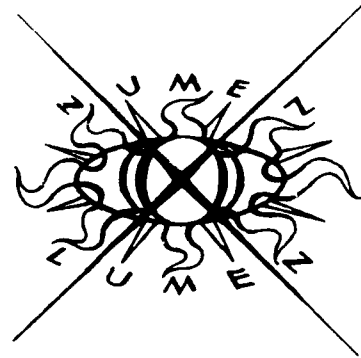


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*MR - PART I
(MRC-TSR-997)*

THE UNIVERSITY OF WISCONSIN
MATHEMATICS RESEARCH CENTER

Contract No. : DA-31-124-ARO-D-462

ON POLYNOMIAL SPLINE FUNCTIONS ON THE
CIRCLE II. MONOSPLINES AND QUADRATURE
FORMULAE

I. J. Schoenberg

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MRC Technical Summary Report #1002
December 1971

Received July 15, 1969

RECEIVED
MAR 1 1972

Madison, Wisconsin

AR 70-31 Unclassified

Security Classification			DOCUMENT CONTROL DATA - R & D	
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)				
1. ORIGINATING ACTIVITY (Corporate author)		11. REPORT SECURITY CLASSIFICATION		
Mathematics Research Center University of Wisconsin, Madison, Wis. 53706		Unclassified		
		12. GROUP		
		None		
3. REPORT TITLE				
ON POLYNOMIAL SPLINE FUNCTIONS ON THE CIRCLE II. MONOSPLINES AND QUADRATURE FORMULAE				
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)				
Summary Report: no specific reporting period.				
5. AUTHOR(S) (First name, middle initial, last name)				
I. J. Schoenberg				
6. REPORT DATE	76. TOTAL NO. OF PAGES	75. NO. OF REFS		
December 1971	19 pp.	4		
80. CONTRACT OR GRANT NO.	80. ORIGINATOR'S REPORT NUMBER(S)			
Contract No. DA-31-124-ARO-D-462	1002			
a. PROJECT NO.	80. OTHER REPORT NO(S) (Any other numbers that may be associated this report)			
None	None			
c.				
d.				
10. DISTRIBUTION STATEMENT				
Distribution of this document is unlimited.				
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY		
None		Army Research Office-Durham, N. C.		
13. ABSTRACT				
This is a continuation of MRC Technical Summary Report #997. The connection between monosplines and quadrature formulae is used to determine the best quadrature formula for $\int f(z)dz$ integrated along the unit circle. The quadrature formula exact for all spline functions is also derived.				

DD FORM 1 NOV 65 1473

Unclassified

Security Classification

AGO 5006A

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ON POLYNOMIAL SPLINE FUNCTIONS ON THE CIRCLE II.

MONOSPINES AND QUADRATURE FORMULAE

I. J. Schoenberg

Introduction

In [4] we have studied the interpolatory properties of the class $\mathcal{S}_{m,k}$ of polynomial spline functions on the circle that were first discussed by Ahlberg, Nilson and Walsh (see [1], also for further references). In the present second paper we discuss quadrature formulae of the form

$$(1) \quad \int_U f(z) dz = \sum_{j=0}^{k-1} C_j f(\omega^j) + Rf, \quad$$

where U is the unit circle $|z| = 1$ described counter-clockwise and $\omega = \exp(2\pi i/k)$. This problem requires a discussion of monospines and will suggest several classes of spline functions and monospines having noteworthy properties. Free use is made of the results of [4], in particular of B-splines and their Fourier series expansions.

1. Monospines and quadrature formulae. Let k and m be positive integers. Our objective is to derive quadrature formulae of the form (1) which are to be exact for polynomials of degree not exceeding $m-1$, i.e.

$$(1.1) \quad Rf = 0 \quad \text{if} \quad f \in \pi_{m-1}.$$

This property is equivalent to the relations

$$(1.2) \quad \int_U z^\nu dz = 0 = \sum_{j=0}^{k-1} C_j \omega^{j\nu}, \quad (\nu = 0, 1, \dots, m-1),$$

and these show that the quadrature formula (1) has the property (1.1) if and only if

$$(1.3) \quad \text{the polynomial } \sum_0^{k-1} C_j x^j \text{ vanishes if } x = 1, \omega, \dots, \omega^{m-1}.$$

We conclude the following: 1. If $m \geq k$ then the polynomial (1.3) vanishes identically, hence $C_j = 0$ for all j . 2. If $m < k$ then the polynomial (1.3) still depends on $k-m$ arbitrary linear parameters. Avoiding trivial cases we shall assume that

$$(1.4) \quad 1 \leq m < k.$$

Expanding the binomials appearing in (1.5) we see that the condition (1.3) is equivalent to the validity of the identity in z

$$(1.5) \quad \sum_0^{k-1} C_j (z - \omega^j)^{m-1} = 0.$$

At this point we reintroduce the class $\mathfrak{S}_{m,k}$ of polynomial spline functions on U of degree $m-1$ having the ω^j as knots, and for a good reason. In fact the identity (1.5) is the precise condition for the existence of a function $S(z) \in \mathfrak{S}_{m,k}$ such that

$$(1.6) \quad C_j = \text{the jump of } S^{(m-1)}(z) \text{ at } z = \omega^j \quad (j = 0, \dots, k-1).$$

Indeed, if $S(z) = Q(z) \in \pi_{m-1}$ on the arc $(\omega^{k-1}, 1)$ then the function on U defined on the arcs $(\omega^{k-1}, 1), (1, \omega), \dots, (\omega^{k-1}, 1)$ by the successive partial sums of the expression

$$Q(z) + \sum_0^{k-1} C_j \frac{(z - \omega^j)^{m-1}}{(m-1)!}$$

is, in view of (1.5), a spline function $S(z)$ with the property (1.6).

The connection of the spline function $S(z)$ with the quadrature formula (1) is close as shown by

Lemma 1. If we write

$$(1.7) \quad K(z) = \frac{1}{m!} z^m - S(z)$$

and if

$$(1.8) \quad f(z) \in C^m(U)$$

then we have the relation

$$(1.9) \quad \int_U f(z) dz = \sum_0^{k-1} C_j f(\omega^j) + (-1)^m \int_U K(z) f^{(m)}(z) dz.$$

Proof. By (1.6) and (1.7) we find that

$$(1.10) \quad -C_j = \text{the jump of } K^{(m-1)}(z) \text{ at } z = \omega^j.$$

Integrations by parts show that

$$\int_U K(z) f^{(m)}(z) dz = - \int_U K'(z) f^{(m-1)}(z) dz = \dots = (-1)^{m-1} \int_U K^{(m-1)}(z) f'(z) dz$$

a point beyond which we can not continue the process because of the discontinuities of $K^{(m-1)}(z)$. However, we may proceed as follows

$$\begin{aligned} \int_U K(z) f^{(m)}(z) dz &= (-1)^{m-1} \sum_j \int_{\omega^j}^{\omega^{j+1}} K^{(m-1)}(z) f'(z) dz \\ &= (-1)^{m-1} \sum_j \left\{ K^{(m-1)}(\omega_-^{j+1}) - K^{(m-1)}(\omega_+^j) f(\omega^j) - \int_{\omega^j}^{\omega^{j+1}} f(z) dK^{(m-1)}(z) \right\} \end{aligned}$$

and using (1.10) we obtain

$$\int_U K(z) f^{(m)}(z) dz = (-1)^{m-1} \sum_j C_j f(\omega^j) + (-1)^m \int_U f(z) dz$$

because $dK^{(m-1)}(z) = dz$ within each of the arcs.

Using a term familiar from the real axis case (see e.g. [3]) we call $K(z)$ a monospline of degree m and denote their class by the symbol $\mathfrak{M}_{m,k}$. We may then summarize our results as follows: To every quadrature formula (1) with the property (1.1) corresponds a monospline $K(z)$ producing the identity (1.9). Conversely, to every monospline $K(z)$ corresponds a quadrature

formula (1), the C_j being described by (1.6). This correspondence is one-to-one up to an element of π_{m-1} that we may add to $K(z)$ without changing our results. Our discussion of the quadrature formula (1) will single out certain spline functions and monosplines for special study.

2. The flower-shaped spline functions. Let $S(z) \in \mathcal{S}_{m,k}$ and let us assume that it satisfies the functional equation

$$(2.1) \quad S(z\omega) = \omega^r S(z), \quad (|z| = 1),$$

where r is an integer, $0 \leq r \leq k-1$. We could say that $S(z)$ is quasiperiodic, or r -quasiperiodic. However, in view of the rotational symmetry of the image of U by $w = S(z)$ we prefer to say that $S(z)$ is flower-shaped and also that $S(z)$ is an r -flower (see Fig. 1 in [4] showing the image of U by a 3-flower for $k = 5, m = 2$).

In terms of the B-spline $M_m(z)$ of [4] we have

Lemma 2. For each $r, 0 \leq r \leq k-1$, there is up to a constant factor a unique r -flower given by the formula

$$(2.2) \quad S(z) = \sum_{j=0}^{k-1} \omega^{jr} M_m(z\omega^{-j}).$$

Proof: By [4, Lemma 1] we may write

$$S(z) = \sum_j c_j M_m(z\omega^{-j}),$$

and (2.1) gives $\sum_j c_j M_m(z\omega^{1-j}) = \sum_j c_j \omega^r M_m(z\omega^{-j})$ or $\sum_{j+1} c_{j+1} M_m(z\omega^{-j}) = \sum_j c_j \omega^r M_m(z\omega^{-j})$, whence $c_{j+1} = c_j \omega^r$ and finally $c_j = c_0 \omega^{jr}$. Setting $c_0 = 1$ we obtain (2.2). Conversely, it is easily established that (2.2) satisfies (2.1) and Lemma 2 is established.

The Fourier expansions of r -flowers are described by

Lemma 3. If $S(z)$ is an r -flower then, up to a constant factor,

$$(2.3) \quad S(z) = z^r \quad \text{if } r = 0, 1, \dots, m-1,$$

and

$$(2.4) \quad S(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+r)(ks+r-1)\dots(ks+r-m+1)} z^{ks+r}$$

if $r = m, m+1, \dots, k-1$.

Proof: We recall [4, formula (2.16)] that

$$(2.5) \quad M_m(z) = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_\nu b_{\nu-1} \dots b_{\nu-m+1} z^\nu$$

where

$$(2.6) \quad b_\nu = \frac{1-\omega^{-\nu}}{\nu} \quad \text{if } \nu \neq 0, \quad b_0 = \frac{2\pi i}{k}.$$

From (2.2) and (2.5)

$$S(z) = \frac{1}{2\pi i} \sum_{\nu} b_\nu \dots b_{\nu-m+1} \left(\sum_{j=0}^{k-1} \omega^{jr} \omega^{-j\nu} \right) z^\nu$$

or

$$(2.7) \quad S(z) = \frac{k}{2\pi i} \sum_{\nu} b_\nu b_{\nu-1} \dots b_{\nu-m+1} z^\nu,$$

where the summation is over all $\nu \equiv r \pmod{k}$, hence $\nu = ks+r$. By (2.6)

we see that $b_\nu = 0$ if ν is a nonvanishing multiple of k . If $0 \leq r \leq m-1$

then (2.7) is seen to reduce to the single term corresponding to $s = 0$ or

$\nu = r$ and this is indeed proportional to (2.3). Also directly it is clear that

$z^r \in \mathcal{S}_{m,k}$ and that z^r is an r -flower.

If $m \leq r \leq k-1$, we may discard on the right side of (2.7) the non-vanishing factor

$$\frac{k}{2\pi i} (1-\omega^{-r})(1-\omega^{-r+1}) \dots (1-\omega^{-r+m-1})$$

and we obtain the expansion (2.4).

3. The monospline of least L_2 -norm. We have already used the fact

that

$$(3.1) \quad S(z) = \sum_{j=0}^{k-1} c_j M_m(z \omega^{-j})$$

represents the most general element of $\mathcal{S}_{m,k}$. Using (2.5) we find its

Fourier expansion to be

$$S(z) = \frac{1}{2\pi i} \sum_{\nu} b_{\nu} b_{\nu-1} \dots b_{\nu-m+1} z^{\nu} \sum_{j=0}^{k-1} c_j \omega^{-\nu j}.$$

However

$$(3.2) \quad \eta_{\nu} = \sum_{j=0}^{k-1} c_j \omega^{-\nu j}$$

in the finite Fourier series representation of an arbitrary periodic sequence

(η_{ν}) of period k . This establishes

Lemma 4. The most general element of $\mathcal{S}_{m,k}$ is given by

$$(3.3) \quad S(z) = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_{\nu} b_{\nu-1} \dots b_{\nu-m+1} \eta_{\nu} z^{\nu}$$

where (η_{ν}) is an arbitrary periodic sequence of numbers of period k .

It is now easy to determine the monospline

$$(3.4) \quad K(z) = \frac{1}{m!} z^m - S(z), \quad S(z) \in \mathcal{S}_{m,k}$$

having least L_2 -norm, or

$$(3.5) \quad \|K\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |K(z)|^2 d\theta \right)^{\frac{1}{2}} = \text{minimum}, \quad (z = e^{i\theta}).$$

Indeed, by Lemma 4 we may write

$$\begin{aligned} K(z) &= \frac{1}{m!} z^m - \frac{1}{2\pi i} \sum_{\nu} b_{\nu} \dots b_{\nu-m+1} z^{\nu} \eta_{\nu} \\ &= \left(\frac{1}{m!} - \frac{1}{2\pi i} b_m \dots b_1 \eta_m \right) z^m - \frac{1}{2\pi i} \sum_{\nu \neq m} b_{\nu} \dots b_{\nu-m+1} \eta_{\nu} z^{\nu} \end{aligned}$$

and using Parseval's relation we obtain

$$\begin{aligned}
(3.6) \quad \frac{1}{2\pi} \int_0^{2\pi} |K(z)|^2 d\theta &= \left| \frac{1}{m!} - \frac{1}{2\pi i} b_m \dots b_1 \eta_m \right|^2 + \frac{|\eta_m|^2}{4\pi^2} \sum_{\substack{\nu \equiv m(k) \\ \nu \neq m}} |b_\nu \dots b_{\nu-m+1}|^2 \\
&+ \frac{1}{4\pi^2} \sum_{\substack{\ell=0 \\ \ell \neq m}}^{k-1} |\eta_\ell|^2 \sum_{\nu \equiv \ell(k)} |b_\nu \dots b_{\nu-m+1}|^2.
\end{aligned}$$

Observe that by (2.6)

$$\sum_{\nu \equiv \ell(k)} |b_\nu \dots b_{\nu-m+1}|^2 > 0 \quad \text{for all integer } \ell.$$

It follows that in minimizing the right side of (3.6) we must have

$$(3.7) \quad \eta_\ell = 0 \quad \text{if} \quad \ell = 0, 1, \dots, m-1, m+1, \dots, k-1$$

Therefore the search reduces to the much narrower class of monosplines of the form

$$(3.8) \quad K(z) = \frac{1}{m!} z^m - \frac{1}{2\pi i} \eta_m \sum_{\nu \equiv m(k)} b_\nu \dots b_{\nu-m+1} z^\nu.$$

In fact, by Lemma 3, we see that (3.8) describes the most general monospline $K(z)$ satisfying the relation

$$(3.9) \quad K(z\omega) = \omega^m K(z).$$

Setting

$$\lambda = \frac{1}{2\pi i} (1-\omega^{-m})(1-\omega^{-m+1}) \dots (1-\omega^{-1}) \eta_m$$

we may write (3.8) as

$$(3.10) \quad K(z) = \frac{1}{m!} z^m - \lambda \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2) \dots (ks+m)} z^{ks+m},$$

and (3.6) becomes

$$(3.11) \quad (m!)^2 (\|K\|_2)^2 = |1-\lambda|^2 + |\lambda|^2 \sum_{s \neq 0} \frac{(m!)^2}{(ks+1)^2 (ks+2)^2 \dots (ks+m)^2},$$

and we are to determine the value of λ that minimizes the right side. The

inequalities $|1-\lambda| \geq |1-\operatorname{Re} \lambda|$, $|\lambda| \geq |\operatorname{Re} \lambda|$ show that this value of λ must be real. Writing

$$(3.12) \quad c = \sum_{s \neq 0} \frac{(m!)^2}{(ks+1)^2 \dots (ks+m)^2}$$

we see from the identity

$$(1-\lambda)^2 + c\lambda^2 = (1+c)\left(\lambda - \frac{1}{1+c}\right)^2 + \frac{c}{1+c}, \quad (c > 0),$$

that the least value of the left-hand side is $c/(1+c)$ and that it is assumed if $\lambda = \lambda_* = 1/(1+c)$. By (3.12) we see that the right side of (3.11) is least if λ equals

$$(3.13) \quad \lambda_* = \left(\sum_{s=-\infty}^{\infty} \frac{(m!)^2}{(ks+1)^2 (ks+2)^2 \dots (ks+m)^2} \right)^{-1}.$$

The least value of the right side of (3.11) we now find to be equal $c/(1+c) = (\lambda_*^{-1} - 1)\lambda_* = 1 - \lambda_*$.

We have therefore established

Theorem 1. The unique monospline (3.4) of least L_2 -norm is

$$(3.14) \quad K_*(z) = \frac{1}{m!} z^m - \lambda_* S_m(z)$$

where

$$(3.15) \quad S_m(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2) \dots (ks+m)} z^{ks+m},$$

while λ_* is given by (3.13). The value of the least L_2 -norm is

$$(3.16) \quad \|K_*\|_2 = \frac{1}{m!} (1 - \lambda_*)^{\frac{1}{2}}.$$

4. The best quadrature formula (1.9). We obtain the best quadrature formula (1.9), by the definition of the term "best" introduced by A. Sard [2], if we choose in (1.9) $K(z) = K_*(z)$, and there remains to determine the

coefficients C_j according to (1.10). However, we have already noticed that $K_*(z)$ satisfies the relation (3.9) which by differentiations yields $K_*^{(\nu)}(z\omega) = \omega^{m-\nu} K_*(z)$ and in particular

$$K_*^{(m-1)}(z\omega) = \omega K_*^{(m-1)}(z) .$$

By (1.10) we see that

$$(4.1) \quad C_j = \omega^j C_0$$

and it therefore suffices to determine the jump $\Delta_0 = -C_0$ of $K_*^{(m-1)}(z)$ at $z = 1$. It is easily obtained as follows.

Differentiating the Fourier series (3.14) $m-1$ times we get

$$(4.2) \quad K_*^{(m-1)}(z) = z - \lambda_* \sum_s \frac{1}{ks+1} z^{ks+1} .$$

Its sum is, as we know, a step function with discontinuities at the ω^j . To determine its jump Δ_0 at $z = 1$, we consider the function $\varphi(\theta)$ of period 2π such that

$$(4.3) \quad \varphi(\theta) = \frac{1}{2}(\pi - \theta) \quad \text{if} \quad 0 < \theta < 2\pi .$$

Defining also $\varphi(0) = 0$, we have the familiar expansion

$$(4.4) \quad \varphi(\theta) = \sum_1^\infty \frac{\sin \nu \theta}{\nu} = \frac{1}{2i} \sum_{s \neq 0} \frac{1}{s} e^{is\theta} \quad \text{for all real } \theta ,$$

from which we derive that

$$\frac{1}{k} \varphi(k\theta) e^{i\theta} = \frac{1}{2i} \sum_{s \neq 0} \frac{1}{ks} z^{ks+1} , \quad (z = e^{i\theta}) .$$

Using (4.2) we find the expansion

$$(4.5) \quad K_*^{(m-1)}(e^{i\theta}) + 2i\lambda_* \frac{1}{k} \varphi(k\theta) e^{i\theta} = (1-\lambda_*)z + \lambda_* \sum_{s \neq 0} \frac{1}{ks(ks+1)} z^{ks+1}$$

which is seen to converge absolutely and uniformly on U . We conclude that

the left side belongs to $C(U)$. The jump of the function $\varphi(\theta)$ being $= \pi$, at $\theta = 0$, by (4.3), we conclude that the jump $2i\lambda_* k^{-1}\pi$ of the second term of the left side of (4.5) cancels the jump Δ_0 of the first term. Therefore

$$(4.6) \quad \Delta_0 = -\frac{2\pi i}{k} \lambda_* .$$

Using the relations (4.1) we obtain

Theorem 2. Let $1 \leq m < k$. Among all quadrature formulae of the form

(1) and having the property (1.1), the best is the quadrature formula

$$(4.7) \quad \int_U f(z) dz = \lambda_* 2\pi i k^{-1} \sum_{j=0}^{k-1} \omega^j f(\omega^j) + (-1)^m \int_U K_*(z) f^{(m)}(z) dz$$

where $K_*(z)$ is the monospline of Theorem 1, while the positive fraction λ_* is defined by (3.13).

Theorem 2 does not state or imply that the familiar average (or Riemann sum)

$$(4.8) \quad A = 2\pi i k^{-1} \sum_{j=0}^{k-1} \omega^j f(\omega^j)$$

is not as good an approximation of the integral as the modified approximation $\lambda_* A$ appearing in (4.7). However, Theorem 2 does state that $\lambda_* A$ is the best approximation in the following sense: For all functions $f(z)$ such that

$$(4.9) \quad \int_0^{2\pi} |f^{(m)}(z)|^2 d\theta \leq H^2 ,$$

where H is a fixed constant, we have the inequality

$$(4.10) \quad \left| \int_U f(z) dz - \lambda_* A \right| \leq J_* \cdot H , \quad J_* = \frac{1}{m!} (2\pi(1-\lambda_*))^{1/2} .$$

For any other inequality

$$\left| \int_U f(z) dz - \sum_{j=0}^{k-1} C_j f(\omega^j) \right| \leq J \cdot H$$

valid for the class (4.9), we must have $J > J_*$.

5. The quadrature formula that is exact if $f \in \mathcal{S}_{m,k}$. We wish to determine the quadrature formula

$$(5.1) \quad \int_U f(z) dz = \sum_{j=0}^{k-1} \tilde{C}_j f(\omega^j) + \tilde{R}f$$

such that

$$(5.2) \quad \tilde{R}f = 0 \quad \text{if} \quad f \in \mathcal{S}_{m,k}.$$

One way to construct such a formula is as follows. We assume that the class $\mathcal{S}_{m,k}$ has the interpolatory property at the knots ω^j . By [4, Theorem 1] we must

$$(5.3) \quad \text{exclude the case that } k \text{ is even and } m \text{ is odd.}$$

We are then assured [4] of the existence of a fundamental function

$L_m(z) \in \mathcal{S}_{m,k}$ such that

$$(5.4) \quad f(z) = \sum_{j=0}^{k-1} f(\omega^j) L_m(z \omega^{-j}) \quad \text{if} \quad f \in \mathcal{S}_{m,k}.$$

Integrating both sides of this identity along U we see that (5.1) has the property (5.2) if we set

$$(5.5) \quad \tilde{C}_j = \int_U L_m(z \omega^{-j}) dz.$$

Again we assume (5.3). A second way of constructing this quadrature formula is as follows. Let $S_0(t)$ be the unique spline function such that the monospline

$$(5.6) \quad K_0(z) = \frac{1}{m!} z^m - S_0(z)$$

has the property

$$(5.7) \quad K_0(\omega^j) = 0 \quad (j = 0, \dots, k-1).$$

The existence and unicity of $S_0(z)$ follow from [4, Theorem 1]. We shall now establish several properties of $K_0(z)$ which will show that this monospline will produce the quadrature formula we are looking for.

I. The monospline (5.6) satisfies

$$(5.8) \quad K_0(z\omega) = \omega^m K_0(z) \quad .$$

Proof: Clearly

$$\omega^{-m} K_0(z\omega) = \frac{1}{m!} z^m - \omega^{-m} S_0(z\omega)$$

is a monospline satisfying the relations (5.7). The unicity of $K_0(z)$ implies the relation (5.8).

II. The quadrature formula

$$(5.9) \quad \int_U f(z) dz = \sum_{j=0}^{k-1} C_j^0 f(\omega^j) + R_0 f \quad ,$$

where $-C_j^0 =$ the jump of $K_0^{(m-1)}(z)$ at $z = \omega^j$, is exact if $f \in \mathcal{S}_{m,k}$.

Proof: By Lemma 1 we may write

$$\pm R_0 f = \int_U K_0(z) f^{(m)}(z) dz = \int_U K_0(z) df^{(m-1)}(z) \quad \text{if } f \in C^m(U) \quad .$$

However, the last form

$$(5.10) \quad \pm R_0 f = \int_U K_0(z) df^{(m-1)}(z)$$

is also applicable (see proof of Lemma 1) if we only assume $f^{(m-2)}(z)$ to be absolutely continuous and $f^{(m-1)}(z)$ of bounded variation on U . We may therefore assume in (5.9), (5.10) that $f(z) \in \mathcal{S}_{m,k}$ and, therefore, that $f^{(m-1)}(z)$ is a step function. If we denote by δ_j its jump at $z = \omega^j$, then (5.7) and (5.10) show that

$$\pm R_0 f = \sum K_0(\omega^j) \delta_j = 0$$

$$\text{III.} \quad C_j^0 = \widetilde{C}_j \quad (j = 0, \dots, k-1) \quad .$$

Proof: Letting $f(z) = L_m(z\omega^{-j})$ in (5.9), then (5.5) shows that

$$C_j^0 = \int_U L_m(z\omega^{-j}) dz = \widetilde{C}_j \quad .$$

The explicit construction of the quadrature formula (5.9) now presents no difficulties. We conclude from the relation (5.8) that $S_0(z)$ is an m -flower and Lemma 3 implies that $K_0(z)$ may be written as

$$(5.11) \quad K_0(z) = \frac{1}{m!} z^m - \lambda_0 \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2)\dots(ks+m)} z^{ks+m}$$

for an appropriate value of the constant λ_0 . The defining properties (5.7), in particular $K_0(1) = 0$ show that

$$(5.12) \quad \lambda_0 = \left(\sum_{s=-\infty}^{\infty} \frac{m!}{(ks+1)(ks+2)\dots(ks+m)} \right)^{-1} \quad .$$

This requires that the sum of the series in (5.12) should not vanish. By [4, Lemma 4] this series can vanish only if $m = k-1$ and m is odd. But this would imply that k is even and m is odd, a situation which can not occur because of our assumption (5.3).

We summarize our results in

Theorem 3. If we exclude the case that

$$(5.13) \quad m = k-1 \quad \text{and} \quad m \text{ is odd} \quad ,$$

then

$$(5.14) \quad \int_U f(z) dz = \lambda_0 2\pi i k^{-1} \sum_0^{k-1} \omega^j f(\omega^j) + (-1)^m \int_U K_0(z) f^{(m)}(z) dz$$

is a quadrature formula that is exact if $f(z) \in \mathcal{S}_{m,k}$, where $K_0(z)$ and λ_0 are described by (5.11) and (5.12). If we replace (5.13) by the more stringent assumption (5.3) then the quadrature formula (5.14) may also be obtained by

integration from the spline interpolation formula (5.4) .

The value $C_0 = 2\pi i \lambda_0 / k$ used in writing the formula (5.14) follow from (4.6) on replacing λ_* by λ_0 .

6. The polynomial components of a certain spline function. The spline function

$$(6.1) \quad S_m(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2)\dots(ks+m)} z^{ks+m}$$

appears in the representation (3.14), (3.15) of $K_*(z)$ as well as in (5.1!) which describes $K_0(z)$. By Lemma 3 it may also be characterized as an m -flower of degree $m-1$. Here we propose to determine explicitly its individual polynomial components. Without loss of generality we may assume that $m = k-1$ and determine the components of

$$(6.2) \quad S_{k-1}(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)\dots(ks+k-1)} z^{ks+k-1}$$

it being clear by differentiations that

$$(6.3) \quad S_m(z) = S_{k-1}^{(k-m-1)}(z), \quad (1 \leq m \leq k-1) .$$

The functional equation

$$(6.4) \quad S_{k-1}(z\omega) = \omega^{-1} S_{k-1}(z)$$

shows that it suffices to determine the polynomial component in the arc $(1, \omega)$.

Setting

$$(6.5) \quad \psi = \exp(\pi i/k)$$

we turn $S_{k-1}(z)$ by ψ^{-1} to obtain, since $\psi^k = -1$, the new spline function

$$(6.6) \quad s(z) = \psi S(z\psi) = \sum_{s=-\infty}^{\infty} \frac{(-1)^{s+1}}{(ks+1)\dots(ks+k-1)} z^{ks+k-1}$$

having its knots in the new locations $\psi\omega^j$. Let

$$(6.7) \quad P_0(z) = \alpha_0 \frac{z^{k-2}}{(k-2)!} + \alpha_1 \frac{z^{k-3}}{(k-3)!} + \dots + \alpha_{k-2}$$

be the component of $s(z)$ in the arc (ψ^{-1}, ψ) . The Laurent series (6.6) having real coefficients it is clear that $\overline{P(z)} = P(\bar{z})$ if z is on the arc (ψ^{-1}, ψ) and therefore also for all z . This shows that $P_0(z)$ is a real polynomial. Also $s(z)$ satisfies the relation $s(z\omega) = \omega^{-1}s(z)$ whence

$$(6.8) \quad s(z) = \omega^{-1} s(z\omega^{-1})$$

If z is in the arc $(\psi, \psi\omega)$, then (6.8) and (6.7) show that the component of $s(z)$ on this arc is

$$(6.9) \quad P_1(z) = \alpha_0 \omega \frac{z^{k-2}}{(k-2)!} + \alpha_1 \omega^2 \frac{z^{k-3}}{(k-3)!} + \dots + \alpha_{k-2} \omega^{k-1}$$

At this point we use that $s(z) \in C^{k-3}(U)$ and obtain at $z = \psi$ the $k-2$ equations

$$(6.10) \quad P_0^{(\nu)}(\psi) = P_1^{(\nu)}(\psi) \quad (\nu = 0, \dots, k-3)$$

Using (6.7) and (6.9) we obtain the system

$$\alpha_0 (\omega^{-1}) \frac{\psi^\nu}{\nu!} + \alpha_1 (\omega^2 - 1) \frac{\psi^{\nu-1}}{(\nu-1)!} + \dots + \alpha_\nu (\omega^{\nu+1} - 1) = 0 \quad (\nu = 1, \dots, k-2)$$

Dividing by $\omega^{\nu+1} - 1$ and using the relations

$$\frac{\omega^{\alpha-1}}{\omega^{\beta-1}} \cdot \frac{\psi^{\beta-\alpha}}{(\beta-\alpha)!} = \frac{1}{(\beta-\alpha)!} \frac{\psi^{\alpha-\psi^{-\alpha}}}{\psi^{\beta-\psi^{-\beta}}} = \frac{1}{(\beta-\alpha)!} \frac{\sin(\pi\alpha/k)}{\sin(\pi\beta/k)}$$

we obtain the system

$$0 = \frac{1}{1!} \alpha_0 \sin(\pi/k) + \alpha_1 \sin(2\pi/k)$$

$$0 = \frac{1}{2!} \alpha_0 \sin(\pi/k) + \frac{1}{1!} \alpha_1 \sin(2\pi/k) + \alpha_2 \sin(3\pi/k)$$

.

$$0 = \frac{1}{(k-2)!} \alpha_0 \sin(\pi/k) + \frac{1}{(k-3)!} \alpha_1 \sin(2\pi/k) + \dots + \alpha_{k-2} \sin((k-1)\pi/k)$$

which in terms of the new unknowns

$$(6.11) \quad \beta_\nu = \alpha_\nu \sin \frac{(\nu+1)\pi}{k} \quad (\nu = 0, \dots, k-2)$$

becomes

$$\begin{aligned} 0 &= \frac{1}{1!} \beta_0 + \beta_1 \\ 0 &= \frac{1}{2!} \beta_0 + \frac{1}{1!} \beta_1 + \beta_2 \\ &\vdots \\ 0 &= \frac{1}{(k-2)!} \beta_0 + \frac{1}{(k-3)!} \beta_1 + \dots + \frac{1}{1!} \beta_{k-3} + \beta_{k-2} \end{aligned}$$

This being the system giving the coefficients of the power series reciprocal to $e^x = \sum x^\nu / \nu!$ we obtain a solution $\beta_\nu = (-1)^\nu / \nu!$ and by (6.11)

$$(6.12) \quad \alpha_\nu = (-1)^\nu / \left(\nu! \sin \frac{(\nu+1)\pi}{k} \right), \quad (\nu = 0, \dots, k-2)$$

From (6.6) and (6.7) we therefore obtain the identity

$$(6.13) \quad \sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1) \dots (ks+k-1)} z^{ks+k-1} = c \cdot \sum_{\nu=0}^{k-2} (-1)^\nu \binom{k-2}{\nu} z^{k-\nu-2} / \sin \frac{(\nu+1)\pi}{k}$$

valid in the arc (ψ^{-1}, ψ) , while c is a constant yet to be determined.

To obtain c we set $z = 1$ to obtain

$$(6.14) \quad \sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1) \dots (ks+k-1)} = c \cdot \sum_{\nu=0}^{k-2} (-1)^\nu \binom{k-2}{\nu} / \sin \frac{(\nu+1)\pi}{k}$$

However, the left side is easily evaluated directly from the partial fraction decomposition

$$(6.15) \quad \frac{1}{(ks+1) \dots (ks+k-1)} = \frac{1}{(k-2)!} \left\{ \frac{1}{(ks+1)} - \binom{k-2}{1} \frac{1}{ks+2} + \dots + (-1)^{k-2} \frac{1}{ks+k-1} \right\}$$

From the classical expansion $\sum (-1)^s / (s+x) = \pi / \sin \pi x$ we obtain

$$\sum \frac{(-1)^s}{ks+\nu+1} = \frac{\pi}{k \sin \frac{(\nu+1)\pi}{k}}$$

If we multiply (6.15) by $(-1)^s$ and sum over all s we have

$$\sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1)\dots(ks+k-1)} = \frac{\pi}{(k-2)!k} \sum_0^{k-2} (-1)^v \binom{k-2}{v} / \sin \frac{(v+1)\pi}{k}.$$

Comparing with (6.14) we find that

$$(6.16) \quad c = \frac{\pi}{(k-2)!k}.$$

Using the relation (6.3) we can also determine the polynomial component of (6.1) within the arc $(1, \omega)$. Indeed

$$\psi^{-m} S_m(z\psi) = \sum_s \frac{(-1)^s}{(ks+1)\dots(ks+m)} z^{ks+m}$$

and differentiating (6.13) $k-m-1$ times we obtain that

$$(6.17) \quad \psi^{-m} S_m(z\psi) = \frac{\pi}{k(m-1)!} \sum_{v=0}^{m-1} (-1)^v \binom{m-1}{v} z^{m-v-1} / \sin \frac{(v+1)\pi}{k}$$

if z is confined to the arc (ψ^{-1}, ψ) .

As an example let us determine $K_*(z)$ using the formulae (3.14), (3.15) for $k=5$ and $m=2$. The rapidly convergent series (3.13) shows that $\lambda_* = .969690$, and (6.17) furnishes the expression

$$\psi^{-2} K_*(z\psi) = \frac{1}{2} z^2 - \lambda_* \frac{\pi}{5 \sin(\pi/5)} \left(z - \frac{1}{2 \cos \frac{\pi}{5}} \right)$$

in the arc (ψ^{-1}, ψ) . Graphing the image of the arc (ψ^{-1}, ψ) by the quadratic polynomial on the right side and turning it by $\psi^2 = \omega$ (or $+72^\circ$) we obtain the image of the arc $(1, \omega)$ by $w = K_*(z)$. The functional equation

$$K_*(z\omega) = \omega^2 K_*(z)$$

allows to complete the image of the entire circle U which is shown in Fig. 1.

The five corners of the curve are the images of the ω^j , Fig. 1 showing in parentheses the corresponding ω^j . Moreover

$$|K(\omega^j)| = .14063, \quad |K(\psi\omega^j)| = .10409.$$

A general discussion and determination of the monsplines $K(z)$ of least L_∞ - norm would be of interest.

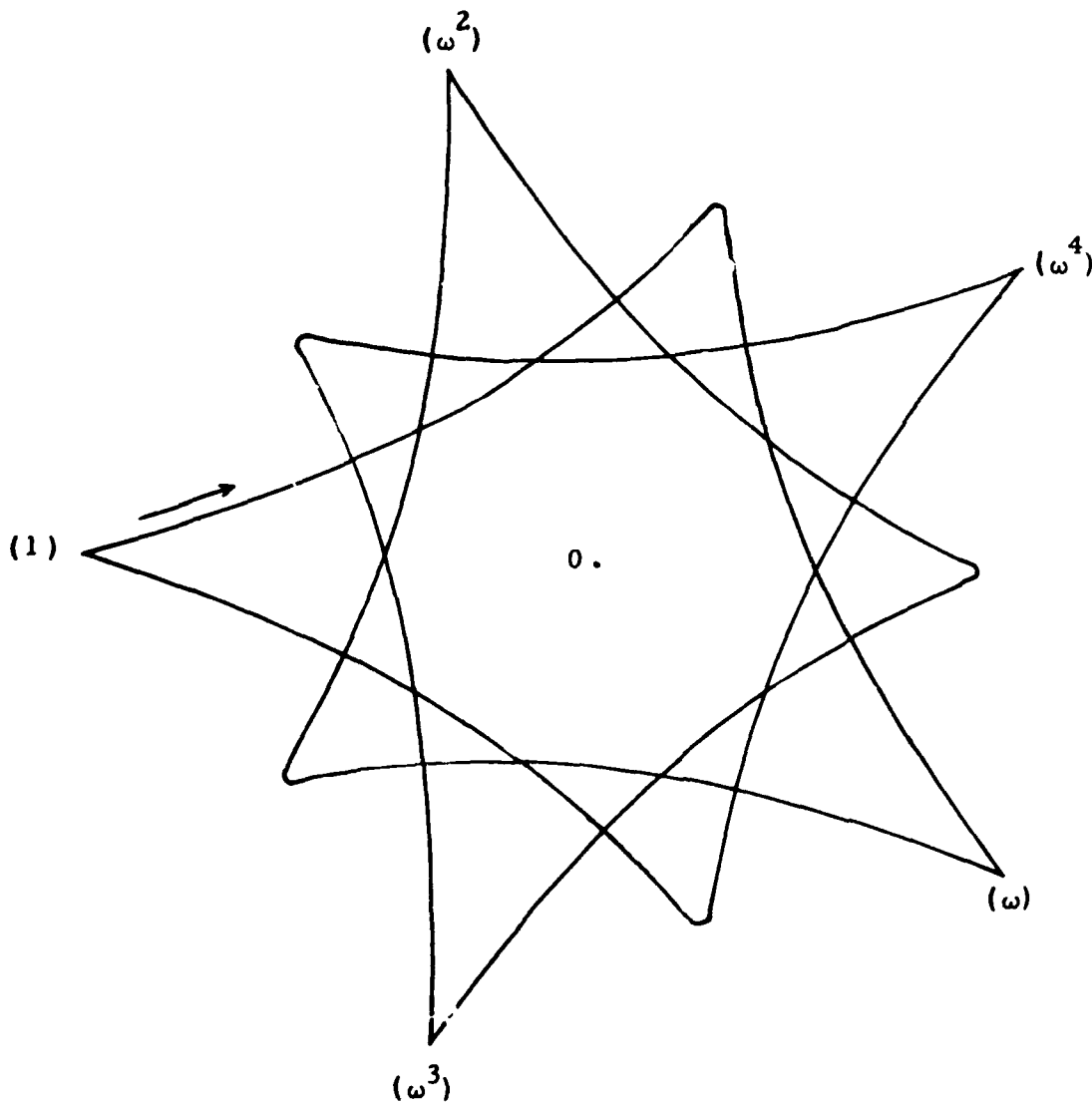


Fig. 1

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