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GENERAL VARIATIONAL METHODS FOR WAVES IN ELASTIC COMPOSITES

by

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GENERAL VARIATIONAL METHODS FOR WAVES IN ELASTIC COMPOSITES[1]

by

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ABSTRACT

General variational theorems in which the displacement, the stress, and the strain in one case, and the displacement and the stress in another case, are given independent variations, and which include appropriate general boundary and discontinuity conditions, are developed with a view toward the application to harmonic waves in elastic composites with periodic structures. The one-dimensional case is first developed in detail, and in order to demonstrate the effectiveness of the results, especially their accuracy in providing the dispersion curve, waves propagating normal to layers in a layered composite are discussed, and numerical results are presented; see Tables I and II. Then the general three-dimensional case is considered, and the results are applied to waves propagating normal to the fibers in a fiber-reinforced composite.

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[1] This work was partly completed while the author was at the University of California, San Diego, La Jolla, California, as a consultant to Grant [redacted]-AFOSR 70-1957, sponsored by the [redacted] United States Air Force, Washington, D. C.

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I. INTRODUCTION

Because of their dispersive and other desirable effects, composite materials have become now an important ingredient in many aerospace structures. The propagation of elastic waves in such materials has, therefore, been discussed by a number of writers using different theories. In [1, 2], for example, a two-term Taylor expansion of the displacement field within each layer in a layered composite is considered together with a certain smoothing process, to arrive at an approximate set of equations which resemble those for a homogeneous continuum with microstructure. Bedford and Stern [3], on the other hand, directly consider a mixture theory to characterize the composite, and proceed to calculate the coupling coefficients in the corresponding constitutive relations, by considering simple static elasticity problems; these authors confine their analysis to a special case where the interacting body forces are assumed to be proportional to the relative displacement of the constituents. In a more recent paper, Hegemier and Nayfeh [4] have used an asymptotic approach and, for waves propagating normal to the layers in a layered composite, have derived systematically mixture-type field equations directly, which, for the harmonic wave, gives the exact dispersion relation, and which can be used (as these authors do) to study the transient waves in such composites.

For harmonic waves in a composite with a periodic structure, one may employ a variational approach. This can particularly become a very effective tool, if one uses a variational statement in which not only the displacement, but also the stress field is given independent variation. Moreover, by permitting discontinuity in the displacement and the stress test functions, one can expect a more accurate reproduction of the local variation in the displacement and the stress fields within and across the constituent materials. An example of such a calculation can be found in a thesis by Wheeler [5] and in a recent article by Kohn, et al. [6], where the classical theorem of stationary potential energy which leads to the well-known Rayleigh quotient for the eigenfrequency, is used. Although the authors in [6] incorporate jump conditions in their variational statement by the addition of certain terms to the classical functional, they do not use this modified form and hence it is not shown how effective it is. Moreover, since the additional terms mentioned above are arrived at by trial and error, their physical significance is not immediately obvious.

In this paper we shall develop general variational statements for harmonic waves in composites, in which the displacement, the stress, and the strain in one case, and the displacement and the stress in another, are given arbitrary variation. In addition we shall develop all the discontinuity and the quasi-periodicity conditions in a straightforward and systematic manner. Of course, we shall base our developments on variational theorems which have been explored by a number of investigators in the mechanics literature, beginning with the work of Hellinger (1914) [7], the unpublished thesis by Prange (1916) [8], contributions by Reissner (1950-53) [9, 10], works by Hu (1954) [11], Washizu (1955) [12], and a large number of other researchers; see the textbook by Washizu [13] for further discussion; see also [14]. In particular, regarding the jump conditions we shall generalize the work by Prager (1967) [15] to include the variation of weighted averaged tractions and displacements on a discontinuity boundary. For composites such a modification proves quite useful, since the elastic constants of the two constituents may differ substantially from each other. Finally, to demonstrate the effectiveness of our results we discuss in some detail, elastic waves propagating normal to the layers in a layered composite. Our numerical results are then compared with those reported in [1, 5, 6] and the exact solution, which comparison reveals the superiority of the variational statements developed herein; see Tables I and II. In fact, accurate results as those presented here, have not been obtained before by any of the methods mentioned above; except, of course, [4] where the exact dispersion relation is obtained. In addition, a second example is worked out, which illustrates the significance of the proposed variational statement that includes a weighted averaged traction on interior discontinuity surfaces. Here, with a suitable choice of the weighting parameter, accurate results are obtained; see Table III.

2. ONE-DIMENSIONAL CASE

In order to stress the essentials we begin by considering an elastic medium whose properties vary periodically in the direction of the propagation of elastic waves, i.e., the x-direction. Let a be the periodicity-length. Then we have $\rho(x+a) = \rho(x)$, and $\eta(x+a) = \eta(x)$, where ρ is the mass-density, and η stands for $\lambda + 2\mu$ when dilatational waves are considered, and for μ when shear waves are considered, λ and μ being the Lamé coefficients.

For harmonic waves with frequency ω, all the field quantities are proportional to $e^{\pm i\omega t}$, where $i = \sqrt{-1}$ and t measures time, and therefore we have

$$\frac{d\sigma}{dx} + \rho\omega^2 u = 0 \quad , \qquad \sigma = \eta\epsilon \quad , \qquad \epsilon = \frac{du}{dx} \quad , \tag{2.1}$$

where σ is the stress, u the displacement, and ϵ the strain.

For a periodic medium Eq. (2.1) has periodic coefficients, and according to the well-known Floquet theory, see for example [16], it admits a solution with the property

$$g(x+a) = g(x)\, e^{iqa} \quad , \tag{2.2}$$

where g stands for any of the dependent variables σ, ϵ or u, and q is the overall wave number.

We wish to establish general variational statements which can be employed together with some appropriate test functions which may or may not be continuous throughout a representative cell, in order to obtain approximate solutions to system (2.1). Since our objective is to apply these variational statements to composite materials whose properties vary substantially from one constituent to the next, we may wish to use different test functions to represent, for example, the displacement in the region occupied by each constituent material. Therefore, within a cell, the test displacement function may suffer a finite discontinuity across the interface between the two materials. Similar remarks apply to the stress field. Hence a general variational method must account for such possible discontinuities.

Let us choose the origin of the coordinate such that $-\frac{a}{2} \leq x \leq \frac{a}{2}$ defines a complete cell. In this region let $x = x_o$ be a point of discontinuity for a test function. If the material to the left is completely disconnected from the material to the right of the discontinuity point, then, since we are dealing with a one-dimensional problem, we may prescribe arbitrarily one boundary condition at $x = x_o^-$ and one boundary condition at $x = x_o^+$, a total of no more than two conditions; $x_o^{\pm} = \lim_{\alpha \to 0} (x_o \pm \alpha)$, $\alpha > 0$. Hence, in general, one may prescribe arbitrarily jumps in the value of two field quntities at $x = x_o$; in the three-dimensional case this number increases to six.

In addition to the jump conditions we must account for the quasi-periodicity condition (2.2) which can be written as

$$u\left(\frac{a}{2}\right) = u\left(-\frac{a}{2}\right) e^{iqa} \quad , \quad \sigma\left(\frac{a}{2}\right) = \sigma\left(-\frac{a}{2}\right) e^{iqa} \quad . \qquad (2.3)$$

As we shall see below, these conditions can be accounted for by either the use of a Lagrangian multiplier, or by simply calculating the work of the tractions at the two boundary points of the cell.

General Variational Statement [7-15]

The most general variational statement for real-valued field quantities with discontinuities, in which the stress, the strain, and the displacement fields are varied independently, may be stated as follows.

$$I = \int_{-a/2}^{a/2} \left[\frac{1}{2}\eta\epsilon^2 - \rho f u - \sigma\left(\epsilon - \frac{du}{dx}\right)\right] dx$$

$$- \left\{Tu, \quad \text{or} \quad \Lambda(u - U)\right\}_{-a/2}^{a/2} - \left\{\bar{\sigma}\langle u\rangle\right\}_{x=x_o} \quad , \qquad (2.4)$$

where f is the body force, T is the prescribed value of traction at the end points, Λ is the Lagrangian multiplier, and U is the prescribed value of the displacement at the end points; at each end-point we can either prescribe the displacement or the stress, but not both. Additionally in (2.4) we have used the notation

$$\bar{\sigma} = \alpha\sigma^{(2)} + (1 - \alpha)\sigma^{(1)} \quad , \quad \langle u\rangle = u^{(2)} - u^{(1)} \quad , \qquad (2.5)$$

where at point x_o, $g^{(2)} = g(x_o^+)$, and $g^{(1)} = g(x_o^-)$, g standing for either σ or u; in (2.5), α is an arbitrary real constant, a weighting parameter.

whose significance will be discussed below.[1] Taking the first variation of (2.4) we obtain

$$\delta I = \int_{-a/2}^{a/2} \left\{ \left[\eta\epsilon - \sigma \right] \delta\epsilon - \left[\epsilon - \frac{du}{dx} \right] \delta\sigma - \left[\frac{d\sigma}{dx} + \rho f \right] \delta u \right\} dx$$

$$+ \left\{ \left[\sigma - T \right] \delta u \quad , \quad \text{or} \quad \left[\sigma - \Lambda \right] \delta \upsilon - \left[u - U \right] \delta \Lambda \right\}_{-a/2}^{a/2}$$

$$+ \left\{ \langle \sigma \rangle \, \delta \bar{u} \, - \, \langle u \rangle \, \delta \bar{\sigma} \right\}_{x = x_o} \quad , \tag{2.6}$$

where

$$\bar{u} = (1 - \alpha) \, u^{(2)} + \alpha u^{(1)} \; . \tag{2.7}$$

From (2.6) we observe that the vanishing of the right-hand side for arbitrary variation of strain, stress, and displacement in the region $-\frac{a}{2} \le x \le \frac{a}{2}$, yields respectively, Hooke's law, the definition of strain, and the momentum equation. From the second term on the right-hand side of (2.6), moreover, we obtain the stress boundary data or the displacement boundary data, depending on whether the traction or the displacement is prescribed at a boundary point; in the latter case the Lagrangian multiplier Λ is the traction corresponding to the prescribed displacement. The last term in the right-hand side of (2.6) corresponds to the jump conditions at the discontinuity point $x = x_o$. At this point one can prescribe arbitrarily the variation of no more than two field quantities. To arrive at a more general case we have implemented the weighting parameter α whose value can be selected arbitrarily. Observe that, for arbitrary variation

[1] When two materials, M^{β}, $\beta = 1,2$, meet at point x_o, the superscript is identified with the corresponding material; i.e., α is assigned to one of the constituents.

of[1] $\bar{\sigma}$, the last term in (2.6) guarantees the continuity of the displacement field across the interface $x = x_0$, whereas, for arbitrary variation of $\bar{u}$ the stress is required to be continuous. For $\alpha = \frac{1}{2}$, the last term in (2.6) becomes

$$\frac{1}{2}(\delta u^{(2)} + \delta u^{(1)})(\sigma^{(2)} - \sigma^{(1)}) - \frac{1}{2}(\delta\sigma^{(2)} + \delta\sigma^{(1)})(u^{(2)} - u^{(1)}) \quad . \tag{2.8}$$

Similarly, for $\alpha = 0$ and $\alpha = 1$ we obtain, respectively,

$$\langle \sigma \rangle \delta u^{(2)} - \langle u \rangle \delta \sigma^{(1)} \; , \qquad \langle \sigma \rangle \delta u^{(1)} - \langle u \rangle \delta \sigma^{(2)} \; . \tag{2.9}$$

These equations indicate the fact that at a discontinuity point different quantities can be assigned arbitrarily. Theoretically, these equations are equivalent. In actual calculation, however, a proper choice for the value of α may lead to more accurate results; we shall exemplify this later on.

Application to Composites

Consider now a composite consisting of layered elastic media bonded together, and let us consider harmonic waves propagating normal to the layers. For simplicity assume that a cell in this composite consists of two materials, M^β, $\beta = 1, 2$, where M^1 occupies the region $-\frac{a}{2} \le x \le -\frac{b}{2}$ and $\frac{b}{2} \le x \le \frac{a}{2}$, and M^2 occupies the region $-\frac{b}{2} \le x \le \frac{b}{2}$; see Fig. 1. The mass density and the elastic constants of these

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[1] Note that $\bar{\sigma}$ is a linear combination of $\sigma^{(2)}$ and $\sigma^{(1)}$ as defined by (2.5). However, it is $\bar{\sigma}$ that has independent variation, and not $\sigma^{(2)}$ and $\sigma^{(1)}$ separately.

materials will be denoted by ρ_β and η_β, $\beta = 1, 2$, respectively, where each constituent may be inhomogeneous in the direction of wave propagation.

To specialize the functional (2.4) to this case, we observe that this functional must remain real-valued, although in the present case the field quantities are complex-valued. Identifying the body force f with $\frac{1}{2}\omega^2 u$, we write[1]

$$I_1 = \int_{-a/2}^{a/2} \left\{ \frac{1}{2}\eta \epsilon \epsilon^* - \frac{1}{2}\rho\omega^2 u u^* - \sigma\left(\epsilon^* - \frac{du^*}{dx}\right) + \text{c.c.} \right\} dx$$

$$- \left\{ \Lambda \left[u^*\left(\frac{a}{2}\right) - u^*\left(-\frac{a}{2}\right) e^{-iqa} \right] \right\} - \left\{ \bar{\sigma} \langle u^* \rangle \right\}_{x=\pm b/2} + \text{c.c.}, \tag{2.10}$$

where the superscript star denotes the complex conjugate, and the term c.c. stands for the complex conjugate of the quantities which precede it.

The second term in the right-hand side of (2.10) corresponds to the constraint on the displacement at the end points imposed by the quasi-periodicity condition $(2.3)_1$; Λ is a Lagrangian multiplier.[2] Instead of using the displacement constraint, we may replace this term by a term which corresponds to the work of the end tractions. The results, however, will be the same, since twice this work is given by

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[1] Here we use the obvious fact that a complex-valued quantity plus its' complex conjugate is equal to twice of its real part.

[2] In the actual calculation, Λ must be set equal to $\sigma\left(\frac{a}{2}\right)$.

$$\sigma\left(\frac{a}{2}\right) u^*\left(\frac{a}{2}\right) - \sigma\left(-\frac{a}{2}\right) u^*\left(-\frac{a}{2}\right) + \text{c.c.}$$

$$= \sigma\left(-\frac{a}{2}\right)\left[u^*\left(\frac{a}{2}\right) e^{iqa} - u^*\left(-\frac{a}{2}\right)\right] + \text{c.c.} \quad , \tag{2.11}$$

where $(2.3)_2$ is used. The comparison of the right-hand side of (2.11) with the second term in the right-hand side of (2.10) reveals that $\Lambda = \sigma\left(\frac{a}{2}\right)$.

Taking the first variation of I_1 we now obtain

$$\delta I_1 = \int_{-a/2}^{a/2} \left\{\left[\eta\epsilon - \sigma\right]\delta\epsilon^* - \left[\epsilon - \frac{du}{dx}\right]\delta\sigma^* - \left[\frac{d\sigma}{dx} + \rho\omega^2 u\right]\delta u^* + \text{c.c.}\right\} dx$$

$$+ \left\{\left[\sigma\left(\frac{a}{2}\right) - \Lambda\right]\delta u^*\left(\frac{a}{2}\right) - \left[\sigma\left(-\frac{a}{2}\right) - \Lambda e^{-iqa}\right]\delta u^*\left(-\frac{a}{2}\right)\right.$$

$$\left. - \left[u^*\left(\frac{a}{2}\right) - u^*\left(-\frac{a}{2}\right) e^{iqa}\right]\delta\Lambda^* + \text{c.c.}\right\}$$

$$+ \left\{\langle\sigma\rangle\,\delta\bar{u}^* - \langle u\rangle\,\delta\bar{\sigma}^* + \text{c.c.}\right\}_{x = \pm b/2} \quad , \tag{2.12}$$

which for arbitrariness of the indicated variations, gives all the field equations and the corresponding boundary and jump conditions.

The functional I_1 may now be specialized to yield other functionals which may be more suitable for calculation in a given context. For example, if we assume that Eqs. $(2.1)_{2,3}$ are satisfied by the test function for the displacement, I_1 reduces to

$$I_2 = \int_{-a/2}^{a/2} \left\{ \eta \frac{du}{dx} \frac{du^*}{dx} - \rho\omega^2 u u^* \right\} dx - \Lambda \left[u^*\left(\frac{a}{2}\right) - u^*\left(-\frac{a}{2}\right) e^{-iqa} \right]$$

$$- \left\{ \left(\overline{\eta \frac{du}{dx}} \right) \langle u^* \rangle \right\}_{x=\pm b/2} + \text{c.c.} \tag{2.13}$$

whose first variation is

$$\delta I_2 = \int_{-a/2}^{a/2} \left\{ - \left[\frac{d}{dx} \left(\eta \frac{du}{dx} \right) + \rho\omega^2 u \right] \delta u^* + \text{c.c.} \right\} dx$$

$$+ \left\{ \left[\eta \frac{du}{dx} \left(\frac{a}{2}\right) - \Lambda \right] \delta u^* \left(\frac{a}{2}\right) - \left[\eta \frac{du}{dx} \left(-\frac{a}{2}\right) - \Lambda e^{-iqa} \right] \delta u^* \left(-\frac{a}{2}\right) \right.$$

$$\left. - \left[u \left(\frac{a}{2}\right) - u \left(-\frac{a}{2}\right) e^{iqa} \right] \delta \Lambda^* + \text{c.c.} \right\}$$

$$+ \left\{ \left\langle \left(\overline{\eta \frac{du}{dx}} \right) \right\rangle \delta \bar{u}^* - \langle u \rangle \, \delta \left(\overline{\eta \frac{du^*}{dx}} \right) + \text{c.c.} \right\}_{x=\pm b/2} , \tag{2.14}$$

where

$$\left(\overline{\eta \frac{du}{dx}} \right) = \alpha \eta_2 \frac{du^{(2)}}{dx} + (1 - \alpha) \eta_1 \frac{du^{(1)}}{dx} , \tag{2.15}$$

where the superscript refers to the corresponding material, i. e., M^1 or M^2.

Other special cases are obvious. For example, if the test function is chosen such that the quasi-periodicity condition is automatically satisfied, the second term in the right-hand side of (2.13) drops out. If, moreover, the test function is continuous throughout the cell, the last term in the right-hand side of (2.13) vanishes. In this very special case we then arrive at the usual Rayleigh quotient which defines the frequency. This is the case which was exemplified in detail in [5, 6]. Observe that the corresponding approximate expression for the stress field obtained in this manner would suffer a discontinuity at the interface of the two constituent materials. The results are, therefore, very poor; see Table II. If, on the other hand, different test functions are used to represent the displacement field within the region occupied by each constituent material, the last term in (2.13) must be retained. This may then result in a more reasonable approximate expression for both the displacement and the stress fields. We therefore wish to stress here that, if one wishes to vary only the displacement field, as is done in [5, 6], one then must use this latter approach, rather than the usual Rayleigh quotient.

In most calculations it is much easier to use the same set of test functions throughout the entire cell. To arrive at sufficiently accurate results, another variational statement must be employed, in which <u>both the displacement and the stress fields can be directly and independently approximated</u>. In the following we shall discuss this case.

<u>Another General Variational Statement [7-15]</u>

Let us denote by D the inverse of the elasticity coefficient η, i.e.,

$$D = \frac{1}{\eta} \quad . \tag{2.16}$$

The most general variational statement for real-valued field quantities with discontinuities, in which the displacement and the stress fields (but not the strain field) are varied independently, may be stated as follows; compare with Eq. (2.4):

$$J = \int_{-a/2}^{a/2} \left[\frac{1}{2} D\sigma^2 - \rho f u - \sigma \frac{du}{dx} \right] dx + \left\{ Tu, \quad \text{or} \quad \Lambda\left(u - U\right) \right\}_{-a/2}^{a/2}$$

$$+ \left\{ \bar{\sigma} \langle u \rangle \right\}_{x=x_o} , \tag{2.17}$$

where all the quantities are defined as before. The first variation of (2.17) is

$$\delta J = \int_{-a/2}^{a/2} \left\{ \left[D\sigma - \frac{du}{dx} \right] \delta\sigma + \left[\frac{d\sigma}{dx} + \rho \omega^2 u \right] \delta u \right\} dx$$

$$- \left\{ \left[\sigma - T \right] \delta u , \quad \text{or} \quad \left[\sigma - \Lambda \right] \delta u \quad \left[u - U \right] \delta \Lambda \right\}_{-a/2}^{a/2}$$

$$- \left\{ \langle \sigma \rangle \, \delta \bar{u} - \langle u \rangle \, \delta \bar{\sigma} \right\}_{x=x_o} \tag{2.18}$$

which yields, for arbitrary variation of the indicated quantities, all the field equations, and the boundary and jump conditions.

Observe that if continuous displacement and stress test functions are used in (2.17), the last term drops out. Nevertheless, the corresponding variational statement will yield reasonable approximate expressions for the displacement and the stress fields.

To apply (2.17) to elastic waves in composites, we proceed as before, and consider the functional

$$J_1 = \int_{-a/2}^{a/2} \left\{ \frac{1}{2} D\sigma\sigma^* + \frac{1}{2}\rho\omega^2 uu^* - \sigma \frac{du^*}{dx} + \text{c.c.} \right\} dx$$

$$+ \left\{ \Lambda \left[u^*\left(\frac{a}{2}\right) - u^*\left(-\frac{a}{2}\right) e^{-iqa} \right] \right\} + \left\{ \bar{\sigma} \langle u^* \rangle \right\}_{x=\pm b/2} + \text{c.c.}$$

(2.19)

whose first variation is

$$\delta J_1 = \int_{-a/2}^{a/2} \left\{ \left[D\sigma - \frac{du}{dx} \right] \delta\sigma^* + \left[\frac{d\sigma}{dx} + \rho\omega^2 u \right] \delta u^* + \text{c.c.} \right\} dx$$

$$- \left\{ \left[\sigma\left(\frac{a}{2}\right) - \Lambda \right] \delta u^*\left(\frac{a}{2}\right) - \left[\sigma\left(-\frac{a}{2}\right) - \Lambda e^{-iqa} \right] \delta u^*\left(-\frac{a}{2}\right) \right.$$

$$\left. - \left[u\left(\frac{a}{2}\right) - u\left(-\frac{a}{2}\right) e^{iqa} \right] \delta\Lambda^* + \text{c.c.} \right\}$$

$$- \left\{ \langle \sigma \rangle \delta \bar{u}^* - \langle u \rangle \delta \bar{\sigma}^* + \text{c.c.} \right\}_{x=\pm b/2} \quad (2.20)$$

It is clear that the vanishing of δJ_1 for arbitrary variation of the indicated quantities, guarantees the satisfaction of the field equations, the quasi-periodicity conditions, and the continuity of the displacement and the stress across the interface of two materials within the cell.

Let us now consider numerical examples before discussing the general three-dimensional case.

Illustrative Examples

I. As our first example let us illustrate the effectiveness of our variational theorem corresponding to functional (2.19). We assume that each constituent in the composite is homogeneous and isotropic.

As our test functions let us consider

$$u = \sum_{n=0}^{\pm N} U_n e^{i(Q+2\pi n)\xi} \quad , \quad \sigma = \sum_{n=0}^{\pm N} S_n e^{i(Q+2\pi n)\xi} \quad , \tag{2.21}$$

where $Q = qa$, $\xi = \frac{x}{a}$, and U_n and S_n are the Fourier coefficients which are to be calculated; $(2.21)_1$ is the test function used in [5, 6].

Since the above test functions are continuous throughout the cell, and moreover satisfy the quasi-periodicity conditions, only the integral in (2.19) survives. Substituting from (2.21) into this integral and carrying out the indicated integrations, we arrive at

$$J_1(N) = \sum_{\substack{n,m=0 \\ n\neq m}}^{\pm N} \left[\Delta\rho\,\omega^2 U_n U_m^* + \Delta D S_n S_m^*\right] \frac{\sin \pi(n-m)\frac{b}{a}}{\pi(n-m)}$$

$$+ \sum_{n=0}^{\pm N} \left[\bar{\rho}\,\omega^2 U_n U_n^* + \bar{D} S_n S_n^* + \frac{i}{a}(Q+2\pi n)(S_n U_n^* - S_n^* U_n)\right], \tag{2.22}$$

where the following notation is employed:

$$\Delta\rho = \rho_2 - \rho_1 \quad , \quad \Delta D = D_2 - D_1 \quad ,$$

$$\bar{\rho} = n_1 \rho_1 + n_2 \rho_2 \quad , \quad \bar{D} = n_1 D_1 + n_2 D_2 \quad , \tag{2.23}$$

$$n_1 = \frac{a-b}{a} \quad , \quad n_2 = \frac{b}{a} \quad .$$

To begin with we immediately observe that, for $N = 0$, (2.22) yields

$$\omega_o = Q/(\bar{\rho}\bar{D})^{1/2} \quad , \quad \bar{D} = \frac{n_1}{\eta_1} + \frac{n_2}{\eta_2} \quad , \tag{2.24}$$

which corresponds to the non-dispersive results that can be obtained by the usual method of calculating the effective mass-density and elastic modulus.

For $N \geq 1$, we introduce the notation

$$\begin{aligned} \underset{\sim}{U} &= \left\{ U_{-N},\ U_{-N+1},\ \ldots,\ U_0,\ \ldots,\ U_N \right\}^T \quad , \\ \underset{\sim}{S} &= \left\{ S_{-N},\ S_{-N+1},\ \ldots,\ S_0,\ \ldots,\ S_N \right\}^T \quad , \end{aligned} \tag{2.25}$$

where superposed T denotes the transpose, and write (2.22) as

$$J_1(N) = \begin{Bmatrix} \underset{\sim}{U}^* \\ \underset{\sim}{S}^* \end{Bmatrix}^T \left[\begin{array}{c|c} \underset{\sim}{\Omega} & \underset{\sim}{H} \\ \hline \underset{\sim}{H}^* & \underset{\sim}{\Phi} \end{array} \right] \begin{Bmatrix} \underset{\sim}{U} \\ \underset{\sim}{S} \end{Bmatrix} \quad , \tag{2.26}$$

where

$$\underset{\sim}{\Omega} = \omega^2 \begin{bmatrix} \bar{\rho} & \Delta\rho \dfrac{\sin \pi \frac{b}{a}}{\pi} & \Delta\rho \dfrac{\sin 2\pi \frac{b}{a}}{2\pi} & \cdots \\ \Delta\rho \dfrac{\sin \pi \frac{b}{a}}{\pi} & \bar{\rho} & \Delta\rho \dfrac{\sin \pi \frac{b}{a}}{\pi} & \cdots \\ \Delta\rho \dfrac{\sin 2\pi \frac{b}{a}}{2\pi} & \Delta\rho \dfrac{\sin \pi \frac{b}{a}}{\pi} & \bar{\rho} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\underset{\sim}{H} = \operatorname{diag} \frac{i}{a} \left\{ Q - 2\pi N\ ,\ Q + 2\pi(n+1)\ ,\ \ldots,\ Q,\ \ldots,\ Q + 2\pi N \right\} \quad , \tag{2.27}$$

and where the matrix $\underset{\sim}{\Phi}$ is obtained by replacing $\omega^2\bar{\rho}$ in the matrix $\underset{\sim}{\Omega}$ by $\bar{D}$, and $\omega^2\Delta\rho$ by ΔD, respectively. From the stationary condition imposed on (2.26) we now immediate arrive at

$$\underset{\sim}{\Omega}\,\underset{\sim}{U} + \underset{\sim}{H}\,\underset{\sim}{S} = \underset{\sim}{0} \quad , \quad \underset{\sim}{H}^*\,\underset{\sim}{U} + \underset{\sim}{\Phi}\,\underset{\sim}{S} = \underset{\sim}{0} \quad ,$$

which yield

$$\underset{\sim}{S} = -\underset{\sim}{H}^{-1}\,\underset{\sim}{\Omega}\,\underset{\sim}{U} \quad , \quad \left[\underset{\sim}{H}^* - \underset{\sim}{\Phi}\,\underset{\sim}{H}^{-1}\,\underset{\sim}{\Omega}\right]\underset{\sim}{U} = \underset{\sim}{0} \quad . \tag{2.28}$$

The second equation in (2.28) provides the eigenfrequencies and the corresponding eigenvectors, and the first equation then gives the Fourier coefficients for the stress field. The characteristic equation therefore is

$$\det\left[\underset{\sim}{H}^* - \underset{\sim}{\Phi}\,\underset{\sim}{H}^{-1}\,\underset{\sim}{\Omega}\right] = 0 \quad . \tag{2.29}$$

For a given value of the wave number Q, the roots of this equation give the corresponding frequencies.

Table I gives typical results with the corresponding exact values for $n_1 = n_2 = \frac{1}{2}$, $\frac{\rho_2}{\rho_1} = 3$, and the indicated values of $\frac{\eta_2}{\eta_1}$. In this table the values of the dimensionless frequency,

$$\nu = a\omega(\bar{\rho}/\bar{\eta})^{1/2}, \quad \bar{\eta} = n_1\eta_1 + n_2\eta_2 \quad , \tag{2.30}$$

are listed for the indicated values of Q and N, and for various eigenmodes. As is seen from this table, our crudest approximation which still manifests dispersion and which corresponds to N = 1, (i.e., 3 plane waves), gives for the first mode, results which are extremely accurate.

For higher modes N must be taken greater than 1, although even for $N = 1$, the second and third modes obtained from (2.29) are still quite accurate; we shall discuss this below. As is seen from Table I, for $N \geq 2$, again extremely accurate results are obtained. Indeed, to our knowledge, no other approximate method has as yet produced such superior results with so little computational effort.

To demonstrate this last assertion, we have compared some of our results with those corresponding to the methods reported in [1] and in [5, 6], in Table IIa. Observe that, even for $N = 5$, (i.e., 11 plane waves), the method of Kohn, et al. [6] gives poorer results than those deduced from Eq. (2.29) for $N = 1$, (i.e., only 3 plane waves), while for a fixed N, Eq. (2.29) requires less computational effort than the corresponding equation in [6]; this is because, in Eq. (2.29) matrix $\underset{\sim}{H}$ is diagonal, whereas in [6] the characteristic equation has the form $\det |\underset{\sim}{A} - \nu^2 \underset{\sim}{B}| = 0$, in which neither $\underset{\sim}{A}$ nor $\underset{\sim}{B}$ is diagonal.

In Table IIb the values of the frequency parameter ν for higher modes, as given by Eq. (2.29), are compared with those obtained in [6].

We note that for $N = 2$, for example, the characteristic Eq. (2.29) is a 5×5 determinant which can yield only the first five eigenfrequencies. As a rule, only the first eigenfrequency is reasonably accurate. A very remarkable feature of the present method, however, is that, even the higher frequencies obtained in this manner, are quite accurate, whereas this is not the case for the method proposed by Kohn, et al. [6]. Table IIc illustrates this fact. In this table all eigenfrequencies are reported for $N = 2$.

II. For our second example we shall illustrate the variational statement (2.13) and especially the significance of the weighting parameter α. Again we consider waves propagating normal to the layers in a layered medium. Here we shall use a different displacement field in each constituent, and set

$$u^{(1)} = \sum_{n=0}^{\pm N} U_n e^{i(Q + 2\pi n)\frac{x}{a}} , \quad -\frac{a}{2} \le x \le -\frac{b}{2} ,$$

$$\text{and} \quad \frac{b}{2} \le x \le \frac{a}{2} ,$$

$$u^{(2)} = \sum_{m=0}^{\pm N} V_m e^{i 2\pi m \frac{x}{b}} , \quad -\frac{b}{2} \le x \le \frac{b}{2} . \tag{2.31}$$

While $u^{(1)}$ satisfies the quasi-periodicity condition, $u^{(2)}$ does not, and therefore (2.31) represents a crude approximation. Nevertheless, if the weighting parameter α is chosen appropriately, reasonable results can be obtained. We shall not report the detailed algebra here, and only give the final equation:

$$I_1 \sim \sum_{\substack{n,m=0 \\ n \neq m}}^{\pm N} \left[-\frac{1}{\bar{\gamma}} (Q + 2\pi m)(Q + 2\pi n) + \frac{\nu^2}{\bar{\theta}} + \frac{4}{\bar{\gamma}} \pi^2 (m-n)^2 \right] \frac{\sin \pi (m-n)\frac{b}{a}}{\pi (m-n)} U_m^* U_n$$

$$+ \sum_{n=0}^{\pm N} \left\{ \left[\frac{1}{\bar{\gamma}} (Q + 2\pi n)^2 n_1 - \frac{\nu^2 n_1}{\bar{\theta}} \right] U_n^* U_n + \left[\frac{(2\pi n)^2 \gamma}{n_2 \bar{\gamma}} - \frac{\nu^2 n_2 \theta}{\bar{\theta}} \right] V_n^* V_n \right\}$$

$$+ 2 \sum_{n,m=0}^{\pm N} \frac{1}{\bar{\gamma}} \left[\frac{2\pi n \alpha \gamma}{n_2} + (1-\alpha)(Q + 2\pi m) \right] \sin \left[\pi (n - m \frac{b}{a}) - Q \frac{b}{2a} \right] \left[U_m^* V_n + U_m V_n^* \right] , \tag{2.32}$$

where $\gamma = \eta_2/\eta_1$, $\theta = \rho_2/\rho_1$, $\bar{\gamma} = n_1 + \gamma\imath$, and $\bar{\theta} = n_1 + \theta n_2$. Table IIIa gives the frequency parameter ν for $\theta = 3$, $n_1 = n_2 = \frac{1}{2}$, $N = 5$, and indicated values of Q and α. As is seen, α can have a significant effect on the accuracy of the result. It should be remarked that, in view of the discontinuity conditions which are involved now, the algebra is more cumbersome than in the previous example. Another point that should be stressed is that $\alpha = \frac{1}{2}$ is not necessarily a good choice for this parameter, as is seen from Table III; for $\alpha = \frac{1}{2}$ our general variational statement (2.13) reduces to the case considered in [6]. The optimum value of α occurs at $\alpha = 2$, and it appears that this optimum α is independent of the value of $\gamma = \eta_2/\eta_1$, for fixed values of n_1 and θ, and for $N = 5$. Further calculation revealed that for sufficiently large N, say, $N \geq 3$, the optimum value of α is always 2, independently of the values of the other parameters. For small N, say, $N \leq 2$, on the other hand, optimum α changes with the other parameters. This is exemplified in Fig. 2, where the variation of ν is plotted against α for indicated values of the other parameters. Note that for $N = 0$, the optimum value of α is 1.0, whereas for $N = 2$, the optimum value is 2.1. It is remarkable that even for $N = 0$ (i.e., $u^{(1)} = U_0 e^{iQ\xi}$ and $u^{(2)} = {}_{0}'$, $\alpha = 1.0$ gives $\nu = 0.280$ as compared with the exact value $\nu = 0.271$. For $N = 2$ and $\alpha = 2.1$, on the other hand, $\nu = 0.274$.

3. THREE-DIMENSIONAL CASE

We shall first develop the relevant general variational statements and then specialize the results for application to harmonic waves in composites with periodic structure. As before, we shall be exclusively concerned with the linear elasticity theory.

Using a rectangular Cartesian coordinate system we denote the displacement, the strain, and the stress components by u_j, ϵ_{jk}, and σ_{jk}, respectively, and assume that the body forces f_j measured per unit mass are given throughout the volume V occupied by an elastic body with mass-density ρ and elasticity coefficients $C_{jk\ell m}(\underset{\sim}{x})$, where $\underset{\sim}{x}$ is the position vector of a typical point in V; $j, k, \ell, m = 1, 2, 3$. In the general anisotropic case $C_{jk\ell m}$ is symmetric with respect to the exchange of j and k, ℓ and m, and jk and ℓm. For the isotropic case, moreover, we have $C_{jk\ell m} = \lambda \delta_{jk} \delta_{\ell m} + 2\mu \delta_{j\ell} \delta_{km}$, where λ and μ are the Lamé coefficients (they may depend on $\underset{\sim}{x}$), and δ_{jk} is the Kronecker delta; we need not confine our discussion to the special isotropic case.

The field equations are:

$$\sigma_{jk,k} + \rho f_j = 0 \quad , \qquad \sigma_{jk} = C_{jk\ell m} \epsilon_{\ell m} \quad ,$$
$$\epsilon_{jk} = u_{(j,k)} \quad , \qquad u_{(j,k)} = \frac{1}{2}(u_{j,k} + u_{k,j}) \quad , \tag{3.1}$$

where a comma followed by an index letter indicates differentiation with respect to the corresponding coordinate variable, and repeated subscripts are to be summed from 1 to 3.

To characterize the boundary data we denote by n_j the exterior unit normal on the regular surface S which bounds the body, and assume that at a typical point on S certain components of the displacement vector, together with the complementary components of the traction vector, are prescribed. We denote by $T_{\underline{i}}$ the prescribed traction components, and by $U_{\underline{\underline{i}}}$ the prescribed displacement components; henceforth, singly and doubly underlined subscript letters refer, respectively, to the prescribed traction and displacement components at a point on S. We thus have

$$\sigma_{\underline{i}k} n_k = T_{\underline{i}} \quad , \qquad u_{\underline{\underline{i}}} = U_{\underline{\underline{i}}} \quad , \quad \text{on } S \quad . \tag{3.2}$$

In our general variational statements we shall admit test functions which may have finite discontinuities across a finite number of isolated surfaces within the volume V. We shall denote the collection of these discontinuity surfaces by Σ, and observe that, at a point on Σ, three boundary data can be prescribed for the material on one side of the surface, and three boundary data for the material on the other side, a total of no more than six conditions. Hence, a jump in the value of at most six quantities may be prescribed arbitrarily at a point on Σ.

With the above preliminaries out of the way we now consider a variational statement in which the displacement, the strain, and the stress can be given arbitrary variation, as follows. Consider the functional

$$K = \int_V \left\{ \frac{1}{2} C_{jk\ell m} \epsilon_{jk} \epsilon_{\ell m} - \rho f_j u_j - \sigma_{jk} \left[\epsilon_{jk} - u_{(j,k)} \right] \right\} dV$$

$$- \int_{S} T_{\underline{j}} u_{\underline{j}} \, dS - \int_{S} T_{\underline{\underline{j}}} \left[u_{\underline{\underline{j}}} - U_{\underline{\underline{j}}} \right] dS - \int_{\Sigma} \overline{T}_j \langle u_j \rangle \, d\Sigma , \tag{3.3}$$

where

$$\overline{T}_j = \alpha T_j^{(2)} - (1 - \alpha) T_j^{(1)} , \quad \langle u_j \rangle = u_j^{(2)} - u_j^{(1)}, \quad \text{on } \Sigma . \tag{3.4}$$

Here the superscripts are defined in the following manner: The discontinuity surface Σ divides the body into subregions with at least a part of Σ as a common surface. Consider two such adjacent subregions, say, regions 1 and 2, and the field quantity g. At a typical point on Σ, the quantity $g^{(\beta)}$, $\beta = 1, 2$, then denotes the limiting value of g as the considered point is approached from the interior of subregion[1] β.

In Eq. $(3.4)_1$, $\alpha = \alpha(\underset{\sim}{x})$ is a real-valued function defined on Σ; this function can be chosen arbitrarily to expedite the numerical calculation in the given context. [For composites, α is assigned to one of the constituents.]

If we now observe that the exterior unit normal to, say, subregion 1, is the negative of the exterior unit normal to subregion 2 at a typical point on Σ, the first variation of (3.3) becomes

$$\delta K = \int_V \left\{ \left[C_{jk\ell m} \epsilon_{\ell m} - \sigma_{jk} \right] \delta\epsilon_{jk} - \left[\epsilon_{jk} - u_{(j,k)} \right] \delta\sigma_{jk} \right.$$

$$\left. - \left[\sigma_{jk,k} + \rho f_j \right] \delta u_j \right\} dV$$

$$+ \int_S \left[\sigma_{\underline{j}k} n_k - T_{\underline{j}} \right] \delta u_{\underline{j}} \, dS - \int_S \left[u_{\underline{\underline{j}}} - U_{\underline{\underline{j}}} \right] \delta T_{\underline{\underline{j}}} \, dS$$

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1 For composites, β refers to the corresponding material, M^{β}.

$$+ \int_{\Sigma} \left\{ \langle T_j \rangle \, \delta \bar{u}_j - \langle u_j \rangle \, \delta \bar{T}_j \right\} d\Sigma \quad , \tag{3.5}$$

where

$$\bar{u}_j = (1 - \alpha) \, u_j^{(a)} + \alpha u_j^{(1)} \quad \text{on} \quad \Sigma \quad . \tag{3.6}$$

We observe that for arbitrary variation of strain, stress, and displacement in V, we obtain Hooke's law, the definition of strain, and the momentum equations, respectively. On the boundary S, moreover, arbitrary variation of $u_{\underline{i}}$ and $T_{\underline{i}}$ gives, respectively, the prescribed traction and displacement boundary data. The last term in (3.5) corresponds to the jump conditions on the discontinuity surface Σ. It indicates that, if weighted averaged displacement and tractions defined by Eqs. (3.6) and $(3.4)_1$, respectively, are given arbitrary variation, then the continuity of the traction and displacement components is guaranteed. Note that, similarly to the one-dimensional case, special forms of these weighted averaged quantities can be obtained by the suitable choice for the arbitrary function α.

Before proceeding to apply the variational statement (3.3) to harmonic waves in composites, let us state another general variational theorem in which only the stress and displacement are given arbitrary variation. To this end consider the functional

$$L = \int_V \left\{ \frac{1}{2} D_{jk\ell m} \, \sigma_{jk} \, \sigma_{\ell m} + \rho f_j \, u_j - \sigma_{jk} \, u_{(j,k)} \right\} dV + \text{B.C.} + \text{J.C.} \quad , \tag{3.7}$$

where B.C. and J.C. denote, respectively, the boundary and the jump conditions as given by the last two terms (with minus sign changed to plus) in the right-hand side of (3.3), and $D_{jk\ell m}$ is the elastic compliance matrix obtained by inverting the matrix $C_{jk\ell m}$. The first variation of L is

$$\delta L = \int_V \left\{ \left[D_{jk\ell m}\, \sigma_{\ell m} - u_{(j,k)} \right] \delta\sigma_{jk} + \left[\sigma_{jk,k} + \rho f_j \right] \delta u_j \right\} dV$$

$$+ \ \mathrm{B.C.} + \mathrm{J.C.} \quad , \tag{3.8}$$

where, again, B.C. and J.C. correspond to the negative of the last three terms in (3.5).

Application to Composites

For the sake of simplicity in presentation, let us consider a composite consisting of a densely packed and completely bounded collection of unit cells in the form of parallelpipeds; our results, however, apply to composites with different periodic structures, without any additional difficulties. Consider a representative cell, and let its three edges be defined by three vectors $\underset{\sim}{\ell}^{\beta}$, $\beta = 1, 2, 3$. We shall assume that the cell with volume V and surface S consists of different material constituents which are separated from each other by the interior surfaces Σ. We observe that, because of the periodic structure, we have $g\left(\underset{\sim}{x} + \underset{\sim}{\ell}^{\beta}\right) = g(\underset{\sim}{x})$, where g stands for any of the material properties.

We shall consider harmonic waves of frequency ω and wave vector q_j propagating in this composite. Because of the periodic structure of the medium, the field equations have periodic coefficients, and therefore, according to the Floquet theory, they admit solutions of the form

$$f\left(\underset{\sim}{x} + \underset{\sim}{\ell}^{\beta}\right) = f(\underset{\sim}{x}) \exp\left\{ i q_j \ell_j^{\beta} \right\} \quad , \tag{3.9}$$

where f stands for any of the field quantities. In particular, for $\underset{\sim}{x}$ on the surface S, Eq. (3.9) defines the quasi-periodicity conditions for the displacement or the tractions, depending on which quantity is identified with f. For the sake of simplicity in presentation let us denote three faces of the parallelpiped which intersect at a common point, by S', and designate a typical point on S' by $\underset{\sim}{\xi}$. Then it is clear that, to each point $\underset{\sim}{\xi}$ on S' there corresponds, for a suitable $\underset{\sim}{\ell}^{\beta}$, a point $\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta}$ on $S - S'$. The quasi-periodicity condition then becomes

$$\begin{aligned} u_j\left(\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta}\right) &= u_j(\underset{\sim}{\xi}) \exp\left\{ i q_k \ell_k^{\beta} \right\} \quad , \\ T_j\left(\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta}\right) &= - T_j(\underset{\sim}{\xi}) \exp\left\{ i q_k \ell_k^{\beta} \right\} \; , \quad \underset{\sim}{\xi} \text{ on } S' \; , \end{aligned} \tag{3.10}$$

where the minus sign in the right-hand side of the last equation occurs, since the exterior unit normals on S at $\underset{\sim}{\xi}$ and $\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta}$, are oppositely oriented.

As in the one-dimensional case, we shall employ the first equation in (3.10) as constraints on the displacement field.

To arrive at a variational statement in which the displacement, the strain, and the stress are varied independently, we now identify the body forces by $\frac{1}{2}\,\omega^2\,u_j$, and from (3.3) obtain

$$K_1 = \int_V \left\{ \frac{1}{2} C_{jk\ell m} \epsilon_{jk} \epsilon^*_{\ell m} - \frac{1}{2} \rho \omega^2 u_j u_j^* - \sigma_{jk} \left[\epsilon^*_{jk} - u^*_{(j,k)} \right] + \text{c.c.} \right\} dV$$

$$- \int_{S'} \left\{ \Lambda_j \left[u_j^* (\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta}) - u_j^* (\underset{\sim}{\xi}) \exp \{ -i q_k \ell_k^{\beta} \} \right] + \text{c.c.} \right\} dS$$

$$- \int_{\Sigma} \left\{ \overline{T}_j \langle u_j^* \rangle + \text{c.c.} \right\} d\Sigma \quad , \tag{3.11}$$

where all the terms are as defined before. Now, taking the first variation of K_1 we obtain

$$\delta K_1 = \int_V \left\{ \left[C_{jk\ell m} \epsilon_{\ell m} - \sigma_{jk} \right] \delta \epsilon^*_{jk} - \left[\epsilon_{jk} - u_{(j,k)} \right] \delta \sigma^*_{jk} \right.$$

$$\left. - \left[\sigma_{jk,k} + \rho \omega^2 u_j \right] \delta u_j^* + \text{c.c.} \right\} dV$$

$$+ \int_{S'} \left\{ \left[T_j \left(\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta} \right) - \Lambda_j \right] \delta u_j^* \left(\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta} \right) \right.$$

$$+ \left[T_j (\underset{\sim}{\xi}) + \Lambda_j \exp \{ -i q_k \ell_k^{\beta} \} \right] \delta u_j^* (\underset{\sim}{\xi})$$

$$\left. - \left[u_j (\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta}) - u_j (\underset{\sim}{\xi}) \exp \{ i q_k \ell_k^{\beta} \} \right] \delta \Lambda_j^* + \text{c.c.} \right\} dS'$$

$$+ \int_{\Sigma} \left\{ \langle T_j \rangle \delta \overline{u}_j^* - \langle u_j \rangle \delta \overline{T}_j^* + \text{c.c.} \right\} d\Sigma \quad . \tag{3.12}$$

As is seen, the vanishing of the first integral in the right-hand side of (3.12) for arbitrary variation of the indicated field quantities, yields all the field equations. The vanishing of the second integral shows that the Lagrangian multiplier Λ_j is given by

$$\Lambda_j = T_j (\underset{\sim}{\xi} + \underset{\sim}{\ell}^{\beta}) = - T_j (\underset{\sim}{\xi}) \exp \{ i q_k \ell_k^{\beta} \} \quad .$$

Thus the second term in the right-hand side of (3.11) is twice the potential of the surface tractions which are constrained to satisfy the quasi-periodicity condition $(3.10)_2$. Finally, the vanishing of the last integral in the right-hand side of (3.12), for arbitrary variation of the weighted averaged displacement and tractions, as defined on Σ by Eqs. $(3.4)_1$ and (3.6), guarantees the continuity of the tractions and the displacements on the discontinuity surface.

Functional (3.11) can now be specialized in the same manner as discussed in connection with the one-dimensional case. For instance, if we wish to vary only the displacement field, we assume that the last four expressions in (3.1) are satisfied, and modify accordingly the integrand of the first integral in the right-hand side of (3.11).

In a similar manner we can immediately write down the functional L_1 from the functional L, which yields, for harmonic waves in composites with periodic structure, the appropriate variational theorem in which the displacement and the stress fields are given independent variation. In this manner we obtain

$$L_1 = \int_V \left\{ \frac{1}{2} D_{jk\ell m} \sigma_{jk} \sigma^*_{\ell m} + \frac{1}{2} \rho \omega^2 u_j u^*_j - \sigma_{jk} u^*_{(j,k)} + \text{c.c.} \right\} dV$$
$$+ \text{Q.C.} + \text{J.C.} \quad , \tag{3.13}$$

where Q.C. and J.C. stand for the quasi-periodic and the jump conditions, respectively, which are the negatives of those occurring in the expression for K_1. As can be immediately verified, the vanishing of the first variation of the L_1, for arbitrary variation of the appropriate quantities, gives all the field equations, the quasi-periodicity conditions, and the jump conditions.

Application to Fiber-Reinforced Composites

As our final illustrative example, we consider harmonic waves propagating normal to the fibers in a fiber-reinforced elastic composite whose fibers are approximately rectangular in cross section, and are placed parallel to each other in a periodic manner. The cross section of a representative cell is shown in Fig. 3. We consider waves propagating in the x_1-direction.

The general variational theorems, Eqs. (3.4) and (3.7), may be applied directly here. As an illustration we shall apply (3.7) which permits variations in the stress and displacement fields. The other case can be handled in a similar manner.

With the geometry defined in Fig. 3 we therefore write

$$L_3 = \int_{-a_1/2}^{a_1/2} \int_{-a_2/2}^{a_2/2} \left\{ D_{\beta\gamma\delta\eta}\,\sigma_{\beta\gamma}\,\sigma_{\delta\eta}^{*} + \rho\,\omega^2 u_\beta u_\beta^{*} - \sigma_{\beta\gamma}\,u_{(\beta,\gamma)}^{*} - \sigma_{\beta\gamma}^{*}\,u_{(\beta,\gamma)} \right\} dx_1\,dx_2$$

$$+ \int_{-a_1/2}^{a_1/2} \left\{ \sigma_{22}\left(x_1, \frac{a_2}{2}\right)\left[u_2^{*}\left(x_1, \frac{a_2}{2}\right) - u_2^{*}\left(x_1, -\frac{a_2}{2}\right)\right]\right.$$

$$\left. + \sigma_{12}\left(x_1, \frac{a_2}{2}\right)\left[u_1^{*}\left(x_1, \frac{a_2}{2}\right) - u_1^{*}\left(x_1, -\frac{a_2}{2}\right)\right] + \text{c.c.} \right\} dx_1$$

$$+ \int_{-a_2/2}^{a_2/2} \Big\{ \sigma_{11}\left(\frac{a_1}{2}, x_2\right)\left[u_1^*\left(\frac{a_1}{2}, x_2\right) - u_1^*\left(-\frac{a_1}{2}, x_2\right) e^{-iqa_1}\right]$$

$$+ \sigma_{12}\left(\frac{a_1}{2}, x_2\right)\left[u_2^*\left(\frac{a_1}{2}, x_2\right) - u_2^*\left(-\frac{a_1}{2}, x_2\right) e^{-iqa_1}\right] + \text{c.c.}\Big\} dx_2$$

$$+ \int_{-b_1/2}^{b_1/2} \Big\{ \bar{\sigma}_{22}\left(x_1, \pm\frac{b_2}{2}\right) \langle u_2^*\left(x_1, \pm\frac{b_2}{2}\right)\rangle + \bar{\sigma}_{12}\left(x_1, \pm\frac{b_2}{2}\right)\langle u_1^*\left(x_1, \pm\frac{b_2}{2}\right)\rangle$$

$$+ \text{c.c.}\Big\} dx_1$$

$$+ \int_{-b_2/2}^{b_2/2} \Big\{ \bar{\sigma}_{11}\left(\pm\frac{b_1}{2}, x_2\right) \langle u_1^*\left(\pm\frac{b_1}{2}, x_2\right)\rangle + \bar{\sigma}_{12}\left(\pm\frac{b_1}{2}, x_2\right)\langle u_2^*\left(\pm\frac{b_1}{2}, x_2\right)\rangle$$

$$+ \text{c.c.}\Big\} dx_2 \quad , \tag{3.14}$$

$\beta, \gamma, \delta, \eta = 1, 2$,

where $D_{\beta\gamma\delta\eta}$ is the elastic compliance matrix for the plane strain case, which for isotropic materials becomes $D_{\beta\gamma\delta\eta} = \frac{1}{2\mu}\left[\delta_{\beta\delta}\delta_{\gamma\eta} - \frac{\lambda}{2(\lambda+\mu)}\delta_{\beta\gamma}\delta_{\delta\eta}\right]$, where $\delta_{\beta\gamma}$ is the two-dimensional Kronecker delta.

The second integral in functional (3.14) corresponds to the periodicity conditions

$$u_\beta\left(x_1, \frac{a_2}{2}\right) = u_\beta\left(x_1, -\frac{a_2}{2}\right) \quad , \qquad \beta = 1, 2 \quad , \tag{3.15}$$

whereas the third integral pertains to the quasi-peridocity conditions

$$u_\beta\left(\frac{a_1}{2}, x_2\right) = u_\beta\left(-\frac{a_1}{2}, x_2\right) e^{iqa_1} \quad ; \tag{3.16}$$

in these integrals the Lagrangian multipliers Λ_β are identified with the corresponding tractions. The last two integrals, moreover, correspond to jump conditions, if different test functions are used in each region for the displacement and stress fields. As we have pointed out before, the inclusion of these jumps, although it may improve on the detailed local variation of the field quantities, results in a more complicated numerical procedure. For this reason it may be desirable to use continuous test functions for the displacement and the stress components, in which case the last two integrals drop out. Simple test-functions of this kind are

$$u_\beta(x_1, x_2) = \sum_{n,m=0}^{\pm N} U_{\beta\gamma}^{(nm)} \exp\left\{i\left[(Q+2\pi n)\xi_1 + 2\pi m \xi_2\right]\right\},$$

$$\sigma_{\beta\gamma}(x_1, x_2) = \sum_{n,m=0}^{\pm N} S_{\beta\gamma}^{(nm)} \exp\left\{i\left[(Q+2\pi n)\xi_1 + 2\pi m \xi_2\right]\right\}, \tag{3.17}$$

where

$$Q = q a_1, \quad \xi_1 = \frac{x_1}{a_1}, \quad \xi_2 = \frac{x_2}{a_2}.$$

Observe that test-functions (3.17) satisfy the continuity conditions, the periodicity and the quasi-periodicity requirements defined, respectively, by (3.15) and (3.16). These functions, however, do not permit for the discontinuity of the stress-components σ_{22} on the interfaces at $x_1 = \pm\frac{b_1}{2}$, and σ_{11} on the interfaces at $x_2 = \pm\frac{b_2}{2}$. Since no exact solution exists for this problem, it is not possible to easily assess the accuracy of the results. For this reason we postpone further discussions until an assessment of their accuracy is made by means of a more refined procedure.

TABLE I

Frequency Parameter $\nu = a\omega(\bar{\rho}/\bar{\eta})^{1/2}$ for Indicated Values of $Q = qa$, N, and $\frac{\eta_2}{\eta_1}$, and for $\frac{b}{a} = \frac{1}{2}$, and $\frac{\rho_2}{\rho_1} = 3$.

FIRST MODE				
Q	$\eta_2/\eta_1 = 4$		$\eta_2/\eta_1 = 50$	
	N = 1	EXACT	N = 1	EXACT
0.5	0.3987	0.3987	0.1379	0.1379
1.0	0.7893	0.7892	0.2710	0.2709
2.0	1.4925	1.4930	0.4969	0.4974
3.0	1.8744	1.8865	0.6039	0.6094

SECOND MODE				
Q	$\eta_2/\eta_1 = 4$		$\eta_2/\eta_1 = 50$	
	N = 2	EXACT	N = 2	EXACT
0.0	5.786	5.795	1.954	1.958
1.0	5.194	5.205	1.902	1.907
2.0	4.506	4.516	1.791	1.797
3.0	4.122	4.125	1.722	1.725

FIFTH MODE				
Q	$\eta_2/\eta_1 = 4$		$\eta_2/\eta_1 = 50$	
	N = 5	EXACT	N = 5	EXACT
0.0	12.51	12.53	6.45	6.49
1.0	12.98	13.00	6.54	6.58
2.0	13.66	13.68	6.76	6.78
3.0	14.05	14.10	7.00	7.03

TABLE IIa

Frequency Parameter $\nu = a\omega(\bar{\rho}/\bar{\eta})^{1/2}$ as given by Eq. (2.29), by Kohn, Krumhansl, and Lee [6], and by Sun, Achenbach, and Herrmann [1], for Indicated Values of $Q = qa$ and N, and for $\frac{b}{a} = \frac{1}{2}$, $\frac{\rho_2}{\rho_1} = 3$, and $\frac{\eta_2}{\eta_1} = 100$.

FIRST MODE					
Q	EXACT	Present Method Eq. (2.29)	Kohn, Krumhansl, and Lee, Ref. [6]		Sun, Achenbach and Herrmann Ref. [1]
		N = 1	N = 1	N = 5	
0.5	0.098	0.098	0.237	0.107	0.099
1.0	0.193	0.193	0.460	0.210	0.196
2.0	0.354	0.354	0.854	0.383	0.381
3.0	0.434	0.430	1.083	0.467	0.546
4.0	0.388	0.380	1.034	0.421	0.687
5.0	0.244	0.238	0.694	0.267	0.805
6.0	0.056	0.055	0.168	0.062	0.901

TABLE IIb

Frequency Parameter $\nu = a\omega(\bar{\rho}/\bar{\eta})^{1/2}$ for Higher Modes, as given by Eq. (2.29), and by Kohn, Krumhansl, and Lee [6], for Indicated Values of $Q = qa$ and N, and for $\frac{b}{a} = \frac{1}{2}$, $\frac{\rho_2}{\rho_1} = 3$, and $\frac{\eta_2}{\eta_1} = 100$.

Q	SECOND MODE, N = 2			THIRD MODE, N = 3		
	EXACT	Present Method Eq.(2.29)	Kohn, Krumhansl, and Lee, Ref. [6]	EXACT	Present Method Eq.(2.29)	Kohn, Krumhansl, and Lee, Ref. [6]
1.0	1.36	1.36	3.79	2.50	2.50	5.47
3.0	1.24	1.24	3.35	2.57	2.56	6.08
5.0	1.34	1.35	3.41	2.51	2.49	6.22

Q	FOURTH MODE, N = 4			FIFTH MODE, N = 5		
	EXACT	Present Method Eq.(2.29)	Kohn, Krumhansl, and Lee, Ref. [6]	EXACT	Present Method Eq.(2.29)	Kohn, Krumhansl, and Lee, Ref. [6]
1.0	3.77	3.76	6.76	4.94	4.94	6.69
3.0	3.70	3.69	6.37	5.02	5.00	7.02
5.0	3.76	3.75	6.28	4.96	4.91	7.13

TABLE IIc

Frequency Parameter $\nu = a\omega(\bar{\rho}/\bar{\eta})^{1/2}$ for First Five Modes given by Eq. (2.29), and by [6] for $N = 2$, $\frac{b}{a} = \frac{1}{2}$, $\frac{\rho_2}{\rho_1} = 3$, and $\frac{\eta_2}{\eta_1} = 50$.

Q	EXACT	Present Method Eq. (2.29)	Kohn, Krumhansl, and Lee. Ref. [6]
1.0	0.27	0.27	0.31
	1.91	1.90	3.95
	3.48	3.46	6.27
	5.26	5.87	12.11
	6.58	7.12	13.90
2.0	0.50	0.50	0.57
	1.80	1.79	3.68
	3.56	3.58	6.63
	5.16	5.47	11.13
	6.78	7.73	14.48

TABLE III

Frequency Parameter $\nu = a\omega(\bar{\rho}/\bar{\eta})^{1/2}$ for Indicated Values of $Q = qa$, α, η_2/η_1, and for $\frac{b}{a} = \frac{1}{2}$, $\frac{\rho_2}{\rho_1} = 3$, and $N = 5$.

FIRST MODE						
	$\eta_2/\eta_1 = 4$			$\eta_2/\eta_1 = 50$		
EXACT	0.789	1.493	1.886	0.271	0.497	0.609
α \ Q	1.0	2.0	3.0	1.0	2.0	3.0
-10.0	0.877	1.586	1.919	0.274	0.503	0.618
- 5.0	0.879	1.590	1.923	0.274	0.504	0.619
0.0	0.929	1.650	1.983	0.292	0.530	0.639
0.5	0.936	1.719	2.101	0.279	0.532	0.677
1.0	1.811	3.622	5.4[illegible]	0.561	1.121	1.682
2.0	0.868	1.568	1.895	0.273	0.499	0.610
10.0	0.873	1.579	1.911	0.274	0.502	0.615

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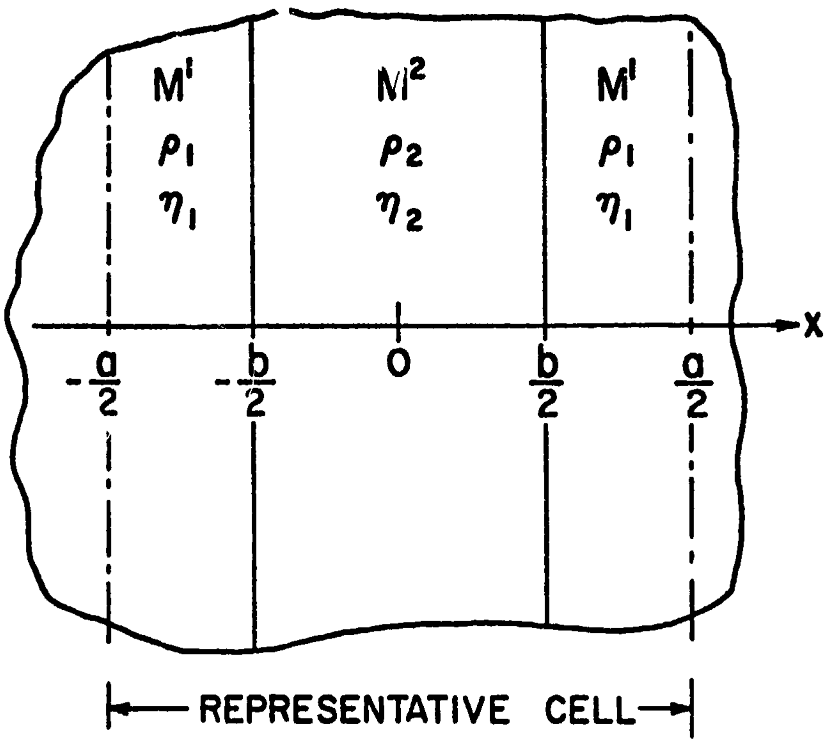


Figure 1

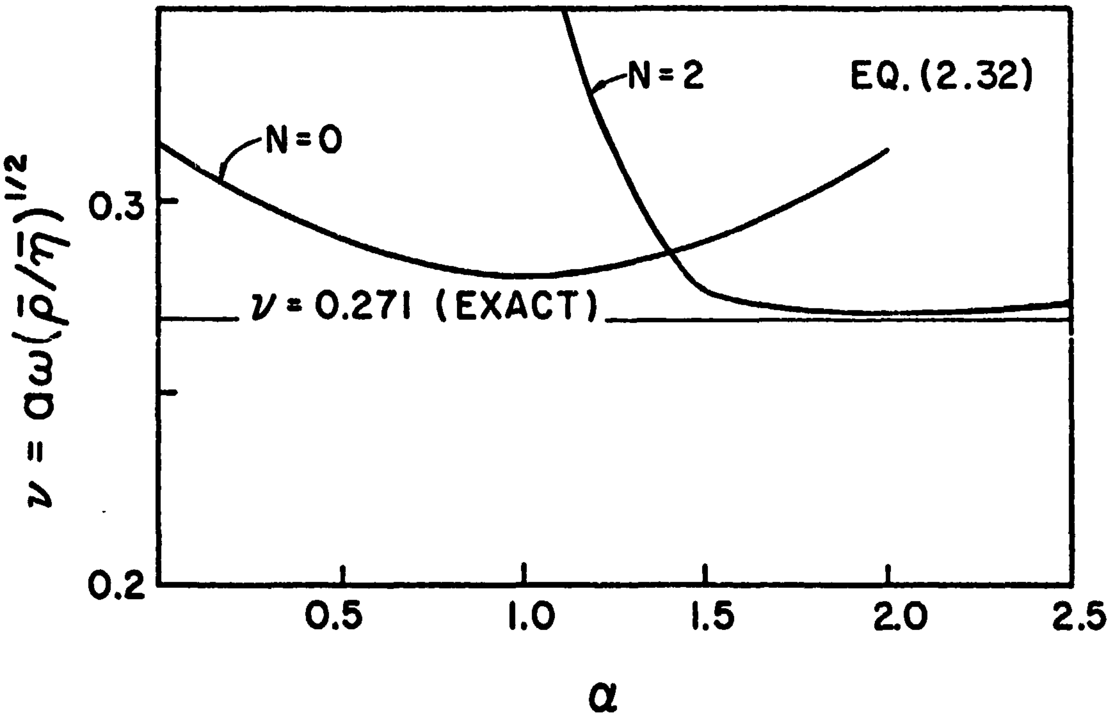
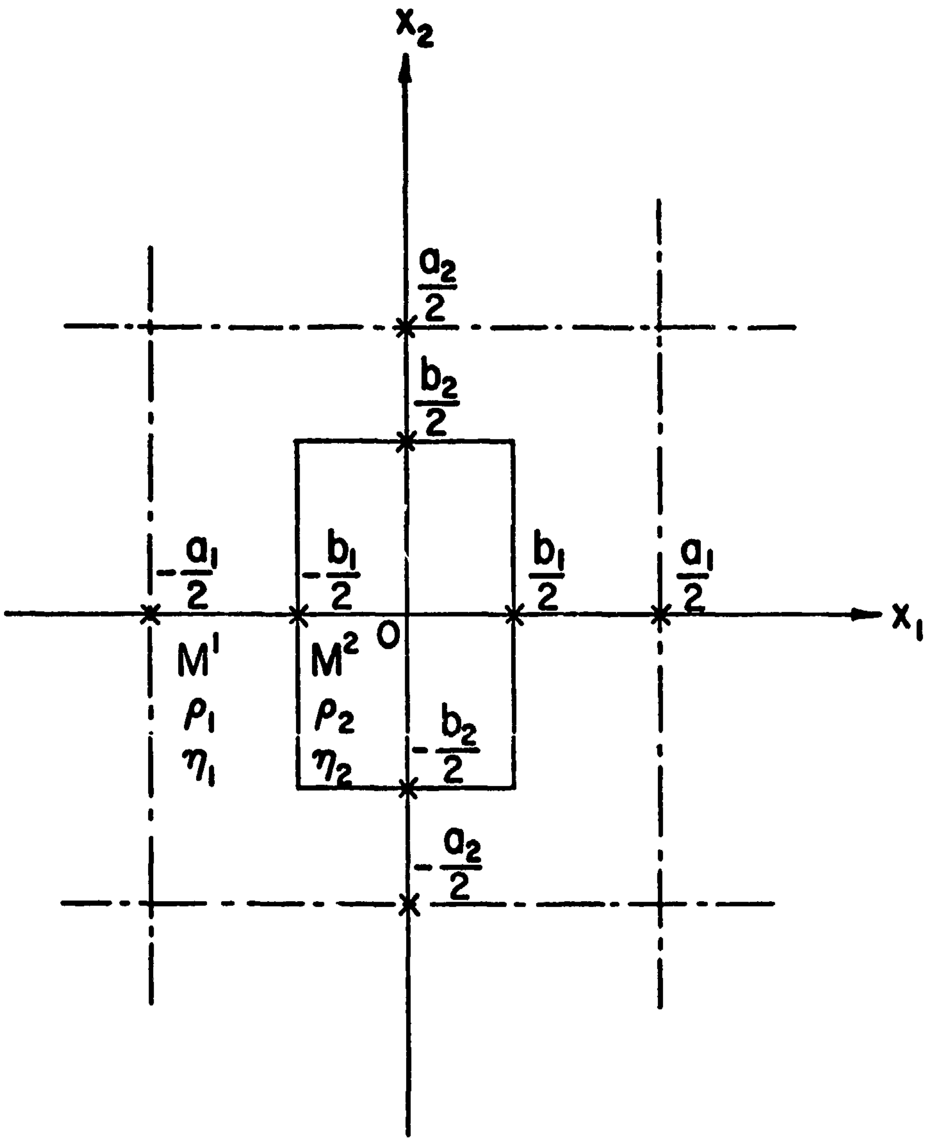


Figure 2

Frequency parameter $\nu = a\omega(\bar{\rho}/\bar{\eta})^{\frac{1}{2}}$ as function of the weighting parameter α; $\frac{b}{a} = \frac{1}{2}$, $\frac{\rho_2}{\rho_1} = 3$, $\frac{\eta_2}{\eta_1} = 50$, and $Q = 1.0$.


x_2
$\frac{a_2}{2}$
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$-\frac{a_1}{2}$
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$\frac{b_1}{2}$
$\frac{a_1}{2}$
x_1
O
M^1
M^2
ρ_1
ρ_2
η_1
η_2
$-\frac{b_2}{2}$
$-\frac{a_2}{2}$

Figure 3