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RADAR REFLECTION COEFFICIENTS FROM A PLASMA GRADIENT INCLUDING THE EFFECTS OF ELECTRON-NEUTRAL AND ELECTRON-ION COLLISIONS

by

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CONTENTS

SECTION		PAGE
	ABSTRACT	i
I	INTRODUCTION	1
II	DERIVATION OF RESULT	3
III	PRESENTATION OF RESULTS	10
APPENDIX	FORTRAN LISTING OF REFLECTION COEFFICIENT PROGRAM	15
	REFERENCES	17

I. INTRODUCTION

A gas will be ionized whenever its temperature is sufficiently high and/or its density is sufficiently low. This state of matter (a plasma) is ubiquitous on the scale of the universe but is comparatively rare on earth. In the vicinity of the earth plasmas occur naturally in the ionosphere and in the wake of meteors, and are produced by lightning. The activities of man create plasmas in gas discharge tubes, in internal combustion engines, during satellite reentry, and upon the detonation of nuclear weapons. Although these examples are by no means inclusive they indicate that the plasma state is sufficiently prevalent to be of interest to astronomers, radio and radar engineers, and many others. In many instances the most important consequence of a plasma is its effect on the propagation of electromagnetic waves. As in many other fields of scientific endeavor the existence of exact mathematical solutions to idealized problems facilitates the development of that precious faculty, "physical intuition."

In this paper we shall discuss an exactly soluble problem concerning the interaction of an electromagnetic wave (polarized normal to the gradient of the electron density) with a plasma whose properties vary only in one direction. The model seems unique in that it includes the effects of both electron-neutral and electron-ion collisions. The electron density profiles considered are $n_e(x) = n_o e^{x/d}$ and $n_e(x) = n_\infty/(1 + e^{-x/d})$. The latter profile is probably a reasonable approximation to the electron density occurring in many physical problems. This feature, and the fact that electron-ion collisions are included as well as the electron-neutral collisions, should make the model of great interest in situations where the electron densities are sufficiently high so that ion-electron collisions are important.

The general form of the equation to be solved is[1]

$$\frac{d^2E}{dx^2} + k^2q^2(x)E = 0 \qquad (1)$$

Solutions of this equation are known for an extremely broad class of functions q, and many examples relevant to the propagation of radio waves in the ionosphere are discussed in Ref. 1. Indeed the example we consider is a special case ($\varepsilon_3 = 0$ in the notation of Ref. 1, p. 376) of the "Epstein" profile. The major contribution of the present paper is the observation that this particular soluble problem has a physically interesting interpretation. In addition the derivation we give is somewhat simpler than is possible in the more general case of an "Epstein" profile.

II. DERIVATION OF RESULT

Consider an infinite plasma such that the electron density $n_e(x)$ depends only on a single coordinate x. An electromagnetic wave plane-polarized in the y direction is incident on the system with angle of incidence Θ_I. The electric field component E_y has the form

$$E_y = e^{ik(ct-Sz)}\, E(x) \tag{2}$$

where we use the symbols C and S for the cosine and sine of the angle of incidence. The function $E(x)$ satisfies the equation[1]

$$\frac{d^2}{dx^2} E = \left(-k^2C^2 + k^2 \frac{n_e(x)}{n_{cr}\left[1 - i\frac{\nu_1}{\omega} - i\frac{\nu_2}{\omega} n_e(x)\right]} \right) E \tag{3}$$

where $\omega = kc$ and ν_1 is the electron-neutral collision frequency, $\nu_2 n_e(x)$ is the electron-ion collision frequency, and n_{cr} is the critical electron density* at the frequency ω.

We now note that in the two cases

$$\text{Case I:} \quad n_e(x) = n_o\, e^{x/d}$$

$$\text{Case II:} \quad n_e(x) = \frac{n_\infty}{1 + e^{-x/d}} \tag{4}$$

the equation is of the form**

$$\frac{d^2}{dx^2} E = -k^2C^2E + k^2 \frac{P}{Q + R\, e^{-x/d}} E \tag{5}$$

---

*Thus $n_{cr} = (f/8970)^2\ cm^{-3}$ with f the frequency in Hertz.
**Where only the ratios P:Q:R are relevant.

where

$$\text{Case I:}\quad P = \frac{n_o}{n_{cr}} \;,\quad Q = -\frac{i\nu_2}{\omega} n_o \;,\quad R = 1 - i\frac{\nu_1}{\omega}$$

$$\text{Case II:}\quad P = \frac{n_\infty}{n_{cr}} \;,\quad Q = 1 - \frac{i\nu_1}{\omega} - \frac{i\nu_2}{\omega} n_\infty \;,\quad R = 1 - \frac{i\nu_1}{\omega} \qquad (6)$$

In order to solve Eq. 5 we introduce the expansion

$$E_\pm(x) = e^{\mp ikCx} \sum_{n=0}^{\infty} a_n^\pm e^{nx/d} \qquad (7)$$

where E_+ corresponds to the incoming wave and E_- is the reflected wave. Substituting this form into Eq. 5 and equating coefficients of $e^{\mp ikCx} \times e^{(n-1)x/d}$ indicates that the coefficients a_n satisfy the recursion relation

$$Q(n - 1)(n - 1 \mp 2ikCd)a_{n-1} + Rn(n \mp 2ikCd)a_n = k^2d^2Pa_{n-1} \qquad (8)$$

or

$$Rn(n \mp 2ikCd)a_n = -Q(n - 1 \mp ikCd + \alpha)(n - 1 \mp ikCd - \alpha)a_{n-1} \qquad (9)$$

where*

$$\alpha^2 = k^2d^2(p - C^2Q)/Q \qquad (10)$$

Thus we have the expansion

$$E_\pm = e^{\mp ikCx} \frac{\Gamma(1 \mp 2ikCd)}{\Gamma(\alpha \mp ikCd)\Gamma(-\alpha \mp ikCd)} \sum_{n=0}^{\infty} \left(-\frac{Q}{R} e^{x/d}\right)^n \frac{\Gamma(n + \alpha \mp ikCd)\Gamma(n - \alpha \mp ikCd)}{\Gamma(n + 1)\Gamma(n + 1 \mp 2ikCd)} \qquad (11)$$

---

*It is obvious, even before the calculation has begun, that in Case I the value of n_o does not influence the absolute value of the reflection coefficient. It is easily seen that α does not depend on n_o. The phase of the reflection coefficient is, however, influenced in a trivial and understandable way by the value of n_o.

where the solution has been normalized so that as $x \to -\infty$

$$E_\pm \sim e^{\mp ikCx} \tag{12}$$

The series converges for $x < d \,\ell n \left|\frac{Q}{R}\right|$.

The physically acceptable solution of Eq. 5 is that linear combination

$$E = E_+ + \mathcal{R} E_- \tag{13}$$

which goes to zero as $x \to +\infty$. Thus the reflection coefficient

$$\mathcal{R} = - \lim_{x\to\infty} \frac{E_+}{E_-} \tag{14}$$

Fortunately the series in Eq. 11 is readily recognizable as a hypergeometric function so that

$$E_\pm = e^{\mp ikCx} F\left(\alpha \mp ikCd, -\alpha \mp ikCd; 1 \mp 2ikCd; -\frac{Q}{R} e^{x/d}\right) \tag{15}$$

which can be analytically continued with the aid of the well known* relation

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F(a, 1-c+a; 1-b+a; z^{-1})$$
$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F(b, 1-c+b; 1-a+b; z^{-1}) \tag{16}$$

If we choose the square root in Eq. 10 so that the real part of α is positive we find** that as $x \to \infty$

$$E_\pm \sim e^{x\alpha/d} \left(\frac{Q}{R}\right)^{\alpha \pm ikCd} \frac{\Gamma(1 \mp 2ikCd)\Gamma(2\alpha)}{\Gamma(\alpha \mp ikCd)\Gamma(\alpha + 1 \mp ikCd)} \tag{17}$$

---

*Reference 2, p. 559. However, the usually reliable Whittaker and Watson gives the result incorrectly (Ref. 3, p. 289).

**Note that $F(a,b;c;0) = 1$.

and

$$R = -\left(\frac{Q}{R}\right)^{2ikCd} \frac{\Gamma(1 - 2ikCd)}{\Gamma(1 + 2ikCd)} \frac{\Gamma(\alpha + ikCd)}{\Gamma(\alpha - ikCd)} \frac{\Gamma(\alpha + 1 + ikCd)}{\Gamma(\alpha + 1 - ikCd)} \tag{18}$$

Since the gamma function is easily calculated numerically, this is generally a very convenient form.

It is interesting to verify that Eq. 18 behaves properly at vanishing collision frequencies. Thus for $\nu_1 = \nu_2 = 0$ and Case II, α is real and positive and $|R| = 1$ provided that $n_\infty \geq n_{cr}C^2$. On the other hand when $n_\infty < n_{cr}C^2$, α is a positive imaginary number and $|R| < 1$. These, of course, are the results to be expected. In Case I it is expected that for $\nu_1 = \nu_2 = 0$, $|R| = 1$.

Verification of this last result is slightly tricky for Case I. When $\nu_2 \rightarrow 0$, $Q_2 \rightarrow 0$ and $\alpha \rightarrow \infty$ so that Eq. 18 is an indeterminate form.* However, as $|\alpha| \rightarrow \infty$

$$\frac{\Gamma(\alpha + \nu)}{\Gamma(\alpha)} \rightarrow \alpha^\nu$$

so that

$$R \rightarrow -\left(\frac{Q\alpha^2}{R}\right)^{2ikCd} \frac{\Gamma(1 - 2ikCd)}{\Gamma(1 + 2ikCd)}$$

$$\rightarrow -\left(\frac{k^2d^2P}{R}\right)^{2ikCd} \frac{\Gamma(1 - 2ikCd)}{\Gamma(1 + 2ikCd)} \tag{19}$$

Thus when $n_e(x) = n_o\, e^{x/d}$ and $\nu_2 = 0$ we have the very simple well-known result**

$$R = -\left(\frac{k^2d^2P}{1 - i\,\frac{\nu_1}{\omega}}\right)^{2ikCd} \frac{\Gamma(1 - 2ikCd)}{\Gamma(1 + 2ikCd)} \tag{20}$$

---

*Equation 8 is also significantly altered so the solutions change their form from a hypergeometric function to a Bessel function.

**Reference 1, p. 357.

and

$$|R| = \exp\left\{-2kdC \arctan\left(\frac{\nu_1}{\omega}\right)\right\} \tag{21}$$

The extreme simplicity of this result makes its use attractive in cases where it may be applicable.*

A very striking property of Eq. 21 is that the reflection coefficient does not go to zero as $\nu_1 \to \infty$ but instead approaches a finite lower bound

$$|R| \xrightarrow[\nu_1 \to \infty]{} \exp(-kdC\pi) \tag{22}$$

As we shall see below, this behavior is unphysical and is due to the mathematical idealization involved in an electron density which becomes infinite as x becomes infinite.

In order to better understand this behavior it is illuminating to consider an alternative derivation of Eq. 21 which does not require an explicit solution of the differential equation, Eq. 5.

Thus in Case I (for the moment we retain $\nu_2 \neq 0$) Eq. 5 may be rewritten

$$\frac{d^2E}{dx^2} = -k^2C^2E + \frac{k^2E}{e^{-(x-x_o)/d} - i\nu_2 n_{cr}} \tag{23}$$

where

$$\frac{x_o}{d} = \ln\left(\frac{n_{cr}}{n_o}\sqrt{1 + \frac{\nu_1^2}{\omega^2}}\right) - i \arctan\left(\frac{\nu_1}{\omega}\right) \tag{24}$$

---

*However, it should be noted that this requires that the electron density exponentially increase to a value much larger than the critical density and yet have the electron-ion collision frequency remain small compared to the electron-neutral collision frequency.

The solutions satisfying the asymptotic condition given by Eq. 12 must have the form

$$E_{\pm} = e^{\mp ikCx_o} f_{\pm}(x - x_o) \tag{25}$$

where $f_{\pm}$ has the asymptotic behavior as $x \to -\infty$

$$f_{\pm}(x) \sim e^{\mp ikCx} \tag{26}$$

and $f_{\pm}(x)$ is <u>independent of</u> ν_1.

The reflection coefficient therefore is*

$$R = -e^{-2ikCx_o} \lim_{x\to\infty} \frac{f_+(x - x_o)}{f_-(x - x_o)} \tag{27}$$

Since $|R| = 1$ when $\nu_1 = \nu_2 = 0$ and since all the dependence on ν_1 is in the exponential factor it is clear that

$$\lim_{\nu_2\to 0} \lim_{x\to\infty} \left| \frac{f_+(x - x_o)}{f_-(x - x_o)} \right| = 1 \tag{28}$$

so that Eq. 21 has been proven.

From the form of the solution (Eq. 25) it is clear that, as ν_1 increases, the region where the reflection occurs moves over to the right with the real part of x_o. That is, the reflections occur from regions where the electron density is of the order of

$$n_{cr}\left(1 + \frac{\nu_1^2}{\omega^2}\right)^{1/2}$$

---

*It is easily verified that Eq. 18 has the general form in Case I. The fact that the line f_+/f_- is independent of x_o follows immediately from the observation that the general solution of Eq. 23 differs from the desired solution by a function which is unbounded as $x \to \infty$.

In all the other cases (Case I, $\nu_2 \to \infty$, and Case II, ν_1 or $\nu_2 \to \infty$) the reflection coefficient goes to zero as the collision frequency goes to infinity.* This type of behavior is more reasonable physically since at high collision frequencies the electric field does not have time to accelerate the electrons and is therefore unable to do any work on them.

*When the real part of α is positive so is the imaginary part. It is then easy to see that $\alpha \to ikCd$ and $R \to 0$ since $\Gamma(0) = \infty$.

III. PRESENTATION OF RESULTS

The reflection coefficient is a function of several variables, so that it is difficult to present the results in a limited space.

For the exponential case the reflection coefficient depends on the dimensionless parameters kd, ν_1/ω, and $\nu_2 n_{cr}/\omega$. The power loss upon reflection ($-20 \log_{10}|R|$) is tabulated in Table 1 for several combinations of these parameters being "small" or "large" (where a decade represents the distinction between nominal values and large or small values).

Similarly for Case II the reflection coefficient depends on the dimensionless combinations kd, ν_1/ω, $\nu_2 n_\infty/\omega$, and n_∞/n_{cr}. The reflection coefficient for a sampling of large and small values of these dimensionless parameters is presented in Tables 2, 3, and 4.

For the convenience of the reader who would like to survey some segment of the parameter space in more detail, the FORTRAN listing of a computer program* which calculates the reflection coefficient is given as an appendix.

*Written for the TRW Timeshare System using a CDC 6500 computer.

TABLE 1

POWER LOSS UPON REFLECTION (dB): EXPONENTIAL PROFILE

ν_1/ω \ $\nu_2 n_{cr}/\omega$	0.01	0.1	1	10	100
kd = 0.1					
0.01	0.31	2.23	11.15	29.91	49.89
0.1	0.46	2.38	11.30	30.06	50.05
1	1.66	3.57	12.49	31.25	51.24
10	2.85	4.77	13.68	32.44	52.43
100	3.00	4.92	13.84	32.60	52.59
kd = 1					
0.01	0.32	1.62	12.57	36.47	57.41
0.1	1.87	3.18	14.13	38.03	58.96
1	13.79	15.09	26.04	49.94	70.88
10	25.70	27.00	37.95	61.86	82.79
100	27.26	28.56	39.51	63.41	84.35
kd = 10					
0.01	2.89	13.32	99.47	236.92	281.60
0.1	18.48	28.90	115.05	252.50	297.18
1	137.60	148.02	234.17	371.62	416.30
10	256.72	267.14	353.30	490.75	534.43
100	272.30	282.72	368.87	506.32	551.00

TABLE 2

POWER LOSS UPON REFLECTION (dB):
"EPSTEIN" PROFILE, $n_\infty/n_{cr} = 0.1$

ν_1/ω \ $\nu_2 n_\infty/\omega$	0.01	0.1	1	10	100
	kd = 0.1				
0.01	32.1	32.0	34.0	50.0	69.9
0.1	32.2	32.1	34.6	50.3	70.1
1	35.4	35.7	38.9	52.2	71.3
10	52.6	52.7	53.4	58.5	73.3
100	72.6	72.6	72.7	73.4	78.6
	kd = 1				
0.01	62.8	61.2	52.8	59.2	77.7
0.1	62.7	61.3	54.1	60.8	79.2
1	66.2	66.0	65.3	73.3	91.2
10	84.5	84.6	85.2	89.7	103.8
100	104.6	104.6	104.7	105.4	110.6
	kd = 10				
0.01	515	498	381	298	306
0.1	513	496	390	314	321
1	515	509	472	433	440
10	553	553	552	552	561
100	575	576	576	576	580

TABLE 3

POWER LOSS UPON REFLECTION (dB):
"EPSTEIN" PROFILE, $n_\infty/n_{cr} = 1$

ν_1/ω \ $\nu_2 n_\infty/\omega$	0.01	0.1	1	10	100
	kd = 0.1				
0.01	1.78	4.10	12.4	30.1	49.9
0.1	4.26	5.66	13.1	30.3	50.1
1	13.8	14.3	18.4	32.2	51.3
10	32.6	32.7	33.4	38.5	53.3
100	52.6	52.6	52.7	53.4	58.6
	kd = 1				
0.01	5.08	10.2	21.4	38.1	57.6
0.1	11.8	14.3	23.5	39.7	59.1
1	34.9	35.4	39.4	52.3	71.1
10	63.5	63.6	64.2	69.2	83.8
100	84.5	84.5	84.6	85.3	90.5
	kd = 10				
0.01	49.9	98.2	186	253	283
0.1	114	136	205	269	299
1	321	325	346	389	418
10	507	507	509	518	538
100	553	553	553	554	559

TABLE 4

POWER LOSS UPON REFLECTION (dB):
"EPSTEIN" PROFILE, $n_\infty/n_{cr} = 10$

ν_1/ω \ $\nu_2 n_\infty/\omega$	0.01	0.1	1	10	100
kd = 0.1					
0.01	0.0536	0.217	1.78	11.05	29.9
0.1	0.372	0.535	2.09	11.3	30.1
1	3.13	3.28	4.64	13.1	31.4
10	13.6	13.6	14.2	18.8	33.3
100	32.6	32.6	32.7	33.4	38.6
kd = 1					
0.01	0.205	0.344	1.73	13.1	36.6
0.1	1.90	2.04	3.43	14.8	38.2
1	15.2	15.3	16.7	27.5	50.2
10	38.5	38.6	39.4	45.6	62.9
100	63.6	63.6	63.7	64.5	70.0
kd = 10					
0.01	1.99	3.13	14.4	105	239
0.1	18.7	19.8	31.2	121	254
1	149	150	161	247	374
10	359	360	365	410	495
100	508	508	508	511	526

App.

APPENDIX *

FORTRAN LISTING OF REFLECTION COEFFICIENT PROGRAM

```
      PROGRAM REFLEX(INPUT,TAPE5=INPUT,OUTPUT)
      REAL NU1,NU2,NINF,NCR
      DATA PI /3.14159265358979324/
      DATA CLT/2.997925E10/
      COMPLEX ZONE
      DATA ZONE/(1.,0.)/
      NAMELIST  /REFDAT/FREQ,NTYPE,NU1,NU2,D,NINF
      DISPLAY*   REFLECTION COEFFICIENT PROGRAM*
      DISPLAY* DATA REQUIRED,NTYPE,FREQ,NU1,NU2,D,NINF*
      DISPLAY*NTYPE<0  PROGRAM TERMINATES*
      DISPLAY*NTYPE =0  EXPONENTIAL*
      DISPLAY*NTYPE >0 "EPSTIEN"*
      DISPLAY* TERMINATE DATA FIELD WITH $ SIGN*
2         CONTINUE
      READ(5,REFDAT)
      NCR=FREQ*FREQ/8.05E7
      ØMEGA=2.*PI*FREQ
      SKD=D*ØMEGA/CLT
      IF(NTYPE)   4,6,8
6         CØNTINUE
      DISPLAY*EXPONENTIAL*
      CALLREFL(SKD,ZØNE,(0.,-1.)*NU2/ØMEGA*NCR,
     A 1.-(0.,1.)*NU1/ØMEGA,AMP,PHS)
10    CONTINUE
      DISPLAY*FREQUENCY*,FREQ,*NCR*,NCR,*NU1*,NU1,
     A  *NU2*,NU2
      DISPLAY*TRANSITION LENGTH*,D
      DISPLAY*AMPLITUDE*,AMP,*PHASE*,PHS
      GØ  TØ  2
8     CØNTINUE
      DISPLAY*  EPSTIEN PROFILE N INFINITY*,NINF
      IF(NINF.EQ.0.)  GØ  TØ  12
      CALL REFL(SKD,(1.,0.)*NINF/NCR,1.-(0.,1.)/ØMEGA
     A *(NU1+NU2*NINF),1.-(0.,1.)*NU1/ØMEGA,AMP,PHS)
      GØ  TØ  10
4     CONTINUE
      CALL EXIT
12    CONTINUE
      DISPLAY*NINF=0  NØ REFLECTED WAVE*
      GØ  TØ  2
      END


      SUBRØUTINE REFL(SKD,A,B,C,AMP,PHS)
      CØMPLEX A,B,C,LØGAM,REF,REFSUB,ALPH
      DATA PI/3.14159265358979324/
      WX=(0.,1.)*LØGAM(1.-(0.,2.)*SKD)
      WX=-WX
      IF(REAL(B).EQ.0..AND.REAL((0.,-1.)*B).EQ.0.)
     A    GØ TØ 2
      IF(SKD.EQ.0.) GØ  TØ  8
      ALPH=SKD*CSQRT((A-B)/B)
      IF(REAL(ALPH).LT.0.) ALPH=-ALPH
      REF=(0.,2.)*SKD*CLØG(B/C)+(0.,2.)*WX
     B+REFSUB(ALPH,SKD)
    4 CØNTINUE
      AMP=20.*REF*0.434294481903251828
      PHS=(0.,-1.)*REF+PI
      RETURN
 2        CØNTINUE
      IF(SKD.EQ.0.)  GØ  TØ  6
      REF=(0.,2.)*WX+(0.,2.)*SKD*CLØG(SKD*SKD*A/C)
      GØ TØ 4
6     CØNTINUE
      AMP=PHS=0.
      RETURN
8     CONTINUE
      ALPH=CSQRT(   (A-B)/B    )
      IF(ALPH.LT.0.)  ALPH=-ALPH
      REF=CLØG(   (ALPH+(0.,-1.) )/(ALPH+(0.,1.)   ))
      GØ  TØ   4
      END
```

---

*Note that inputs must be in CGS units.

App.

```
      COMPLEX FUNCTION REFSUB(AL,SKD)
      COMPLEX AL,LOGAM
      REFSUB=LOGAM(AL+1.+(0.,1.)*SKD)-LOGAM(AL+1.-(0.,
     A 1.)*SKD)
      REFSUB=CLOG((AL-(0.,1.)*SKD)/(AL+(0.,1.)*SKD))
     B +2.*REFSUB
      RETURN
      END
```

```
      COMPLEX FUNCTION LOGAM(Z)
      COMPLEX Z,ZW,ZWSQM
      DATA PI/3.14159265358979324/
      DATA CA/.91893853320467274/
      JNEG=0
      LOGAM=0
      R=ZW=Z
      IF(R.GT.0.) GO TO 2
      ZW=1.-ZW
      JNEG=1
2           CONTINUE
      IF(R.GT.10.) GO TO 4
      LOGAM=LOGAM-CLOG(ZW)
      R=ZW=ZW+1
      GO TO 2
4           CONTINUE
      ZWSQM=1./ZW/ZW
      LOGAM=(ZW-0.5)*CLOG(ZW)-ZW+CA+
     A 1./ZW*(1./12.-ZWSQM*(1./360-ZWSQM*(1./
     B 1260.-ZWSQM/1680.)))+LOGAM
      IF(JNEG.EQ.0) RETURN
      LOGAM=CLOG(PI/CSIN(PI*Z))-LOGAM
      RETURN
      END
```

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