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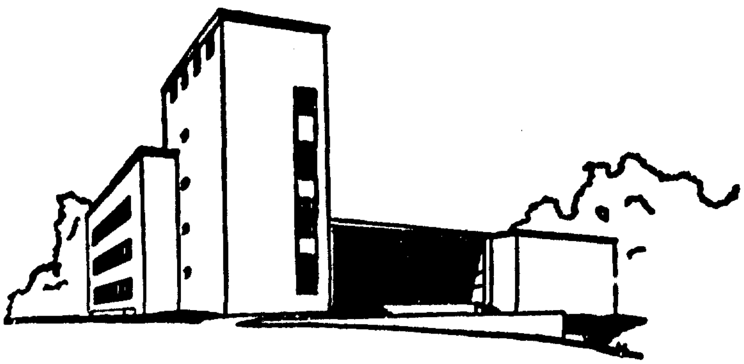

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AN ALGORITHM FOR ASSIGNING USES TO SOURCES IN A SPECIAL CLASS OF TRANSPORTATION PROBLEMS

V. Srinivasan*

and

G. L. Thompson **

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* The University of Rochester
** Carnegie-Mellon University

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13. ABSTRACT

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ABSTRACT

This paper considers a special class of transportation problems in which the needs of each user are to be supplied entirely by one of the available sources. We first show that an optimum solution to this special transportation problem is a basic feasible solution to a slightly different standard transportation problem. A branch and bound solution procedure for finding the desired solution to the latter is then presented and illustrated with an example. We then consider an extension of this problem by allowing the possibility of increasing (at a cost) the source capacities. The problem formulation is shown to provide a generalization to the well-known assignment problem. The solution procedure appears to be relatively more efficient when the number of uses greatly exceeds the number of sources.

1. INTRODUCTION

In a recent paper [4], DeMaio and Roveda consider a special class of transportation problems with a set of sources $I = \{1,2,\ldots,i,\ldots,W\}$ having known capacities b_i and a set of uses $J = \{1,2,\ldots,j,\ldots,M\}$ with known demands r_j for a homogeneous material (the b_i and r_j are assumed to be strictly positive). The objective is to minimize the total transportation cost Z subject to the constraints that (i) each user's demand is fulfilled by exactly one of the sources, and (ii) the total amount shipped from each source does not exceed its capacity. Denoting by c_{ij} the cost of transporting all the r_j units from the ith source to the jth use and defining x_{ij} to be 1 or 0 depending on whether or not use j is assigned to source i, the problem is to

$$\text{minimize} \quad Z = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \tag{1}$$

subject to the constraints:

$$\sum_{j \in J} r_j x_{ij} \leq b_i \quad \text{for} \quad i \in I, \tag{2}$$

$$\sum_{i \in I} x_{ij} = 1 \quad \text{for} \quad j \in J, \text{ and} \tag{3}$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for} \quad i \in I \text{ and } j \in J. \tag{4}$$

The authors of [4] present an implicit enumeration approach to solving this problem. In Section 2 we show that an optimal solution to this problem can be characterized as a basic feasible solution to a slightly modified transportation problem and that such a solution can be obtained by an algorithm similar to the subtour elimination method for solving traveling salesman problems [5, 7]. Since our algorithm utilizes the underlying structure of the transportation problem, it is believed to be computationally more efficient than the implicit enumeration approach. As will be seen in Section 2 the present algorithm appears to be particularly suitable when the number of uses far exceeds the number of sources. Furthermore, our approach can be easily extended to problems where capacity expansion for warehouses is a possibility. In Section 3 we consider this extension and provide other practical applications covered by this model.

2. THE ALGORITHM

To bring the problem (1)-(4) into the standard transportation format, we first make the transformations

$$y_{ij} = r_j x_{ij} \quad \text{for} \quad i \in I \quad \text{and} \quad j \in J, \text{ and} \tag{5}$$

$$d_{ij} = c_{ij}/r_j \quad \text{for} \quad i \in I \quad \text{and} \quad j \in J; \tag{6}$$

i.e., y_{ij} denotes the amount shipped from source i to use j at unit cost d_{ij}. To convert the inequalities (2) into equations, we adopt the usual procedure [3] of adding a slack use $M + 1$ and setting

$$J' = J \cup (M + 1), \tag{7}$$

$$c_{i,M+1} = 0 \quad \text{for} \quad i \in I, \quad \text{and} \tag{8}$$

$$r_{M+1} = \sum_{i \in I} b_i - \sum_{j \in J} r_j. \tag{9}$$

The problem (1)-(4) can then be verified to be equivalent to:

$$\text{minimize} \quad Z = \sum_{i \in I} \sum_{j \in J'} d_{ij} y_{ij} \tag{10}$$

subject to the constraints;

$$\sum_{j \in J'} y_{ij} = b_i \quad \text{for } i \in I, \tag{11}$$

$$\sum_{i \in I} y_{ij} = r_j \quad \text{for} \quad j \in J', \tag{12}$$

$$y_{ij} \geq 0 \quad \text{for} \quad i \in I \quad \text{and} \quad j \in J', \text{ and} \tag{13}$$

$$y_{ij} = 0 \text{ or } r_j \quad \text{for } i \in I \quad \text{and} \quad j \in J. \tag{14}$$

The problem (10)-(13) is a standard transportation problem and hence can be solved by the primal transportation algorithm (also known as the MODI method [3]). We assume that the reader is familiar with the usual terminology that a cell is an index pair (i,j) with row (source) i and column (use) $j \in J$; a basis B to the problem (10)-(13) is a collection of $(W + M)$ cells without cycles (loops or stepping-stone tours) and such that every row $i \in I$ and column $j \in J'$ has at least one cell. A solution $\{y_{ij}\}$ is basic if $y_{ij} = 0$ for $(i,j) \notin B$. A basic solution is feasible if the $\{y_{ij}\}$ satisfy the constraints (11)-(13). It is well known [3] that the MODI method yields a basic optimal solution (i.e., a basic feasible solution for which cost Z is minimal) to the problem (10)-(13).

DEFINITION 1. We define P to be the standard transportation problem (10)-(13) and P' to be the special transportation problem (10)-(14). We define a basis to be row-unique, if corresponding to every column $j \in J$, there is an unique row i_j such that $(i,j) \in B$ if and only if $i = i_j$.

By definition, B has $W + M$ cells. Since a row-unique basis has exactly one cell for each column $j \in J$, it follows that the $M + 1^{st}$ column has W cells; i.e., $(i,M+1) \in B$ for every $i \in I$. (Such a basis cannot have cycles since only the $M + 1^{st}$ column contains more than one cell.)

Theorem 1 below establishes the connection between the problems P and P'.

THEOREM 1. There is a one-to-one correspondence between feasible solutions to P' and row-unique basic feasible solutions to P.

PROOF. Consider any feasible solution $\{y_{ij}\}$ to P'. By (12)-(14) and from the assumption that $r_j > 0$, it follows that corresponding to every use $j \in J$ there is an unique source i_j such that $y_{ij} > 0$ if and only if $i = i_j$. Corresponding to this solution we define B to be the set of $W + M$

cells $\{(i_j,j) \text{ for } j\epsilon J \cup (i,M+1) \text{ for } i\epsilon I\}$. Consequently, the solution $\{y_{ij}\}$ is basic since $y_{ij} = 0$ for $(i,j)\notin B$. It is a feasible solution for P since $\{y_{ij}\}$ is feasible for P'. Since B is defined uniquely, this correspondence is unique.

To prove the converse, assume that we have a row-unique basic feasible solution $\{y_{ij}\}$ to P. By (12)-(13) and row-uniqueness it follows that corresponding to every column $j\epsilon J$ there is an unique row i_j such that $y_{ij} = r_j$ if $i = i_j$ and zero otherwise. Consequently (14) is satisfied and from (11)-(13) it follows that $\{y_{ij}\}$ is feasible to P' as well. Furthermore this correspondence is unique thus completing the proof.

By Theorem 1 and from the fact that the problems P and P' share a common objective function (10) it now follows that:

COROLLARY 1: There is a one-to-one correspondence between optimal solutions to P' and the optima among the row-unique basic solutions to P.

A solution procedure to the problem P' now easily follows somewhat along the lines of the subtour elimination algorithms for the traveling-salesman problem [5, 7].

This algorithm is basically a branch-and-bound procedure which begins by partitioning the set of row-unique basic feasible solutions and then calculating lower bounds on the costs of all solutions in a subset. The initial bound is found by solving the standard transportation problem P. If the basic optimal solution to P is row-unique then we are finished in the sense that we have an optimal solution to P' as well (Corollary 1). Suppose on the contrary that the basic optimal solution to P is not row-unique. Let us denote by j' one of the columns $j\epsilon J$ which has more than than one cell belonging to B and let (i',j') be one such basic cell.

(Though any such (i',j') can be chosen, we discuss below a heuristic for choosing a 'good' (i',j') from the point of view of computational efficiency.) We now branch into two subproblems (a) the subset in which (i',j') is a cell in the optimum row-unique basic-optimal solution; and (b) the subset in which (i',j') is not a cell in the optimum solution. The two new transportation problems corresponding to (a) and (b) are solved to determine the lower bounds for all row-unique basic optimal solutions in their respective subsets. If the optimal solution corresponding to any one subset is row-unique and the cost of this solution is less than or equal to the lower bounds on all other subsets then such a solution is optimal. If not, then one selects that subset having the smallest lower bound and branches again into two subproblems. Eventually one is assured of finding an optimum row-unique basic optimal solution and consequently an optimum to P' (by Corollary 1).

Several comments on the above algorithm are now in order. First, it is obvious that the procedure for branching on a non row-unique basis excludes that basis from the two subsets but does not exclude any row-unique basis. The algorithm converges in a finite number of steps since the total number of bases is finite and since at least one basis is excluded at every iteration. Second, the branching procedure results in a partition of the row-unique basic feasible solutions in that subset and hence the algorithm can be expected to be efficient. Third, for the subproblem with (i',j') constrained to be in the optimal solution, by row-uniqueness it follows that (i',j') is the only cell in column j'. Consequently, we can drop column j' from further consideration, modify $b_{i'}$ to $b_{i'} - r_{j'}$ and solve a smaller transportation problem. This reduction in $b_{i'}$ may further simplify the problem since the routes (i',j') for which r_j is greater than the new value of $b_{i'}$ cannot possibly be in the optimum solution (such cells can be eliminated by defining $d_{i'j'} = \infty$). Fourth, the optimal solutions to the subproblems can be efficiently obtained by the operator theory of parametric programming developed in [8, 9] rather than re-solving. Moreover, the backtracking steps of the branch and bound procedure may also be done this way. Finally, a non-row-unique basis can have at most W columns which have more than one cell (this follows from the fact that a basis has $W + N$ cells with at least one cell for each of the N columns). Consequently the fraction of the maximum number of columns which do not satisfy row-uniqueness is W/N. Thus the proposed algorithm can be expected to be relatively more efficient for problems where the number of uses greatly exceeds the number of sources.

We now consider the question of choosing the cell (i',j') upon which to make the computational procedure branch. Let us denote by J^* the set of columns that have two or more cells of the basis $(J^* \subset J)$. Given a column $j \in J^*$, we suggest branching on that basic cell (i,j) for which d_{ij} is the smallest. Then along the branch in which (i,j) is excluded from the optimal solution the cost can be expected to increase approximately by $\Delta_j = (d_{\ell j} - d_{ij}) y_{ij}$ where $d_{\ell j}$ is the next smallest cost of a basic cell in column j and y_{ij} is the amount shipped via the smallest cost basic cell (i,j). Consequently, for branching, we can choose the column $j' \in J^*$ for which Δ_j is the largest and branch on (i',j') where (i',j') has the lowest cost among all basic cells in column j'.

We summarize the above results in Algorithm 1 for solving the special transportation problem.

ALGORITHM 1. For finding an optimal solution to the special transportation problem (1)-(4).

1. Set up the problem P defined by (10)-(13). Let P_1 denote the problem P, $\Omega_1 = \emptyset$ denote the set of cells constrained to be included in the optimum solution and $\Psi_1 = \emptyset$ denote the of cells excluded from the optimum solution. Let Y_1 be the optimum solution to P_1 with basis B_1 and cost Z_1 (this may be obtained by the prima. (MODI) method). Let $S = \{1\}$ denote the set of problems under consideration and let $m = 1$ denote the total number of problems generated so far.

2. Choose the problem P_k for which Z_k is the smallest for $k\epsilon S$. If B_k is row-unique go to (5). Otherwise go to (3).

3. (a) Find the set of columns J^* for which the basis B_k has two or more basic cells in that column. For each column $j\epsilon J^*$ find the two basic cells (i,j) and (ℓ,j) for which the unit costs are the smallest and the second smallest respectively. Define $\Delta_j = (d_{\ell j}-d_{ij})y_{ij}$ and choose the $j'\epsilon J^*$ for which Δ_j is the largest. Select the lowest cost basic cell (i',j') in column j', for branching.

   (b) Define P_{m+1} as the problem obtained from P_k by constraining (i',j') to be an additional basic cell i.e., $\Omega_{m+1} = \Omega_k \cup \{(i',j')\}$ and let $\Psi_{m+1} = \Psi_k$. The problem P_{m+1} can be obtained from P_k by dropping column j' and defining $b_{i'}$ to be $b_{i'} - r_{j'}$. For columns j such that $r_j > b_{i'}$, define $d_{i'j} = \infty$.

   (c) Define P_{m+2} as the problem obtained from P_k by excluding (i',j') from the optimal basis. Set $\Psi_{m+2} = \Psi_k \cup \{(i',j')\}$ (i.e., $d_{i'j'} = \infty$) and let $\Omega_{m+2} = \Omega_k$.

   (d) Denote the basic optimal solutions to P_{m+1} and P_{m+2} (obtained by the MODI method) to be Y_{m+1} and Y_{m+2} with bases B_{m+1} and B_{m+2}. Define Z_{m+1} = optimal cost to $P_{m+1} + \sum_{(i,j)\epsilon\Omega_{m+1}} r_j d_{ij}$ and

$$Z_{m+2} = \text{optimal cost to } P_{m+2} + \sum_{(i,j)\epsilon\Omega_{m+2}} r_j d_{ij}.$$

4. Drop k from the set S and add $(m+1)$ and $(m+2)$ to S. Redefine m as $m+2$. Go to (2).

5. The optimal solution to the special transportation problem (1-4) is given by Y_k and Ω_k with the associated cost = optimal cost to $P_k + \sum_{(i,j)\in\Omega_k} r_j d_{ij}$. Stop.

We illustrate below the application of Algorithm 1 with the same example as was solved in [4]. Fig. 1a shows this problem with four sources S1,...,S4, five uses U1,...,U5, costs c_{ij}, capacities b_i and demands r_j as shown. Note that since $r_1 > b_4$ and $r_2 > b_4$, the cells (1,4) and (2,4) cannot possibly be in the optimum solution. Consequently $c_{14} = c_{24} = \infty$.

Figure 1a - 1d about here

At step (1) of Algorithm 1 we set up the transportation problem $P_1 = P$ as shown in Fig. 1b, by adding a dummy use U6 with demand $r_6 = \sum_{i=1}^{4} b_i - \sum_{j=1}^{5} r_j = 14 - 11 = 3$ (eqn. (9)) and defining costs d_{ij} as per equations (6) and (8). For the problem P_1 none of the cells are constrained to be included or excluded in the optimum solution so that $\Omega_1 = \Psi_1 = \emptyset$. The optimum solution to P_1 obtained by the primal method (the capacities b_i were perturbed slightly to prevent cycling [3]) is also shown in Fig. 1a where the circled cells denote the basic cells with the amounts y_{ij} written over the circles ($y_{ij} = 0$ for non-basic cells). The optimum value for the objective function can be verified to be $Z_1 = 19/3$. We now set $m = 1$ and $S = \{1\}$. In step (2) of the algorithm, we find that the basis of Fig. 1b is not row-unique so that we proceed to step (3).

In step 3(a) we find $J^* = \{1,2\}$ so that $\Delta_1 = (2/3 - 1/3) \times 1 = 1/3$ and $\Delta_2 = (1 - 1/3) \times 2 = 4/3$ so that $j' = 2$ and $(i',j') = (2,2)$. In step 3(b) problem P_2 is defined as problem P_1 with (2,2) constrained to be included in the basis (i.e., $\Omega_2 = \{(2,2)\}$ and $\Psi_2 = \emptyset$). Consequently, we drop use U2 and change b_2 to $b_2 - r_2 = 4 - 3 = 1$. Since r_1, r_3 and r_4 are greater than b_2, the cells (2,1), (2,3), (2,4) can not be in the optimal solution so that we set $d_{21} = d_{23} = d_{24} = \infty$ and obtain P_2. The optimal solution to P_2, shown in Fig. 1c., was obtained by the primal method. The optimal cost of the solution to P_2 can be verified from Fig. 1c to be 23/3 so that $Z_2 = (23/3) + \sum_{(i,j)\in\Omega_2} r_j d_{ij} = (23/3) + (3 \times 1/3) = 26/3$. Similarly P_3 is obtained from P_1 by excluding (2,2) from the optimal solution ($\Omega_3 = \emptyset$, $\Psi_3 = \{(2,2)\}$). Consequently d_{22} is set equal to ∞ in Fig. 1d. The optimal solution to P_3 is also shown in this figure with $Z_3 = 25/3$. The branching of P_1 to P_2 and P_3 on the basis of cell (2,2) can be seen in Fig. 2 as well. We now set $S = \{2,3\}$ and m=3 and return to step (2).

Since $Z_3 < Z_2$ and since B_3 is not row-unique we now branch the problem P_3 into two subsets. From Fig. 1d. $J^* = \{1,4\}$, $\Delta_1 = 2/3$ and $\Delta_4 = 1$ so that $j' = 4$ and $(i',j') = (3,4)$. In step 3(b) we define P_4 to be the same as P_3 but with (3,4) constrained to be included in the optimal solution (i.e., $\Omega_4 = \{(3,4)\}$ and $\Psi_4 = \{(2,2)\}$). Consequently we drop U_4 and change b_3 to $b_3 - r_4 = 3 - 2 = 1$. Since r_1, r_2, r_3 are greater than b_3 we make $d_{31} = d_{32} = d_{33} = \infty$ to obtain Fig. 1e. The optimal solution to P_4 is shown in Fig. 1e with cost 25/3 so that $Z_4 = 25/3 + (r_4 \times d_{34}) = 28/3$.

Figures 1e - 1h about here

Fig. 1f shows the problem P_5 obtained from P_3 by constraining (3,4) to be excluded from the optimal solution (i.e., $\Psi_5 = \{(2,2), (3,4)\}$). We mark $d_{34} = \infty$ and obtain the row-unique basic optimal solution of Fig. 1f. with $Z_5 = 9$. In step (4), S becomes $\{2,4,5\}$ and m = 5.

We now return to step (2) of the algorithm to find that Z_2 is the smallest among the problems in S so that we branch P_2 to P_6 and P_7 on the basis of cell (3,1) as shown in Figs. 1g and 1h. Now S becomes {4,5,6,7} so that Z_5 is the smallest cost. Since Y_5 is row-unique, the optimal solution to the special transportation problem is given by Fig. 1f with cost $Z_5 = 9$. This optimal solution assigns the uses U1, U2, U3, U4, U5 to sources S3, S1, S2, S4 and S1 respectively, the same solution as in [4]. Figure 2 shows the branch-and-bound tree at the end of the computation.

Figure 2 about here

From a computational point of view, it is not necessary to store the problems P_k for $k \epsilon S$. It is enough if we store the sets Ω_k and Ψ_k for $k \epsilon S$. To construct P_1 from the original problem P, we first set $d_{ij} = \infty$ for $(i,j) \epsilon \Psi$. Next, for every $(i,j) \epsilon \Omega_k$ we drop column j and modify b_i to $b_i - r_j$. Finally we eliminate those cells (i,j) with $j \epsilon J$ for which $r_j > b_i$ (by defining $d_{ij} = \infty$).

It is interesting to compare our algorithm to the implicit enumeration approach in [4]. The latter starts out with the solution of the total cost Z' obtained when each use j is assigned to that source i with the least cost c_{ij}. The feasibility condition (14) is satisfied at every step but not (11). On the other hand, our procedure starts with the least cost optimal solution to P of cost Z'' and maintains the feasibility condition (11) but not (14). Denoting by Z^* the optimal cost of the special transportation problem, the relative efficiency of the algorithms will vary across problems depending on whether Z' or Z'' is closer to Z^*. As mentioned earlier, for problems with a large M/N the infeasibility of (14) is relatively small so that our algorithm is better suited for such problems. On the other hand for problems for which the ratio M/W is small, the all zero-one algorithm of [4] can be expected to be more efficient.

Finally it should be pointed out that although this algorithm has been developed using the primal method as a subroutine for solving transportation problems, other methods such as the primal-dual methods could also be used.

3. EXTENSIONS AND APPLICATIONS

We first formulate an extension of the special transportation problem where the capacities b_i can be increased by unit cost g_i. Denoting by u_i the additional capacity of source i, equations (10) and (11) are modified to become (15)-(16) below:

$$Z = \sum_{i \in I} [g_i u_i + \sum_{j \in J'} d_{ij} y_{ij}] \quad \text{and} \tag{15}$$

$$\sum_{j \in J} y_{ij} = b_i + u_i \quad \text{for} \quad i \in I \tag{16}$$

Let us denote by h_i the maximum additional capacity that can be added to source i (if there is no such constraint, h_i can be set equal to a very large number). As a further generalization let p_i denote the unit cost of not utilizing the capacity of source i (if this involves a unit saving then p_i would be negative) and let q_i' (≥ 0) denote the minimum utilization level for source i. Defining $q_i = b_i - q_i'$ the following relations hold for the slack use $M + 1$:

$$d_{i,M+1} = p_i \quad \text{for} \quad i \in I, \quad \text{and} \tag{17}$$

$$y_{i,M+1} \leq q_i \quad \text{for} \quad i \in I. \tag{18}$$

The additional capacities u_i can be thought of as a surplus use $(M + 2)$. We now define

$$J'' = J' \cup \{(M+2)\}, \text{ and} \tag{19}$$

$$y_{i,M+2} = h_i - u_i \quad \text{for } i \in I. \tag{20}$$

Furthermore, since $0 \leq u_i \leq h_i$, we have

$$0 \leq y_{i,M+2} \leq h_i \quad \text{for } i \in I. \tag{21}$$

The objective function (15) becomes

$$Z = Z_0 + \sum_{i \in I} \sum_{j \in J''} d_{ij} y_{ij} \tag{22}$$

where $$Z_0 = \sum_{i \in I} g_i h_i \quad \text{and} \tag{23}$$

$$d_{i,M+2} = -g_i \quad \text{for } i \in I. \tag{24}$$

The constraints (16) become

$$\sum_{j \in J''} y_{ij} = b_i + h_i \quad \text{for } i \in I. \tag{25}$$

The constraints (12) hold as usual for $j \in J$. But for the dummy uses (M+1), (12) should be modified to

$$\sum_{i \in I} y_{i,M+1} = \sum_{i \in I} (b_i + u_i) - \sum_{j \in J} r_j$$

so that from (20) we obtain

$$\sum_{i \in I} y_{i,M+1} + \sum_{i \in I} y_{i,M+2} = \sum_{i \in I} (b_i + h_i) - \sum_{j \in J} r_j \tag{26}$$

The constraint (26) is not a regular transportation constraint since it involves variables from two columns. To bring it to standard transportation form we define

$$y_{W+1,M+1} = N_1 - \sum_{i \in I} y_{i,M+1} \quad \text{and} \tag{27}$$

$$y_{W+1,M+2} = N_2 - \sum_{i \in I} y_{i,M+2} \tag{28}$$

where N_1 and N_2 are large positive numbers so that $y_{W+1,M+1}$ and $y_{W+1,M+2}$ are nonnegative. Consequently (26) becomes

$$y_{W+1,M+1} + y_{W+1,M+2} = \sum_{j \in J} r_j - \sum_{i \in I} (b_i + h_i) + N_1 + N_2 \qquad (29)$$

Consequently if we define $y_{W+1,j} = 0$ for $j \in J$ (by setting $d_{W+1,j} = \infty$ for $j \in J$), (29) becomes

$$\sum_{j \in J''} y_{W+1,j} = \sum_{j \in J} r_j - \sum_{i \in I} (b_i + h_i) + N_1 + N_2 \qquad (30)$$

Finally, by defining

$$I' = I \cup \{(W+i)\} \qquad (31)$$

the constraints (27)-(28) can be rewritten as

$$\sum_{i \in I'} y_{i,M+1} = N_1 \quad \text{and} \qquad (32)$$

$$\sum_{i \in I'} y_{i,M+2} = N_2 \,. \qquad (33)$$

Figures 3 summarizes the capacitated (or upper bounded) transportation formulation of this problem. The special transportation problem has the additional constraint (14) that each use $j \in J$ has to be supplied by only one (possibly different) source $i \in I$.

An algorithm for this generalized problem should be obvious. We can utilize the same branch and bound procedure of Section 2 with the capacitated transportation formulation of Figure 3. However, the implicit enumeration algorithm of [4] is not capable of such an easy extension (although DeMaio and Roveda [4] in their concluding discussion suggest the problem generalization considered here).

Though the special transportation model concerns itself with sources and uses typically considered as warehouses and markets, we wish to point out that it offers an important generalization to assignment models. (For other interesting and important assignment problem generalizations see the paper [2] by Charnes, Cooper, Niehaus and Stedry.) Consider, for instance, assigning jobs to machines in a case when it may be prohibitive to do the same job on more than one machine (perhaps because of set-up cost considerations). Denoting by r_j the time required to perform the job j, b_i the time available on machine i and c_{ij} the cost of performing job j on machine i, we obtain the special transportation problem (1)-(4). Similarly, this model can also be utilized in assigning workers to supervisors (or students to advisors) where r_j is the time needed to supervise the j-th worker. These applications suggest a further extension of problem (1)-(4) where (2) is replaced by

$$\sum_{j\in J} r_{ij} x_{ij} \leq b_i \quad \text{for} \quad i\in I \ , \tag{34}$$

i.e., r_{ij} is not necessarily constant for all $i\in I$; in other words job j might be done with differing efficiencies by each of the machines. The branch and bound procedure of Section 2 would then have to be applied to a generalized transportation problem [1,6] with column demands equal to unity.

4. CONCLUSIONS

In this paper we have considered a special class of transportation problems of assigning uses to sources and provided a branch and bound solution procedure with the standard transportation problem as a subroutine. Compared to the implicit enumeration approach in [4] this algorithm appears to be computationally more efficient particularly for problems where the number of uses greatly exceeds the number of sources.

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	U1	U2	U3	U4	U5	b
S1	2	3	4	7	1	5
S2	4	1	1	8	8	4
S3	1	7	11	1	6	3
S4	∞	∞	10	3	5	2
r	3	3	2	2	1	

Fig. 1a

Original Problem

	U1	U2	U3	U4	U5	U6	b
S1	2/3 ^{2}	1 ^{1}	2	7/2	1 ^{1}	0 ^{1}	5
S2	4/3	1/3 ^{2}	1/2 ^{2}	4	8	0	4
S3	1/3 ^{1}	7/3	11/2	1/2 ^{2}	6	0	3
S4	∞	∞	5	3/2	5	0 ^{2}	2
r	3	3	2	2	1	3	

Fig. 1b

Problem $P_1 = P$

$\Omega_1 = \emptyset$, $\Psi_1 = \emptyset$, $Z_1 = 19/3$

	U1	U3	U4	U5	U6	b
S1	2/3 ^{2}	2 ^{2}	7/2	1 ^{1}	0 ^{0}	5
S2	∞	∞	∞	8	0 ^{1}	1
S3	1/3 ^{1}	11/2	1/2 ^{2}	6	0	3
S4	∞	5	3/2	5	0 ^{2}	2
r	3	2	2	1	3	

Fig. 1c

Problem P_2

$\Omega_2 = \{(2,2)\}$, $\Psi_2 = \emptyset$, $Z_2 = 26/3$

	U1	U2	U3	U4	U5	U6	b
S1	2/3 ^{1}	1 ^{3}	2	7/2	1 ^{1}	0	5
S2	4/3	∞	1/2 ^{2}	4	8	0 ^{2}	4
S3	1/3 ^{2}	7/3	11/2	1/2 ^{1}	6	0	3
S4	∞	∞	5	3/2 ^{1}	5	0 ^{1}	2
r	3	3	2	2	1	3	

Fig. 1d

Problem P_3

$\Omega_3 = \emptyset$, $\Psi_3 = \{(2,2)\}$, $Z_3 = 25/3$

Fig. 1a - 1d

Transportation Tableaus for the Example

	U1	U2	U3	U5	U6	b
S1	$2/3^{1}$	1^{3}	2	1^{1}	0	5
S2	$4/3^{2}$	∞	$1/2^{2}$	8	0^{0}	4
S3	∞	∞	∞	6	0^{1}	1
S4	∞	∞	5	5	0^{2}	2
r	3	3	2	1	3	

Fig. 1e

Problem P_4

$\Omega_4 = \{(3,4)\}$, $\Psi_4 = \{(2,2)\}$, $Z_4 = 28/3$

	U1	U2	U3	U4	U5	U6	b
S1	2/3	1^{3}	2	7/2	1^{1}	0^{1}	5
S2	4/3	∞	$1/2^{2}$	4	8	0^{2}	4
S3	$1/3^{3}$	7/3	11/2	∞	6	0^{0}	3
S4	∞	∞	5	$3/2^{2}$	5	0^{0}	2
r	3	3	2	2	1	3	

Fig. 1f

Problem P_5

$\Omega_5 = \emptyset$, $\Psi_5 = \{(2,2),(3,4)\}$, $Z_5 = 9$

	U3	U4	U5	U6	b
S1	2^{2}	7/2	1^{1}	0^{2}	5
S2	∞	∞	8	0^{1}	1
S3	∞	∞	∞	0^{0}	0
S4	5	$3/2^{2}$	5	0^{0}	2
r	2	2	1	3	

Fig. 1g

Problem P_6

$\Omega_6 = \{(2,2),(3,1)\}$, $\Psi_6 = \emptyset$, $Z_6 = 10$

	U1	U3	U4	U5	U6	b
S1	$2/3^{3}$	2^{1}	7/2	1^{1}	0	5
S2	∞	∞	∞	8	0^{1}	1
S3	∞	11/2	$1/2^{2}$	6	0^{1}	3
S4	∞	5^{1}	3/2	5	0^{1}	2
r	3	2	2	1	3	

Fig. 1h

Problem P_7

$\Omega_7 = \{(2,2)\}$, $\Psi_7 = \{(3,1)\}$, $Z_7 = 12$

Fig. 1e - 1h

Transportation Tableaus fo. the Example.

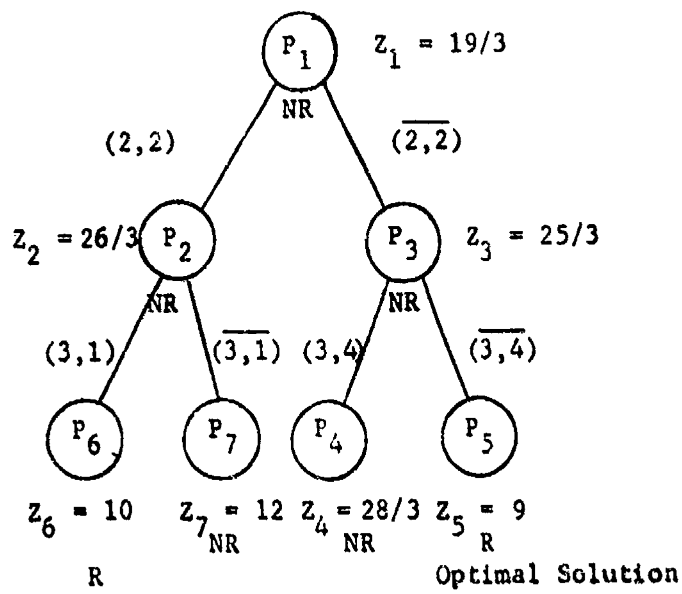

Fig. 2

Branch and Bound Tree Diagram at Optimum

Note 1. R = Row-unique optimal basis; NR = Non row-unique optimal basis

Note 2. The label (2,2) connecting P_1 and P_2 indicates that P_2 is obtained from P_1 by constraining its optimum solution to include the cell (2,2). Similarly P_5 is obtained from P_3 by excluding the cell (3,4) from its optimal solution (denoted by $(\overline{3,4})$).

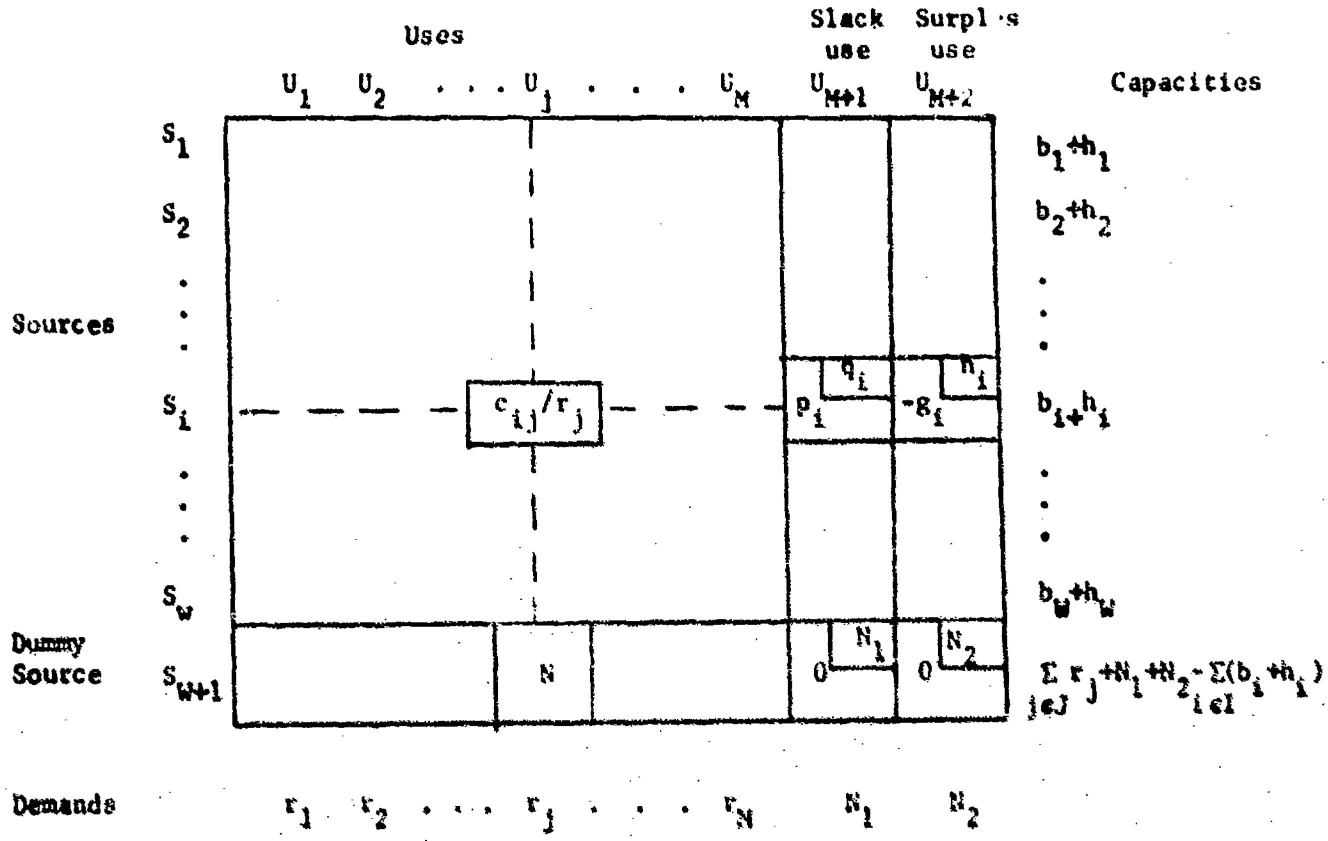

Fig. 3

Transportation Format for the Generalized Problem

Note: 1. In each cell the number at the center denotes the unit cost d_{ij}. The number at the opper righthand corner denotes an upper bound for the cell (if this is blank this implies that there is no upper bound).

2. N, N_1, N_2 denote very large positive numbers.