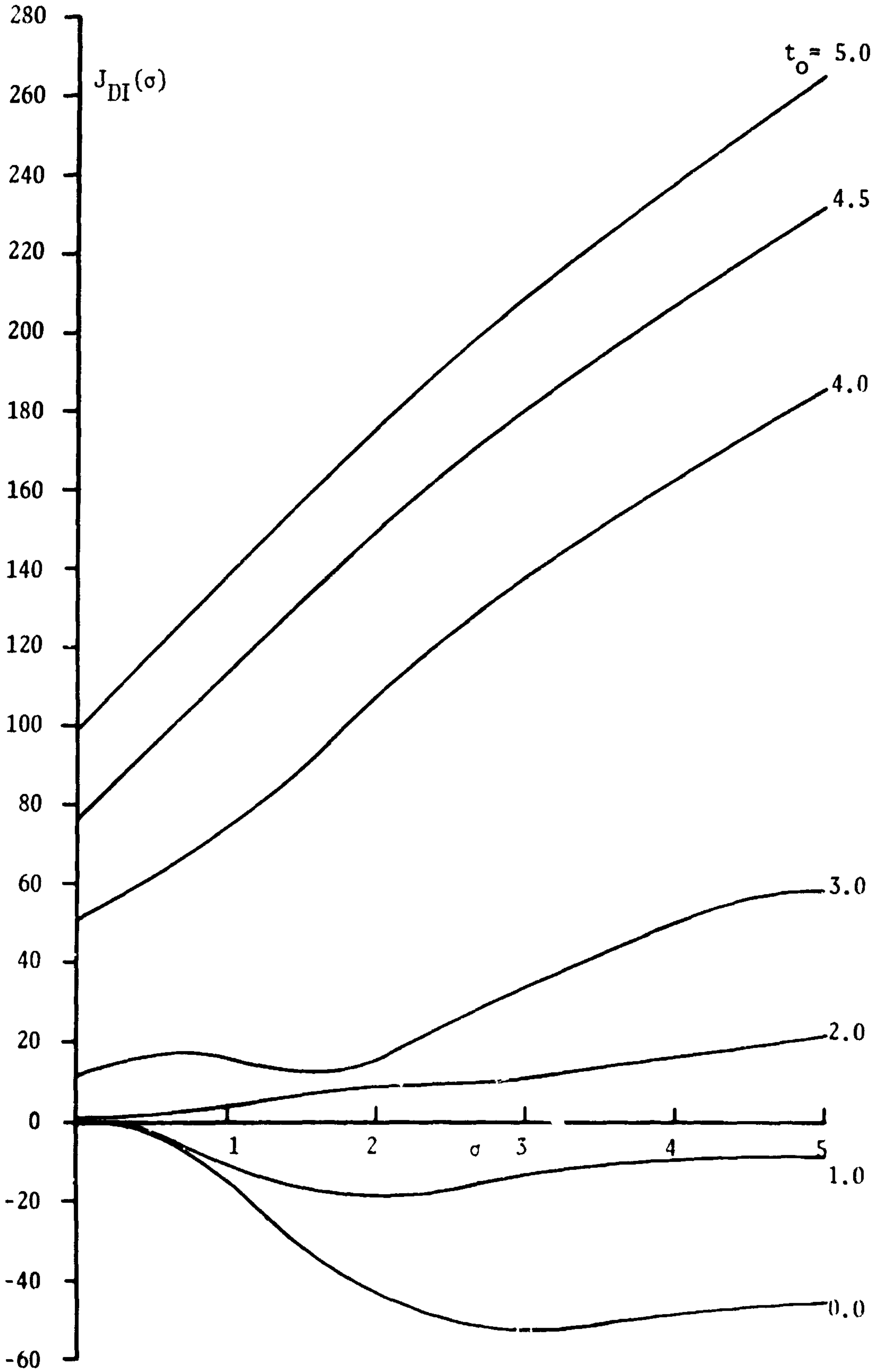

Fig. 22. $J_{DI}(\sigma)$ vs. σ.

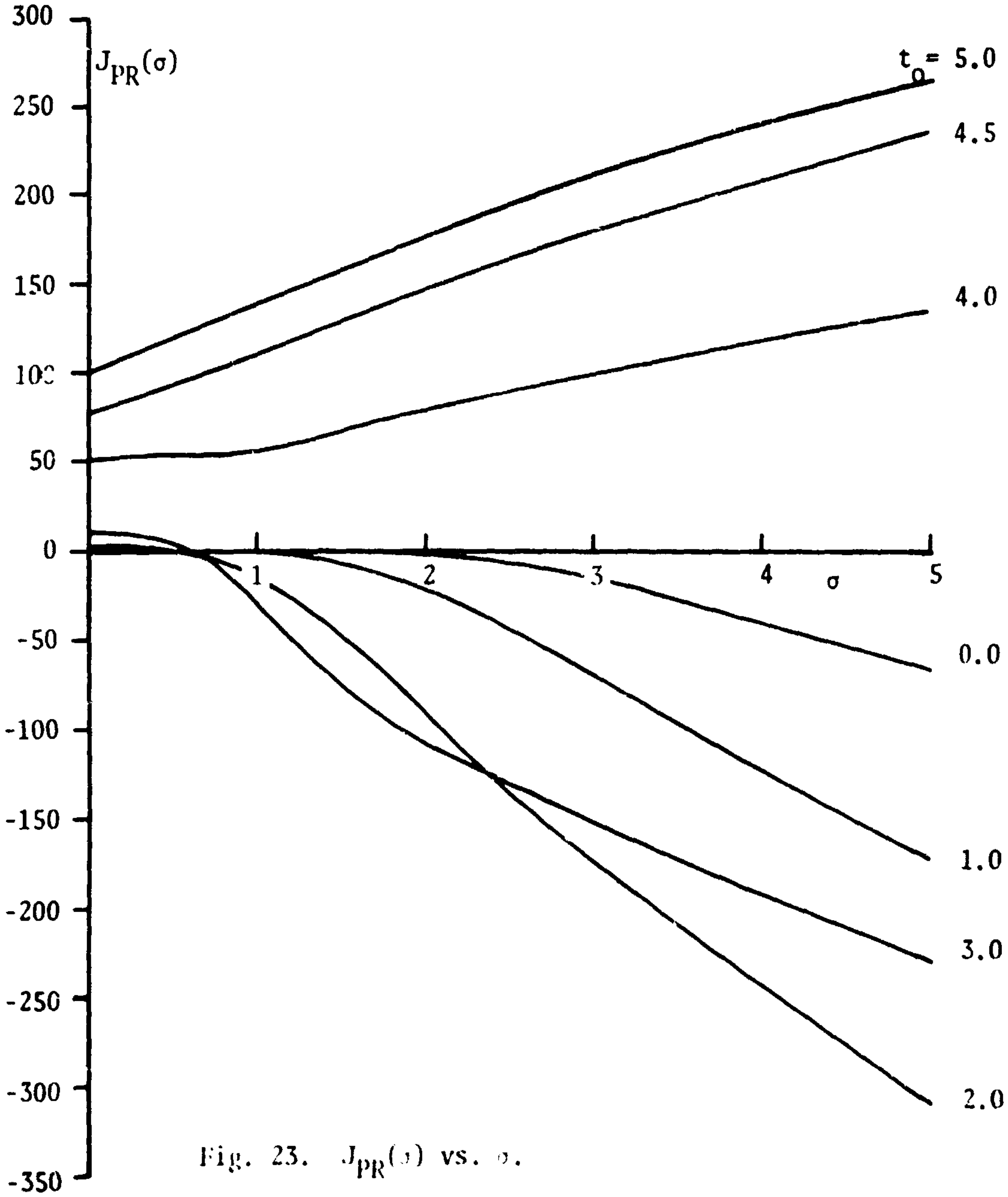

Fig. 23. $J_{PR}(\sigma)$ vs. σ.

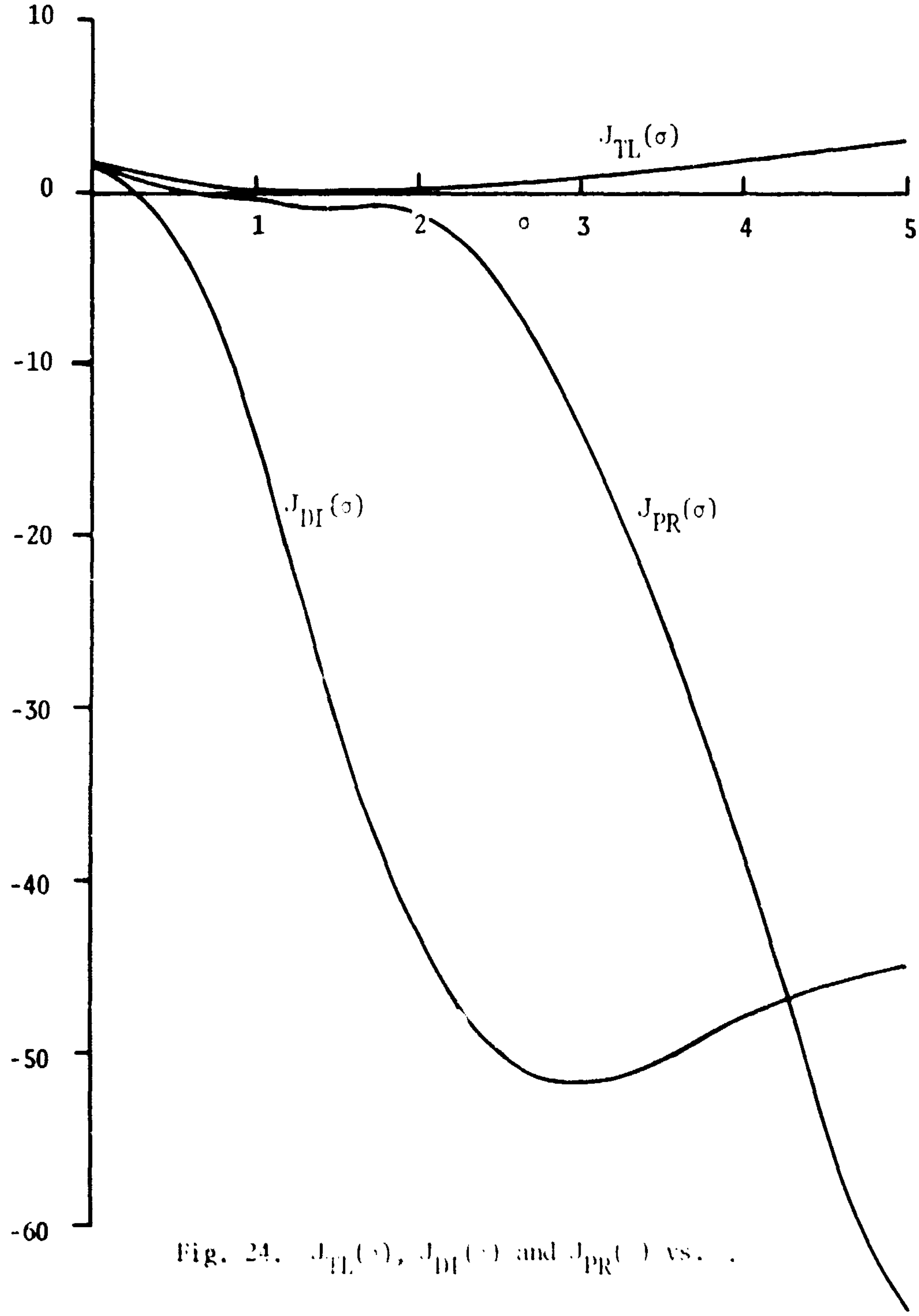

Fig. 24. $J_{TL}(\sigma)$, $J_{DI}(\sigma)$ and $J_{PR}(\sigma)$ vs. σ.

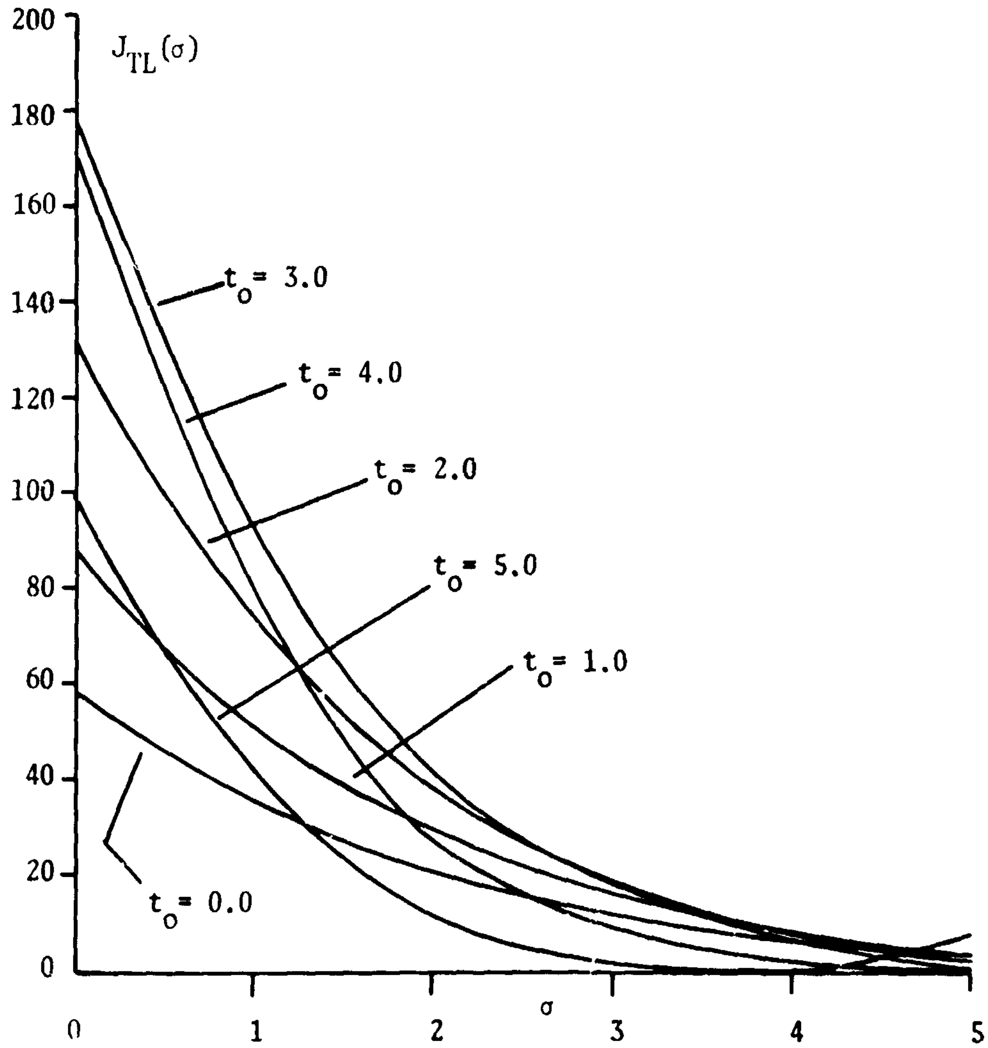

Fig. 25. $J_{TL}(\sigma)$ vs. σ.

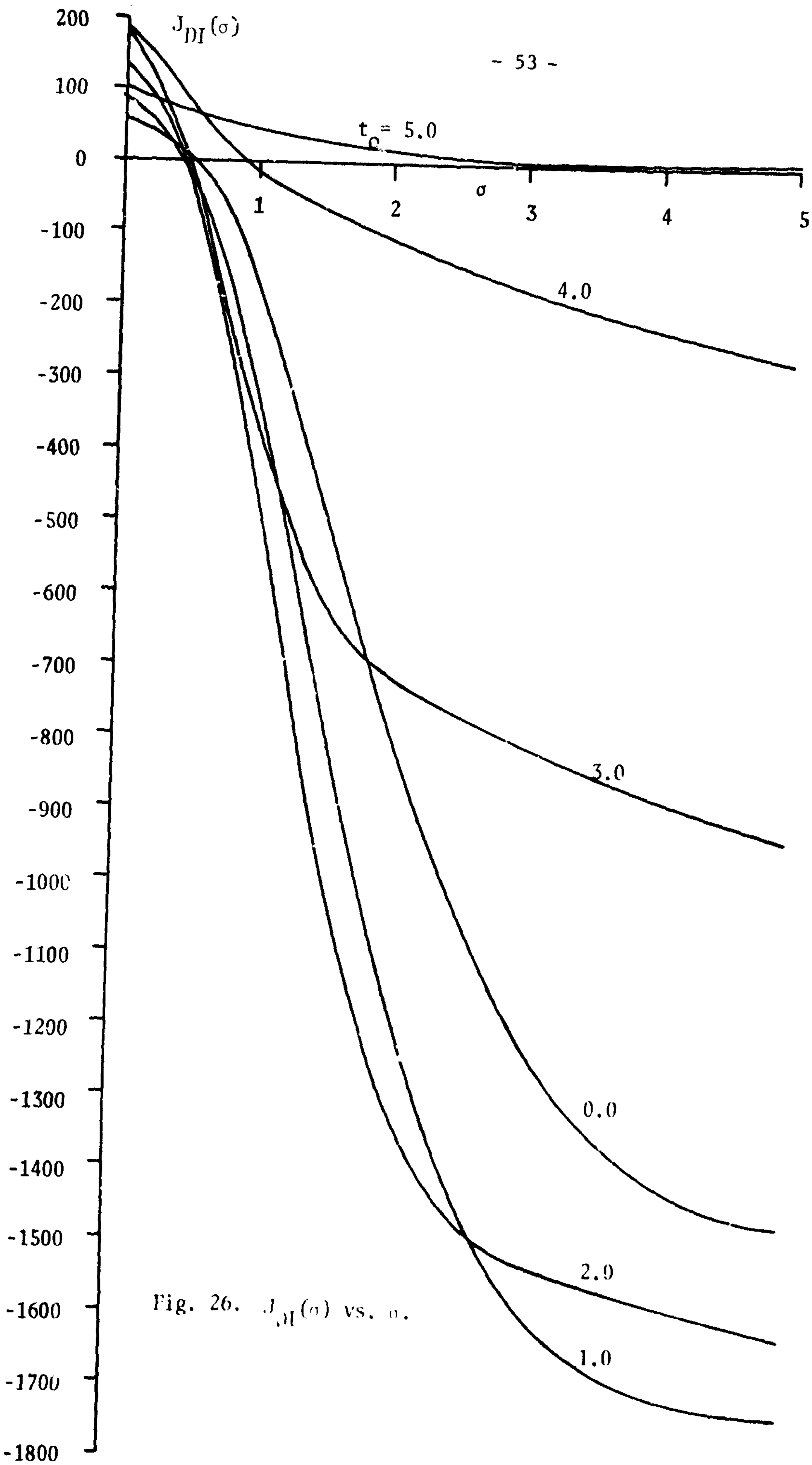

Fig. 26. $J_{DI}(\sigma)$ vs. σ.

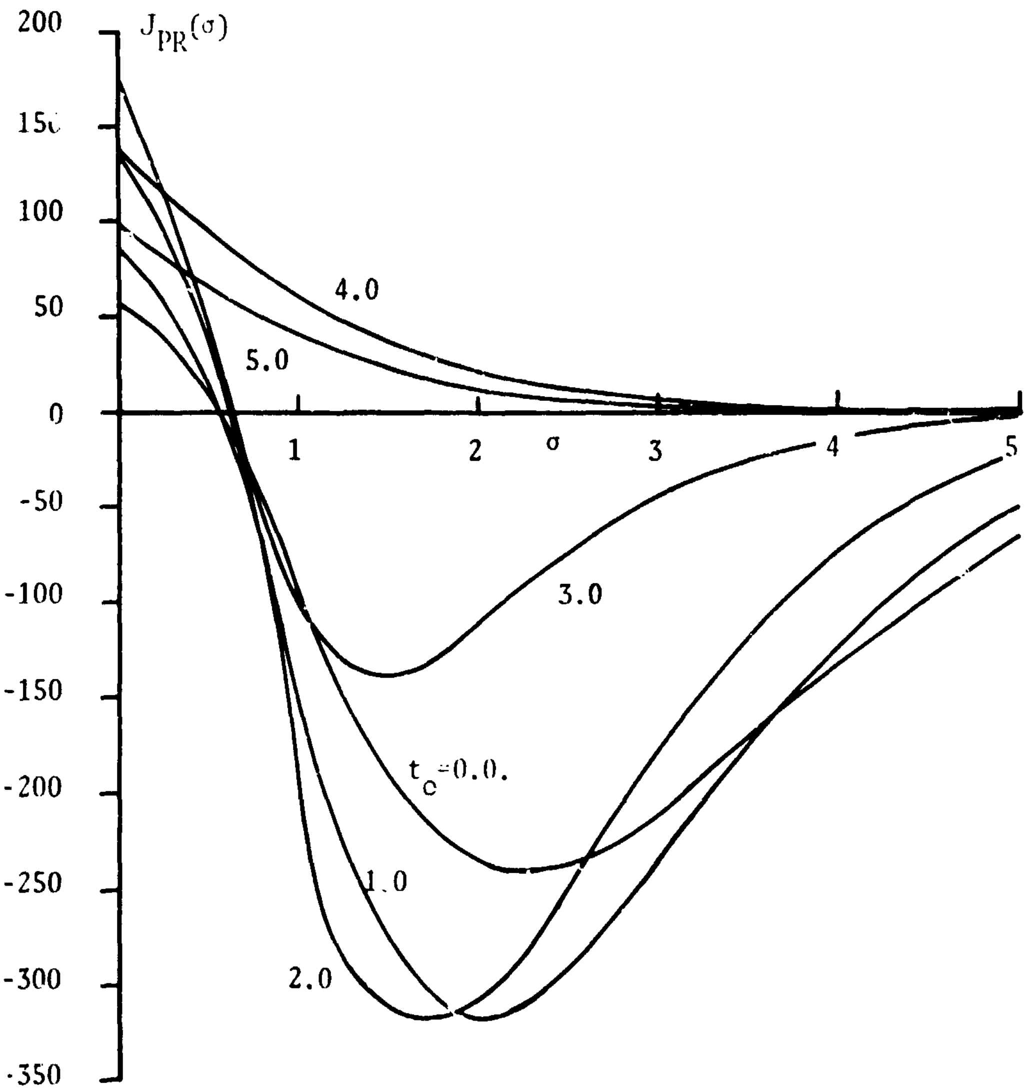

Fig. 27. $J_{PR}(\sigma)$ vs. σ.

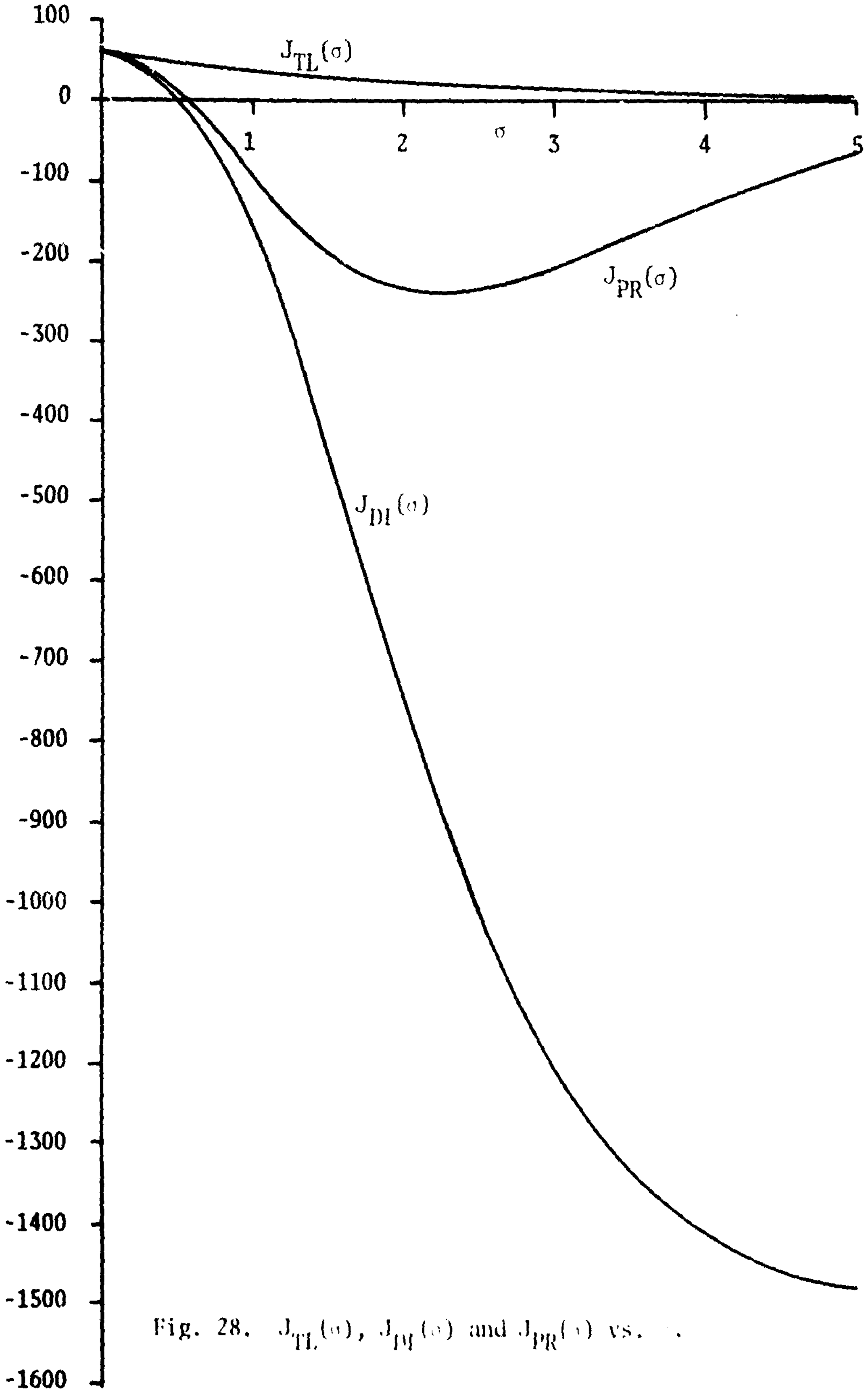

Fig. 28. $J_{TL}(\sigma)$, $J_{DI}(\sigma)$ and $J_{PR}(\sigma)$ vs. σ.