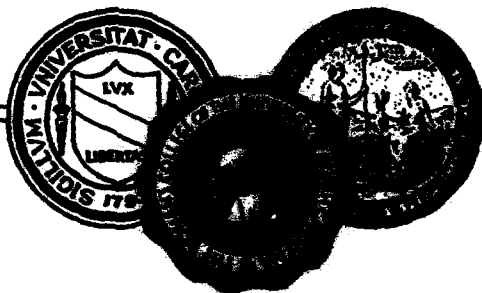


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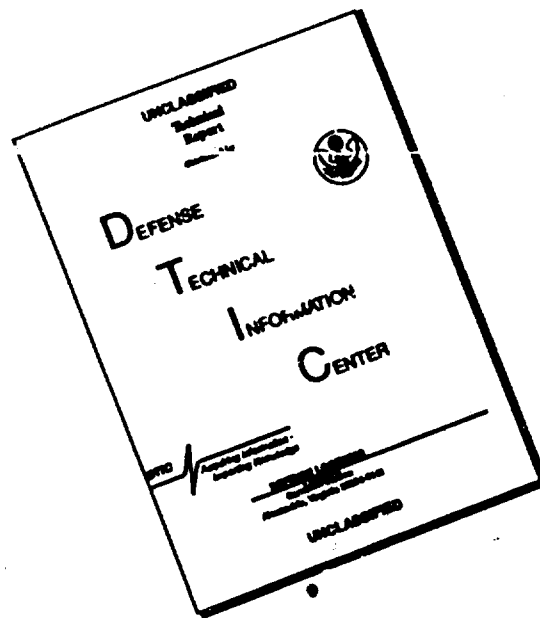
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**STATIONARY STOCHASTIC POINT PROCESSES II**

*by Johannes Kerstan and Klaus Matthes*

**STATIONARY STOCHASTIC POINT PROCESSES III**

*by P. Franken, A. Liemant and K. Matthes*

Translated by

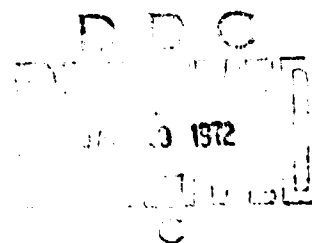
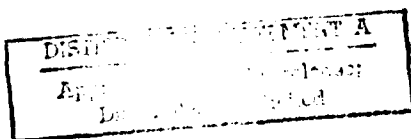
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*Department of Statistics*

*University of North Carolina at Chapel Hill*

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## Prefatory Remarks Concerning This Translation

This paper is a translation from German of the second and third of the series "Stationary stochastic point processes I, II and III", written respectively by K. Matthes; J. Kerstan and K. Matthes; P. Franken, A. Liemant and K. Matthes in *Deutsche Math. Vereinigung*, Vols 66, 67. (A list of contents of the three papers appears below.) Part I has been translated previously by R. J. Serfling (Statistics Report M190, Florida State University).

The present translation of Parts II and III was done by C. E. Jeffcoat, University of North Carolina, and D. J. Daley of the Australian National University. In particular, Dr. Daley has added footnotes where clarification seemed desirable. The notation of Serfling has been used - in replacing Gothic letters by script.

It should be remarked that these papers are of unquestionable importance in the Point Process literature. They treat a point process in the main as a special case of a *marked* point process. This gives a very useful and general framework, which does, of course, involve some notational complexity due to the "mark" structure. This notation can be simplified when marks are not relevant.

The numbering of the references and chapters carries throughout the three-part series, which is comprised of the following chapters:

Part I. (by K. Matthes) 66, 66-79.

1. Foundations
2. Intensity and Palm distribution
3. Stationary Poisson point processes.

Part II. (by J. Kerstan and K. Matthes) 66, 106-118.

4. Stationary infinitely divisible distributions
5. Regular and singular infinitely divisible distributions.

Part III. (by P. Franken, A. Liemant and K. Matthes) 67, 183-202.

6. A new approach to  $\tilde{P}_{\lambda, Q}$
7. Palm distributions in a generalized sense
8. The distributions  $(\tilde{P})_0$
9. Generalization of a result of Palm and Khinchin
10. Limit distributions for infinitesimal arrays of stationary renewal processes
11. Some relations between  $P$  and  $P_L$ .

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## STATIONARY STOCHASTIC POINT PROCESSES, II\*

Johannes Kerstan and Klaus Matthes

### 4. Stationary infinitely divisible distributions

Let  $\mathcal{M}_K$  denote the smallest  $\sigma$ -algebra of subsets of  $M_K$  with respect to which the mapping<sup>1</sup>  $\phi \longrightarrow N(\phi \cap (I \times L))$  is measurable for all intervals  $I$  contained in  $[-k, k)$  and all  $L$  in  $K$ . We say that  $[K, K]$  has *Property (a)* when the following is true:

- (a) If a finite measure  $\mu_k^U$  is defined on each  $\mathcal{M}_K$  such that  $\mu_{k''}^U$  is an extension of  $\mu_{k'}^U$  whenever  $k'' > k'$ , then there exists a common extension of the  $\mu_k^U$ 's to a measure  $U$  on  $M_K$ .

$U$  is uniquely determined by the  $\mu_k^U$ 's because the algebra  $\bigcup_{k=1}^{\infty} \mathcal{M}_K$  generates  $M_K$ .

Property (a) imposes no severe restrictions on  $[K, K]$ . It is certainly satisfied if  $K$  is countable and  $K$  is the family of all subsets of  $K$ , or if  $K = \mathbb{R}^n$  and  $K$  consists of the Borel subsets of  $\mathbb{R}^n$ .

In this and the three subsequent chapters, we assume there is given a fixed mark space  $[K, K]$  with Property (a). We define a stationary distribution  $P$  on  $M_K$  to be *infinitely divisible* if there exist finite sequences  $P_{n,1}, \dots, P_{n,s_n}$  of stationary distributions on  $M_K$  such that

$$(1) (P_{n,1} * \dots * P_{n,s_n}) \implies P$$

\* Stationäre zufällige Punktfolgen, II. *Jber. Deutsch. Math.-Verein.* 66 (1964) 106-118. Part I in same *J.* 66 (1963) 66-79.

<sup>1</sup> Always here,  $N(\cdot) = \text{card}(\cdot) = (\text{German}) \text{Ans}(\cdot)$ .

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and

(2) for all positive  $t, \epsilon$  there exists  $n_{t,\epsilon}$  such that for all

$$n \geq n_{t,\epsilon},$$

$$\max_{1 \leq i \leq s_n} P_{n,i}(\phi \cap ((0,t) \times K) \neq \phi) < \epsilon.$$

From this definition it follows immediately that the convolution  $P_1 * P_2$  of stationary infinitely divisible  $P_1, P_2$  is also stationary infinitely divisible. Also, if  $P$  is stationary and for every positive integer  $n$  there exists  $U_n$  stationary with  $(U_n)^n = P$ , then for all positive  $t$

$$U_n(\phi \cap ((0,t) \times K) \neq \phi) \rightarrow 0 \quad (n \rightarrow \infty)$$

and  $P$  is infinitely divisible.

For all stationary distributions  $S$  on  $M_K$  and all  $\lambda > 0$  we define the stationary distribution  $E_{\lambda,S}$  by

$$E_{\lambda,S} = \exp(-\lambda) \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} S^m$$

where we define  $S^0 = S_\phi$ . Then

$$E_{\lambda_1, S_1} * E_{\lambda_2, S_2} = E_{\lambda_1 + \lambda_2, (\lambda_1 S_1 + \lambda_2 S_2) / (\lambda_1 + \lambda_2)}$$

and, in particular,  $(E_{\lambda/n, S})^n = E_{\lambda, S}$ . Thus every  $E_{\lambda, S}$  is infinitely divisible.

Observe that  $E_{\lambda, S} = \delta_\phi$  if and only if  $S = \delta_\phi$ . For  $S \neq S_\phi$  we have by 1.2 the relation

$$E_{\lambda, S} = E_{\lambda S(\psi \neq \phi), S(\cdot | M'_K)}.$$

We now give a simple model from which a second class of stationary infinitely divisible  $P$ 's can be derived.

Let  $Q$  be any distribution on  $M_K$  and  $\lambda > 0$ . Let  $\phi \in M'$  have the Poisson distribution  $P_\lambda$  and let each  $\psi_n \in M_K$  ( $-\infty < n < \infty$ ) have the distribution  $Q$ . Further suppose the  $\psi_n$ 's to be mutually independent and independent of  $\phi$ . For  $\phi = \{x_i\}_{-\infty < i < \infty}$  set<sup>2</sup>

$$\chi = \bigcup_{-\infty < i < \infty} T_{-x_i} \psi_i.$$

In this way each  $x_i \in \phi$  defines a cluster ("Schauer")  $T_{-x_i} \psi_i$  and  $\chi$  consists of the superposition of all the clusters.

Clearly the mapping  $[\phi, (\psi_n)_{-\infty < n < \infty}] \rightarrow N(\chi \cap (I \times L))$  is measurable for all  $L \in K$  and all bounded intervals  $I$ , where, of course, the value  $+\infty$  is allowed. Our construction leads to  $\chi \in M_K$  with probability one precisely when for all bounded  $I$ ,  $N(\chi \cap (I \times K)) < \infty$  with probability one. On account of the stationarity of  $P_\lambda$  it is enough to verify this condition for some non-empty  $I$ . Observe that then the distribution of  $\chi$  is stationary; denote it by  $P_{\lambda, Q}$ .

For every bounded interval  $I$ ,  $Q(\psi_n(I+x) \times K) \neq \phi$  is a measurable non-negative function of  $x$ . If it has a finite integral for some non-empty bounded  $I$  then the same is true for every bounded interval. By the Borel-Cantelli lemma, it follows (cf. [27]) that

$$4.1. \quad P_{\lambda, Q} \text{ is defined if and only if } \int_{-\infty}^{\infty} Q(\psi_n((I+x) \times K) \neq \phi) dx < \infty.$$

Hence  $Q(N(\psi) < \infty) = 1$  is a necessary but not sufficient condition.

For  $Q = \delta_\phi$ , and only in this case,  $P_{\lambda, Q} = \delta_\phi$ . On the basis of the first invariance property<sup>3</sup> of the Poisson distribution given in Chapter 3, we have the

<sup>2</sup>  $T_t \phi$  is defined in 1.1.

<sup>3</sup> This is a reference to §3.3.

relation for  $Q \neq \delta_\phi$

$$P_{\lambda, Q} = P_{\lambda', Q(\cdot | \Psi \neq \phi)} \quad \text{where} \quad \lambda' = \lambda Q(\Psi \neq \phi).$$

Now let  $Q(\Psi \neq \phi) = 1$ . Then with probability one  $\Psi$  has the form  $\{[z_1, k_1], \dots, [z_n, k_n]\}$  with  $z_1 < \dots < z_n$ . Let  $Q^*$  denote the distribution of  $\Psi^* = T_{z_1} \Psi$ . Then clearly

$$Q^*(\Psi^* n((-\infty, 0) \times K) \neq \phi) = 0 \quad \text{and}^4 \quad Q^*(\Psi^* n(\{0\} \times K) \neq \phi) = 1.$$

On the basis of a sharper form (see 6.2) of the second invariance property of  $P_\lambda$  given in Chapter 3, we have

$$P_{\lambda, Q} = P_{\lambda, Q^*}.$$

We can, therefore, restrict the above model without loss of generality by requiring that the clusters be subject to the causality principle<sup>5</sup> and that the observed reaction to an impulse occurring at time  $t$  begin immediately at  $t$ .

Since  $P_{\lambda_1, Q_1} * P_{\lambda_2, Q_2} = P_{\lambda_1 + \lambda_2, (\lambda_1 Q_1 + \lambda_2 Q_2) / (\lambda_1 + \lambda_2)}$ ,  $(P_{\lambda/n, Q})^n = P_{\lambda, Q}$ . Thus every  $P_{\lambda, Q}$  is infinitely divisible.

Each  $E_{\lambda, S}$  can be represented as the limit of a weakly convergent sequence of distributions of the form  $P_{\lambda, Q}$ . For let  $S$  be an arbitrary stationary distribution on  $M_K$  and  $\lambda > 0$ . If now we denote by  $S_n$  the distribution of  $\Psi_n((0, n) \times K)$  induced by  $S$ , then

$$4.2. \quad P_{\lambda/n, S_n} \Rightarrow E_{\lambda, S} \quad (n \rightarrow \infty).$$

We now return to the general theory. Let  $P$  be a stationary infinitely divisible distribution on  $M_K$ . For all finite sequences<sup>6</sup>  $\underline{I} \equiv \{I_1, \dots, I_n\}$ ,

<sup>4</sup> The original has  $Q^*(\Psi^* n(\{0\} \times K) = \phi) = 1$ .

<sup>5</sup> i.e., the Poisson process triggers a cluster in which all events (= points) occur on or after the triggering event.

<sup>6</sup> The notation  $\underline{I}$  and  $\underline{L}$  seems useful.

$\underline{L} \equiv \{L_1, \dots, L_n\}$  of bounded intervals  $I_i$  and measurable subsets  $L_i$  of  $K$ , the distribution  $p_{\underline{I}; \underline{L}}$  of the vector  $[N(\phi \cap (I_1 \times L_1)), \dots, N(\phi \cap (I_n \times L_n))]$  is infinitely divisible and therefore

$$4.3. \quad p_{\underline{I}; \underline{L}} \text{ has the representation } p_{\underline{I}; \underline{L}} = \exp(-\lambda) \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} q^m.$$

In 4.3,  $\lambda \geq 0$  and  $q$  is the distribution of a random vector with non-negative integer-valued components.

For  $p_{\cdot; \cdot} \neq \delta_{[0, \dots, 0]}$  the representation 4.3 is uniquely determined by the requirement that  $q(\{[0, \dots, 0]\}) = 0$ . We then set  $\lambda = \lambda_{\underline{I}; \underline{L}}$  and  $q = q_{\underline{I}; \underline{L}}$ . If on the other hand  $p_{\cdot; \cdot} = \delta_{[0, \dots, 0]}$  then we set  $\lambda = 0$  and  $q = \delta_{[0, \dots, 0]}$ .

Set  $q_{\cdot; \cdot} = \sum_x q(\{x\}) \delta_x$  where  $x$  runs through all the vectors of  $R^n$  with non-negative integer-valued coordinates. Then  $p_{\cdot; \cdot}$  is the convolution of

$$p_{\cdot; \cdot; x} = \exp(-\lambda_{\cdot; \cdot; x}) \cdot \sum_{m=0}^{\infty} \frac{(\lambda_{\cdot; \cdot; x})^m}{m!} \delta_{mx} \quad \text{where } \lambda_{\cdot; \cdot; x} = \lambda_{\cdot; \cdot} \cdot q_{\cdot; \cdot}(\{x\}).$$

These  $\lambda_{\cdot; \cdot; x}$  are uniquely determined by  $P$ . Conversely the  $\lambda_{\cdot; \cdot; x}$  collectively determine  $p_{\cdot; \cdot}$  and therefore  $P$ . In the convolution of stationary infinitely divisible distributions the corresponding values of the  $\lambda_{\cdot; \cdot; x}$ 's are added.

From the  $\lambda_{\cdot; \cdot; x}$ 's we now use an extension procedure common in measure theory to derive a measure which will be of considerable assistance in studying the structure of infinitely divisible distributions.

4.4. *To each stationary infinitely divisible distribution  $P$  on  $M_K$  corresponds exactly one (possibly infinite) measure  $\tilde{P}$  on  $M_K$  with the properties*

(a) *for all finite sequences  $\underline{I}$  and  $\underline{L}$  of bounded intervals and*

measurable subsets of  $K$  respectively and all non-null vectors

$\underline{m} = [m_1, \dots, m_n]$  with non-negative integer-valued coordinates

$$\tilde{P}(N(\chi \cap (I_i \times L_i)) = m_i, i=1, \dots, n) = \lambda_{\underline{L}; \underline{L}; \underline{m}};$$

$$(\beta) \quad \tilde{P}(\{\phi\}) = 0;$$

The correspondence  $P \rightarrow \tilde{P}$  gives a single-valued invertible mapping of the class of all stationary infinitely divisible distributions on  $M_K$  onto the class of those measures on  $M_K$  which are stationary relative to  $T_t$  and in addition to  $(\beta)$  have the property

(\gamma) for all bounded intervals  $I$  the measure of  $\{\chi: \chi \cap (I \times K) \neq \phi\}$  is finite.

Proof. Let  $H$  denote the totality of those subsets of  $M_K$  that can be put in the form  $H = \{\phi: N(\phi \cap (I_i \times L_i)) = m_i, i = 1, \dots, n\}$  with  $[m_1, \dots, m_n] \neq [0, \dots, 0]$ . The  $\sigma$ -ring generated by  $H$  (cf. [26], §5) contains every  $X$  in  $M_K$  with the property  $\phi \in X$ . In this proof, we call a function  $Y$  on  $H$   $H$ -finitely additive if for all  $H$  in  $H$

$$Y(H) = \sum_{k=0}^{\infty} Y(\{\phi: N(\phi \cap (I_{n+1} \times L_{n+1})) = k\} \cap H).$$

Let  $P$  be a stationary distribution on  $M_K$  for which each of the vectors  $[N(\phi \cap (I_i \times L_i)), i=1, \dots, n]$  has a joint distribution of the form 4.3. Then there can be at most one measure  $\tilde{P}$ , since the values on the  $\sigma$ -ring generated by  $H$  are uniquely determined by  $(\alpha)$  and hence the values on all of  $M_K$  are uniquely determined because of  $(\beta)$ . It is easily seen from the definition of the  $\lambda_{\cdot, \cdot, \cdot; \cdot}$ 's that the equation in  $(\alpha)$  leads to a set-function  $H$ -finitely additive. Let  ${}_k H$  denote the ensemble of those  $X$  in  $H$  for which  $I_1, \dots, I_n$  are contained in  $[-k, k]$ . For every  $X$  in  ${}_k H$ ,

$$\begin{aligned} P(X) &= P(N(\phi \cap (I_i \times L_i)) = m_i, i = 1, \dots, n) \\ &\geq \exp(-\lambda_{\cdot, \cdot, \cdot; \cdot}) \lambda_{\cdot, \cdot, \cdot; \cdot} q_{\cdot, \cdot, \cdot}(\{\underline{m}\}) = \exp(-\lambda_{\cdot, \cdot, \cdot; \cdot}) \lambda_{\cdot, \cdot, \cdot; \cdot} \end{aligned}$$

i.e.  $\exp(\lambda_{\underline{I};\underline{L}})P(X) \geq \tilde{P}(X)$ , and hence a fortiori

$$\exp(\lambda_{[-k,k];K}) P(X) \geq \tilde{P}(X).$$

On the basis of this inequality, it follows that the  $H$ -finitely additive set-function induced on  $\mathcal{H}_k$  by means of  $(\alpha)$  can be extended in exactly one way to a measure  $\nu_k$  on the  $\sigma$ -ring  $\mathcal{R}$  generated by  $\mathcal{H}_k$ . For  $k'' > k'$ ,  $\nu_{k''}$  is an extension of  $\nu_{k'}$ . For each  $m$  and all  $k > m$  we define a finite measure on  $M_K$  by

$$\nu_m^Q(\cdot) = \nu_k^V(\cdot \cap \{\chi: \chi \cap ((-m,m) \times K) \neq \emptyset\}).$$

For  $k'' > k'$ ,  $\nu_{k''}^Q$  is an extension of  $\nu_{k'}^Q$ . Because of property (a) of the mark space  $[K,K]$  the  $\nu_m^Q$  ( $k = m+1, m+2, \dots$ ) may be regarded as restrictions of a well-defined finite measure  $Q_m$  on  $M_K$ . The  $Q_m$  are monotone increasing in  $m$ . Thus  $\limsup_m Q_m$  is a measure with the properties (a) and (b).

Thus the existence of  $\tilde{P}$  is established.  $\tilde{P}$  is clearly stationary and because of (a) satisfies condition (y).

Let  $H$  be defined as before with  $x = [m_1, \dots, m_n] \neq [0, \dots, 0]$ . Then

$$P(H) = p_{\underline{I};\underline{L}}(\{x\}) = \exp(-\lambda_{\underline{I};\underline{L}}) \sum_{m=1}^{\infty} \frac{(\lambda_{\underline{I};\underline{L}})^m}{m!} (q_{\underline{I};\underline{L}})^m(\{x\}).$$

We now write this equation in another form. Set

$$\tilde{P}_{\underline{I};\underline{L}}(\cdot) = \tilde{P}(\cdot \cap \{\chi: \sum_{i=1}^n N(\chi \cap (I_i \times L_i)) \neq 0\}).$$

By using direct products, we can now form the convolution powers of the finite measure  $\tilde{P}$  as in the case of distributions. Then for all  $m \geq 1$ ,

$$(\lambda_{\underline{I};\underline{L}})^m (q_{\underline{I};\underline{L}})^m(\{x\}) = (\tilde{P}_{\underline{I};\underline{L}})^m(H).$$

Thus we obtain (for  $H$  as defined)

$$(*) \quad P(H) = \exp \left[ -\tilde{P} \left( \sum_{i=1}^n N(X \cap (I_i \times L_i)) \neq 0 \right) \right] \sum_{m=1}^{\infty} \frac{1}{m!} (\tilde{P}_{\tilde{L}; \tilde{L}})^m(H).$$

Suppose now that  $V$  is a stationary measure on  $M_K$  satisfying both of conditions  $(\beta)$  and  $(\gamma)$ . Replacing  $\tilde{P}$  in  $(*)$  by  $V$  yields an  $H$ -finitely additive set-function  $Z$ . For all  $H \in \mathcal{H}$ ,  $Z(H) \leq 1$ . If  $H \in \mathcal{H}_k$ , then

$$P(H) \leq \sum_{m=1}^{\infty} \frac{1}{m!} (V_{(-k, k]; K})^m(H).$$

Thus for all  $k$  a measure  ${}_k Z'$  on  ${}_k R$  is given. For  $k'' > k'$ ,  ${}_{k''} Z'$  is an extension of  ${}_{k'} Z'$ . We now set  ${}_k Z'_m(\cdot) = {}_k Z'(\cdot \cap \{\phi \in (-m, m) \times K\} \neq \emptyset)$  and thus obtain for each  $k > m$  a finite measure  ${}_k Z'_m$  on  ${}_k M_K$ . The  ${}_k Z'_m$  ( $k = m+1, m+2, \dots$ ) may be regarded as the restrictions of some  $Z'_m$  on  $M_K$ . The  $Z'_m$  are monotonic increasing with  $m$ , and their upper limit  $Z'$  is a measure on  $M_K$  with the property  $Z'(M_K) \leq 1$ . We now set

$$P(X) = \begin{cases} Z'(X) & \text{for } \phi \notin X \\ Z'(X) + (1 - Z'(M_K)) & \text{for } \phi \in X, \end{cases}$$

and thus obtain a stationary distribution  $P$  on  $M_K$  which for all  $H \in \mathcal{H}$  is related to the given measure  $V$  by  $(*)$ . Then the representation 4.3 is valid generally for  $P$ , and  $\tilde{P} = V$ . The correspondence  $P \rightarrow \tilde{P}$  thus gives a one-one mapping of the set of those stationary  $P$ 's for which 4.3 is valid onto the set of all stationary  $V$  with the properties  $(\beta)$  and  $(\gamma)$ .

It remains to be shown that  $P$  is infinitely divisible if 4.3 holds generally. For this we set  $V = \frac{1}{n} \tilde{P}$  and denote the associated distribution by  $Q_n$ . Since  $\overline{(\tilde{P}_1 * \tilde{P}_2)} = \tilde{P}_1 + \tilde{P}_2$ , we have  $(Q_n)^n = P$ . From the existence of these  $Q_n$ 's however it follows that  $P$  is infinitely divisible. This completes the proof of 4.4.

For  $\lambda > 0$  and  $S(M'_K) = 1$ ,  $E_{\lambda,S} = \lambda S$ . In this case therefore,  $\tilde{P}(M'_K) < \infty$ . Conversely, let  $P(\neq \delta_\phi)$  be any stationary infinitely divisible distribution with the property  $\gamma \equiv \tilde{P}(M'_K) < \infty$ . Then  $\tilde{E}_{\gamma, \gamma^{-1}\tilde{P}} = \tilde{P}$ , and hence  $P = E_{\gamma, \gamma^{-1}\tilde{P}}$ . From this we see the validity of\*

4.5. A stationary infinitely divisible distribution  $P$  has the form  $E_{\lambda,S}$  with  $\lambda > 0$  and  $S(M'_K) = 1$  if and only if  $P \neq \delta_\phi$  and  $\tilde{P}(M'_K) < \infty$ . Both  $\lambda$  and  $S$  are uniquely determined by  $P$  by means of  $\lambda = \tilde{P}(M'_K)$  and  $S = \frac{1}{\lambda} \tilde{P}$ .

If a sequence  $(P_n)$  of stationary infinitely divisible distributions converges weakly to  $P$ , then the representation 4.3. is possible for  $P$  in general, i.e.  $P$  is also infinitely divisible. The  $\lambda_{\cdot, \cdot, \cdot, x}$  values  ${}^n \lambda_{\cdot, \cdot, \cdot, x}$  with respect to  $P_n$  all tend to the corresponding  $\lambda_{\cdot, \cdot, \cdot, x}$ 's with respect to  $P$ . Moreover,  $\sum_{k>m} {}^n \lambda_{I;K;k} \rightarrow 0$  as  $m \rightarrow \infty$  for every bounded interval  $I$  uniformly in  $n$ . On account of the convergence of  ${}^n \lambda_{\cdot, \cdot, \cdot, x}$  to  $\lambda_{\cdot, \cdot, \cdot, x}$  this relation is equivalent to

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} {}^n \lambda_{I;K;k} = \sum_{k=1}^{\infty} \lambda_{I;K;k}.$$

Both the above conditions are also sufficient for the convergence to  $P$ . In other words,  $P_n \Rightarrow P$  if and only if for all  $H \in \mathcal{H}$ ,  $\tilde{P}_n(H) \rightarrow \tilde{P}(H)$  and moreover for all bounded intervals  $I$

$$\lim_{n \rightarrow \infty} \tilde{P}_n(\chi \cap (I \times K) \neq \phi) = \tilde{P}(\chi \cap (I \times K) \neq \phi).$$

Using known results on the distributions of sums of independent random vectors, we obtain the result

\* Note added in proof: From equation (\*) we have a further characterization of the distributions  $E_{\lambda,S}$ : 4.5'. A stationary infinitely divisible distribution  $P$  has the form  $E_{\lambda,S}$  if and only if  $P(\{\phi\})$  is positive.

4.6. Let  $P_{n,1}, \dots, P_{n,s_n}$  be a triangular array of stationary distributions on  $M_K$  satisfying condition (2) of the definition of infinite divisibility.

Then the weak convergence of  $(P_{n,1} * \dots * P_{n,s_n})$  to  $P$  as  $n \rightarrow \infty$  is equivalent to  $(E_{1,P_{n,1}} * \dots * E_{1,P_{n,s_n}}) \Rightarrow P$ .

A distribution  $P$  is stationary and infinitely divisible if and only if it can be represented as the limit of a weakly convergent sequence of distributions of the form  $E_{\lambda,S}$ . We have namely

4.7. For all stationary infinitely divisible  $P$ ,  $E_{n,P}^{1/n}$  converges weakly to  $P$  as  $n \rightarrow \infty$ .

We define here the fractional convolution power  $P^{m/n}$  by means of  $\sqrt[m/n]{P} = \frac{m}{n} \tilde{P}$ .

## 5. Regular and singular infinitely divisible distributions

We call a stationary infinitely divisible distribution  $P$  on  $M_K$  *singular infinitely divisible* if for every  $t > 0$

$$P(\phi \cap ((-t, t) \times K)) \neq \phi \mid \phi \cap (((-t-m, -t) \cup (t, t+m)) \times K) = \phi \rightarrow 0 \quad (m \rightarrow \infty).$$

This definition is intuitively clear albeit formally complicated. We shall now give it in a simple abstract form.

Let  $P \neq \delta_\phi$  be stationary and infinitely divisible. Consider the vectors

$$[N(\phi \cap ((-t-m, -t) \times K)), N(\phi \cap ((-t, t) \times K)), N(\phi \cap ((t, t+m) \times K))].$$

Let  $r$  denote the convolution of all distributions

$$\exp(-\lambda \cdot; x) \sum_{m=0}^{\infty} ((\lambda \cdot; x)^m / m!) \delta_{mx} \quad \text{where } \lambda \cdot \text{ refers to the three intervals in the}$$

vector just given and  $x$  runs through those vectors  $[m_1, m_2, m_3]$  with non-negative integer-valued coordinates for which  $m_1 = m_3 = 0$  and  $m_2 > 0$ . Then it follows that

$$\sum_x \lambda_{.;x} = \tilde{P}(\chi \cap (((-t-m, -t) \cup (t, t+m)) \times K) = \phi, \chi \cap ((-t, t) \times K) \neq \phi)$$

is monotonically decreasing in  $m$ . Denote the convolution of the remaining

$$\exp(-\lambda_{.;y}) \sum_{m=0}^{\infty} \frac{(\lambda_{.;y})^m}{m!} \delta_{my} \quad (y = [k_1, k_2, k_3] \text{ with } k_1 + k_3 > 0)$$

by  $s$ . Then  $p_{.} = r * s$ . The conditional probability in the definition above is precisely  $r(x \neq [0, 0, 0])$ . This tends to zero as  $m \rightarrow \infty$  if and only if  $\sum_x \lambda_{.;x} \rightarrow 0$  as  $m \rightarrow \infty$ . Hence we have the following characterization.

5.1. *A stationary infinitely divisible distribution  $P$  on  $M_K$  is singular infinitely divisible if and only if  $\tilde{P}(N(\chi) < \infty) = 0$ .*

Every  $E_{\lambda, S}$  is singular infinitely divisible: for  $S = \delta_{\phi}$  we have  $E_{\lambda, S} = \delta_{\phi}$  and for  $S(M'_K) = 1$  all the mass of  $E_{\lambda, S}$  is concentrated on  $M'_K$ . However, not all singular infinitely divisible  $P$  necessarily have this form. For example, the convolution  $P$  of  $E_{1/n, P_{1/n}}$  ( $n = 1, 2, \dots$ ) is singular infinitely divisible but  $\tilde{P} = \sum_{n=1}^{\infty} \frac{1}{n} P_{1/n}$  is no longer finite.

There are even singular infinitely divisible  $P$  which cannot in any way be expressed in the form  $P = E_{\lambda, S} * H$  with  $S(M'_K) = 1$  and  $H$  singular infinitely divisible. Distributions of this type can be constructed in the following manner. Let  $F$  be a distribution function with the properties  $F(0+) = 0$ ,  $\int_0^{\infty} x dF(x) = \infty$ . Then there exists no stationary renewal process  $P_F$ . However, an analogous stationary measure  $Q_F$  on  $[M'_K, M'_K]$  with infinite total mass can be defined.  $Q_F$  satisfies conditions  $(\beta)$  and  $(\gamma)$ . All of its

mass lies on the measurable subset

$$\{\chi: \lim_{t \rightarrow \infty} t^{-1} N(\chi \cap (0, t)) = 0\}.$$

A stationary infinitely divisible distribution  $P$  on  $M_K$  is called *regular infinitely divisible* if the conditional distribution

$$P(\cdot | \phi \cap (((-n-m, -n) \cup (n, n+m)) \times K) = \phi)$$

converges weakly to  $P$  as  $m, n \rightarrow \infty$ .

We shall now give an alternative form for this definition also. We consider the vectors

$$[N(\phi \cap ((-n-m, -n) \times K)), N(\phi \cap ((-t, t) \times K)), N(\phi \cap ((n, n+m) \times K))]$$

for fixed positive  $t$ . The regular infinite divisibility of  $P$  is then equivalent to the validity of

$$\begin{aligned} \lim_{m, n \rightarrow \infty} \sum \lambda_{j,x} &= \lim_{m, n \rightarrow \infty} \tilde{P}(\phi \cap (((-n-m, -n) \cup (n, n+m)) \times K) = \phi, \phi \cap ((-t, t) \times K) \neq \phi) \\ &= \tilde{P}(\phi \cap ((-t, t) \times K) \neq \phi) \end{aligned}$$

for all  $t > 0$ , and we have

5.2. A stationary infinitely divisible distribution  $P$  on  $M_K$  is regular infinitely divisible if and only if  $\tilde{P}(N(\chi) = \infty) = 0$ .

From 5.1 and 5.2, there follows

5.3. Each stationary infinitely divisible distribution  $P$  on  $M_K$  can be expressed in exactly one way as the convolution  $P_1 * P_2$  of a regular infinitely divisible stationary  $P_1$  with a singular infinitely divisible  $P_2$ .

We denote the measurable subset  $\{\chi: 0 < N(\chi) < \infty\}$  of  $M_K$  by  $M''_K$ . Each  $\chi$  in  $M''_K$  can be put in the form  $\{[z_1, k_1], \dots, [z_n, k_n]\}$  with  $z_1 < \dots < z_n$ . Set  $M''_K \cap M_K = M''_K$ . Then  $[\psi, t] \rightarrow T_{-t} \psi$  induces a one-one measurable (both ways) mapping of the direct product of

$$M^*_K = \{\chi: \chi \cap ((-\infty, 0) \times K) = \emptyset, \chi \cap (\{0\} \times K) \neq \emptyset\} \cap M''_K$$

and  $R^1$  onto  $M''_K$ . Put  $M^*_K = M^*_K \cap M_K$ . Then to each measure on  $M''_K$  there corresponds a well-defined measure on  $M^*_K \times \mathcal{B}_1$  where  $\mathcal{B}_1$  is the  $\sigma$ -algebra of Borel sets in  $R^1$ .

Now let  $Q$  be a distribution on  $M^*_K$  with the property

$$\int_0^\infty Q(\phi \cap ((I+x) \times K) \neq \emptyset) dx < \infty$$

and suppose  $\lambda > 0$ . Then by 4.7  $E_{n, P_{\lambda/n, Q}} \Rightarrow P_{\lambda, Q}$  as  $n \rightarrow \infty$ , and hence also, for all  $H$  in  $\mathcal{H}$ ,

$$\overline{E_{n, P_{\lambda/n, Q}}}(H) = n P_{\lambda/n, Q}(H) \rightarrow \tilde{P}_{\lambda, Q}(H) \quad (n \rightarrow \infty).$$

On the other hand,  $n P_{\lambda/n, Q}$  tends to the measure induced on  $M''_K$  by  $Q \times \lambda \mu$  where  $\mu$  denotes Lebesgue measure on  $\mathcal{B}_1$ . Thus we have determined  $\tilde{P}_{\lambda, Q}$  and it is clear that all  $P_{\lambda, Q}$  are regular infinitely divisible.

Now suppose that we are given a stationary measure  $V$  on  $M''_K$  satisfying the conditions (B) and (Y). If we denote the corresponding measure on  $M^*_K \times \mathcal{B}_1$  by  $V^\circ$ , then for all  $S$  in  $M^*_K$  and bounded Borel sets  $B$  we have a product representation  $V^\circ(S \times B) = G(S) \mu(B)$  with  $G(S) < \infty$ .  $G$  is a finite measure on  $M^*_K$ . We have  $V^\circ = G \times \mu = Q \times \lambda \mu$  with  $\lambda = G(H^*_K)$  and  $Q = \frac{1}{\lambda} G$ . Because of (Y),  $\int_0^\infty Q(\phi \cap ((I+x) \times K) \neq \emptyset) dx < \infty$ . Hence we obtain

5.4. A stationary distribution  $P \neq \delta_\phi$  on  $M_K$  is regular infinitely divisible if and only if there exist  $\lambda > 0$  and a distribution  $Q$  on  $M_K^*$  with the property

$$\int_0^\infty Q(\phi_n((I+x) \times K) \neq \phi) dx < \infty$$

such that  $P = P_{\lambda, Q}$ . Both  $\lambda$  and  $Q$  are uniquely determined by  $P$ .

Let  $M_K^r$  and  $M_K^l$  denote the sets of infinite  $\phi \in M_K$  for which  $N(\phi_n((-\infty, 0) \times K)) < \infty$  and  $N(\phi_n((0, \infty) \times K)) < \infty$  respectively. Both sets are invariant with respect to  $T_t$  and are contained in  $M_K$ . If  $\tilde{P}(M_K^r) > 0$  for some stationary infinitely divisible  $P$  then we can proceed to a new distribution  $S$  with  $\tilde{S} = \tilde{P}(\cdot) \cap M_K^r$  and apply the arguments used in the proof of 5.4. But these lead to a contradiction because the corresponding distribution  $Q$  on  $M_K^*$  then cannot satisfy the condition  $\int_0^\infty Q(\phi_n((I+x) \times K) \neq \phi) dx < \infty$ . Hence we have  $\tilde{P}(M_K^r) = 0$ . Similarly  $\tilde{P}(M_K^l) = 0$ . These equations enable us to formulate the basic definitions of this chapter "one-sidedly":

5.5. An infinitely divisible stationary distribution  $P$  on  $M_K$  is regular (resp. singular) infinitely divisible if and only if

$$P(\cdot | \phi_n((-n-m, -n) \times K) = \phi) \Rightarrow P \text{ as } m, n \rightarrow \infty$$

(resp., for all  $t > 0$ ,

$$P(\phi_n((0, t) \times K) \neq \phi | \phi_n((-m, 0) \times K) = \phi) \rightarrow 0 \text{ as } m \rightarrow \infty).$$

The Poisson process distributions  $P_\lambda$  are all regular infinitely divisible, for  $P_\lambda = P_{\lambda, \delta_{\{0\}}}$  for  $\lambda > 0$  and  $P_\lambda = \delta_\phi$  for  $\lambda = 0$ . On account of 5.4 we then have for  $\lambda > 0$  the relations  $\tilde{P}_\lambda(N(\chi) \neq 1) = 0$ ,  
 $\tilde{P}_\lambda(\chi_n(0, t) \neq 0) = \lambda t$ .

Hence a sequence  $\{P_n\}$  of stationary infinitely divisible distributions converges to  $P_\lambda$  if and only if for all  $t > 0$ ,  $\tilde{P}_n(N(\chi_n(0,t)) > 1) \rightarrow 0$  and  $\tilde{P}_n(N(\chi_n(0,t)) = 1) \rightarrow \lambda t$  as  $n \rightarrow \infty$ . With the aid of 4.6 we now obtain (cf. [25], [28])

5.6. Suppose given a triangular array  $P_{n,1}, \dots, P_{n,s_n}$  of stationary distributions on  $M$  satisfying condition (2) at the beginning of Chapter 4. Then  $(P_{n,1} * \dots * P_{n,s_n}) \Rightarrow P_\lambda$  if and only if for all  $t > 0$ ,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{s_n} P_{n,i}(N(\chi_n(0,t)) > 1) = 0,$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{s_n} P_{n,i}(N(\chi_n(0,t)) = 1) = \lambda t.$$

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## STATIONARY STOCHASTIC POINT PROCESSES, III\*

P. Franken, A. Liemant and K. Matthes

### 6. A new approach to $\overline{P}_{\lambda, Q}$

We begin with a generalization of the construction of  $P_{\lambda, Q}$  given in Chapter 4.

Let there be given a random mapping from  $k \in K$  to a cluster  $\Psi$  with marks in  $[K', K']$ , i.e. with each  $k \in K$  there is associated a distribution  $Q_{(k)}$  on  $M_K$ , such that for all  $C \in M_K$ , the real function  $Q_{(k)}(C)$  is measurable with respect to  $K$ . We shall assume that the marks of all marked points  $[x, k]$  of  $\phi \in M_K$  generate clusters  $\Psi_{[x, k]}$  distributed independently according to  $Q_{(k)}$ . Let  $G_1$  denote the measurable set of those  $\phi$  for which with probability one the sets  $O_{[x, k]}$  of positions of elements of  $T_{-x} \Psi_{[x, k]}$  are pairwise disjoint. By  $G_2$  we denote the measurable set of those  $\phi$  for which, with probability one, every bounded interval contains only finitely many positions of elements of  $X_\phi = \cup_{[x, k] \in \phi} T_{-x} \Psi_{[x, k]}$ . Clearly if  $\phi_1, \phi_2 \in G_2$ , then  $G_2$  contains  $\phi_1 \cup \phi_2$  if  $\phi_1 \cup \phi_2 \in M_K$ . Set  $G = G_1 \cap G_2$ . If  $\phi \in G$ , then  $G$  also contains all subsets of  $\phi$ .

For  $\phi \in G$  denote the distribution of the marked point process  $X_\phi$  by  $P_{(\phi)}$ . Then for all  $X \in M_K$ , the real function  $P_{(\cdot)}(X)$  defined on  $G \in M_K$  is measurable with respect to  $M_K$ , so we can associate with every measure  $V$  on  $M_K$  satisfying the condition  $V(G) > 0$  a measure  $V_{Q(\cdot)}$  on  $M_K$ , by means of

$$V_{Q(\cdot)}(X) = \int_G P_{(\phi)}(X) V(d\phi).$$

\* *Jahr. Deutsch. Math.-Verein.* 67 (1965) 183-202.

If  $Q_{(k)}$  does not depend on  $k$  we write simply  $V_Q$ . Thus  $P_{\lambda, Q} = (P_{\lambda})_Q$ , and it should be noted that  $P_{\lambda}(G_1) = 1$  always.

From the definitions above, the following results are immediate.

- (a) If the positions of the elements of  $\phi_1 \in G$  are all different from those of  $\phi_2 \in G$ , then  $\phi_1 \cup \phi_2 \in G$  if and only if  $P_{(\phi_1)} * P_{(\phi_2)}$  exists and  $P_{(\phi_1 \cup \phi_2)} = P_{(\phi_1)} * P_{(\phi_2)}$ .
- (b)  $V_{Q(\cdot)}(M_K) = V(M_K)$ .
- (c) If  $V$  is stationary, so is  $V_{Q(\cdot)}$ .
- (d) If every  $c_i > 0$ , then the existence of  $(\sum_1 c_i V_i)_{Q(\cdot)}$  is equivalent to the existence of every  $(V_i)_{Q(\cdot)}$ , and  $(\sum_1 c_i V_i)_{Q(\cdot)} = \sum_1 c_i (V_i)_{Q(\cdot)}$ .

From (a) we obtain

- (e) Let  $A, B$  be finite measures on  $M_K$  for which  $A*B$  is meaningful. Then  $(A*B)_{Q(\cdot)} = A_{Q(\cdot)} * B_{Q(\cdot)}$  in the sense that the existence of one side implies the existence of the other and equality.

From (a), (b) and (e) we obtain

- (f)  $P_{Q(\cdot)}$  is a stationary infinitely divisible distribution if  $P$  is. We retain the notation  $E_{\lambda, S}$  for  $\sum_{m=0}^{\infty} \exp(-\lambda) \frac{\lambda^m}{m!} S^m$  even if  $S$  is not stationary. From (d) and (e) follows

- (g) If  $E_{\lambda, S}$  exists, then  $(E_{\lambda, S})_{Q(\cdot)} = E_{\lambda, S_{Q(\cdot)}}$  in the sense that the existence of one side implies the existence of the other.

The main result of this chapter is

6.1. Let  $P$  be a stationary infinitely divisible distribution on  $M_K$ . Then  $P_{Q(\cdot)}$  exists if and only if  $(\tilde{P})_{Q(\cdot)}$  exists and satisfies condition (γ) of 4.4. For all  $X \in M_K$ , such that  $\phi \notin X$  we have

$$\overline{P_{Q(\cdot)}}(X) = (\tilde{P})_{Q(\cdot)}(X).$$

**Proof:** We proceed in several steps.

(1°) An analysis of its proof shows that the assertion 4.4 also holds for non-stationary infinitely divisible distributions  $V$ . The measures  $\tilde{V}$  thereby satisfy requirements (β) and (γ) of 4.4. We have always that  $\tilde{V}(\phi \cap (\{x\} \times K) \neq \emptyset) = 0$ . Also in the non-stationary case equation (\*) holds as well as the correspondingly modified assertion 4.5.

(2°) For all bounded intervals  $I$ ,  $\phi \rightarrow \phi \cap (I \times K)$  gives a measurable mapping of  $M_K$  into itself. Every measure  $V$  is carried by this mapping into a measure  ${}_I V$ .  ${}_I V$  is infinitely divisible if  $V$  is, and for all  $X \in M_K$  such that  $\phi \not\subset X$  we have  $(\int_I \tilde{V})(X) = \int_I (\tilde{V})(X)$ . Clearly the measure  $\int_I \tilde{V}$  is always finite.

(3°) We now make the additional assumption that there exists  $a > 0$  with the property

$$Q(k) \{ \phi \subseteq ((-a, a) \times K) \} = 1 \quad (k \in K).$$

Then every infinitely divisible  $V$  satisfies  $V(\bar{G}_2) = \tilde{V}(\bar{G}_2) = 0$ .

Moreover, it now follows from the existence of  $(\tilde{V})_{Q(\cdot)}$  that this measure also satisfies condition (γ) of 4.4.

Suppose  $r < s$ . Set  $I = (r-a, s+a)$ . From 4.5,  ${}_I P$  has the form  $E_{\lambda, S}$  with  $\lambda = \int_I P(M_K)$ ,  $S = \lambda^{-1} \int_I P$ . According to (g), the existence of  $({}_I P)_{Q(\cdot)}$  is equivalent to the existence of  $(\int_I P)_{Q(\cdot)}$ , and  $({}_I P)_{Q(\cdot)} = E_{\lambda, S_{Q(\cdot)}}$ . Under our assumptions, the existence of  $P_{Q(\cdot)}$  is equivalent to the existence of  $({}_I P)_{Q(\cdot)}$ . The corresponding result holds for  $(\tilde{P})_{Q(\cdot)}$ . Hence the existence statement of 6.1 is deduced.

From  $({}_I P)_{Q(\cdot)} = E_{\lambda, S_{Q(\cdot)}}$  it follows that

$$({}_I P)_{Q(\cdot)} = E_{\lambda, S_{Q(\cdot)}} (\mid \phi \neq \emptyset),$$

i.e.,

$$\overline{((I P)_{Q(\cdot)})}(\cdot) = (\overline{I P})_{Q(\cdot)}(\cdot) \cap \phi \neq \phi).$$

We deduce from this that for all measurable  $X$  with the property that  $\phi \notin X$ ,

$$\overline{((I P)_{Q(\cdot)})}(X) = (\overline{I P})_{Q(\cdot)}(X).$$

The remainder of the proposition follows from the fact that for all

$$H = \{\phi: N(\phi \cap (I_1 \times L_1)) = m_1, \dots, N(\phi \cap (I_n \times L_n)) = m_n\}$$

with  $[m_1, \dots, m_n] \neq [0, \dots, 0]$ ,  $L_1, \dots, L_n \in K'$  and  $I_1, \dots, I_n \subseteq (r, s)$ , both

$$\overline{((I P)_{Q(\cdot)})}(H) = \overline{P}_{Q(\cdot)}(H) \text{ and } (\overline{I P})_{Q(\cdot)}(H) = (\tilde{P})_{Q(\cdot)}(H) \text{ hold.}$$

(4°) Let  $Q(\cdot)$  be arbitrary. Then for every  $I = (-n, n)$ ,  $I^{Q(\cdot)} \equiv$

$I^{Q(\cdot)}$  satisfies the additional assumption used in (3°). For notational convenience here in (4°), we use  $I$  to denote only intervals of the form  $(-n, n)$ .

Assume now that  $P_{Q(\cdot)}$  exists. Then a fortiori every  $P_{I^{Q(\cdot)}}$  exists and hence so does  $(\tilde{P})_{I^{Q(\cdot)}}$ . For every

$$Y = \{\phi: N(J_1 \times L_1) \geq m_1, \dots, N(J_n \times L_n) \geq m_n\}$$

with  $[m_1, \dots, m_n] \neq [0, \dots, 0]$ ,  $P_{I^{Q(\cdot)}}(Y)$  increases monotonically to  $P_{Q(\cdot)}(Y)$  ( $n \rightarrow \infty$ ). Hence  $P_{I^{Q(\cdot)}} \Rightarrow P_{Q(\cdot)}$  as  $n \rightarrow \infty$ . From the continuity assertion given in II under 4.5, each  $(\tilde{P})_{I^{Q(\cdot)}}(Y) = (\tilde{P})_{I^{Q(\cdot)}}(Y)$  converges to  $(\tilde{P})_{Q(\cdot)}(Y)$ .

Let  $I G_1$  be the analogue of  $G_1$  with respect to the cluster distribution  $I^{Q(\cdot)}$ . Each  $\tilde{P}_{I G_1}$  increases monotonically to  $\tilde{P}(G_1)$ . Since  $\tilde{P}_{I G_1} = 0$  for all  $n$ ,  $\tilde{P}(G_1) = 0$  also. Now if  $\tilde{P}(G_2) > 0$ , we cannot have  $(\tilde{P})_{I^{Q(\cdot)}}(N(\phi \cap (J \times K')) \geq m)$  converging uniformly in  $I$  to zero as  $m \rightarrow \infty$ . Hence the existence of  $(\tilde{P})_{Q(\cdot)}$  is established.

For all  $Y$ ,  $\overline{P}_{I^{Q(\cdot)}}(Y)$  increases monotonically to  $\tilde{P}_{Q(\cdot)}(Y)$ .  
 Therefore  $\tilde{P}_{Q(\cdot)}(X) = \overline{P}_{Q(\cdot)}(X)$  for all  $X \in M_K$ , with the property  $\phi \in X$ . In particular,  $\tilde{P}_{Q(\cdot)}$  satisfies condition  $(\gamma)$  of 4.4.

(5°) Conversely, we now assume that  $\tilde{P}_{Q(\cdot)}$  exists and satisfies condition  $(\gamma)$  of 4.4. Then the same is true for  $\tilde{P}_{I^{Q(\cdot)}}$ . Hence  $P_{I^{Q(\cdot)}}$  exists also. Let  $V$  be the infinitely divisible distribution corresponding to  $\tilde{P}_{Q(\cdot)}$  ( $(\cdot) \cap \phi \neq \phi$ ). For all  $Y$   $\tilde{P}_{I^{Q(\cdot)}}(Y)$  increase monotonically to  $\tilde{P}_{Q(\cdot)}(Y)$  as  $n \rightarrow \infty$ . By the continuity assertion already used in (4°) we see that  $P_{I^{Q(\cdot)}}$  converges weakly to  $V$  as  $n \rightarrow \infty$ . The existence of  $P_{Q(\cdot)}$  follows from the same results used in (4°) to show the existence of  $\tilde{P}_{Q(\cdot)}$ . The second statement of 6.1 has already been proved in (4°). Thus the proof of 6.1 is complete.

In general,  $\tilde{P}_{Q(\cdot)}(\{\phi\}) > 0$ , so the equation in 6.1 cannot hold for all measurable  $X$ . The restriction  $\phi \notin X$  is superfluous if  $Q_{(k)}(\psi \neq \phi) = 1$  for all  $k \in K$ .

From 6.1 and the easily determined structure of  $\tilde{P}_\lambda$  there follows as a special case the structural assertion for  $\overline{P}_{\lambda, Q}$  derived towards the end of Chapter 5.

Also from 6.1, we obtain the following sharpening of the second invariance property of  $P_\lambda$  given in Chapter 3:

6.2. Let  $P$  be the distribution of an independently marked stationary Poisson process. If  $Q_{(k)}(N(\psi)=1) = 1$  for all  $k \in K$ , then  $P_{Q(\cdot)}$  exists and is also the distribution of an independently marked stationary Poisson process.

We have already used 6.2 after 4.1 to prove that  $P_{\lambda, Q} = P_{\lambda, Q^*}$ .

## 7. Palm distributions in a generalized sense

Let  $S_K$  denote the class of those measures on  $M_K$  stationary with respect to  $T_t$  and satisfying condition ( $\gamma$ ) of 4.4. Although any  $Q \in S_K$  need not be finite, a substantial part of the remarks in Chapter 2 remain valid for such measures.

In contrast to Chapter 2, we will not distinguish here any  $L \in K$ , i.e. we develop ideas in this chapter only for the special case  $L=K$ . Accordingly, the intensity  $\rho_Q$  of  $Q \in S_K$  will always be understood to be the integral  $\int N(\phi \cap ((0,1) \times K)) Q(d\phi)$ . For  $0 < \rho_Q < \infty$ , we can then introduce  $Q_0$ , the Palm distribution of  $Q$ , by

$$Q_0(S) = \frac{1}{\rho_Q} \int N(\{x: 0 < x < 1, (\{x\} \times K) \cap \phi \neq \emptyset, T_x \phi \in S\}) Q(d\phi)$$

for  $S \in M_K$ .  $Q_0$  is always a distribution on  $M_K$ , and in the case where  $Q(M'_K) = 1$ ,  $Q(\{\emptyset\}) = 0$ , it coincides with the Palm distribution (with respect to  $K$ ) introduced in Chapter 2.

Set  $M^o_K = \{\phi: \phi \in M_K, (\{0\} \times K) \cap \phi \neq \emptyset\}$ ,  $M^o_K = M^o_K \cap M_K$ . Every Palm distribution  $Q_0$  can be regarded as a distribution on  $M^o_K$ .

The following formulae for computations clearly hold.

7.1. Let  $Q = \sum_i Q_i$  with  $Q_i \in S_K$ ,  $0 < \rho_{Q_i}$  and  $\rho_Q = \sum_i \rho_{Q_i} < \infty$ . Then  $Q_0 = \frac{1}{\rho_Q} \sum_i \rho_{Q_i} (Q_i)_0$ .

7.2. For all positive  $\alpha$  and all  $Q \in S_K$  such that  $0 < \rho_Q < \infty$ ,  $(\alpha Q)_0 = Q_0$ .

7.3. For all  $Q \in S_K$  such that  $0 < \rho_Q < \infty$ ,  $(Q(\{ \cdot \} \cap \phi \neq \emptyset))_0 = Q_0$ .

By 7.2  $\rho_Q$  can take any positive value for a given Palm distribution  $Q_0$ .

Let  $W_K$  be the set of all  $Q \in S_K$  such that  $Q(\{\phi\}) = 0$ . From the results of Chapter I of [30], we then have the following inversion formula:

7.4. For all  $Q \in W_K$  with finite positive intensity, all  $S \in M_K$  and all non-empty bounded open intervals  $I = (\alpha, \beta)$ ,

$$Q(S \cap (\phi \cap (I \times K) \neq \emptyset)) = \rho_Q \int \left[ \int_{\max(\alpha, \beta - x_0(\phi))}^{\beta} k_S(T_{-\tau} \phi) d\tau \right] Q_0(d\phi)$$

where  $x_0(\phi) = \infty$  when no element in  $\phi$  has its position in  $(0, \infty)$ .

Each  $Q \in W_K$  such that  $0 < \rho_Q < \infty$  is thus uniquely determined by  $\rho_Q$  and  $Q_0$ .

We denote by  $\rightarrow$  the type of convergence in  $W_K$  introduced in II following 4.5. If  $Q$  as well as all the  $Q_n$ 's are distributions, then  $Q_n \rightarrow Q$  ( $n \rightarrow \infty$ ) is equivalent to  $Q_n \Rightarrow Q$  ( $n \rightarrow \infty$ ).

The transformation from  $Q$  to  $Q_0$  is continuous in the following sense (cf. [29]).

7.5. Suppose  $Q, Q_n \in W_K$  with  $0 < \rho_Q, \rho_{Q_n} < \infty$  ( $n = 1, 2, \dots$ ), and  $\rho_{Q_n} \rightarrow \rho_Q$  ( $n \rightarrow \infty$ ). Then  $Q_n \rightarrow Q$  ( $n \rightarrow \infty$ ) is equivalent to  $(Q_n)_0 \Rightarrow Q_0$  ( $n \rightarrow \infty$ ).

Let  $W_K^S$  denote the set of all  $Q \in W_K$  such that  $Q(\phi \in M'_K) = 0$ . For all  $Q \in W_K^S$  with  $0 < \rho_Q < \infty$  the inversion formula<sup>1</sup> 2.7 is valid. In [30], the following generalization of 2.9 is proved:

<sup>1</sup> The proof of 2.7 given in I contains a gap: both sides of (5) there may be infinite. In the proof of 2.7 however, we may restrict attention to such  $S$  for which  $\phi \in S$  always implies  $y_0(\phi) - y_{-1}(\phi) \leq c_S$  with  $c_S < \infty$ . Then  $\int [\int_0^{y_0} k_S(T_{-\tau} \phi) d\tau] P(d\phi) \leq c_S$ .

7.6. A distribution  $P$  on  $M^0_K$  is the Palm distribution of some  $Q \in W^S_K$  with  $0 < \rho_Q < \infty$  if and only if  $P(M^0_K) = 1$  and the restriction of  $P$  to  $M^0_K \cap M^1_K$  is invariant with respect to  $\theta$ .

Let  $W^F_K$  denote the set of all  $Q \in W_K$  with the property  $Q(\phi \text{ is infinite}) = 0$ . The Palm distributions of measures in  $W^F_K$  can also be characterized by symmetry properties (cf. [30]).

We denote the set of all finite  $\phi$  in  $M^0_K$  by  $M^F_K$ . Each  $\phi \in M^F_K$  is of the form  $\phi = \{[z_1, k_1], \dots, [z_n, k_n]\}$  with  $z_1 < \dots < z_n$ . We define a shift operator  $\gamma$  on  $M^F_K$  by

$$\gamma\{[z_1, k_1], \dots, [z_n, k_n]\} = \begin{cases} T_{z_{i+1}-z_i} \phi & \text{if } z_i = 0 \text{ and } i < n, \\ T_{z_1} \phi & \text{if } z_n = 0. \end{cases}$$

Then we have

7.7. A distribution  $P$  on  $M^0_K$  is the Palm distribution of some  $Q \in W^F_K$  with  $0 < \rho_Q < \infty$  if and only if  $P(\phi \text{ is finite}) = 1$  and the restriction of  $P$  to  $W^F_K$  is invariant with respect to  $\gamma$ .

7.6 and 7.7 give us a general view of the class of all Palm distributions of measures in  $W_K$  since, as we saw in Chapter 5, every  $Q \in W_K$  is uniquely expressible in the form  $Q = Q_1 + Q_2$  with  $Q_1 \in W^F_K$ ,  $Q_2 \in W^S_K$ .

We now introduce a simple class of Palm distributions. Let  $F$  be a left-continuous distribution function of a positive (possibly infinite) random variable. Thus we require  $F(0+0) = 0$  but allow  $F(\infty) < 1$ . Let  $(\xi_i)_{-\infty < i < \infty}$  be a doubly infinite sequence of independent positive random variables with distribution  $F$ . If  $F(\infty) = 1$  then with probability 1 all  $\xi_i$ 's are finite so that we can form the point process (as under 2.9 in I)

$$\{ \dots, -(\xi_{-n} + \dots + \xi_{-1}), \dots, -\xi_{-1}, 0, \xi_0, \dots, (\xi_0 + \dots + \xi_n), \dots \}.$$

However, if  $F(\infty) < 1$ , then with probability one there exists a least non-negative index  $n_1$  such that  $\xi_{n_1} = \infty$ , and a greatest negative index  $n_2$  such that  $\xi_{n_2} = \infty$ . We then consider the point process

$$\{-(\xi_{n_2-1} + \dots + \xi_{-1}), \dots, -\xi_{-1}, 0, \xi_0, \dots, (\xi_0 + \dots + \xi_{n_1-1})\}.$$

Thus with every  $F$  we have associated a distribution  $H_F$  on  $M^0$ . On account of 7.6 and 7.7 every  $H_F$  is a Palm distribution of a measure  $Q$  on  $W$ . If  $F(\infty) < 1$  then  $Q \in W^r$ ; otherwise  $Q \in W^s$ .  $Q$  is finite if and only if  $\int_0^\infty x dF(x) < \infty$ . In this case, we can set  $Q = P_F$ . For  $F(\infty) = 1$  and  $\int_0^\infty x dF(x) = \infty$ ,  $Q$  is an element of the family<sup>2</sup> of measures  $Q_F$  given in II under 5.1.

## 8. The distributions $(\tilde{P})_0$

From now on we again assume that condition (a) of Chapter 4 holds. Then by 4.4 we have an invertible one-one mapping of  $W_K$  onto the class  $U_K$  of stationary infinitely divisible distributions on  $M_K$ . Here the classes  $W_K^r$  ( $W_K^s$ ) correspond to the regular (resp. singular) infinitely divisible  $P$  in  $U_K$ .

By simple computation, we have (cf. [29])

$$8.1. \quad \rho_P = \rho_{\tilde{P}} \text{ whenever } 0 < \rho_P < \infty.$$

Thus in the case  $0 < \rho_P < \infty$ , we can form  $(\tilde{P})_0$  as well as  $P_0$ . Because of 4.4 and 7.4 each  $P \in U_K$  with finite positive intensity is uniquely

<sup>2</sup> Note that  $Q_F$  was uniquely determined only up to a positive constant multiplier.

determined by  $\rho_P$  and  $(\tilde{P})_0$ . For a given  $(\tilde{P})_0$ ,  $\rho_P$  can assume any positive value. From 7.5 and the remarks in II following 4.5, we obtain

8.2. Let  $P, P_n \in U_K$  with  $0 < \rho_P, \rho_{P_n} < \infty$  ( $n = 1, 2, \dots$ ) and suppose  $\rho_{P_n} \rightarrow \rho_P$  ( $n \rightarrow \infty$ ). Then  $P_n \Rightarrow P$  is equivalent to  $(\tilde{P}_n)_0 \Rightarrow (\tilde{P})_0$ .

For  $P = E_{\lambda, S}$  with  $\lambda > 0$ ,  $0 < \rho_S < \infty$ , we find by computation that  $\rho_P = \lambda \rho_S$ ,  $(\tilde{P})_0 = S_0$ ,  $P_0 = P * (\tilde{P})_0$ . Thus if  $F(\infty) = 1$  and  $\int_0^\infty x dF(x) < \infty$ , then the class of those  $P$  in  $U$  such that  $(\tilde{P})_0 = H_F$  is precisely the set of  $E_{\lambda, P_F}$  with  $\lambda > 0$ . On the other hand, we do not know any simple representation of the singular infinitely divisible  $P \in U$  such that  $(\tilde{P})_0 = H_F$  for  $F(\infty) = 1$ ,  $\int_0^\infty x dF(x) < \infty$ .

For all  $P \in U_K$  with finite positive intensity we have by 4.7 the relation  $n^{n/P} \rightarrow \tilde{P}$  ( $n \rightarrow \infty$ ), and hence using 7.5 we have the convergence

8.3.  $(n^{n/P})_0 \Rightarrow (\tilde{P})_0$  as  $n \rightarrow \infty$ .

Now let  $P$  be any distribution in  $U_K$  with  $0 < \rho_P < \infty$ . Then because of 4.7 and 8.2 we have the relation  $(E_{n, n^{n/P}})_0 \Rightarrow P_0$  as  $n \rightarrow \infty$ . However,  $(E_{n, n^{n/P}})_0 = (E_{n, n^{n/P}}) * (n^{n/P})_0$ , and here the first factor converges to  $P$  and the second to  $(\tilde{P})_0$ . Using 1.3 we obtain

8.4. For all  $P \in U_K$  such that  $0 < \rho_P < \infty$ ,

$$P_0 = P * (\tilde{P})_0.$$

Now let  $P \in U_K$  be regular infinitely divisible and distinct from  $\delta_\phi$ . Then by 5.4,  $P = P_{\lambda, Q}$  with  $\lambda > 0$  and  $Q(M_K^*) = 1$ , whence  $\rho_P = \rho_{\tilde{P}} = \lambda \int N(\Psi) Q(d\Psi)$ . For the time being, we make the additional assumption that  $Q(N(\Psi) = n) = 1$ . Let  $r$  be a random variable with  $p(r=i) = 1/n$  ( $i = 1, \dots, n$ ) independent of

the marked point process  $\Psi$  with distribution  $Q$ . With probability one each  $\Psi$  is of the form

$$\Psi = \{[z_1, k_1], \dots, [z_n, k_n]\} \text{ with } 0 = z_1 < \dots < z_n.$$

Now let  $Q^+$  denote the distribution of  $T_{z_r(\Psi)}\Psi$ . Then  $(\overline{P_{\lambda, Q}})_0 = Q^+$ . In particular, for  $\lambda > 0$ ,  $(\tilde{P}_\lambda)_0$  is precisely  $\delta_{\{0\}}$ .

We now turn to the weaker restriction  $0 < \rho_P < \infty$ . Then

$$P_{\lambda, Q} = \sum_n Q(\Psi=n) \tilde{P}_{\lambda, Q}(\Psi=n) \text{ and hence}$$

$$8.5. (\overline{P_{\lambda, Q}})_0 = (\int N(\Psi) Q(d\Psi))^{-1} \sum_n n Q(N(\Psi)=n) (Q(\Psi=n))^+,$$

where the summation is taken over all  $n$  such that  $Q(N(\Psi)=n) > 0$ .

We already know that any  $P \in U$  such that  $(\tilde{P})_0 = H_P$  for  $F(\infty) < 1$  is regular infinitely divisible, and accordingly all such  $P$  are of the form  $P_{\lambda, Q}$ . Clearly  $Q$  is uniquely determined by  $F$  if we require  $Q(M^*) = 1$ . On the other hand  $\lambda$  may take any positive value. From 8.5 the form of  $Q$  is easily obtained:

$$8.6. Q \text{ is the distribution of } \{0, \xi_0, \dots, (\xi_0 + \dots + \xi_{n_1-1})\}.$$

Equation 8.4 is easily interpreted for  $P$  regular infinitely divisible. We see that it can even be used as a characterization of infinite divisibility, viz. (cf. [30]) we have

8.7. If for a stationary distribution  $P$  on  $M_K$  with  $0 < \rho_P < \infty$  there exists a distribution  $H$  on  $M_K^0$  such that  $P_0 = P * H$ , then  $P \in U_K$  and  $H = (\tilde{P})_0$ . In particular a stationary distribution  $P$  on  $M$  such that  $0 < \rho_P < \infty$  is of the form  $P_\lambda$  if and only if  $P_0 = P * \delta_{\{0\}}$ .

The particular case in 8.7 has already been given by Slivnyak [18].

### 9. Generalization of a result of Palm and Khinchin

An array  $(P_{n,i})$  ( $1 \leq i \leq s_n; n = 1, 2, \dots$ ) of stationary distributions on  $M_K$  is called *infinitesimal* if it satisfies condition (2) at the beginning of Chapter 4. From 4.6 and 8.2 we have

9.1. Let  $(P_{n,i})$  be an infinitesimal array of stationary distributions on  $M_K$  with finite positive intensities  $\rho_{n,i}$ . If there exists a distribution  $P$  such that  $0 < \rho_P < \infty$ ,  $\sum_{i=1}^{s_n} \rho_{n,i} \rightarrow \rho_P$  ( $n \rightarrow \infty$ ) and  $(P_{n,1} * \dots * P_{n,s_n}) \rightarrow P$  ( $n \rightarrow \infty$ ), then  $P \in U_K$  and  $(\sum_{i=1}^{s_n} \rho_{n,i})^{-1} (\sum_{i=1}^{s_n} \rho_{n,i} (P_{n,i})_0) \rightarrow (\tilde{P})_0$ .

Given  $(P_{n,1} * \dots * P_{n,s_n}) \rightarrow P$ ,  $0 < \rho_P < \infty$ , and  $\sum_{i=1}^{s_n} \rho_{n,i} \rightarrow \rho$  as  $n \rightarrow \infty$ , it does not necessarily follow that  $\rho = \rho_P$ . For example, if  $(P_{n,i})$  satisfies all the assumptions of 9.1, then the array formed by adding  $P_{n,s_n+1} = E_{n-1, P_n^\lambda}$  ( $\lambda > 0$ ) is also infinitesimal and  $(P_{n,1} * \dots * P_{n,s_n+1}) \rightarrow P$  ( $n \rightarrow \infty$ ). However,  $\sum_{i=1}^{s_n+1} \rho_{n,i} \rightarrow \rho_P + \lambda$  ( $n \rightarrow \infty$ ). In general, we may conclude from  $0 < \rho_{n,i} < \infty$  and  $(P_{n,1} * \dots * P_{n,s_n}) \rightarrow P$  only that  $\rho_P \leq \liminf_{n \rightarrow \infty} \sum_{i=1}^{s_n} \rho_{n,i}$ . However, under the assumptions of 9.1,  $(P_{n,i})$  is infinitesimal in a stronger sense:

9.2. If  $(P_{n,i})$  satisfies all the assumptions of 9.1, then  $\max_{1 \leq i \leq s_n} \rho_{n,i} \rightarrow 0$  ( $n \rightarrow \infty$ ).

**Proof.** Suppose that the maximum  $\not\rightarrow 0$  ( $n \rightarrow \infty$ ). Then without loss of generality, we can assume that  $\rho_{n,1} \geq c > 0$  for all  $n$ . Then since

$(P_{n,2} * \dots * P_{n,s_n}) \Rightarrow P$ ,  $\rho_P \leq \liminf_{n \rightarrow \infty} \sum_{i=1}^{s_n} \rho_{n,i} \leq \rho_P^{-c}$ , which is a contradiction.

A sharper converse of 9.1 is

9.3. Let  $(P_{n,i})$  be an infinitesimal array of stationary distributions on  $M_K$  with finite positive intensities. Suppose that  $\sum_{i=1}^{s_n} \rho_{n,i} \rightarrow \rho$  ( $n \rightarrow \infty$ ) where  $0 < \rho < \infty$  and that  $(\sum_{i=1}^{s_n} \rho_{n,i})^{-1} (\sum_{i=1}^{s_n} \rho_{n,i} (P_{n,i})_0) \Rightarrow X$  ( $n \rightarrow \infty$ ). Then  $(P_{n,1} * \dots * P_{n,s_n})$  converges weakly to some  $P$  with intensity  $\rho$  as  $n \rightarrow \infty$ .

The proof follows from 4.6, 8.2 and the following lemma.

9.4. The class of all Palm distributions of measures on  $W_K$  with finite positive intensities is closed under weak convergence.

Proof. Let a sequence  $P_n$  of Palm distributions converge weakly to a distribution  $X$ . With each  $P_n$  we associate the corresponding  $Q_n \in W_K$  with intensity 1. Because of 8.2, it is sufficient to show that  $Q_n$  converges to some  $Q \in W_K$  with intensity 1.

By a formal application of the inversion formula 7.4 we define for all  $S$  in the  $\sigma$ -ring  ${}_k R$  defined following 4.4

$${}_k Q(S) = \int \left[ \int_{\max(-k, k-x_0(\phi))}^k k_S(T_{-\tau}\phi) d\tau \right] X(d\phi),$$

thus obtaining a finite measure on  ${}_k R$ .

For  $I_1, \dots, I_\ell \subseteq (-k, k)$ ,  $L_1, \dots, L_\ell \in K$ , and  $[m_1, \dots, m_\ell] \neq [0, \dots, 0]$ , let

$$S = \{\phi: N(\phi \cap (J_1 \times L_1)) \geq m_1, \dots, N(\phi \cap (J_\ell \times L_\ell)) \geq m_\ell\}.$$

Then the bounded integrand in square brackets is in a certain defined sense (see

[29]) continuous almost everywhere with respect to  $X$ . Thus, for all  $S$  of the form specified,  $Q_n(S) \rightarrow_k Q(S)$  ( $n \rightarrow \infty$ ).

From this it follows that  $_{k+1}Q$  is always an extension of  $_kQ$ . Therefore since (a) holds, there exists a common extension of every  $_kQ$  to a measure  $Q$  on  $M_K$  such that  $Q(\{\phi\}) = 0$ . On the basis of the definition of  $_kQ$ ,  $Q$  also satisfies condition ( $\gamma$ ) of 4.4.

The convergence statement given above can now be put in the form  $Q_n \rightarrow Q$  ( $n \rightarrow \infty$ ). Therefore  $Q$  is also stationary and hence in  $W_K$ . Clearly

$0 < \rho_Q \leq 1$ . Now

$$\lim_{t \rightarrow 0} \frac{1}{t} Q(\phi_n((0,t) \times K) \neq \phi) = \lim_{t \rightarrow 0} \frac{1}{t} \int \left[ \int_{(t-x_0(\phi))^+}^t d\tau \right] X(d\phi) = 1.$$

If we had used here the representation 7.4 for  $Q$ , then the right-hand term would have been  $\rho_Q \cdot 1$ . Hence  $\rho_Q = 1$ , and the proof of 9.4 is complete.

Clearly a sequence of Palm distributions  $Y_n$  of distributions in  $W$  converges to  $(\tilde{P}_\lambda)_0 = \delta_{\{0\}}$  ( $\lambda > 0$ ) if and only if for all  $t > 0$  the sequence of distribution functions  $F_n$  of  $x_0(\phi)$  induced by  $Y_n$  converges to zero. If we note that to a mixture of Palm distributions there is a corresponding mixture of distribution functions of  $x_0(\phi)$ , then we obtain from 9.1 and 9.3 the sharper form derived in [23] of the result of Palm and Khinchin on convergence to  $P_\lambda$  (proved in [4]):

9.5. Let  $(P_{n,i})$  be an infinitesimal array of stationary distributions on  $M$  with finite positive intensities  $\rho_{n,i}$ . Suppose  $\sum_{i=1}^s \rho_{n,i} \rightarrow \lambda$  ( $n \rightarrow \infty$ ),  $0 < \lambda < \infty$ . Then the convergence  $(P_{n,1} * \dots * P_{n,s_n}) \rightarrow P_\lambda$  is equivalent to

$$\sum_{i=1}^{s_n} \rho_{n,i} F_{n,i}(t) \rightarrow 0 \quad \text{for all } t > 0,$$

where  $F_{n,i}$  is the distribution function of  $x_0(\phi)$  induced by  $(P_{n,i})_0$  (cf. 2.7 of I).

### 10. Limit distributions for infinitesimal arrays of stationary renewal processes

Let  $G$  denote the class of all left-continuous distribution functions  $G$  of non-negative (possibly infinite) random variables. We have  $G(0) = 0$  always, but it is not necessary that either  $G(0+0) = 0$  or  $G(\infty) = 1$ . Write  $G_n \Rightarrow G$  ( $n \rightarrow \infty$ ) if for all continuity points  $t$  of  $G$ ,  $G_n(t) \rightarrow G(t)$  ( $n \rightarrow \infty$ ).

Let  $F$  denote the class of all  $F \in G$  such that  $F(0+0) = 0$ . In Chapter 7, we established a one-one correspondence between each  $F \in F$  and certain distributions  $H_F$  on  $M^0$ . The convergence just defined for distribution functions corresponds to weak convergence of the corresponding distributions on  $M^0$ :

10.1. For a sequence of distributions  $F_n \in F$ ,  $H_{F_n} \Rightarrow X$  ( $n \rightarrow \infty$ ) if and only if  $F_n \Rightarrow F$  ( $n \rightarrow \infty$ ) for some  $F \in F$  and also  $X = H_F$ .

The proof follows from the simple lemma 10.2 below.

Each  $\phi \in M^0$  has the form (cf. 1.1 of I)  $\{x_i\}_{-m-1 < i < n}$ , where  $x_{-1} = 0$ . Let  $d_j$  denote the "lifetimes"  $x_{j+1} - x_j$ . Then  $\phi \in M^0$  is also uniquely determined by  $(d_j)_{-m-1 < j < n-1}$ . Write the event

$$-n-1 < r < s < n-1, \quad d_i < t_i \quad \text{for} \quad r \leq i \leq s$$

more briefly as  $\{d_i < t_i, r \leq i \leq s\}$ . Then from the characterization of weak convergence derived in [29] we have the lemma

10.2. A sequence  $(Y_n)$  of distributions on  $M^0$  converges weakly to  $Y$  if and only if for all  $r < s$  such that  $r \leq 0 \leq s$  and all continuity points  $\{t_r, \dots, t_s\}$  of  $Y\{d_i < t_i, r \leq i \leq s\}$ ,  $Y_n\{d_i < t_i, r \leq i \leq s\} \rightarrow Y\{d_i < t_i, r \leq i \leq s\}$  ( $n \rightarrow \infty$ ).

With the aid of 9.1, 9.3 and 10.1 we may summarize the limiting behaviour of infinitesimal arrays  $(P_{n,i})$  with constant "rows"  $(P_{n,i})_{1 \leq i \leq n}$  of stationary renewal distributions. We note here the relation derived under 2.9 of I that  $(\rho_{P_F})^{-1} = \Delta_F = \int_0^\infty x dF(x)$ .

10.3. Let  $(F_n)$  be a sequence of elements of  $F$  such that  $F_n(\infty) = 1$ ,  $\Delta_{F_n} < \infty$ ,  $\Delta_{F_n} \rightarrow \infty$  ( $n \rightarrow \infty$ ). Further, let  $(S_n)$  be a given sequence of non-negative integers such that  $S_n(\Delta_{F_n})^{-1} \equiv S_n \rho_{F_n} \rightarrow \rho$  ( $n \rightarrow \infty$ ),  $0 < \rho < \infty$ . Then  $(P_{F_n})^{S_n} \Rightarrow P$  ( $n \rightarrow \infty$ ) is equivalent to  $F_n \Rightarrow F$  ( $n \rightarrow \infty$ ),  $F \in F$ ,  $\rho_P = \rho$ , and  $(\tilde{P})_0 = H_F$ .

The  $F$  appearing in 10.3 may be any element of  $F$  on account of

10.4. To every  $F \in F$  there exists a sequence  $(F_n)$  in  $F$  such that  $F_n \Rightarrow F$  ( $n \rightarrow \infty$ ),  $F_n(\infty) = 1$ ,  $\Delta_{F_n} < \infty$ ,  $\Delta_{F_n} \rightarrow \infty$ .

Proof. Suppose  $F \in F$  and  $F(t_0) = 1$ . Set

$$F_n(t) = \begin{cases} (1 - \frac{1}{n}) F(t) & \text{for } t \leq t_n, \\ 1 & \text{for } t > t_n, \end{cases}$$

where  $t_n$  is chosen sufficiently large so that  $\Delta_{F_n} > n$  (e.g.  $t_n = n^2$ ). If, on the other hand,  $F \in F$  has  $F(t) < 1$  for all  $0 < t < \infty$ , then set

$$F_n(t) = \begin{cases} F(t) & t \leq n \\ F(n) & n < t \leq t_n \\ 1 & t_n < t \end{cases}$$

where  $t_n$  is chosen sufficiently large so that  $\Delta_{F_n} > n$  (and hence  $t_n > n$ ).

This choice of  $F_n$  suffices to prove the assertion of 10.4.

The examination of arbitrary infinitesimal arrays of stationary renewal distributions  $P_{F_n,1}$  requires some preliminaries.

First we recall the concept of weak convergence of distributions (cf. [31], [32]). Let  $E$  be a complete separable metric space and  $\mathcal{E}$  the  $\sigma$ -field of Borel sets of  $E$ . A sequence  $(P_n)$  of distributions on  $E$  converges weakly to a distribution  $P$  if  $\int f dP_n \rightarrow \int f dP$  ( $n \rightarrow \infty$ ) for all bounded continuous real functions  $f$  on  $E$ . Thus weak convergence depends only on the topology in  $E$ . From  $P_n \Rightarrow P$  ( $n \rightarrow \infty$ ) follows the "pointwise" convergence  $P_n(L) \rightarrow P(L)$  for all continuity sets relative to  $P$ , i.e., for those sets  $L \in \mathcal{E}$  whose boundaries have zero probability relative to  $P$ . We can always introduce on the class of all distributions on  $E$  a metric with respect to which this becomes a complete separable metric space in which the metric convergence coincides with the weak convergence introduced above.

For  $E = [0, \infty]$  the distributions on  $E$  are in one-one correspondence with the distributions in  $G$  whereby weak convergence corresponds to the convergence  $G_n \Rightarrow G$  ( $n \rightarrow \infty$ ). Hence there exists in  $G$  a metric with the properties given above. Because of the compactness of  $E$ ,  $G$  is also compact. We denote by  $\underline{G}$  the family of Borel sets of  $G$ . It is generated by the sets  $\{G: G(t) < c\}$ .

Clearly all of  $H_F(d_1 < t_1, r \leq s)$  and hence all  $H_F(S)$  ( $S \in M^0$ ) depend measurably (with respect to  $\underline{G}$ ) on  $F$ , so that we can form the mixture

$\int_{\mathcal{F}} H_{\mathcal{F}}(\cdot) q(d\mathcal{F})$  for all distributions  $q$  on  $\underline{G}$  such that  $q(\mathcal{F}) = 1$ . Then we have

10.5.  $q$  is uniquely determined by  $\int_{\mathcal{F}} H_{\mathcal{F}}(\cdot) q(d\mathcal{F})$ .

Proof. Let  $h$  be a function on  $\mathcal{G}$  of the form  $h(G) = \prod_{v=0}^n G(t_v)$ .

Here we may choose for  $(t_v)$  an arbitrary finite sequence of positive numbers.

Then

$$\int_{\mathcal{F}} h(\mathcal{F}) q(d\mathcal{F}) = \int_{\mathcal{F}} H_{\mathcal{F}}\{d_{i-2} < t_i \text{ for } 1 \leq i \leq n\} q(d\mathcal{F}).$$

The expected value of  $h$  with respect to  $q$  can therefore be expressed by means of  $\int_{\mathcal{F}} H_{\mathcal{F}} q(d\mathcal{F})$ .

We now consider functions  $f(G) = \prod_{i=1}^m \left( \int_0^{t_i} G(\tau) d\tau \right)$  where  $(t_i)$  is once more an arbitrary finite sequence of positive numbers. Each  $f$  is the uniform limit of a sequence of linear combinations of functions  $h$ . Hence

$\int_{\mathcal{F}} f(\mathcal{F}) q(d\mathcal{F})$  is also uniquely determined by  $\int_{\mathcal{F}} H_{\mathcal{F}} q(d\mathcal{F})$ .

The same is also true for all finite linear combinations

$g = \alpha_0 + \sum_{k=1}^s \alpha_k f_k$ . The class of all such  $g$  forms a sub-algebra of the algebra of all continuous real functions on  $G$ . Clearly this sub-algebra separates points in  $G$ , i.e.,  $G_1 \neq G_2$  implies the existence of some  $g$  such that  $g(G_1) \neq g(G_2)$ . By the Stone-Weierstrass theorem (see e.g. [33] p. 47), every continuous real function on  $G$  is the uniform limit of a sequence  $(g_n)$ .

Thus we see that the integral with respect to  $q$  of each continuous real function on  $G$  is uniquely determined by  $\int_{\mathcal{F}} H_{\mathcal{F}} q(d\mathcal{F})$ . Hence our proposition 10.5 follows.

The transformation from  $q$  to  $\int_{\mathcal{F}} H_{\mathcal{F}} q(d\mathcal{F})$  is continuous in the following sense:

10.6. Let  $(q_n)$  be a sequence of distributions on  $\underline{G}$  such that  $q_n(F) = 1$  ( $n = 1, 2, \dots$ ). Then  $\int_F H_F q_n(dF) \Rightarrow X$  ( $n \rightarrow \infty$ ) is equivalent to  $q_n \Rightarrow q$  ( $n \rightarrow \infty$ ),  $q(F) = 1$ ,  $X = \int_F H_F q(dF)$ .

Proof. Suppose  $q_n \Rightarrow q$ ,  $q(F) = 1$ . Choose a finite sequence of bounded open intervals  $I_i = (a_i, b_i)$  ( $i = 1, \dots, m$ ) such that

$\int_F H_F(\phi \cap \{a_1, \dots, a_m, b_1, \dots, b_m\} \neq \phi) q(dF) = 0$ . Then, relative to  $q$ ,

$H_F(\phi \cap \{a_1, \dots, a_m, b_1, \dots, b_m\} \neq \phi) = 0$  for almost all  $F$ . Hence by 10.1,

$H_F(N(\phi \cap I_i) = n_i, 1 \leq i \leq m)$  is continuous  $q$ -almost everywhere. Hence (cf. [32]

Theorem 1.8)

$$\lim_{n \rightarrow \infty} \int_F H_F(N(\phi \cap I_i) = n_i, 1 \leq i \leq m) q_n(dF) = \int_F H_F(N(\phi \cap I_i) = n_i, 1 \leq i \leq m) q(dF),$$

i.e.,  $\int_F H_F q_n(dF) \Rightarrow \int_F H_F q(dF)$  ( $n \rightarrow \infty$ ).

Now suppose that  $q_n \Rightarrow q$ ,  $q(F) < 1$ . Then it is impossible that

$\int_F H_F q_n(dF) \Rightarrow X$ , for otherwise by 10.2, apart from a countable set of  $t$

values  $\int_F H_F(d_{-1} < t) q_n(dF) \rightarrow X(d_{-1} < t)$ , i.e.,  $\int_F F(t) q_n(dF) \rightarrow X(d_{-1} < t)$ . The

right-hand side converges monotonically to zero as  $t \downarrow 0$ . On the other hand,

however,

$$\lim_{t \downarrow 0} \lim_{n \rightarrow \infty} \int_F F(t) q_n(dF) \geq \int_F F(0+0) q(dF) > 0.$$

Now suppose  $\int_F H_F q_n(dF) \Rightarrow X$  ( $n \rightarrow \infty$ ). By the compactness of  $\underline{G}$ ,  $(q_n)$  is relatively compact in the set of all distributions on  $\underline{G}$ . Choose any convergent subsequence  $q_{n_m} \Rightarrow q$  ( $m \rightarrow \infty$ ). Then  $\int_F H_F q_{n_m}(dF) \Rightarrow X$  ( $m \rightarrow \infty$ ).

By the first step in the proof, we have  $X = \int_F H_F q(dF)$  and  $q(F) = 1$ .

By 10.5, all convergent subsequences of relatively compact sequences  $(q_n)$

have the same limit, i.e.,  $q_n \Rightarrow q$  ( $n \rightarrow \infty$ ) and  $q(F) = 1$ . This completes

the proof of 10.6.

From 9.1, 9.3 and 10.6 we have

10.7. Let  $(P_{n,i})$  be an infinitesimal array of stationary renewal distributions such that  $\sum_{i=1}^{s_n} \rho_{n,i} \rightarrow \rho$  ( $0 < \rho < \infty$ ). Then  $(P_{F_{n,1}} * \dots * P_{F_{n,s_n}})$   
 $\Rightarrow P$  ( $n \rightarrow \infty$ ) is equivalent to

$$\left( \sum_{i=1}^{s_n} \rho_{n,i} \right)^{-1} \left( \sum_{i=1}^{s_n} \rho_{n,i} \delta_{\{F_{n,i}\}} \right) \Rightarrow q, \quad q(F) = 1, \quad \rho_p = \rho, \quad (\tilde{P})_0 = \int_F H_F q(dF).$$

We end this chapter with

10.8. In 10.7,  $q$  can be any distribution on  $\underline{G}$  such that  $q(F) = 1$ .

**Proof.** In the space of all distributions on  $\underline{G}$  fix a metric with the properties indicated earlier. Moreover, let  $q$  on  $\underline{G}$  with  $q(F) = 1$  be given.

We now construct a corresponding array  $(P_{F_{n,i}})$  by rows.

Let  $n$  be any non-negative integer. First we choose a mixture

$\sum_{i=1}^m a_i \delta_{\{F_i\}}$ ,  $F_i \in F$ , whose distance to  $q$  is less than  $1/n$ . Here the coefficients can always be chosen to be of the form  $a_i = r_i/k$  where the  $r_i$  are non-negative integers and  $k > n$ . The proof of 10.4 shows that the  $F_i$ 's may be replaced by distribution functions  $L_i \in F$  such that  $L_i(\infty) = 1$ ,

$\int_0^\infty x dL_i(x) = kr$  for a non-negative integer  $r$ , and the distance from

$\sum_{i=1}^m \frac{r_i}{rk} \delta_{\{L_i\}}$  to  $q$  is still less than  $1/n$ . We can now construct the  $n^{\text{th}}$

row of the desired array as follows: let  $s_n = \sum_{i=1}^m r_i r$ , and set  $r_i r$  of the

$P_{n,i}$ 's equal to  $P_{L_i}$ . As  $n$  increases, the mixture converges to  $q$ .

## 11. Some relations between $P$ and $P_L$

In what follows  $P$  is a distribution on  $M_K$ ,  $L \in K$  and  $0 < \lambda(L) < \infty$ .

Moreover, we assume that  $P(\phi_L \in M') = 1$ . We use the notation of Chapter 2 throughout.

First we start from the footnote following 2.3. Let  $S$  be any set in  $M^L$ . Then with probability one, the limit  $\eta_S$  of

$$t^{-1} N(\{x: 0 < x < t, (\{x\} \times L) \cap \phi \neq \emptyset, T_x \phi \in S\})$$

as  $t \rightarrow \infty$  exists. We set  $\eta_{M^L} = \sigma$  and obtain by virtue of

$$P_{*,L}(S) = M \left( \frac{\eta_S}{\sigma} \right)$$

a distribution  $P_{*,L}$  on  $M^L$ . The measures  $P_L, P_{*,L}$  are equivalent. They coincide if and only if  $\sigma$  equals  $\lambda(L)$  almost everywhere. Thus, if  $P$  is ergodic,  $P_L = P_{*,L}$ .

From the definition of  $P_{*,L}$  we have

$$11.1. \quad \frac{1}{n} \sum_{i=0}^{n-1} P(T_{y_i(\phi)} \phi \in S) \rightarrow P_{*,L}(S) \quad (n \rightarrow \infty) \quad \text{for all } S \in M^L.$$

From the Ergodic Theorem follows the existence for almost all  $\phi \in M^L$  relative to  $P_L$  of the limit  $\omega$  of  $\frac{1}{n} \sum_{i=0}^{n-1} (y_i(\phi) - y_{i-1}(\phi))$  as  $n \rightarrow \infty$ .

$$11.2. \quad \frac{dP_{*,L}}{dP_L} = (\lambda(L))\omega.$$

Proof. By 2.7

$$\begin{aligned} P(T_{y_i(\phi)} \phi \in S) &= \lambda(L) \int \left[ \int_0^{y_0(\phi)} k_S(T_{y_i(\phi)} \phi) d\tau \right] P_L(d\phi) \\ &= \lambda(L) \int (y_0(\phi) - y_{-1}(\phi)) k_S(T_{y_i(\phi)} \phi) P_L(d\phi) \\ &= \int \lambda(L) (y_{-1}(\phi) - y_{-1-1}(\phi)) k_S(\phi) P_L(d\phi) \end{aligned}$$

and hence

$$\frac{1}{n} \sum_{i=0}^{n-1} P(T_{y_i(\phi)} \phi \in S) = \int \left( \frac{\lambda(L)}{n} \sum_{i=0}^{n-1} (y_{-1}(\phi) - y_{-1-1}(\phi)) \right) k_S(\phi) P_L(d\phi)$$

from which the assertion 11.1 follows by the ergodic theorem.

From 11.2 we see (cf. [18], Theorem 5) that

$$11.3. \text{ For all } S \in M_K, \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P_{*,L}(T_\tau \phi \in S) d\tau = P(S).$$

For mixing  $P$  we can deduce a simpler assertion. For brevity, we denote by  $\phi_t$  the restriction of  $T_{-t}$  to  $M_K$ ; then for mixing  $P$  we have the convergence result

$$11.4. P_L | \phi_t \Rightarrow P \quad (t \rightarrow \infty).$$

PROOF. Let  $I_1, \dots, I_\ell$  be bounded open intervals and  $V_1, \dots, V_\ell$  sets in  $K$ . Set

$$H = \{ \phi : N(\phi \cap (I_1 \times V_1)) = n_1, \dots, N(\phi \cap (I_\ell \times V_\ell)) = n_\ell \}.$$

By 2.4, to every  $\varepsilon > 0$  there exists  $\tau_\varepsilon > 0$  such that uniformly in  $t$  and for  $0 < \tau \leq \tau_\varepsilon$ ,

$$|P_L(\phi_t(H)) - P(\phi_t^L \in \phi_t(H) | x_L < \tau)| < \varepsilon/3.$$

If  $\tau$  is chosen sufficiently small then for  $t$  sufficiently large

$$|P(\phi_t^L \in \phi_t(H) | x_L < \tau) - P(\phi \in \phi_t(H) | x_L < \tau)| < \varepsilon/3.$$

On the other hand, for large  $t$

$$|P(\phi \in \phi_t(H) | x_L < \tau) - P(\phi \in H)| < \varepsilon/3.$$

We can also sharpen 11.1 under suitable assumptions:

11.5. If  $P_L$  is mixing with respect to  $\theta$ , then  $P(T_{y_n(\phi)} \phi \in S)$  converges as  $n \rightarrow \infty$  to  $P_L(S)$  for all  $S \in M^L$ .

The proof follows from the equation

$$P(T_{y_n(\phi)} \phi \in S) = \lambda(L) \int (y_{-n}(\phi) - y_{-n-1}(\phi)) k_S(\phi) P_L(d\phi).$$

The following example shows that the conclusion of 11.5 does not hold for all mixing  $P$ .

Let  $\lambda_1$  and  $a$  be positive and  $(\zeta_i)_{-\infty < i < \infty}$  a doubly-infinite sequence of independent positive random variables distributed according to  $F(x) = 1 - \exp(-\lambda x)$  ( $x \geq 0$ ). Let  $X$  denote the distribution of

$$\{\dots, -(2a + \zeta_{-1}), -(a + \zeta_{-1}), 0, a, 2a + \zeta_0, 3a + \zeta_0, 4a + \zeta_0 + \zeta_1, \dots\}$$

and  $Y$  the distribution of

$$\{\dots, -(3a + \zeta_{-1}), -(2a + \zeta_{-1}), -a, 0, a + \zeta_0, 2a + \zeta_0, 3a + \zeta_0 + \zeta_1, \dots\}.$$

By 2.9 the mixture  $\frac{1}{2}(X+Y)$  is the Palm distribution of a distribution  $P$  on  $M'$ . It is clear that  $P$  is mixing, but the conclusion of 11.5 does not hold.

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