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EFFECTS OF TURBULENCE INSTABILITIES ON LASER PROPAGATION

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EFFECTS OF TURBULENCE INSTABILITIES ON LASER PROPAGATION

D. A. de WOLF

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PRINCIPAL INVESTIGATOR AND
PROJECT ENGINEER: DR. DAVID A. de WOLF
PHONE: 609 452-2700 EXT. 3023

CONTRACTING ENGINEER:
RAYMOND P. URTZ, JR.
PHONE: 315 330-3443

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This technical report has been reviewed and is approved.

Raymond P. Urtz Jr.

RADC Project Engineer

FOREWORD

This Quarterly report was prepared by RCA Laboratories, Princeton, New Jersey under Contract No. F30602-71-C-0356. It describes work performed from 9 June to 8 September 1971 in the Communications Research Laboratory, K. H. Powers, Director. The principal investigator and project engineer is D. A. de Wolf.

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The Air Force Program Monitor is Raymond P. Urtz, Jr.

ABSTRACT

The average effective area of the focus of a laser beam in turbulent air (diffraction-limited in free space) has been computed. The results show that the focal spot can be decreased only to a certain extent by increasing the transmitting aperture up to a critical size determined by the turbulence structure constant C_n^2. Critical-aperture radius $r_{oc} = 1/2kL^{1/2}\ \kappa_m^{1/6} C_n$ (k is wavenumber, L is pathlength, κ_m is inner-scale wavenumber). If the aperture is made larger, the focal-spot area will fluctuate around a constant value close to a minimum value L/kr_{oc} independent of the aperture size. This minimum is an atmospheric limit to the focussing power of a laser for applications in which illumination is not shorter than a typical fluctuation time.

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SECTION I

SUMMARY

When images are formed from laser beams propagating through turbulent air, a variety of scintillation phenomena occurs: beam wander, intensity fluctuations, hot- and cold-spot formation, image blurring, spot broadening, etc. The purpose of this project is to study these effects analytically, and thus to interpret measurements and predict performance in future laser systems.

The difficulties of an analytic approach - aside from obvious mathematical ones - lie in the present status of theoretical knowledge of the basic optics in turbulent air. The statistics of the irradiance (the prime quantity of interest) are uncertain, and in particular its variance and higher-order measures of fluctuations are not clearly established.

Rather than starting a basic investigation in this first quarter into the full statistics of the irradiance of an idealized wave, we preferred to define and solve a more limited problem: the limitations imposed by atmospheric turbulence upon the focussing of a laser beam. It is quite clear that a primary factor in reconnaissance and weapons performance of laser beams is the ability to focus the beam. In contrast to the non-linear and self-induced effects such as thermal blooming and distortion, atmospheric turbulence creates beam distortions that are *not* a function of beam strength. Furthermore the effects are *random* and therefore unpredictable.

Previous work on this problem appeared to us to be incomplete, or - perhaps - not correct. We have defined an effective focal-spot area of a laser beam, and have computed its average area in the presence of atmospheric turbulence. The analysis utilizes only those irradiance statistics that are well-established and agreed upon by most workers in this field. The results indicate that for sufficiently small aperture radii r_o one can get large focal spots with radius $r_L \sim L/kr_o$ (L is pathlength, k is wavenumber) which can be made smaller by increasing r_o until the aperture radius approaches a critical size determined by the atmospheric-turbulence strength parameter C_n^2, namely $r_{oc} = 1/2kL^{1/2}\kappa_m^{1/6}C_n$ (k is wavenumber, L is pathlength, κ_m is inner-scale wavenumber). There may be no advantage in increasing r_o beyond this critical size because the focal spot then fluctuates in size and on the average does not decrease in radius below L/kr_{oc}. Details of the analysis are appended in technical-report form in subsequent sections of this Quarterly Report.

The results - given for so-called Kolmogorov turbulence - can be tested. Comparisons with data - to be made available yet - are contemplated in the future. If necessary, the results can be adapted to other forms of turbulence.

SECTION II

FOCUSSED LASER BEAM IN TURBULENT AIR: INTRODUCTION

The topic of this report is the effect of turbulent air upon the focussing of a laser beam aimed horizontally within a few meters of earth's or sea's surface of a receiver located a distance L from the effective optical aperture (with radius r_o) of the beam-producing laser system.

In free space, the focal spot is diffraction limited. Its shape is dependent upon the way the illuminating flux varies over the aperture located in the plane z = 0. Common to all practical cases is the property that the focal spot has an effective radius determined by the distance $r_L = L/kr_o$ (k is the wavenumber of the laser-beam radiation, which may be considered monochromatic without undue restriction). It cannot be made smaller without modifying L, k, or r_o. In turbulent air, however, the phase relationships between rays emanating from different parts of the transmitting aperture to the focus are modified and the result is spot broadening beyond the diffraction limit.

A naive calculation of the effect can be given first. Consider the aperture of the laser-optics system producing a focussed beam as a lens. By exploiting the refractive properties of the lens, i.e., by choosing its material and shape, one attempts to direct all the rays penetrating it to a focal point. However, ray bending, or diffraction of rays, occurs around the edges of the lens and deflection of rays by an angle $\theta \sim (kr_o)^{-1}$ occurs. As a result, light from the edges of the lens which is intended for the focal point on the central axis at distance L is misaimed by an angle of the order of θ. A blurring of the focal point into a spot of thickness $r_L \sim L\theta$ occurs. Of course this picture is only one of a number of other, equivalent, ways of describing diffraction. Now, when the rays must propagate through a random geometrical-optical atmosphere (i.e., a medium which is quite transparent but which deflects rays continuously with local radius of curvature very large compared to wavelength in randomly varying directions) the misaiming is by an angle $\theta + \delta\theta$, and the blurring occurs over a thickness $r_L \sim L(\theta + \delta\theta)$. Because the average $\langle L\ \delta\theta \rangle = 0$, the average blurring radius in one direction is as before: diffraction limited. The average area $\pi \langle r_L^2 \rangle$ is a measure of the net effect of the atmosphere, and it is easily observed that

$$\langle r_L^2 \rangle = L^2 \left[\theta^2 + \langle(\delta\theta)^2\rangle\right] \tag{1}$$

The angular variance $\langle(\delta\theta)^2\rangle$ in homogeneously turbulent air can be estimated by geometrical-optics formulas for ray curvature. Although the calculation is not new, it is not easily accessible and we provide a short derivation in Appendix II. We also set $\theta = (kr_o)^{-1}$ and find

$$\langle r_L^2 \rangle \sim (L/kr_o)^2 \left[1 + 3.9\ C_n^2\ k^2\ L\ \kappa_m^{1/3}\ r_o^2\right] \tag{2}$$

This formula predicts that the effects of turbulence are negligible unless C_n^2 approaches a critical value determined by the parameters

$$\left(C_n^2\right)_{crit.} \sim \kappa_m^{-1/3} L^{-1} r_o^{-2} k^{-2} \tag{3}$$

In words, the broadening effect of turbulence is increased by utilizing higher frequencies and larger apertures. Lawrence and Strohbehn[1], however, have utilized another type of naive argument to predict a critical C_n^2, and their result amounts to

$$\left(C_n^2\right)_{crit.} \sim L_o^{-3} r_o^{7/3} \tag{4}$$

In contrast to the first line of reasoning, this result predicts a frequency-independent broadening that *decreases* with increasing aperture. There are perhaps some reasons to be apprehensive, especially of this latter line of reasoning, but a simplified argument such as the above has little value other than to provide an intuitive interpretation for a rigorous derivation. It should be accompanied by a rigorous derivation. In the following sections, Equation (2) will be derived rigorously for a focussed laser beam in turbulent air governed by a Kolmogorov spectrum of refractive-index fluctuations for propagation paths that exceed those for which the irradiance saturates[2].

SECTION III

FOCUSSED LASER BEAM IN TURBULENT AIR: BASIC FORMULATION

Consider the aperture of a laser system in the z = 0 plane. A point on the aperture is denoted by $\vec{r}_1 = (\vec{\rho}_1, 0)$. Schmeltzer[3] has shown that the electric field $E_o(r)$ at location $r = (\vec{\rho}, L)$ in free space is given by the aperture integral

$$E_o(\vec{r}) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int d^2\rho_1\, G_o\left(\vec{r}-\vec{r}_1\right) U_o\left(\vec{r}_1\right) \exp\left(-ik\rho_1{}^2/2R\right) . \qquad (5)$$

Here, $G_o(\Delta r) = [\exp(ik\Delta r)] / \Delta r$, is the field of a spherical wave emanated at an origin and received at a vectorial location $\vec{\Delta r} = \vec{r}-\vec{r}_1$ from that origin; R is the radius of curvature of the transmitted wavefront, and U_o is the aperture function weighting the amplitude of the transmitted field [it is chosen as a Gaussian in Equation (5) to conform with laser beams, but it can also be chosen differently to correspond to other optical systems that produce a positively or negatively (divergent) focussed beam]. It can be seen that the approximation $\partial G_o/\partial z \approx ikG_o$ is excellent in Equation (5). This approximation ignores terms of order $(kL)^{-1}$ and of order $L/kr_o{}^2$ in the electric field; it does not ignore the phase effects of the exponent of G_o.

The formulation [Equation (5)] with $\partial G_o/\partial z$ replaced by ikG_o adapts itself excellently to the problem of a laser beam in turbulent air. The fact that Equation (5) is a linear integral equation with kernel G_o over an effective source function immediately establishes that in turbulent air

$$E(\vec{r}) = -\frac{ik}{2\pi}\int d^2\rho_1\, G\left(\vec{r},\vec{r}_1\right) U_o\left(\vec{r}_1\right) \exp\left(-ik\rho_1{}^2/2R\right) , \qquad (6)$$

where $G(\vec{r},\vec{r}_1)$ is the spherical-wave field at $\vec{r}$ when a spherical wave is emitted from r_o in turbulent air. The field $G(\vec{r},\vec{r}_1)$ reduces to $G_o(\vec{r}-\vec{r}_1)$ as the turbulence vanishes and the medium is thus reduced to free space, consequently we find it useful to set $G(\vec{r},\vec{r}_1) \equiv B(\vec{r},\vec{r}_1)G_o(\vec{r}-\vec{r}_1)$ so that $B \rightarrow 1$ as $\delta\varepsilon \rightarrow 0$.

If the beam is focussed at L, i.e., R = L one can approximate $G_o(\vec{r}-\vec{r}_1)$ to first order in the exponent and to zero-th order in the denominator to obtain,

$$E(\vec{r}) = L^{-1} \exp\left[ik\left(L + \rho^2/2L\right)\right] \cdot \frac{ik}{2\pi}\int d^2\rho_1 B\left(\vec{r},\vec{r}_1\right)U_o\left(r_1\right) \exp\left(ik\vec{\rho}\cdot\vec{\rho}_1/L\right) . \qquad (7)$$

This formulation, which also ignores terms of order kr_o^4/L^3 is very convenient for our purpose. The received power at r is proportional $|E|^2$. We therefore form the irradiance $I = |E|^2$ to find,

$$I(\vec{r}) = \left| (k/2\pi L) \int d^2\rho_1 \; B\left(\vec{r},\vec{r}_1\right) \; U_o\left(\vec{r}_1\right) \; \exp\left(ik\vec{\rho}\cdot\vec{\rho}_1/L\right) \right|^2 \tag{8}$$

The irradiance distribution over the focus, i.e., the dependence of $I(\vec{r})$ upon $\vec{\rho}$ for fixed L, determines the properties of the focal spot. The free-space properties are immediately apparent from Equation (8) because $B = 1$. In that case, the insertion of Equation (5) for $U_o(\vec{r}_1)$ yields an easily performed Fourier integral inside the modulus signs of Equation (8):

$$I_o(\vec{r}) = \left(kr_o^2/L\right)^2 \exp\left[-\left(kr_o\rho/L\right)^2\right], \tag{9}$$

and it can be seen by making the integration-variable $\vec{\rho}_1$, non-dimensional for more general $U_o(\vec{r}_1)$ that $I_o(\vec{r})$ depends only upon $kr_o\rho/L$, and so that $I_o(\vec{r})$ decreases (albeit not monotonically) with increasing $kr_o\rho/L$. It is obvious from Equation (9) and its generalization that the focal spot has a halfwidth r_L that is easily related to L/kr_o. The difficulty is in extending the concept to the case $B \neq 1$, i.e., to turbulent air, because it is not possible in general to perform the random Fourier integral.

SECTION IV

DEFINITION OF FOCAL-SPOT RADIUS IN TURBULENT AIR

As with so many other randomly varying quantities, one can only hope to extract more information about the behavior of $I(\vec{r})$ with the atmospheric and system parameters by calculating statistics. The random quantity $I(\vec{r})$ in Equation (8) is determined by the statistics of $B(\vec{r},\vec{r}_1)B^*(\vec{r},\vec{r}_2)$ at two aperture locations $\vec{r}_1$ and $\vec{r}_2$. The statistics of the normalized spherical-wave field $B(\vec{r},\vec{r}_1)$ are uncertain to date, but the average of the above product, $\langle B(\vec{r},\vec{r}_1)B^*(\vec{r},\vec{r}_2)\rangle$ the *mutual coherence factor* (mcf), has been determined by a variety of methods[4-6] for long propagation distances.

$$\langle B(\vec{r},\vec{r}_1)B^*\left(\vec{r},\vec{r}_2\right)\rangle = \exp\left\{ - \frac{k^2\epsilon^2 L}{8\pi}\int_0^\infty dK K\Phi(K)\left[1 - L^{-1}\int_0^L dz\, J_0(2Kz\Delta\rho/L)\right]\right\} \quad (10)$$

$$\Delta\rho = |\vec{\rho}_1-\vec{\rho}_2|$$

Here, $J_0(K\Delta\rho)$ is a Bessel function (zero-th order), and the reciprocity theorem has been invoked to interpret $B(\vec{r},\vec{r}_1)$ also as the normalized field for propagation from $\vec{r}$ to $\vec{r}_1$. The validity of Equation (10) is limited only by the conditions $\epsilon^2 \ll k^{-2}\ell_i^{-2}$ and $L \ll \ell_i\epsilon^{-2}$ aside from the usual optical (far-field and sagittal) assumptions. The region of validity includes that in which $\langle[I(\vec{r})]^2\rangle$ is observed to saturate.

Thus, $\langle I(\vec{r})\rangle$ appears promising as a statistic which yields information about r_L. Unfortunately the computation of $\langle I(\vec{r})\rangle$ after inserting Equation (10) into the average of Equation (8) is still very difficult. Consider Equation (10). For propagation over more than several meters under at least moderately turbulent conditions ($C_n^2 > 10^{-15}m^{-2/3}$) the quantity $k^2\epsilon^2 L\,\ell_i > 1$. It therefore follows that the coordinate $|\vec{\rho}_1-\vec{\rho}_2|$, which we abbreviate by $\Delta\rho$, need be specified in the integrand only for $\Delta\rho < L_0$ because $\langle B(1)B^*(2)\rangle$ (another obvious abbreviation) becomes negligibly small otherwise. For $\ell_0 < \Delta\rho < L_0$ it then follows that the exponent in Equation (10) is proportional to $k^2C_n^2L(\Delta\rho)^{5/3}$, and to $k^2C_n^2L\kappa_m^{1/3}(\Delta\rho)^2$ for $\Delta\rho < \ell_0$. Even so, it appears that $\langle I(\vec{r})\rangle$ can be evaluated further only numerically.

However, the form of Equation (8) suggests strongly that an integration over $d^2\rho$, the plane of the focal spot, will simplify matters because a delta function in $\Delta\rho$ (or something like it) follows when $B(1)B^*(2)$ depends weakly upon $\vec{r}$. The first radius-like quantity we can form is

$$\langle r_L^2\rangle \equiv \int d^2\rho\, \langle I(\vec{r})\rangle \Big/ \langle I(0,L)\rangle \quad (11)$$

The numerator is easily evaluated - we shall do so presently - but the denominator is subject to the problems discussed in the preceding paragraph. Hence we define an effective radius with the next non-zero higher order weighted mean:

$$\langle r_L{}^2\rangle \equiv \int d^2\rho\ \rho^2 \langle I(\vec{\rho},L)\rangle \Big/ \int d^2\vec{\rho} \langle I(\vec{\rho},L)\rangle \tag{12}$$

The denominator of Equation (12) is easily computed because the $d^2\rho$ integration introduces a two-dimensional delta function in $\vec{\rho}_1-\vec{\rho}_2$. Consequently $\langle B(1)B^*(2)\rangle$ reduces to $\langle B(1)B^*(1)\rangle = 1$. What is left yields

$$\int d^2\rho \langle I(\vec{\rho},L)\rangle = \pi r_o{}^2 \tag{13}$$

just as we expect from energy conservation ($\pi r_o{}^2$ is the power radiated toward the focus by the aperture).

The numerator of Equation (12) requires somewhat more work. After performance of the $d^2\rho$ integral, we obtain

$$-\frac{L^2}{k^2}\int d^2\rho_1 \int d^2\rho_2 \langle B(1)B^*(2)\rangle \exp\left[-\left(\rho_1{}^2+\rho_2{}^2\right)/2r_o{}^2\right] \times \left(\frac{\partial^2}{\partial \Delta x^2}+\frac{\partial^2}{\partial \Delta y^2}\right)\delta_2\left(\vec{\rho}_1-\vec{\rho}_2\right) \tag{14}$$

We perform partial integration with respect to x_1 and y_1 twice in order to remove derivatives from the delta function. Because the boundary terms at $x_1 = \pm\infty$, $y_1 = \pm\infty$ are zero, the net effect is to transfer the differential operator in Equation (14) to the two other factors of the integrand. The presence of a delta function then requires that we know $\langle B(1)B^*(2)\rangle$ only for infinitesimal (but non-zero) $\Delta\rho$ in order to obtain the derivatives and then set $\Delta\rho = 0$. This is a considerable simplification. Referring to Appendix I, we observe that

$$\begin{aligned}\lim_{\Delta\rho\to 0} \langle B(1)B^*(2)\rangle &= \lim_{\Delta\rho\to 0} \exp\left[-\left(k^2\epsilon^2 L/24\pi\right) M_3\cdot\left(\Delta\rho\right)^2\right]\\ &= \lim_{\Delta\rho\to 0} \exp\left[-1.3\, k^2 L\, \kappa_m{}^{1/3}\, C_n{}^2 \left(\Delta\rho\right)^2\right]\end{aligned} \tag{15}$$

Here, we have utilized Equations (I-6) and I-7) in the second step. Let us define $\beta \equiv 1.3\, k^2 L \kappa_m^{1/3} C_n^2$, so that Equation (14) reduces to

$$\frac{L^2}{k^2} \int d^2\rho_1 \int d^2\rho_2\, \delta_2\left(\vec{\rho}_1 - \vec{\rho}_2\right) \cdot \left(\frac{\partial^2}{\partial \Delta x} + \frac{\partial^2}{\partial \Delta y}\right) \times \exp\left[-\beta\left(\Delta\rho\right)^2 - \left(\rho_1^2 + \rho_2^2\right)/2r_o^2\right] \tag{16}$$

The differentiations are now easily performed, and after then setting $\Delta\rho = 0$ we find

$$\int d^2\rho\, \rho^2\, \langle I\left(\vec{\rho}, L\right)\rangle = \pi\left(L/k\right)^2\left(1 + 4\beta r_o^2\right) \tag{17}$$

The averaged effective radius defined in Equation (12) is thus given by Equations (13) and (17), and the result is

$$\langle r_L^2\rangle = \left(L/kr_o\right)^2\left(1 + 4\beta r_o^2\right)$$
$$\beta \equiv 1.3\, k^2\, L\, \kappa_m^{1/3}\, C_n^2 \tag{18}$$

It is interesting to note that this is roughly equal to Equation (2) [the numerical coefficient of $k^2 L \kappa_m^{1/3} C_n^2 (\Delta\rho)^2$ is $3\,\gamma\,(\Phi)\,\gamma\,(\epsilon^2)/8\pi \approx 3.9$ in the former case]. The result shows the following: Given a laser system (L and k specified) and atmospheric conditions (C_n^2 specified), it follows that the focal spot can be made smaller by increasing r_o until it approaches a critical value $r_{oc} = 1/2\beta^{1/2}$. There is, however, no further decrease of average focal-spot area when increasing r_o above r_{oc}, because $\langle r_L^2\rangle$ saturates rapidly to the value $(L/kr_{oc})^2$ which is independent of r_o.

The derivation can be extended to non-Gaussian $U(\vec{r}_1)$ by defining an aperature radius

$$r_o^{-2} \equiv -\int d^2\rho\, U_o\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)U_o \Bigg/ \int d^2\rho\, U_o\, U_o \tag{19}$$

The same result is obtained as before, namely Equation (18) with the above new definition of r_o, and the derivation does not differ essentially from that for Gaussian $U_0(\vec{r}_1)$. The only requirement is that r_o as defined in Equation (19) is finite.*

*The parameter r_o^{-2} defined in Equation (19) is identical to the width of the spatial power spectrum of $U_o(\vec{r})$ defined as the normalized second moment of $|\tilde{U}_o(\vec{K})|^2$, where $\tilde{U}_o(\vec{K})$ is the Fourier transform of $U_o(\vec{r})$. It is well known that certain idealizations of $U_o(\vec{r})$ - such as $U_o(\vec{r})$ = const. for $r \leq r_o$, and $U_o(\vec{r}) = 0$ for $r > r_o$ - give difficulties in computing finite spectral widths.

SECTION V

CONCLUSIONS

In the previous section we derived rigorously an expression for an irradiance-weighted focal-spot area on the average. This area is broadened by atmospheric turbulence beyond the diffraction-limited area $\pi\,(L/kr_o)^2$. Surprisingly enough, the simple argument of Section I yielded exactly the same answer although it ignores correlation of rays from different parts of the transmitting aperture and utilizes the ray concept as if there were no ray crossings and/or caustic effects before the focal point is reached. We would like to understand this somewhat more. Consider the electric field $E(\vec{\rho},L)$ received in the focal plane, and Fourier analyze it in terms of transverse random components $dE(\vec{K},L)$

$$E(\vec{\rho},L) = \int dE(\vec{K},L)\,\exp(-i\vec{K}.\vec{\rho})$$

with

$$\langle dE(\vec{K}_1,L)dE^*(\vec{K}_2,L)\rangle = (4\pi)^{-1}d^2K_1d^2K_2\;\delta_2(\vec{K}_1-\vec{K}_2)\tilde{I}(\vec{K},L) \qquad (20)$$

This is the customary Fourier analysis for a homogeneous isotropic random process and $\tilde{I}(K,L)$ is the two-dimensonal spectral density of $E(\vec{\rho},L)$. From Equation (20) and the definition (12) it is then not difficult to find

$$\langle r_L{}^2\rangle = -\left[\partial^2\tilde{I}(\vec{K},L)/\partial K^2\right]_{K=0}/\,\tilde{I}\,(0,L) \qquad (21)$$

i.e., $\langle r_L{}^2\rangle$ is determined by smallest-wavenumber components of the spectral density of $E(\vec{\rho},L)$ in the focal plane. The main effect determined by small-wavenumber $\tilde{I}(\vec{K},L)$ can only be due to a gross deflection of the entire beam. Apparently this is why the effect of atmospheric turbulence on $\langle r_L{}^2\rangle$ is so aptly described by a ray angle $\delta\theta$ as in Section I.

If, then, the quantity $\langle r_L{}^2\rangle$ is any measure of the average effective area of the focal spot in turbulent air, one finds that the turbulence strength $C_n{}^2$ must exceed a critical value given parametrically in Equation (3) in order for appreciable broadening beyond the diffraction limit. This differs from the conclusion of Lawrence and Strohbehn[1] in Equation (4). A corollary of our result is that the only restriction on aperture size for validity of the approximations is $L \ll kr_o{}^2$. We have used this restriction in developing Equation (6). It has also been used in ignoring the fact in Equation (10) that $\vec{r}$ makes an angle $\theta_r \lesssim r/L \sim (kr_o)^{-1} \ll (kL)^{-1/2}$ with the central axis (terms of order $k^{-1}L^{-1}$ are discarded). The results [Equation (18)] are displayed in Figures 1 and 2 for wavelengths $\lambda = 0.6\ \mu m$ and $\lambda = 10.6\ \mu m$ in the following fashion: We defined a critical-aperture radius $r_{oc} = (4\beta)^{-1/2}$ so that for given $C_n{}^2$ and L the average focal-spot area is just twice the diffraction area. Let $r_{Lc} \sim L/kr_{oc}$. If $r_o < r_{oc}$ then the diffraction-limited radius $r_L \sim L/kr_o$ is a good approximation, and $r_L > r_{Lc}$. When $r_o > r_{oc}$, then Equation (18) shows that r_L is slightly larger than r_{Lc}. Consequently $\langle r_L{}^2\rangle$ is the smallest

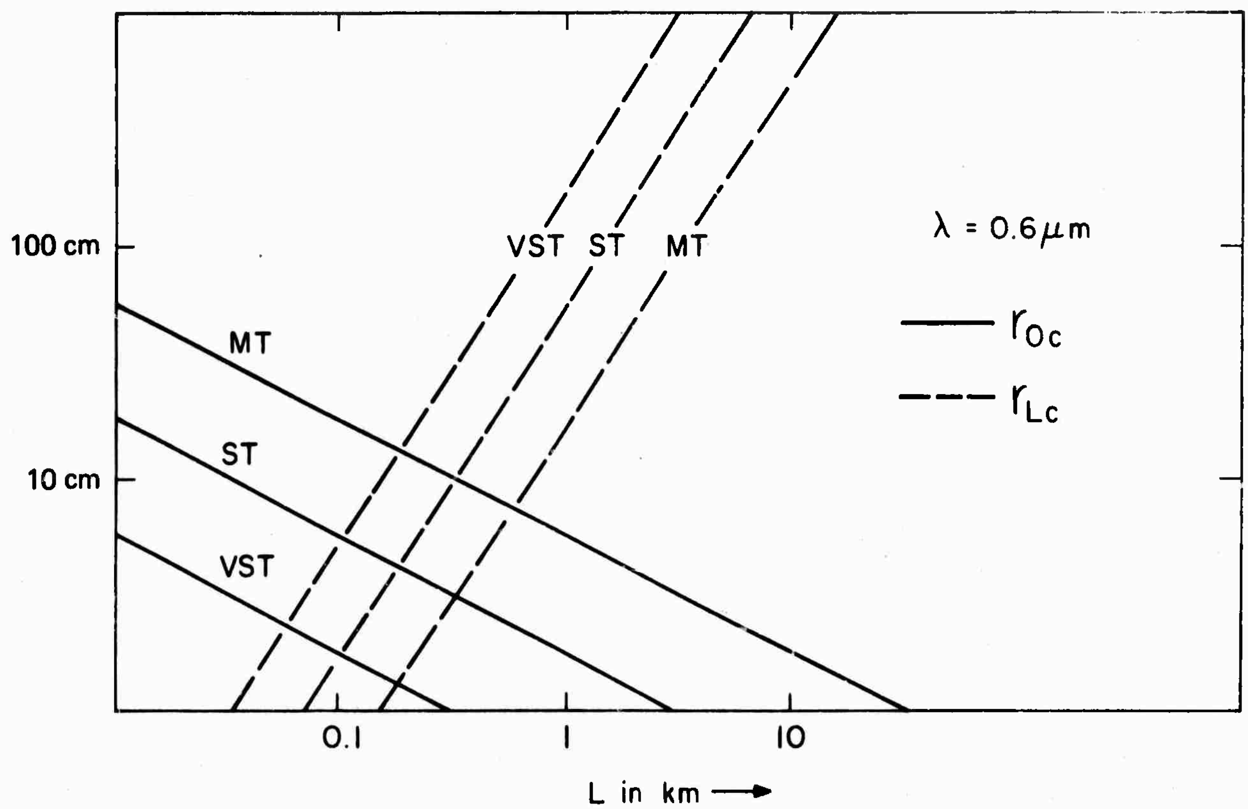

Figure 1. Maximal aperture radius r_{oc} required to obtain a minimal focal-spot area with effective radius r_{Lc}. Radii r_{oc} and r_{Lc} as functions of path-length L for λ = 0.6 μm, κ_m^{-1} = 7 mm, and $C_n^2 = 10^{-14}$(MT), 10^{-15}(ST), and 10^{-16}(VST) in $m^{-2/3}$ units.

average area of the focal spot to be obtained under atmospheric conditions prescribed by given values of C_n^2. The definitions yield

$$r_{oc} = 1/2kL^{1/2}\kappa_m^{1/6} C_n \qquad (22)$$
$$r_{Lc} = 2L^{3/2}\kappa_m^{1/6} C_n$$

and these quantities are plotted for $\kappa_m^{-1} \sim 7$ mm in Figures 1 and 2 for three values of C_n^2 corresponding to moderate (MT), strong (ST), and very strong (VST) turbulence.

Finally, we compare our results to previous work. Gebhardt and Collins[7] have computed r_L/r_o vs kr_o^2/L. Their results show the same trend as ours for

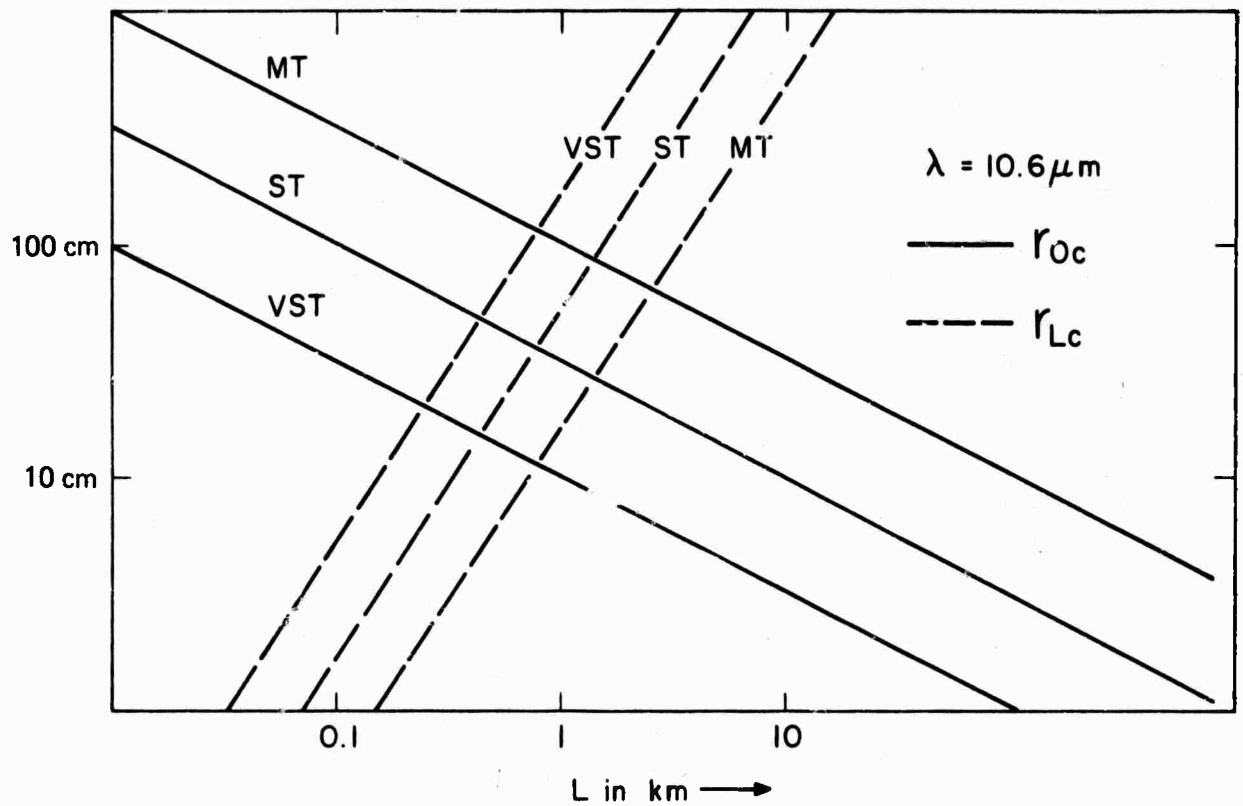

Figure 2. Maximal aperture radius r_{oc} required to obtain a minimal focal-spot area with effective radius r_{Lc}. Radii r_{oc} and r_{Lc} as functions of path-length L for $\lambda = 10.6$ μm, $\kappa_m^{-1} = 7$ mm, and $C_n^2 = 10^{-14}$(MT), 10^{-15}(ST), and 10^{-16}(VST) in $m^{-2/3}$ units.

$r_o << r_{oc}$, but there is no saturation for larger r_o. As a consequence, they conclude - in sharp contrast to us - that better focussing can be achieved by increasing r_o no matter how large C_n^2 is. The difference in their work appears to arise from the use of weak-amplitude theory and a number of assumptions regarding the average log-amplitude. Lutomirski and Yura[8] compute a half-power angle $\sim L^{1/2}\kappa_m^{1/6}C_n$ which is similar to our result for r_L/L (also a half-power angle) when $r_o > r_{oc}$. The result appears to agree with ours only for large paths $L >> k^{-2}C_n^{-2}\kappa_m^{-5/3}$, however. Their choice of aperture (equivalent to that discussed in the footnote on page 9) may make further comparison difficult.

Note added in proof: after the completion of this work, it came to our attention that Varvatsis and Sancer[11] have found the same result.

APPENDIX I

TURBULENCE QUANTITIES REQUIRED IN WAVE-PROPAGATION CALCULATIONS

Different authors use differing nomenclature and normalizations of turbulence scales and spectra. To avoid confusion, we tabulate our own definitions for future reference.

The basic quantity is the deviation from the mean dielectric permittivity $\delta\varepsilon(\vec{r})$. Its lowest non-zero statistic is its variance $[\delta\varepsilon(r)]^2$ which may or may not be dependent upon location $\vec{r}$. In the case that it is not, we abbreviate the variance by the symbol ^{2}.

In the case of homogeneous turbulence, the two-point correlation $\langle \delta\varepsilon(r_1)\ \delta\varepsilon\ (\vec{r}_2)\rangle$ depends only upon the difference vector $\Delta\vec{r} = \vec{r}_1 - \vec{r}_2$. We write this two-point correlation as $^2C(\Delta\vec{r})$, thus insuring the normalization $C(0) = 1$, and define the turbulence spectrum $\Phi(\vec{K})$ as

$$\Phi(\vec{K}) \equiv \int d^3\Delta r\ C\ (\Delta\vec{r})\ \exp(i\vec{K}.\Delta\vec{r}) \qquad \text{(I-1)}$$

When there is no preferred direction (isotropic turbulence), the spectrum depends only on the length of the vector $\vec{K}$, and we will denote the argument as a scalar quantity. For some purposes, we prefer to give special attention to the direction of propagation: the z axis in our work. Vector $\vec{K}$, for example, will then be denoted as $\vec{K}_T$, K_z in functional arguments (where $\vec{K}_T$ is the projection of $\vec{K}$ upon the z = 0 plane).

Definition (I-1) implies for the case of isotropy that

$$M_2 \equiv \int_0^\infty dKK^2\Phi(K) = 2\pi^2 \qquad \text{(I-2)}$$

The Kolmogorov spectrum $\Phi(K)$ corresponding to the "two-thirds" law of the temperature structure function in the inertial subrange of turbulence, and modified to incorporate small and large wavenumber properties is

$$\Phi(K) = \gamma(\Phi)\ L_o^{\,3} \left(1 + K^2L_o^{\,2}\right)^{-11/6} \exp\left(-K^2/\kappa_m^{\,2}\right) \qquad \text{(I-3)}$$

L_o: macroscale of turbulence
ℓ_o: microscale of turbulence ($\ell_o = 5.92\ \kappa_m^{-1}$)
$\gamma(\Phi)$: a normalizing constant obtained from Equation (A-2) and given to good accuracy by $4\pi^2\Gamma(11/6)/\Gamma(3/2)\Gamma(1/3) \approx 15.7$.

An important parameter is the integral scale ℓ_i, defined as the first moment M_1 divided by 16 π

$$\ell_i \equiv \frac{1}{16\pi} \int_0^\infty dKK\Phi(K) = \gamma(\ell_i)L_o \tag{I-4}$$

$\gamma(\ell_i)$, a constant obtained by inserting Equations (I-3) into (I-4) and given to good accuracy by $3\,\gamma(\Phi)/80\pi \approx 1.88$

For some applications, it will be useful to have an expression for the first moment M_1 itself:

$$M_1 \equiv \int_0^\infty dKK\Phi(K) = \frac{3}{5}\gamma(\Phi)L_o \tag{I-5}$$

The third moment M_3 is obtained by inserting Equation (I-3) into its definition. Upon ignoring an error of order $2\kappa_m^{-2}L_o^{-2}$, we obtain

$$M_3 \equiv \int_0^\infty dKK^3\Phi(K) \approx 2.79\gamma(\Phi)\kappa_m^{1/3}L_o^{-2/3} \tag{I-6}$$

There is a connection between ϵ^2 and the structure constant C_n^2. By forming the refractive-index structure function $D(\vec{r})$ by Tatarski's[9] Equations (1.38) we obtain in terms of our normalizations

$$D(\vec{r}) = \frac{\epsilon^2}{16\pi^2}\int d^3K\,\Phi(K)\left[1-\cos\left(\vec{K}.\vec{r}\right)\right]$$

with $\Phi(K)$ given by Equation (I-3). We compare this to Tatarski's[9] Equations (1.41) and (3.24) and obtain

$$\epsilon^2 \equiv \gamma\left(\epsilon^2\right)C_n^2L_o^{2/3}$$

$$\gamma\left(\epsilon^2\right) = 32\pi^3 \times 0.033/\gamma(\Phi) \approx 2.08 \tag{I-7}$$

APPENDIX II

RAY BENDING IN TURBULENT AIR: ANGULAR VARIANCE

A brief derivation of the angular variance $<(\delta\theta)^2>$ for ray bending over distance L in turbulent air is given here. In geometrical optics[11] the local radius of curvature is the inverse length of the vector $n^{-1} \vec{\nabla}_T n$ where n is the refractive index and ∇_T is the gradient in the plane through the local portion of the ray, normal to the local propagation direction. In turbulent air, this vector is approximated well by $\vec{\nabla}_T \delta\varepsilon/2$, because $n = (\varepsilon)^{1/2} \approx 1 + \delta\varepsilon/2$ with $\delta\varepsilon << 1$. The geometry of Figure II-1 shows that $ds = R_c d\delta\theta$ and consequently

$$\vec{\delta\theta} = \int_0^L ds\, \hat{R}_c / R_c(s) = \frac{1}{2} \int_0^L ds\, \vec{\nabla}_T\, \delta\varepsilon(s) \qquad \text{(II-1)}$$

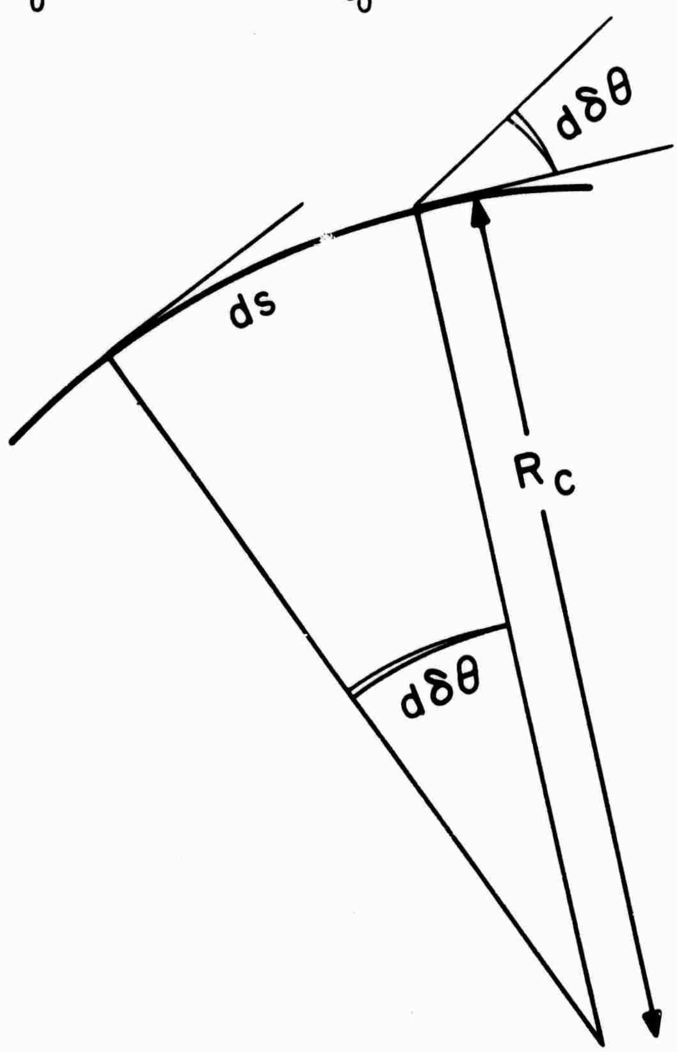

Figure II-1. Radius of curvature $R_c(s)$ at pathlength parameter s.

The vector notation ($\hat{R}_c$ is a unit vector) indicates an integration over direction of $\delta\theta$ (i.e., over change of plane in which $\delta\theta$ lies) as well as over the change in curvature. We form $<(\delta\theta)^2>$ from (II-1), replace the gradient operators at points s_1 and s_2 of the double integral by the gradient operator with respect to s_1-s_2, then apply the formulas of Appendix I to find

$$<(\delta\theta)^2> = (\epsilon^2 L/8\pi) \int_0^\infty dKK^3\Phi(K)$$

$$\approx \left[3\gamma(\Phi)\gamma(\epsilon^2)/8\pi\right] C_n^2 L\, \kappa_m^{1/3} \approx 3.9\, C_n^2 L\, \kappa_m^{1/3} \qquad \text{(II-2)}$$

The above equation is a geometrical-optics estimate of the mean square angular deviation of a ray in turbulent air with a modified Kolmogorov spectrum of refractive-index fluctuations.

REFERENCES

1. R. S. Lawrence and J. W. Strohbehn, Proc. IEEE 58, 1523 (1970).
2. M. E. Gracheva and A. S. Gurvich, Sov. Radiophysics 8, 511 (1965).
3. R. A. Schmeltzer, Quart. Appl. Math. 24, 339 (1967).
4. R. E. Hufnagel and N. R. Stanley, J. Opt. Soc. Am. 54, 52 (1964).
5. W. P. Brown, Jr., IEEE Trans. Ant. Prop. AP-15, 81 (1967).
6. D. A. de Wolf, Radio Sci. 2, 1379 (1967); 3, 308 (1968).
7. F. G. Gebhardt and S. A. Collins, Jr., J. Opt. Soc. Am. 59, 1139 (1969).
8. R. F. Lutomirski and H. T. Yura, Appl. Opt. 10, 1652 (1971).
9. V. I. Tatarski, Wave Propagation in a Turbulent Medium, transl. R. A. Silverman, McGraw-Hill, New York, N.Y. (1961).
10. M. Born and E. Wolf, Principles of Optics, 3rd Ed. Pergamon Press, Oxford (1965).
11. A. D. Varvatsis and M. I. Sancer, Can. J. Phys., 49, 1233 (1971).