THE APPLICATION OF AN ABSORBING MARKOV CHAIN IN PREDICTING LEARNING

A project

by

William Kanne, Jr.

Submitted in Partial Fulfillment of

I. En. 685

to

The Department of Industrial Engineering

Texas A&M University

May 1969

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) USAMC Intern Training Center - USALMC Red River Army Depot ATTN: AMXMC-ITC-E-M Texarkana, Texas 75501	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP N/A

3. REPORT TITLE
THE APPLICATION OF AN ABSORBING MARKOV CHAIN IN PREDICTING LEARNING.

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
N/A

5. AUTHOR(S) (First name, middle initial, last name)
William Kanne, Jr.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May, 1969	33	5

8a. CONTRACT OR GRANT NO. N/A b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) N/A 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) N/A

10. DISTRIBUTION STATEMENT
Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
N/A	Dir for Maint Hdqrs, USAMC Wash, D. C. 20315

13. ABSTRACT

A method is presented that utilizes the theory of Markov chains in predicting learning for manual activities.

The model is applied to a realistic example and the results compared with learning curve theory. The results illustrate that a Markov chain approach to learning can give a good approximation to a real life situation.

Recommendations are made for further applications of this model to actual situations, such as production lines. Areas of additional research are also discussed.

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT

ABSTRACT

A method is presented that utilizes the theory of Markov chains in predicting learning for manual activities.

The model is applied to a realistic example and the results compared with learning curve theory. The results illustrate that a Markov chain approach to learning can give a good approximation to a real life situation.

Recommendations are made for further applications of this model to actual situations, such as production lines. Areas of additional research are also discussed.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Learning curve models have been used as a means of forecasting productivity, or production time per item, for many years. The learning curve concept was originally applied by the aircraft industry during the second world war for a variety of purposes ranging from contract negotiation to production scheduling. (1), (2) Since then it has been used to estimate the production of many diversified products. (3), (4)

The concept of learning by repetition, and the corresponding decrease in task time, can be explained by the careful examination of a work task. When a task has been repeated a number of times, a worker becomes familiar with the tools and procedures used in performing the task. Usually a coordinated pattern is learned which reduces the total task time, resulting in greater efficiency.

One characteristic noticeable in almost all learning processes is the fact that initially there is a rapid decrease in the task time. After a number of repetitions the task time gradually levels to a steady state value.

Many attempts have been made to mathematically model a learning process. Almost all of these models assume some form of an exponentially decreasing function, which after a period of time is assumed to level to a constant value. This is illustrated in Figure 1.

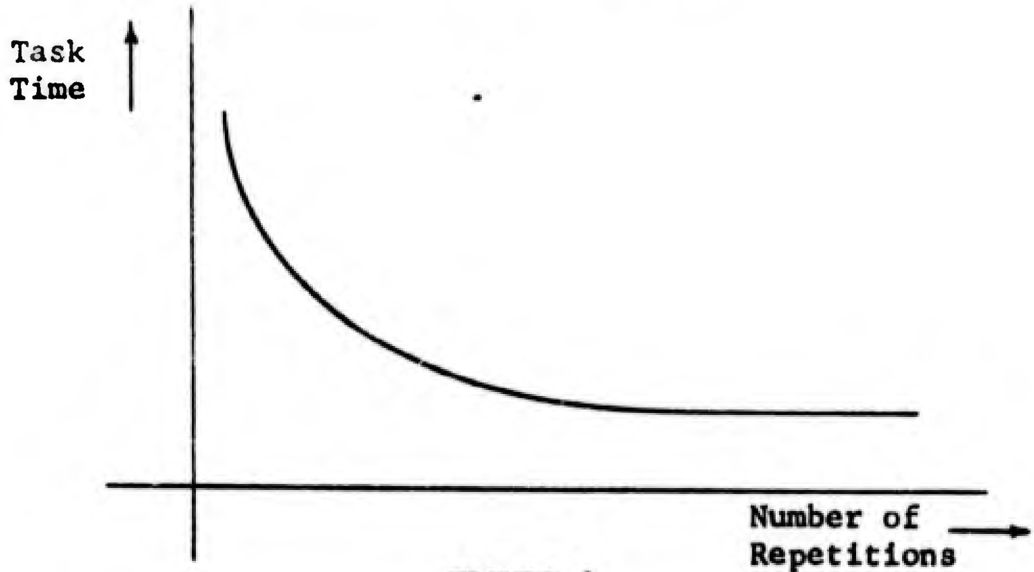


FIGURE 1

In most exponentially decreasing models, empirical data has shown that learning proceeds at a constant rate each time the number of repetitive tasks is doubled.

To illustrate this point, let T(1) be the task time for the first repetition, T(2) the time for the second, etc.. Then the experience factor, b, is represented by the ratio

$$(1) \qquad b = \frac{T(2)}{T(1)} = \frac{T(4)}{T(2)} = \frac{T(6)}{T(3)} \cdots = \frac{T(2X)}{T(X)} .$$

It will be shown that the experience factor is directly related to the slope of the learning curve.

The general form of the learning curve can be expressed as

$$(2) \qquad t(x) = t(1)x^{-N} \quad ,$$

where $t(x)$ is the time required for the x^{th} repetition, x is the number of repetitions, and N is the slope of the curve. Sometimes it is more convenient to write equation (2) in logarithmic form:

$$(3) \qquad \log t(x) = \log t(1) - N \log x \quad .$$

Equation (3) plots as a straight line on log-log paper as illustrated in Figure 2.

The relationship between the experience factor, b, and the slope, N, will now be determined. From Figure 2, the slope of the learning curve can be calculated.

$$(4) \qquad -N = \frac{\log t(2x) - \log t(x)}{\log 2x - \log x} \quad .$$

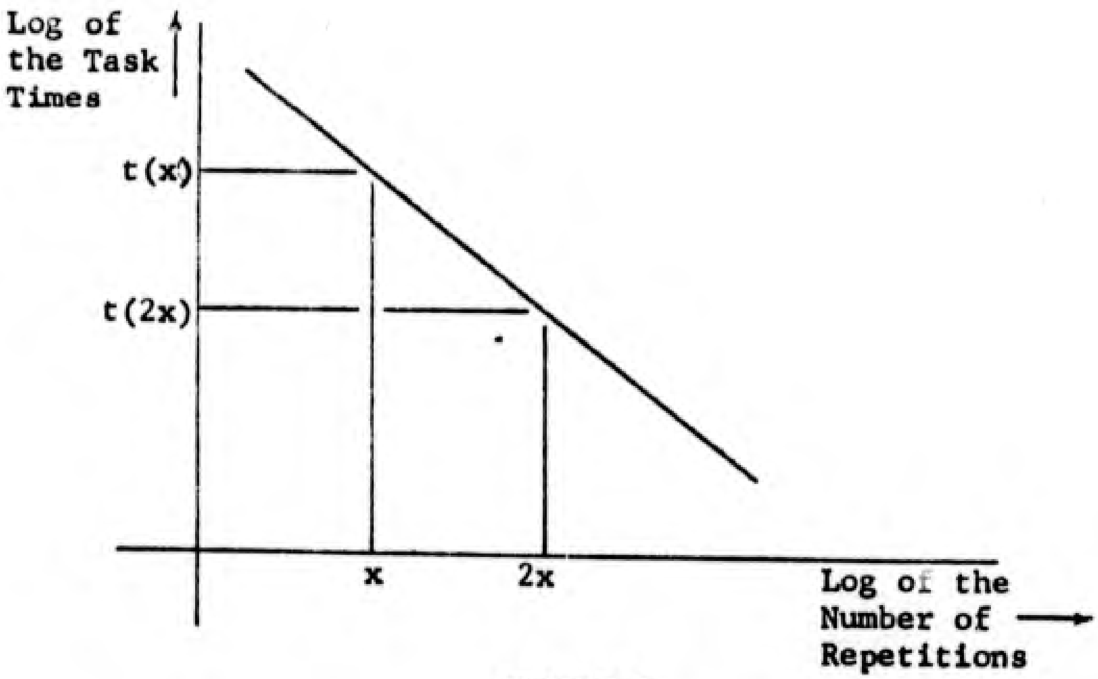


FIGURE 2

Rearranging equation (4);

$$(5) \qquad -N = \frac{\log \frac{t(2x)}{t(x)}}{\log \frac{2x}{x}} = \frac{\log \frac{t(2x)}{t(x)}}{\log 2} \; .$$

Substitute:

$$b = \frac{t(2x)}{t(x)} \qquad ,$$

$$(6) \qquad -N = \frac{\log b}{\log 2} \quad .$$

Substituting equation (6) into equation (2) gives the learning curve equation using the experience factor.

$$(7) \qquad t(x) = t(1)\ x^{\frac{\log b}{\log 2}} \quad .$$

Equation (7) will be used throughout this paper.

This paper attempts to mathematically model the learning curve process with the use of a Markov chain. The basic assumptions of an exponentially decreasing task time and the use of an experience factor are still retained. It will be possible with the model to evaluate the probability of having a specific task time, given the repetition number. This probability can be calculated knowing only the first few repetition times. The probability of learning will be determined and a transition matrix constructed. The purpose, then, of this paper is to provide an alternate method of modeling the learning curve process.

In Chapter 2 the thoery of Markov chains is presented, and a procedure is introduced for estimating the probability of learning in each state.

In Chapter 3 a specific example is presented and the results evaluated in Chaper 4.

Chapter 5 contains the concluding remarks and recommendations for further study. The two appendices contain the computer programs and results used in this model.

CHAPTER 2

THEORY

This section will develop the concept of a Markov process as used in this model. (5)

Consider a task which consists of a series of states which completely describe the task. The outcome at the n^{th} state is allowed to depend on the outcomes of the previous states. This type of relationship is usually represented by a probability tree. The path through the tree can be described by a series of states which ultimately gives an outcome. Let f_j be a function with a domain the set of all paths through the probability tree. The value of f_j will be the outcome at the j^{th} state. The functions f_1, f_2, f_3, ... f_n can then be called outcome functions. The set of functions f_1, f_2, f_3, ... f_n is called a stochastic process. A finite Markov process is a finite stochastic process such that the probability that the outcome of the n^{th} state given the output of the $(n - 1)^{th}$ state is independent of any state before the $(n - 1)^{th}$. In other words, for a Markov process, knowing the outcome of the last experiment one can neglect any other information about the past in predicting the future.

All the probabilities defining the movement from one state to another are contained in a transition matrix.

A finite Markov chain is a finite Markov process in which the transition probabilities are independent of n, n being the number of states in the process. The model developed here will be a finite Markov chain.

The Markov properties can be represented as follows:

(8) $$M_1 = M_0 P \quad ,$$

where M_0 is a row matrix defining the initial state probabilities, P is the transition matrix, and M_1 a row matrix defining the probabilities of being in any state after the first 'link' of the Markov chain.

It follows then that

(9) $$M_2 = M_1 P = (M_0 P) P = M_0 P^2 \quad .$$

In general

(10) $$M_n = M_0 P^n \quad .$$

To illustrate this property, consider the following powers of a transition matrix P.

P =

	s_1	s_2	s_3	s_4	s_5
s_1	0.2	0.8	0.0	0.0	0.0
s_2	0.0	0.2	0.8	0.0	0.0
s_3	0.0	0.0	0.2	0.8	0.0
s_4	0.0	0.0	0.0	0.2	0.8
s_5	0.0	0.0	0.0	0.0	1.0

P^2 =

	s_1	s_2	s_3	s_4	s_5
s_1	0.04	0.32	0.64	0.00	0.00
s_2	0.00	0.04	0.32	0.64	0.00
s_3	0.00	0.00	0.04	0.32	0.64
s_4	0.00	0.00	0.00	0.04	0.96
s_5	0.00	0.00	0.00	0.00	1.00

P^3 =

	s_1	s_2	s_3	s_4	s_5
s_1	0.008	0.096	0.384	0.512	0.000
s_2	0.000	0.008	0.096	0.384	0.512
s_3	0.000	0.000	0.008	0.096	0.896
s_4	0.000	0.000	0.000	0.008	0.992
s_5	0.000	0.000	0.000	0.000	1.000

Assume the process started in state s_2. Then M_0 is the matrix (0 1 0 0 0). It is desired to find the probability of being in state s_5 after three iterations, (links), of the Markov chain.

(11) $M_3 = M_0 P^3$

(12) $M_3 = (0.000 \quad 0.008 \quad 0.096 \quad 0.384 \quad 0.512)$

From equation (12), the probability of being in state s_5 is 0.512. Notice that M_3 is identical to the second row of P^3. Thus the n^{th} power of the transition matrix gives the probabilities of being in any state, given the starting state.

Markov chains can be classified as either ergodic or absorbing. In an ergodic Markov chain it is possible to go from any state to some other state in the chain. An absorbing Markov chain has the property that it has at least one state which, upon entering, can never be left. The chain will terminate in this absorbing state.

The model presented here will be an absorbing Markov chain because the steady state portion of the learning curve will be considered as an absorbing state.

The remaining portion of this chapter will be devoted to defining a relationship for the probability of learning.

Consider a task which is repeated many times on a certain piece of equipment. The decrease in the task time due to learning is to be studied. The time for the entire task, T_T, can be divided as follows:

(13) $$T_T = T_L + T_R \quad ,$$

where T_L is the learning time, and T_R is the unlearnable portion of the total task time. When the task is done for the first time T_L is large, but it decreases every time that learning takes place. Some portion of the task time will not decrease even after many repetitions. An example of this unlearnable portion might be the actual drilling of a hole in a metal plate. There are mechanical limitations on the speed of the drill boring through the metal, i.e., bit speed, available torque, etc.. Thus the drilling time would remain approximately constant for any number of repetitions. The actual identification of T_L and T_R is not necessary. This discussion is presented simply to identify the portion of the task time which changes as learning takes place.

A state, s_i, in this model will be defined as the interval of time for the i^{th} repetition. An example is shown in Figure 3. In this example, the process is in state s_1 during the first repetition, and is in state s_2 during the second repetition, etc.. The steady state region has been defined as the absorbing state. In this example it is state s_{21}.

The probability of learning will be defined as

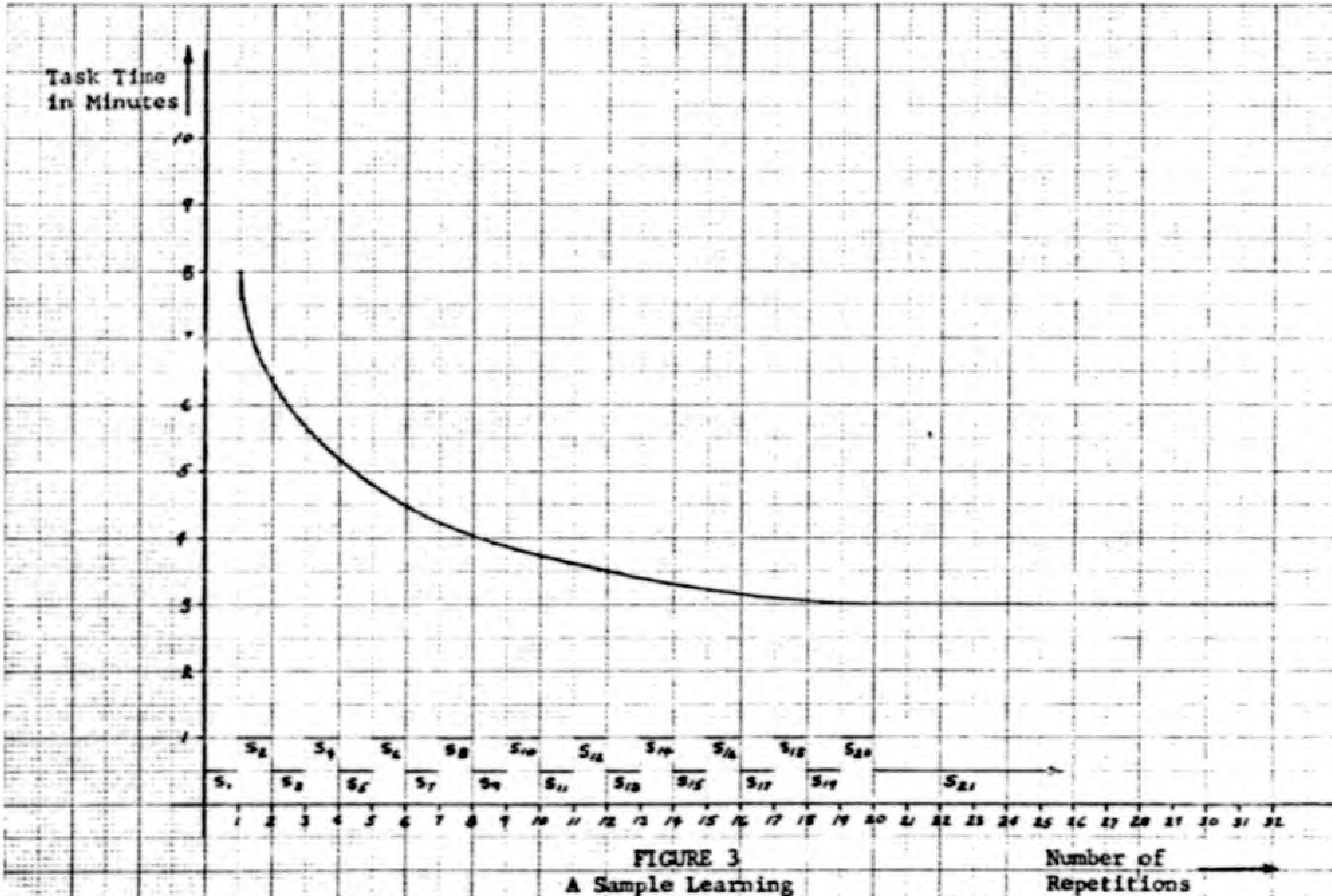


FIGURE 3
A Sample Learning
Curve

the probability that the time to perform a task for the n^{th} time is less than the time to perform the same task for the $(n - 1)^{th}$ time. This definition is good for all states in the Markov chain except the absorbing or steady state portion, where no learning is defined to exist.

The maximum probability of learning will occur during the first few repetitions of a task, because of the rapid decrease in task times in this portion of the curve. After more repetitions are made, the decrease in task time per repetition becomes smaller, and the probability of learning will also decrease.

Consider for a moment the employees performing the task. Each of them has different capabilities for learning because of differences in experience, coorindation, and motivation. Each will have different maximum probabilities of learning for this reason. For the average worker this maximum probability will be considered to range from 0.80 to 0.98. For assembly lines the maximum probability of learning will be the average of the employees working on the line.

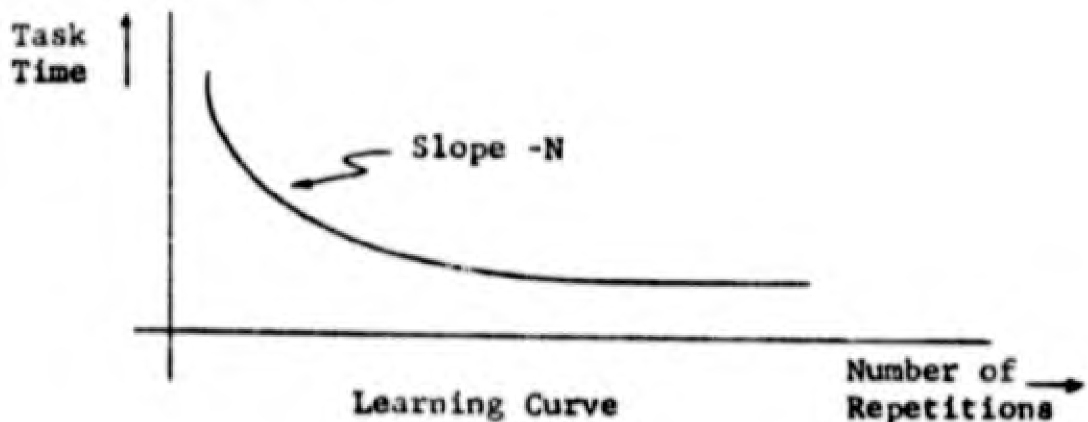
The transition matrix in this model will have one state for each repetition until the steady state portion of the learning curve, and one absorbing state for all units produced during the steady state section. In the example in Figure 3, the first twenty states correspond to the first twenty repetitions, and the twenty-first state is assumed to be the absorbing state. Each row of the transition matrix will have two probabilities, the sum of both probabilities being one.

The first of these probabilities will be the probability of performing the task and not learning from the repetition, (no decrease in task time). In the transition matrix this will be the probability of remaining in the same state during the next link of the Markov chain.

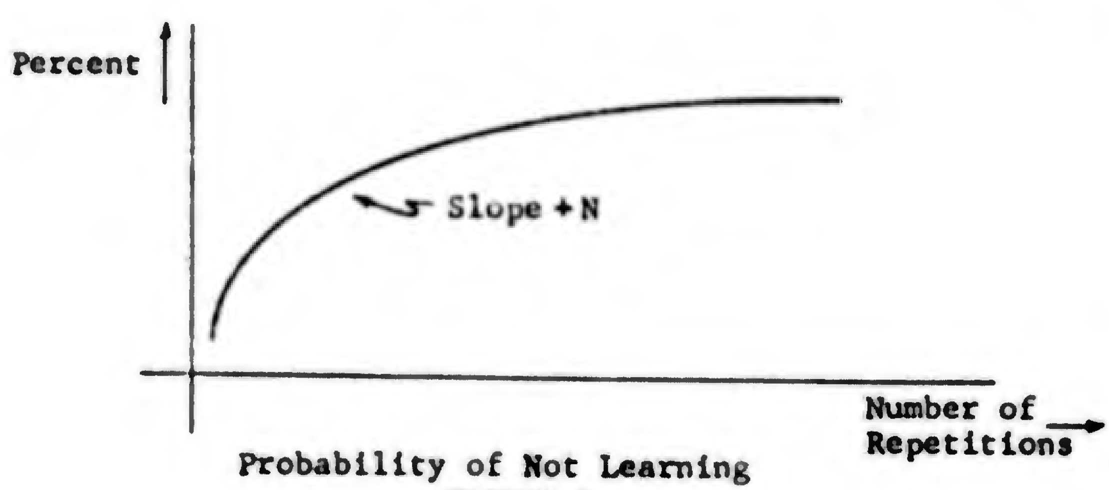
The second probability in each row of the transition matrix will be the probability of learning while performing a task, (a decrease in task time). This will be represented in the matrix by the probability of advancing a state in the Markov chain.

In this model the probability of not learning will be calculated, and the probability of learning will be determined knowing the sum of both is equal to one.

In general, it would seem reasonable that the probability of not learning would have characteristics just opposite that of the learning curve. In other words, the probability of not learning would start at a small value during the first few repetitions, and increase to a maximum value just before the steady state portionl For this model, the probability of not learning will be assumed to have the same general form as the learning curve equation, with an identical but positive slope. See Figure 4 and Figure 5.



Learning Curve
FIGURE 4



Probability of Not Learning
FIGURE 5

The probability of not learning for the x^{th} state, $P_{NL}(x)$, will have the form

(15) $$P_{NL}(x) = P_{NL}(1)\ x^N \quad ,$$

where

$$N = \frac{\log b}{\log 2} \quad .$$

$P_{NL}(1)$, the probability of not learning for the initial repetition, will be one minus the maximum probability of learning of the individual or group. Knowing $P_{NL}(1)$ and N, the probability of not learning for any state x can be calculated using equation (15).

The probability of learning for any state x, $P_L(x)$, is then equal to

(16) $$P_L(x) = 1 - P_{NL}(x) \quad .$$

CHAPTER 3

APPLICATION TO A SPECIFIC EXAMPLE

An example of this Markov chain model will now be presented.

Consider the following hypothetical situation. An industrial engineer is working for a large manufacturer who has contracted to perform modifications on a tank radio for the Army. He has set up a trial production line on which the modifications are to be done. A trial run of four radios is scheduled and run. Having only the modification times for the first four radios, the engineer now wants to estimate the probability that after twenty radios have been modified the process will stabilize. The final modification time per radio is also required.

A summary of the steps required are listed below:

1. Calculate as many experience factors as the data allows from equation (1), and find the average.
2. Calculate the slope of the learning curve from equation (6).
3. Calculate points on the learning curve using equation (2).
4. Plot the learning curve and determine where to

approxinate the curve with the steady state portion.

5. Assign the states one per repetition and an absorbing state for the steady state portion.

6. Consider the person or persons performing the task and assume a value for the maximum probability of learning.

7. Calculate the probability of not learning for each state from equation (15).

8. Construct a transition matrix, and find the powers of this matrix to evaluate the probabilites desired.

For this example, the first four modification times are as follows: T_1 = 83.0 minutes, T_2 = 70.5 minutes, T_3 = 64.1 minutes, and T_4 = 59.0 minutes. Knowing these initial task times the experience factor can be calculated.

$$b = \frac{t(2x)}{t(x)}$$

let x = 1

$$b = \frac{70.5}{83.0} = 0.850$$

let x = 2

$$b = \frac{59.0}{70.5} = 0.837$$

$$\bar{b} = \frac{0.850 + 0.837}{2} = 0.844$$

The learning curve for this experience factor is represented by:

$$t(x) = t(1)\ x^{-N} \quad , \quad \text{where } -N = \frac{\log b}{\log 2} \quad .$$

Now $$-N = \frac{\log 0.844}{\log 2.000} = -0.2448 \quad ,$$

and $t(x) = t(1)\ x^{-0.2448}$.

In this case $t(1) = 83.0$,

so $t(x) = (83.0)\ x^{-0.2448}$.

Using the computer program shown in Appendix I, and letting x = 1, 2, 3, ... 25, points on the curve were calculated and plotted in Figure 6. Note that state s_{21} is assumed to be the absorbing state, and all units after the twentieth will be contained in this state.

For this example the maximum probability of learning for the modification assembly line will be assumed to be 0.95.

So $P_{NL}(1) = 1.00 - 0.95 = 0.05$.

$$P_{NL}(x) = P_{NL}(1)\ x^{N} \quad , \quad N = 0.2448$$

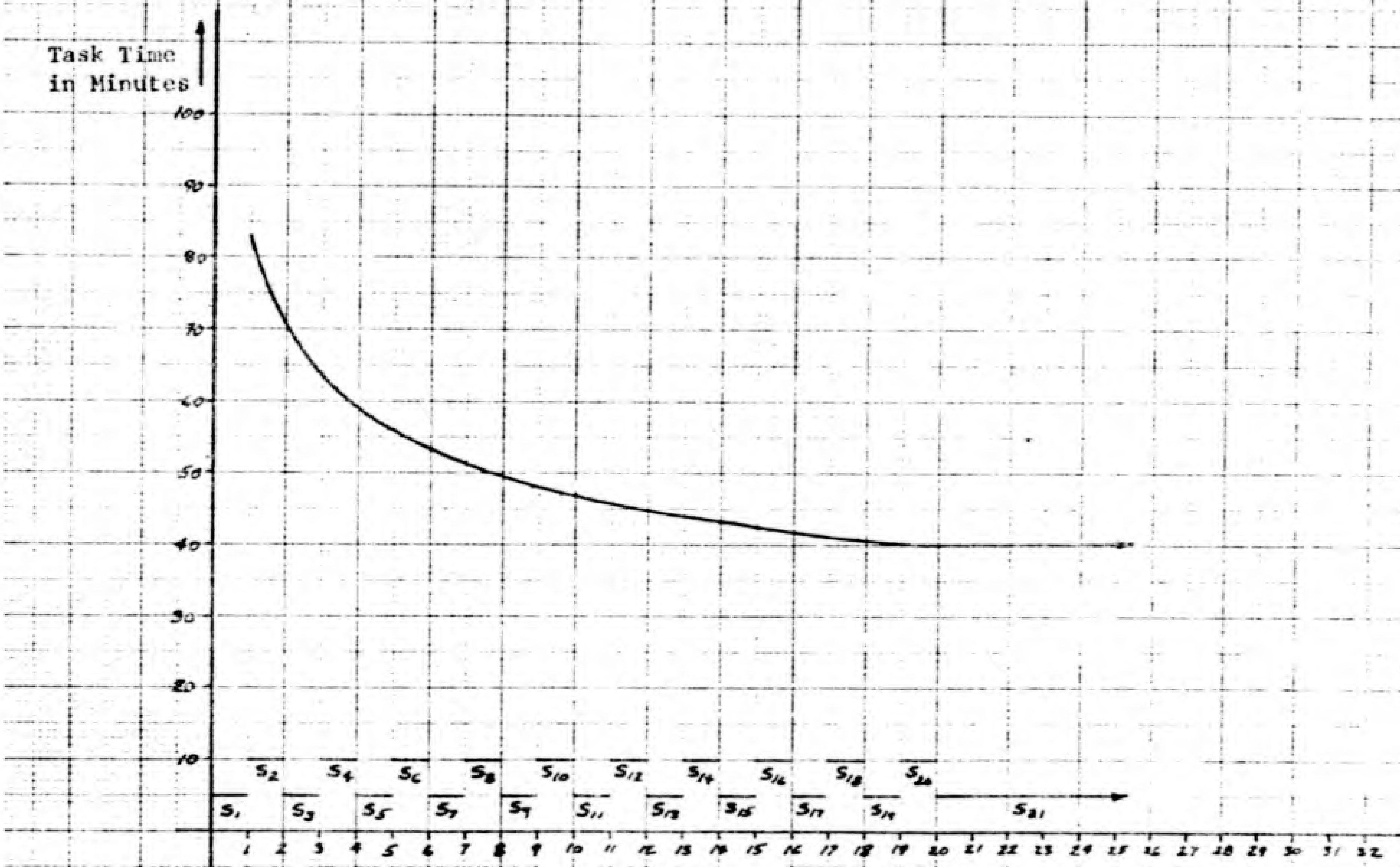


FIGURE 6
Learning Curve for the Modification
of Tank Radios

$$P_{NL}(x) = (0.05)\ x^{0.2448} \quad .$$

Values of the probability of not learning, $P_{NL}(x)$, letting $x = 1, 2, 3, \ldots. 25$, were obtained by the computer program in Appendix I. The values of $P_{NL}(x)$ are plotted in Figure 7.

The probability of learning, $P_L(x)$, for any state x can be calculated from

$$P_L(x) = 1.00 - P_{NL}(x) \quad .$$

The probability of learning in each state can now be determined except for the first state, and the final absorbing state. In state one, the probability of moving to state two is assumed to be one. State s_{21} is the absorbing state, so the probability of staying in this state is one.

All of the above information is summarized in Table I. The transition matrix, resulting from the probabilities in Table I, is contained in Table II.

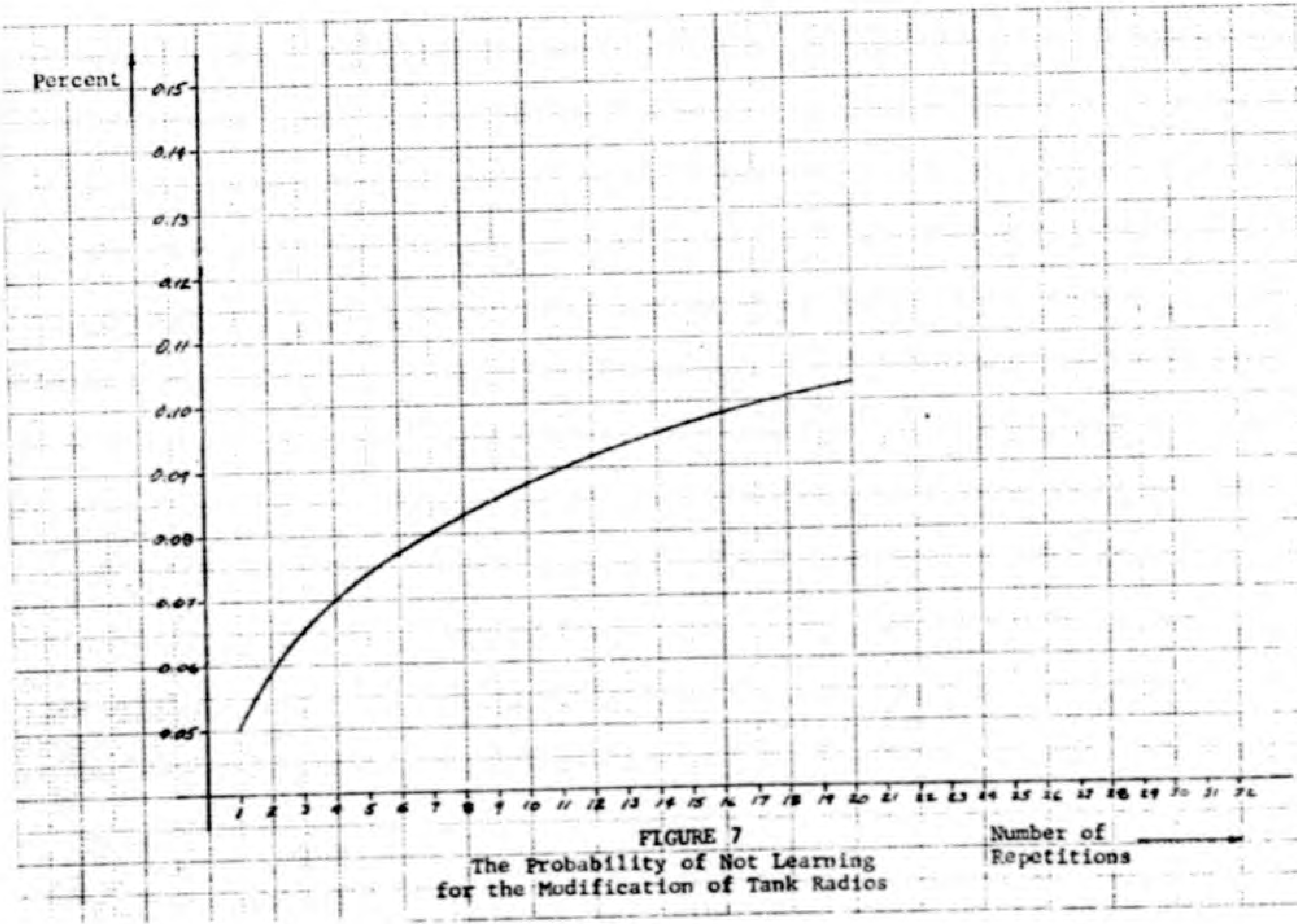


FIGURE 7
The Probability of Not Learning
for the Modification of Tank Radios

TABLE I

State	Task Time	P_{NL}	P_L
1	83.00	0.0000	1.0000
2	70.05	0.0500	0.9500
3	63.43	0.0592	0.9408
4	59.11	0.0654	0.9346
5	55.97	0.0702	0.9298
6	53.53	0.0741	0.9259
7	51.55	0.0775	0.9225
8	49.89	0.0805	0.9195
9	48.47	0.0831	0.9169
10	47.24	0.0856	0.9144
11	46.15	0.0878	0.9122
12	45.17	0.0899	0.9101
13	44.30	0.0918	0.9082
14	43.50	0.0936	0.9064
15	42.77	0.0953	0.9047
16	42.10	0.0970	0.9030
17	41.48	0.0985	0.9015
18	40.90	0.1000	0.9000
19	40.36	0.1014	0.8986
20	40.04	0.1028	0.8976
21	40.00	0.0000	1.0000

TABLE II

THE TRANSITION MATRIX, M

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}
s_1	0	1.0000	0	0	0	0	0	0	0	0	0
s_2	0	0.0500	0.9500	0	0	0	0	0	0	0	0
s_3	0	0	0.0592	0.9408	0	0	0	0	0	0	0
s_4	0	0	0	0.0654	0.9346	0	0	0	0	0	0
s_5	0	0	0	0	0.0702	0.9298	0	0	0	0	0
s_6	0	0	0	0	0	0.0741	0.9259	0	0	0	0
s_7	0	0	0	0	0	0	0.0775	0.9225	0	0	0
s_8	0	0	0	0	0	0	0	0.0805	0.9195	0	0
s_9	0	0	0	0	0	0	0	0	0.0831	0.9169	0
s_{10}	0	0	0	0	0	0	0	0	0	0.0856	0.9144
s_{11}	0	0	0	0	0	0	0	0	0	0	0.0878
s_{12}	0	0	0	0	0	0	0	0	0	0	0
s_{13}	0	0	0	0	0	0	0	0	0	0	0
s_{14}	0	0	0	0	0	0	0	0	0	0	0
s_{15}	0	0	0	0	0	0	0	0	0	0	0
s_{16}	0	0	0	0	0	0	0	0	0	0	0
s_{17}	0	0	0	0	0	0	0	0	0	0	0
s_{18}	0	0	0	0	0	0	0	0	0	0	0
s_{19}	0	0	0	0	0	0	0	0	0	0	0
s_{20}	0	0	0	0	0	0	0	0	0	0	0
s_{21}	0	0	0	0	0	0	0	0	0	0	0

TABLE II, cont.

THE TRANSITION MATRIX, M

	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}	s_{17}	s_{18}	s_{19}	s_{20}	s_{21}
s_1	0	0	0	0	0	0	0	0	0	0
s_2	0	0	0	0	0	0	0	0	0	0
s_3	0	0	0	0	0	0	0	0	0	0
s_4	0	0	0	0	0	0	0	0	0	0
s_5	0	0	0	0	0	0	0	0	0	0
s_6	0	0	0	0	0	0	0	0	0	0
s_7	0	0	0	0	0	0	0	0	0	0
s_8	0	0	0	0	0	0	0	0	0	0
s_9	0	0	0	0	0	0	0	0	0	0
s_{10}	0	0	0	0	0	0	0	0	0	0
s_{11}	0.9122	0	0	0	0	0	0	0	0	0
s_{12}	0.0899	0.9101	0	0	0	0	0	0	0	0
s_{13}	0	0.0918	0.9082	0	0	0	0	0	0	0
s_{14}	0	0	0.0936	0.9064	0	0	0	0	0	0
s_{15}	0	0	0	0.0953	0.9047	0	0	0	0	0
s_{16}	0	0	0	0	0.0970	0.9030	0	0	0	0
s_{17}	0	0	0	0	0	0.0985	0.9015	0	0	0
s_{18}	0	0	0	0	0	0	0.1000	0.9000	0	0
s_{19}	0	0	0	0	0	0	0	0.1014	0.8986	0
s_{20}	0	0	0	0	0	0	0	0	0.1028	0.8972
s_{21}	0	0	0	0	0	0	0	0	0	1.0000

Using the transition matrix in Table II, the probabilities associated with the Markov chain can be evaluated by taking powers of the transition matrix, M. The probabilities associated with the second link of the Markov chain are contained in M^2, and in general the probabilities associated with the n^{th} link of the Markov chain are contained in M^n.

A computer program to evaluate the powers of the transition matrix is shown in Appendix II. The second through the twenty-fifth power of the transition matrix have been calculated, and are shown in Appendix II.

The results are analyzed in the next chapter.

CHAPTER 4
ANALYSIS OF RESULTS

There are two basic probabilities which can be calculated from this Markov learning model.

A. Given any starting state s_i, find the probability of being in the steady state region, or absorbing state, after n repetitions.

B. Given any starting state s_i, find the probability of being in any other state after n repetitions.

Using the example problem developed in Chapter 3 and the results in Appendix II, these two probabilities will be illustrated.

The following Markov property, which was derived previously, will be used to evaluate all the probabilities in this model; the i^{th} row of the n^{th} power of the transition matrix gives the probability of being in each of the various states after n repetitions, assuming the process started in s_i.

To illustrate the first type of probability, the probability of being in the steady state after the 21^{st} unit is modified will be calculated. The desired information is on page 57, Appendix II, which contains the 21^{st}

power of the transition matrix. The desired probability will be located in the first row, (the process started in s_1), and in the 21^{st} column, (the absorbing state is s_{21}). In Appendix II, the probabilities are listed seven to a row, so it actually takes three rows to make one row of the transition matrix. Each row of the transition matrix is in brackets. Thus the probability of being in the absorbed state by the 21^{st} repetition is 0.4866.

By comparison, see Figure 6, Chapter 3, the graph of the learning curve for this example. This graph predicts that the 21^{st} unit and all which follow will be in the steady state region.

Table III shows the corresponding probabilities of being in the steady state, s_{21}, after the listed number of repetitions. These probabilities are from Appendix II under row one, column 21 of the corresponding power of the transition matrix.

From the table it can be seen that the Markov model is slighty more conservative in predicting the steady state region. Table III shows that it takes at least 24 repetitions for the probability of this event to become very large.

TABLE III

Number of Repetitions	Probability of Being in the Steady State Region
21	0.4866
22	0.7398
23	0.8897
24	0.9596
25	0.9869

An example of the second type of probability that can be evaluated using this model will be the following; estimate the probability that by the 10^{th} repetition the task time will have decreased at least 30 minutes. From Figure 6, Chapter 3, the initial task time is 83.0 minutes. A decrease of 30 minutes will be 53.0 minutes. From Figure 6, a task time of 53.0 minutes occurs in state s_7. We now wish to calculate the probability of being in state s_7 or greater after ten repetitions, given the starting state was s_1. The probabilities required are on page 46, Appendix II. Since the procedure was assumed to start in state s_1, the first row of the 10^{th} power of the transition matrix will contain the desired information. The probability of having at least a 30 minute decrease will be the probability of being in state s_7, plus the probability of being in any state higher than state s_7. The total probability, P_t, will be the following:

$$P_t = P(s_7) + P(s_8) + P(s_9) \quad \ldots + P(s_{21})$$

$$P_t = 0.001755 + 0.017762 + 0.107873 + 0.360521$$

$$+ 0.511987 + 0.000000 \quad \ldots + 0.000000$$

$$P_t = 0.99898$$

Thus the Markov model predicts that there is a probability of almost one of reducing the modification task time by at least 30 minutes by the tenth repetition.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

The Markov chain representation of the learning curve has been shown to display the same characteristics as the standard learning curve. This model has the advantage of developing probabilities of events, which can be used to make decisions. The basic assumption of the model, that the probability of not learning increases at the same rate the learning curve decreases, seems to be a fairly accurate assumption.

When attempting to model any real life situation, especially in the human factors area, it is impossible to mathematically describe actual outcomes. Human responses to the same stimuli vary over such a wide range that predicting the correct response a majority of the times can be considered an achievement.

The author believes that this Markov chain model will give a reasonably accurate probability of learning in a majority of cases to which it is applied. Thus it can be useful as a tool in predicting the effects of learning on new production lines.

To gain more confidence in the model, an actual task should be performed a number of times, and the results compared to the probabilities predicted with the model.

Another interesting aspect of this model would be to consider the existance of a probability of forgetting in each state. Each row in the transition matrix would then contain three probabilities. The first would be the probability of moving back a state, or the probability of forgetting. The second would be the probability of remaining in the same state, which would imply no learning. The third would be the probability of advancing a state which would imply learning.

Some relationship between the time elapsed since the last repetition, and the number of states the process should regress will have to be assumed or discovered. The general procedure, however, will be the same as the model presented.

LIST OF REFERENCES

(1) Andress, F., 'The Learning Curve as a Production Tool', Harvard Business Review, vol. 32, 1954. pp 87-97.

(2) Cole, R.R., 'Increasing Utilization of the Cost-Quantity Relationship in Manufacturing', Journal of Industrial Engineering, vol. 9, no. 3, May-June 1958.

(3) Kottler, J.J., 'Learning Curve-Case History', Journal of Industrial Engineering, July-August 1964. pp 176-180.

(4) Kneip, Jerre, 'The Maintenance Progress Function', Journal of Industrial Engineering, Nov.-Dec. 1965. pp 398-400.

(5) Kemeny, J.G., Snell, J.L., "Finite Markov Chains", Van Nostrand, Inc., 1964.