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### **A NOTE ON ADF TESTS FOR SUBHYPOTHESES IN MULTIPLE LINEAR REGRESSION**

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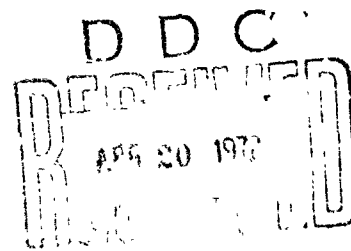
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**A NOTE ON ADF TESTS FOR SUBHYPOTHESES  
IN MULTIPLE LINEAR REGRESSION**

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## FOREWORD

This is an interim report of the work done under Contract F33615-71-C-1927 by the University of North Carolina. The work done by P. K. Sen in this report is sponsored by the Aerospace Research Laboratories under the above contract; it is accomplished on Project 7071 "Research in Applied Mathematics" and is technically monitored by P. R. Krishnaiah. The work of M. L. Puri is supported by the National Science Foundation under Grant NSF-GP-27715.

## ABSTRACT

The purpose of this note is to show that asymptotically distribution-free (ADF) rank tests proposed by Koul [Ann. Math. Statist. 41(1970), 1273-1281] for testing subhypotheses in multiple linear regression when based on symmetric scores are asymptotically totally inefficient and can even be inconsistent for symmetric parent distributions. Also, the condition given in his Theorem 2.4 for a test to be ADF is not necessary.

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1. Introduction. For every  $n (\geq 1)$ , let  $\{ Y_{n1}, \dots, Y_{nn} \}$  be a sequence of independent random variables, such that

$$(1.1) \quad F_{ni}(x) = P\{ Y_{ni} \leq x \} = F(x - \beta_1 x_{ni}(1) - \beta_2 x_{ni}(2)), \quad i = 1, \dots, n;$$

$x_{ni}(j)$ ,  $i = 1, \dots, n$ ,  $j = 1, 2$  are known constants, and  $\beta_1, \beta_2$  are unknown parameters. Our problem is to test

$$(1.2) \quad H_0: \beta_1 = 0 \quad \text{vs.} \quad H_1: \beta_1 \neq 0 \quad (\text{or } < \text{ or } > 0),$$

where  $\beta_2$  is treated as a nuisance parameter.

Koul(1970) considered an arbitrary estimator  $\hat{\beta}_n$  of  $\beta_2$ , such that  $|n^{\frac{1}{2}}(\hat{\beta}_n - \beta_2)| = O_p(1)$ , and considered the following procedure. Let

$$(1.3) \quad Y_{ni}(b) = Y_{ni} - b x_{ni}(2), \quad \hat{Y}_{ni} = Y_{ni}(\hat{\beta}_n), \quad i = 1, \dots, n,$$

and let  $R_{ni}(b)$  (or  $\hat{R}_{ni}$ ) be the rank of  $Y_{ni}(b)$  (or  $\hat{Y}_{ni}$ ) among the  $n$  values in

$$(1.3). \quad \text{Let } \bar{x}_n(j) = n^{-1} \sum_{i=1}^n x_{ni}(j), \quad j = 1, 2, \text{ and } \dots$$

$$(1.4) \quad S_n^{(1)}(b) = n^{-1} \sum_{i=1}^n (x_{ni}(1) - \bar{x}_n(1)) \phi(R_{ni}(b)/(n+1)), \quad \hat{S}_n^{(1)} = S_n^{(1)}(\hat{\beta}_n),$$

where  $\phi(u)$  is absolutely continuous and bounded inside  $[0,1]$ . He showed that under certain conditions,  $\hat{S}_n^{(1)}$  is ADF under  $H_0$  in (1.2), and hence, a test for  $H_0$  can be based on  $\hat{S}_n^{(1)}$ . He showed that if  $F$  is double exponential and  $\psi(u)$  is Freund-Ansary or Mood type (which are both symmetric), then the corresponding  $\hat{S}_n^{(1)}$  is ADF. To this result, we add the following.

Theorem 1. For the entire class of bounded, absolutely continuous and symmetric scores ( i.e., for  $\phi(u) = \phi(1-u)$ ,  $u \in [0,1]$ ), Koul's test based on  $S_n^{(1)}$  is ADF, but is totally asymptotically inefficient ( in the sense that for local alternatives, the power is equal to the level of significance) when the underlying  $F$  is symmetric.

Theorem 2. Whenever  $\phi(u) = \phi(1-u)$ ,  $u \in [0,1]$ , Koul's ADF tests can also be inconsistent for a fixed  $\beta_1 \neq 0$ , no matter whether  $F$  is symmetric or not.

Let us now define

$$(1.5) \quad S_n^{(2)}(b) = n^{-1} \sum_{i=1}^n (x_{ni}(2) - \bar{x}_n(2)) \phi(R_{ni}(b)/(n+1)),$$

so that when  $\phi(u)$  is non-decreasing,  $S_n^{(2)}(b)$  is non-increasing in  $b: -\infty < b < \infty$ .

Let then

$$(1.6) \quad \hat{\beta}_{n,1} = \sup\{ b: S_n^{(2)}(b) > 0 \}, \quad \hat{\beta}_{n,2} = \inf\{ b: S_n^{(2)}(b) < 0 \};$$

$$(1.7) \quad \hat{\beta}_n = [ \hat{\beta}_{n,1} + \hat{\beta}_{n,2} ]/2.$$

Finally, let

$$(1.8) \quad \phi(u, f) = -f'(F^{-1}(u))/f(F^{-1}(u)), \quad u \in [0, 1],$$

$$(1.9) \quad b_n(\phi, f) = [n^{-1} \sum_{i=1}^n (x_{ni}(1) - \bar{x}_n(1))(x_{ni}(2) - \bar{x}_n(2))] b(\phi, f);$$

$$(1.10) \quad b(\phi, f) = \int_0^1 \phi(u, f) \phi(u) du = -\int_{-\infty}^{\infty} f'(x) \phi(F(x)) dx.$$

In his theorem 2.4, Koul (1970) has shown that a sufficient condition for  $\hat{S}_n^{(1)}$  to be ADF is that  $b_n(\phi, f) = 0$ . In this context, we have the following:

**Theorem 3.** Whenever  $\phi(u)$  is monotonic in  $u \in [0, 1]$ , and  $\hat{\beta}_n$  is defined by (1.6)-(1.7),  $\hat{S}_n^{(1)}$  is ADF, no matter whether  $b_n(\phi, f)$  is equal to 0 or not.

The proofs of the theorems are sketched in section 2. It clearly follows from the first two theorems that symmetric scores (such as Mood's where  $\phi(u) = (u - 1/2)^2$ ) are unsuitable for this problem, while from the last theorem, it follows that for monotonic scores (such as Wilcoxon's where  $\phi(u) = u - 1/2$ ), Koul's condition that  $b_n(\phi, f) = 0$  can easily be avoided when the estimator  $\hat{\beta}_n$  is based on the same scores.

2. Outline of the proofs. (i) Theorem 1.  $\phi$  symmetric implies that  $\phi(u) = \phi(1-u)$ ,  $u \in [0, 1]$ , while  $F$  symmetric implies that  $\phi(u, f) = -\phi(1-u, f)$ ,  $u \in [0, 1]$ . Hence, by (1.10),

$$(2.1) \quad b(\phi, f) = 0 \quad \Rightarrow \quad b_n(\phi, f) = 0.$$

Consider then the sequence of local alternatives  $\{H_n\}$ , where

$$(2.2) \quad H_n: \beta_1 = \beta_1^{(n)} = n^{-1/2}\theta, \quad \theta \neq 0.$$

Now, by Theorem 2.3 of Koul (1970), under (2.1),

$$(2.3) \quad n^{1/2} \hat{S}_n^{(1)} = n^{1/2} S_n^{(1)}(\beta_2) + o_p(1), \text{ as } n \rightarrow \infty,$$

where  $S_n^{(1)}(\beta_2)$  is defined by (1.4) for  $b = \beta_2$ . Since by (1.1) and (1.3),  $Y_{ni}(\beta_2)$  has the distribution  $F(x - \beta_1 x_{ni}(1))$ , for  $i = 1, \dots, n$ , by the well-known results of Hájek(1962), under (2.2),

$$(2.4) \quad \mathcal{L}(n^{1/2} S_n^{(1)}(\beta_2) - \theta b_n(\phi, f) \mid H_n) \rightarrow \mathcal{N}(0, A^2),$$

where

$$(2.5) \quad A^2 = \int_0^1 \phi^2(u) du - \left( \int_0^1 \phi(u) du \right)^2.$$

As shown in (2.1), under the hypothesis of theorem 1,  $b_n(\phi, f) = 0$ , and hence, by (2.3) and (2.4), we conclude that for all finite  $\theta$ , the asymptotic power of the test based on  $\hat{S}_n^{(1)}$  remains equal to  $\alpha$ , the level of significance. That is, the test is ADF but totally inefficient. Q.E.D.

(ii) Theorem 2. Let  $\tau_\alpha$  be the upper 100 $\alpha$ % point of the standard normal distribution.

Then, we are to show that

$$(2.6) \quad P\{ |n^{1/2} \hat{S}_n^{(1)}| > A\tau_{\alpha/2} \mid \beta_1 \neq 0 \}$$

does not necessarily converge to 1 as  $n \rightarrow \infty$ . This we show by means of the following example.

Let  $n = 2m$ ,  $x_{n1}(1) = \dots = x_{nm}(1) = -1$ ,  $x_{nm+1}(1) = \dots = x_{nn}(1) = 1$ ,  $\beta_1 = 2 + \epsilon$ ,  $\epsilon > 0$ , and let  $F(x)$  be such that  $F(-1) = 0$ ,  $F(1) = 1$ , [the later can always be made by choosing first an absolutely continuous df  $G$  with finite Fisher information, and letting then  $F(x) = [G(x) - G(-1)]/[G(1) - G(-1)]$  for  $x \in [-1, 1]$ , and  $F(x) = 0$  for  $x \leq -1$ ,  $F(x) = 1$ , for  $x \geq 1$ .] Then, proceeding as in Koul(1970), we have

$$(2.7) \quad \max_{1 \leq i \leq n} |[Y_{ni} - \hat{\beta}_n x_{ni}(2)] - [Y_{ni} - \beta_2 x_{ni}(2)]| = o_p(1), \text{ as } n \rightarrow \infty,$$

and hence, for every  $\epsilon > 0$ , there exists an  $n_\epsilon$ , such that for  $n \geq n_\epsilon$ , the right hand side of (2.7) is less than  $\epsilon$ , with probability approaching to unity, as  $n \rightarrow \infty$ .

Since the variation of  $F$  is contained entirely within the range  $(-1, 1)$  and  $\beta_1 =$

$2 + \epsilon$ ,  $\epsilon > 0$ , it follows that for  $n \geq n_\epsilon$ ,

$$(2.8) \quad Y_{ni} - \hat{\beta}_n x_{ni}(2) < Y_{nj} - \hat{\beta}_n x_{nj}(2), \quad i=1, \dots, m, \quad j=m+1, \dots, n,$$

with a probability arbitrarily close to unity as  $n \rightarrow \infty$ . Hence,

$$(2.9) \quad P\{ |\hat{S}_n^{(1)} - n^{-1} \sum_{i=1}^m [\phi((i+m)/2m) - \phi(i/2m)]| = 0 \} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Since  $\phi(u) = \phi(1-u)$  for all  $u \in [0,1]$ , by (2.9),

$$(2.10) \quad n^{1/2} \hat{S}_n^{(1)} = 0 \text{ with probability tending to 1 as } n \rightarrow \infty,$$

which shows that (2.6) does not converge to 1 as  $n \rightarrow \infty$ , so the test is inconsistent.

The same proof holds for  $\beta_1 < -2$  or for  $|\beta_1| > 2$ . Q.E.D.

(iii) Theorem 3. Let  $C_{jj'}^{(n)} = n^{-1} \sum_{i=1}^n (x_{ni}(j) - \bar{x}_n(j))(x_{ni}(j') - \bar{x}_n(j'))$ , for  $j, j' = 1, 2$ . Then, by Theorem 2.3 of Koul (1970), as  $n \rightarrow \infty$ ,

$$(2.11) \quad n^{1/2} [\hat{S}_n^{(1)} - S_n^{(1)}(\beta_2)] = -n^{1/2} (\hat{\beta}_n - \beta_2) b_n(\phi, f) + o_p(1).$$

On the other hand, under (1.2),  $\beta_1 = 0$ , so that when  $\hat{\beta}_n$  is defined by (1.6) and (1.7), by the fundamental results of Jurečková (1969), as  $n \rightarrow \infty$ ,

$$(2.12) \quad n^{1/2} | S_n^{(2)}(\beta_2) - (\hat{\beta}_n - \beta_2) b_n(\phi, f) | = o_p(1).$$

Consequently, by (2.11) and (2.12), under (1.2), as  $n \rightarrow \infty$ ,

$$(2.13) \quad n^{1/2} \hat{S}_n^{(1)} = n^{1/2} [ S_n^{(1)}(\beta_2) - S_n^{(2)}(\beta_2) ] + o_p(1).$$

Finally, under (1.2),  $n^{1/2} [ S_n^{(1)}(\beta_2), S_n^{(2)}(\beta_2) ]$  jointly has asymptotically a bivariate normal distribution with mean  $(0,0)$  and covariance matrix  $A^2 \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ , where  $C_{jj'} = \lim_{n \rightarrow \infty} C_{jj'}^{(n)}$ ,  $j, j' = 1, 2$ . Hence, by (2.13), under  $H_0$  in (1.2),

$$(2.14) \quad \mathcal{L}_n(n^{1/2} \hat{S}_n^{(1)}) \rightarrow \mathcal{N}(0, A^2(C_{11} + C_{22} - 2C_{12})), \text{ as } n \rightarrow \infty,$$

and as  $A^2$  as well as the  $C_{jj'}$  are known quantities, the result follows. Q.E.D.

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