The Design of a Tunable Notch Filter With Variable Q

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13. ABSTRACT

In electro-mechanical servo systems one often encounters undesired resonances which contribute to the system instability. One method used to combat such a resonance is to place a notch filter in the feedback loop to reduce gain at the resonance frequency, while allowing frequencies outside the notch to pass without appreciable phase shift or attenuation. The notch filter described in this paper is particularly useful in this application since it can be tuned to the resonant frequency while in the circuit, the Q can be adjusted to cover the resonance, and the notch attenuation can be adjusted for essentially zero transmission at the center frequency.

The filter is of simple design and easily constructed. The most critical restriction is that the operational amplifiers used in the circuit be stable at unity gain. However, this is not particularly difficult to realize.

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INTRODUCTION

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The filter is of simple design and easily constructed. The most critical restriction is that the operational amplifiers used in the circuit be stable at unity gain. However, this is not particularly difficult to realize.

BASIC THEORY

In Figure 1, the (+) input is the non-inverting input to the operational amplifier; the (-) input, the inverting input. In the following discussion it will be assumed that the operational amplifiers used are "ideal"[1]; i.e.

1. $Z_{in} = \infty$

2. $Z_{out} = 0$

3. Gain $= \infty$

4. Zero voltage difference between the two input terminals gives a zero output, (infinite common mode rejection and zero offset voltage).

Typical operational amplifiers have Z_{in} of 10^6 ohms, Z_{out} of less than 5×10^3 ohms, gains greater than 80 db, common mode rejection of greater than 80 db, and offset voltages less than 1 millivolt. Such operational amplifiers cause deviation of only a few percent from the results calculated using ideal models.

Referring to Figure 1, and measuring the voltages with respect to ground, the output of the filter is given by

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$$e_{out} = -\frac{R_5}{R_3}\left(e_2 - e_1\right) + \frac{R_4 + R_5}{R_4} \quad . \tag{1}$$

For the bridged-T network, it can be shown (see appendix) that for low source impedance,

$$e_1 = e_2 \frac{1}{6}\left(5 + e^{i\alpha}\right) \tag{2a}$$

where

$$\cos\alpha = \frac{x^2 - 9}{x^2 + 9}$$

$$\sin\alpha = \frac{6x}{x^2 + 9}$$

and

$$x = \frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} \tag{2b}$$

$$\omega_o = 1/RC, \quad \omega = \text{input frequency.}$$

From (2a) it may be seen that

$$e_2 - e_1 = e_2 \frac{1}{6}\left(1 - e^{i\alpha}\right) . \tag{2c}$$

Inserting (2a, c) into (1),

$$6\,\frac{e_{out}}{e_2} = -\frac{R_5}{P_3} + \frac{R_4 + R_5}{R_4} + \left[\frac{R_5}{R_3} + \frac{R_4 + R_5}{R_4}\right] e^{i\alpha}. \tag{3a}$$

When $e_{out} = 0$ at $e^{i\alpha} = -1$, using (3a),

$$\frac{R_5}{R_3} = 2\,\frac{R_4 + R_5}{R_4} \quad . \tag{3b}$$

With (3b) and (2a), (1) reduces to

$$e_{out} = \frac{R_4 + R_5}{R_4}\,\frac{e_2}{2}\left[1 + e^{i\alpha}\right] \tag{4}$$

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For the first amplifier the output is

$$e_2 = -\left[\frac{R_2}{R_1} \ e_{in} + \frac{R_2}{R_6} \ e_{out}\right].$$ (5)

Finally, combining (4) and (5),

$$\frac{e_{out}}{e_{in}} = -\left[\frac{R_2}{R_1}\right]\left[\frac{R_4 + R_5}{R_4}\right]\left[\frac{1 + e^{i\alpha}}{2 + \frac{R_2}{R_6}\left(\frac{R_4 + R_5}{R_4}\right)\left(1 + e^{i\alpha}\right)}\right].$$ (6)

To remove the imaginary terms from the denominator and convert (6) to a more usable form, let

$$B = \frac{R_2}{R_6}\left(\frac{R_4 + R_5}{R_4}\right)$$ (7a)

and

$$\phi = \tan^{-1}\frac{\sin\alpha}{(1 + B)\ (1 + \cos\alpha)}$$

$$= \tan^{-1}\left(\frac{-3}{X(1 + B)}\right)$$ (7b)

so that

$$\frac{1 + e^{i\alpha}}{2 + B + Be^{i\alpha}} = \left[\frac{1 + \cos\alpha}{B^2(1 + \cos\alpha) + 2B(1 + \cos\alpha) + 2}\right]^{\frac{1}{2}} e^{i\phi}.$$ (7c)

Then, finally the output vs the input of the filter is

$$\frac{e_{out}}{e_{in}} = -\frac{R_6}{R_1} \ B\left[\frac{1 + \cos\alpha}{B^2(1 + \cos\alpha) + 2B(1 + \cos\alpha) + 2}\right]^{\frac{1}{2}} e^{i\phi}.$$ (8)

3

To define the characteristics of the filter better, the theoretical gain and Q must be established. Equation (2b) shows that as ω gets much greater or much smaller than $\cdot$, x approaches $+\infty$ or $-\infty$ respectively. For either value, $\cos\alpha$ approaches 1 while ψ approaches zero. Setting the gain of the filter equal to the gain in these limits, and applying the limits to (8),

$$A = \left| \lim_{\substack{\omega \to \infty \\ \omega \to 0}} \frac{e_{out}}{e_{in}} \right| = \left| -\frac{R_6}{R_1} \frac{B}{B+1} \right|. \qquad (9)$$

Since at the notch frequency the filter output is zero, I now chose to define a modified Q in terms of the bandwidth at the points where

$$\left| \frac{e_{out}}{e_{in}} \right| = A/2:$$

$$\overline{Q} = \frac{\omega_o}{\omega_+ - \omega_-} . \qquad (10)$$

ω_+ is the frequency greater than ω_o such that $\left| \dfrac{e_{out}}{e_{in}} \right| = A/2$;

ω_- is the frequency less than ω_o such that $\left| \dfrac{e_{out}}{e_{in}} \right| = A/2$.

Solving (8) for these 6 db points,

$$\frac{\omega_\pm - \omega_o}{\omega_o} = \pm \left[\frac{3}{4(B+1)^2} \right]^{\frac{1}{2}} - \left[\sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!} \left(\frac{-3}{4(B+1)^2} \right)^n \right] \qquad (11a)$$

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and

$$\bar{Q} = \frac{B + 1}{\sqrt{3}} \qquad \text{(11b)}$$

where

$$(2n-3) \, !! \equiv 1.3.5.7.......(2n-3)$$

$$(2n) !! \equiv 2.4.6....(2n)$$

and for

$$\left[\sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!} \left(\frac{-3}{4(B+1)^2} \right)^n \right] = 1 - \left(1 + \frac{3}{4(B+1)^2} \right)^{\frac{1}{2}} = E.$$

E is a measure of the lack of symmetry of the 6 db points about the notch frequency; it represents a 33% shift in $\left(\omega_{\pm} - \omega_o \right)$ for B = 0, but only about 4% for B = 2. Therefore, for B greater than 2.0 the notch is essentially symmetrical in nature.

The phase shift at ω_o is $\mp 90°$; at $\omega = 0$, $180°$; at $\omega \gg \omega_o$, $180°$. Setting

$$x_{\pm} = \left(\frac{\omega_{\pm}}{\omega_o} - \frac{\omega_o}{\omega_{\pm}} \right) = \pm \frac{\sqrt{3}}{B+1}$$

and using (7b),

$$\phi_{\pm} = \tan^{-1} \sqrt{3} = \pm 60°.$$

Thus for all B, $\phi_{\pm} = \pm 60°$ and the overall phase shift is $180° \pm 60°$. Thus the overall gain should be less than 2 to insure stability of the circuit.

If $180°$ of phase shift is unacceptable in the circuit and an odd number of filters is to be used, the circuit in Figure 1 may be replaced by circuit in Figure 2. For this circuit, equation 5 becomes

$$e_2 = - \left[\frac{R_6 + R_2}{R_6} \frac{R_8}{R_7 + R_8} e_{in} + \frac{R_2}{R_6} e_{out} \right]. \qquad \text{(12)}$$

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Equation (12) may be rewritten as

$$e_2 = -\left[\frac{R_2}{R_1'} \, e_{in} + \frac{R_2}{R_6} \, e_{out}\right]$$

where

$$R_1' = -\frac{R_6\left(R_7 + R_8\right)}{R_8\left(R_2 + R_6\right)} \, R_2 \, .$$

With this modification equation (8) becomes

$$\frac{e_{out}}{e_{in}} = \frac{R_8\left(R_2 + R_6\right)}{R_6\left(R_7 + R_8\right)} \; B \; \left[\frac{1 + \cos\alpha}{B^2(1+\cos\alpha) + 2B(1+\cos\alpha) + 2}\right]^{\frac{1}{2}} e^{i\phi}, \tag{13}$$

introducing the desired 180° phase shift. The gain at $\omega \to 0, \infty$ now becomes

$$A_{non-invert} = \underset{\substack{\omega \to \infty \\ \omega \to 0}}{L} \; \frac{e_{out}}{e_{in}} = \frac{R_8\left(R_2 + R_6\right)}{R_6\left(R_7 + R_8\right)} \; \frac{B}{B + 1} \; . \tag{14}$$

All the rest of the argument remains identical.

SUMMARY

1. Notch frequency $= \omega_o = 1/RC$

2a. Gain $= A_1 = -\dfrac{R_6}{R_1} \dfrac{B}{B + 1} \cdot$ (inverting)

2b. Gain $= A_{NI} = \dfrac{R_7\left(R_2 + R_6\right)}{R_2\left(R_7 + R_8\right)}$ (non-inverting)

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3. The 50% gain points occur at $\omega_{\pm}$, where

$$\omega_{\pm} = \omega_0 \pm \frac{\omega_0}{2} \frac{\sqrt{3}}{(B+1)} - \omega_0 \left[\sum_{1}^{\infty} \frac{|2n-3|!!}{(2n)!!} \left(\frac{-3}{4(B+1)^2} \right)^n \right].$$

4. The condition for null at the notch frequency is

$$\frac{1}{R_3} = 2 \left(\frac{1}{R_4} + \frac{1}{R_5} \right).$$

5a. The notch frequency is set by R

 b. The Q is set by R_6

 c. Inverting gain is set by R_1

 d. Non-inverting gain is set by R_7 and R_8.

6. $\overline{Q} = \dfrac{\omega_0}{\omega_+ - \omega_-} = \dfrac{B+1}{\sqrt{3}}$

DESIGN PROCEDURE

1. Choose the desired frequency range, and select suitable operational amplifiers.

2. Choose, R_2, R_4, and R_5 compatible with the op amps being used and the circuit being driven. If more drive is needed, a buffer stage may be added.

3. From equation 10 and 11b, and using the center frequency ω_0 and the average 6 db width desired, B may be calculated:

$$B = \frac{\sqrt{3} \; \omega_0}{\omega_+ - \omega_-} - 1, \quad \overline{Q} = \frac{B+1}{\sqrt{3}} \; .$$

4. Using the values in 2 and 3 and knowing the desired gain (which should be less than 2), the equation for R_3, R_6, and R_1 are:

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$$R_3 = \tfrac{1}{2} \; \frac{R_4 \, R_5}{R_4 + R_5} \qquad \text{(from (3b))}$$

$$R_6 = \frac{R_2}{B} \; \frac{R_4 + R_5}{R_4} \qquad \text{(from (7a))}$$

$$R_1 = \frac{R_6}{A} \; \frac{B}{B + 1} \qquad \text{(from (9))}$$

5. Compute RC for the desired frequency. Since ganged pots track about $\pm 5\%$, it is best to use a fixed R which gives the maximum frequency desired and a dual pot which when added to the fixed R gives the minimum frequency desired. The capacitors should be of the low drift mica type.

6. In order to compensate for resistor tolerances and to set the notch center frequency output to zero, about 1/4 of R_3 should be obtained using a variable pot of value $R_3/2$.

7. Making R_6 variable allows one to adjust the Q of the notch filter. To compensate for the effect of R_6 on the overall gain, R_1 should be made variable also.

DESIGN EXAMPLES

1.A.1. The amplifiers to be used are Philbrick type K2-W with $\pm$ 100 volts output at $\pm$ 1 mA, f_o = 100 cps, gain of 2, input signal $< \pm$ 10 volts.

   2. Set R_2 = 30 K

        R_4 = 50 K

        R_5 = 200 K

   3. For 6 db points at $f_o \pm$ 10 cps,

$$B = \frac{\sqrt{3} \; \omega}{\omega_+ - \omega_-} - 1 = \sqrt{3} \; \frac{100}{20} - 1 = 7.66; \quad \overline{Q} = 5.00$$

   4. $R_3 = \tfrac{1}{2} \dfrac{R_4 \, R_5}{R_4 + R_5} = \tfrac{1}{2} \dfrac{10,000}{250}$ K = 20 K (use a 25 K pot)

$$R_6 - \frac{R_2}{b} \quad \frac{R_4 + R_5}{R_4} = \frac{1}{7.66} \quad \frac{30}{50} \quad 250 \text{ K} = 19.58 \text{ K}$$

$$R_1 = \frac{19.58K}{2} \quad \frac{7.66}{8.66} = 9.66 \text{ K}$$

. 5 $C = .09 \ \mu\text{fd (measured)}$

$R = \frac{107}{180} = 17.7$ K (use a 25 K ganged pot.)

B.1.   Experimental Results

At + R = Max $\simeq$ 25 K, f_0 = 71 Hz

At + R $\approx$ 4.2 K, $f_0 \approx$ 400 Hz.

For R < 4 K, the ganged pots did not track well, and $\overline{Q}$ and notch attenuation began to fall off.

2. For f_0 = 100 Hz, R $\approx$ 17.5 K

For Gain = 2.0 @ 10 Hz and 1 KHz, $R_1 \approx$ 8.4 K.

For notch attenuation > 50 db, $R_3 \approx$ 20 K

For Gain = 1, f $\approx$ 90 Hz and 112 Hz; $\overline{Q}$ = 4.6.

II.A.1.   Two notch filters are to be built using integrated circuits. The amplifiers to be used are Fairchild μA739's. The filters are to be unity gain with _ iter frequencies of 96 Hz respectively.

2.   Set R_2 = 30 K

    R_4 = 51 K

    R_5 = 100 K

3.   For the first filter, the 6 db points are at $\approx \pm$ 6 Hz; for the second, $\approx \pm$ 11 Hz:

$$B_1 = \sqrt{3} \ \frac{96}{12} - 1 = 12.85$$

$$B_2 = \sqrt{3} \ \frac{187}{22} - 1 = 13.72$$

Use B = 13.3, $\overline{Q}$ = 8.3.

4. $R_3 = \frac{1}{2} \cdot \frac{5100}{151} = 16.9$ K

$R_6 = \frac{30 \text{ K}}{13.3} \cdot \frac{151}{51} = 6.7$ K

$R_1 = (6.6\text{K}) \frac{13.3}{14.3} = 6.1$ K

5. C - .01 μfd

R(96 Hz) = 166 K

R(187 Hz) - 85.2 K.

B. **Experimental Results**

1. Figure 3 shows the final circuit; Figure 4 shows the gain vs the frequency.

2. For f_o = 96 Hz

For Gain = 1 @ 10 & 100 Hz, R_1 = 6.13 K

For notch attenuation > 44 db, R_3 = 16.5 K

Gain = $\frac{1}{2}$ @ f_- = 90.5 Hz and f_+ = 101.1 Hz, R_6 = 6.2 K, $\overline{Q}$ = 9.06

For f_o = 187 Hz

Gain = 1 @ 10 & 100 Hz, R_1 = 5.8 K

Notch attenuation > 43 db R_3 = 16.5 K

Gain = $\frac{1}{2}$ @ f_- = 176 Hz and f_+ = 199.5 Hz, R_6 = 7.2 K, $\overline{Q}$ = 7.96

3. Figure 5 shows the maximum stable $\overline{Q}$ vs the $\overline{Q}$ for $R_6 = \infty$.

4. Figure 6 shows the range of $\overline{Q}$ available using a 20 K ohm potentiometer for R_6.

5. Figure 7 shows picture, completed dual filter.

REFERENCES

1. Burr-Brown Research Corporation, "Handbook of Operational Amplifier Applications",
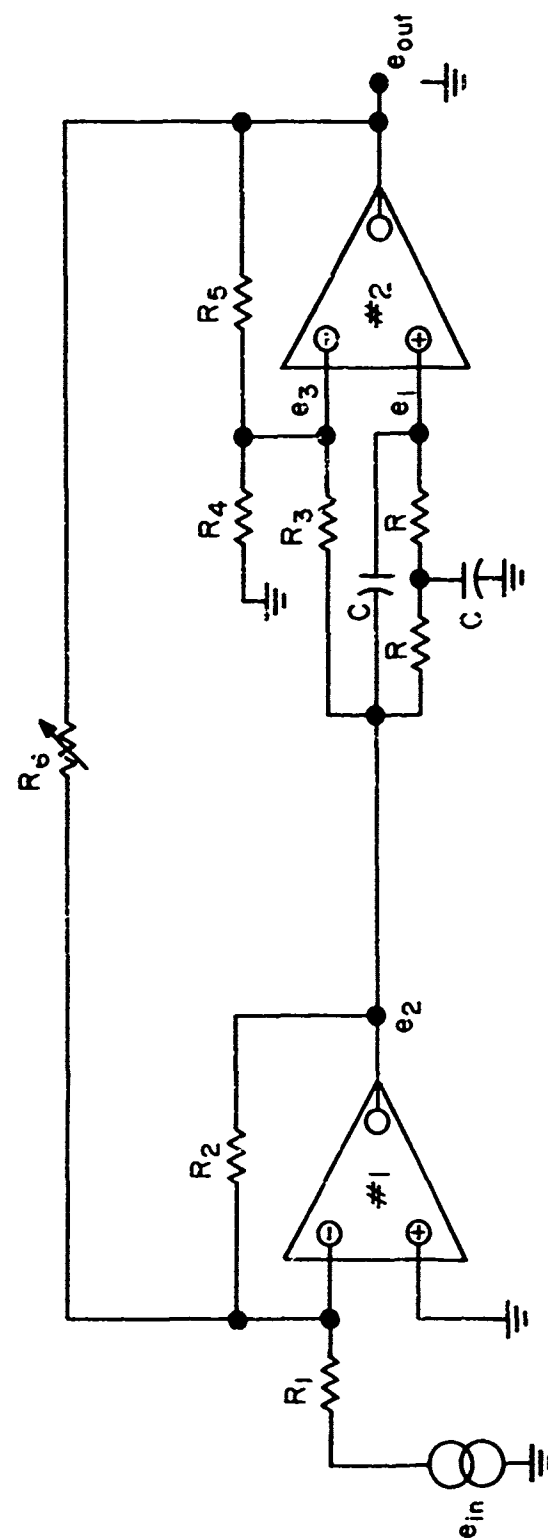   1963, pp. 1-44.

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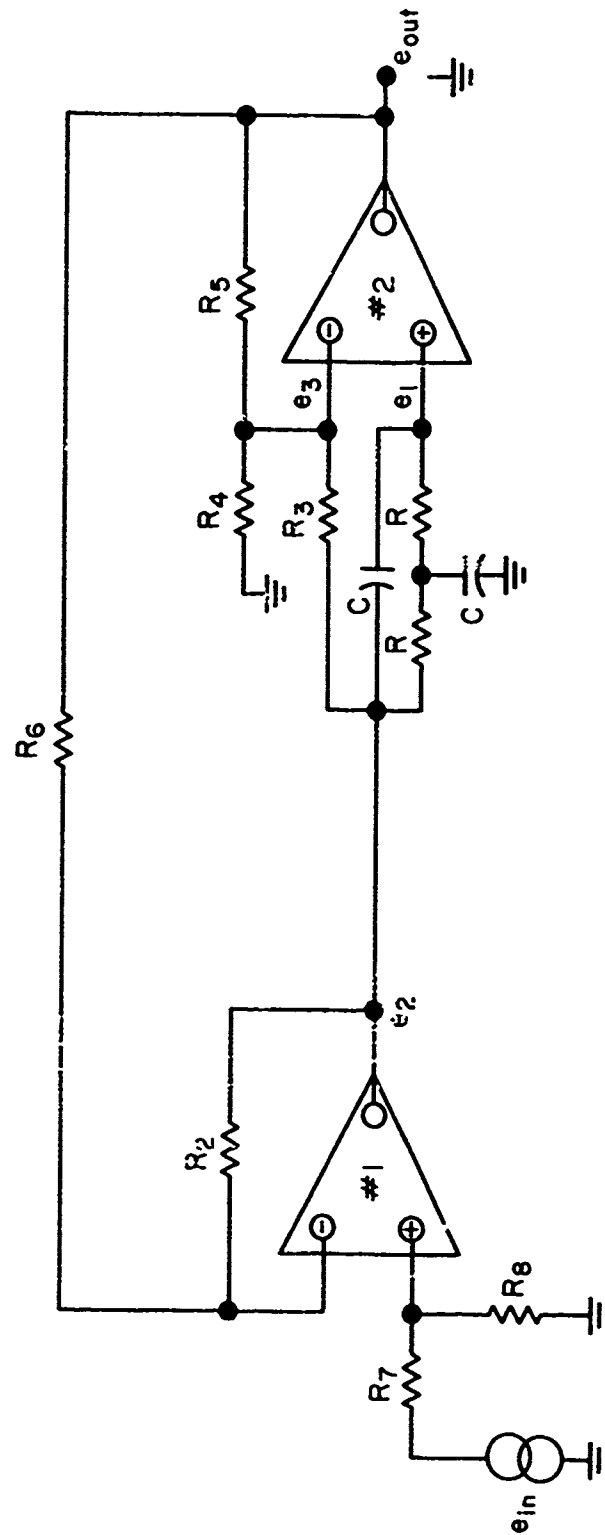Fig. 1 – Schematic diagram of inverting notch filter.

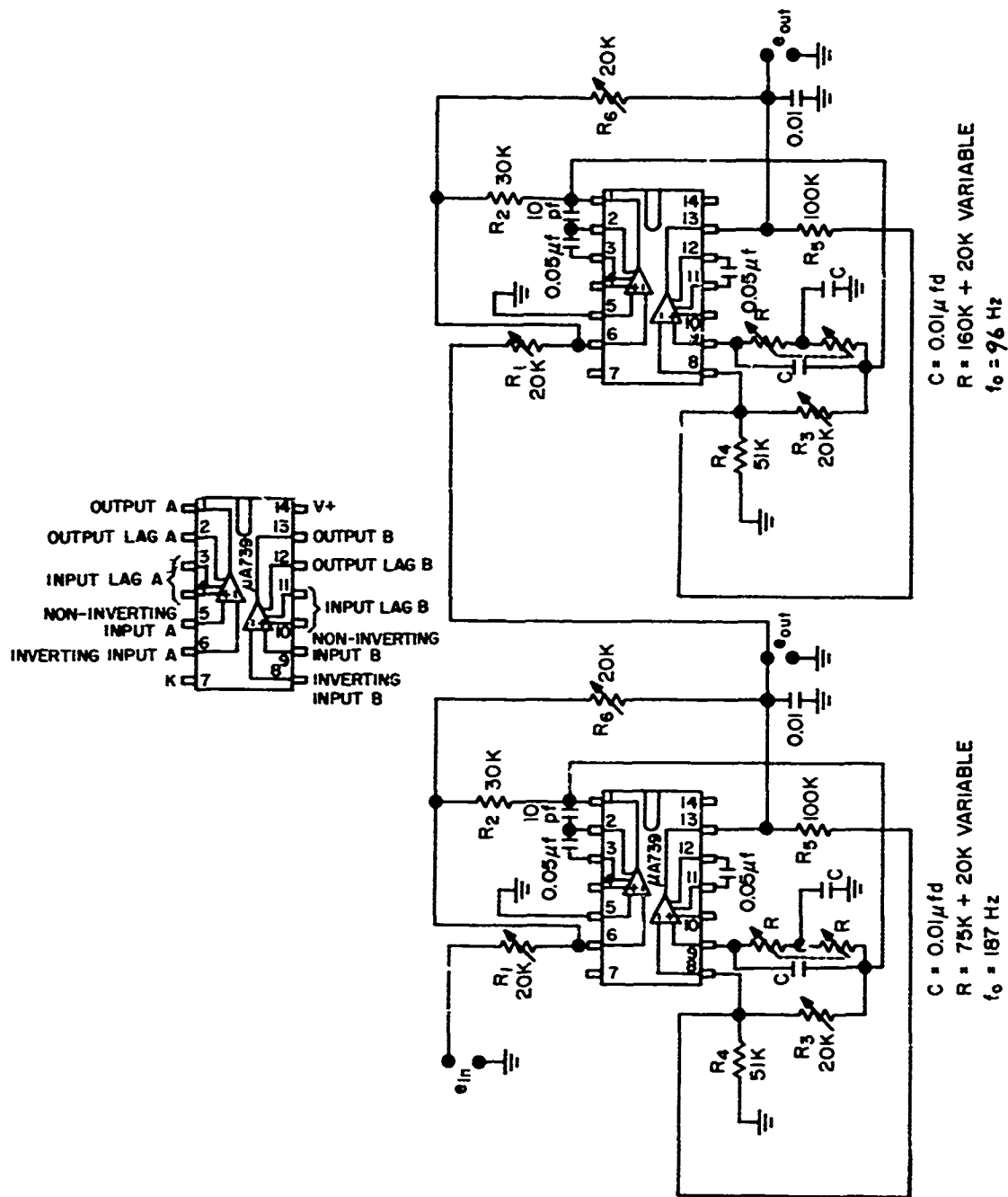Fig. 2 — Schematic diagram of non-inverting notch filter.

OUTPUT A — 1 | 14 — V+
OUTPUT LAG A — 2 | 13 — OUTPUT B
INPUT LAG A — 3 | 12 — OUTPUT LAG B
| μA739C |
NON-INVERTING INPUT A — 5 | 11 — INPUT LAG B
INVERTING INPUT A — 6 | 10 — NON-INVERTING INPUT B
K — 7 | 9 — INVERTING INPUT B
| 8 |

R₂ 30K
R₆ 20K
R₅ 100K
R₁ 20K
R₄ 51K
R₃ 20K
10 pf
0.05μf
0.05μf
0.01
μA739

C = 0.01μfd
R = 160K + 20K VARIABLE
f_o = 96 Hz

R₂ 30K
R₆ 20K
R₅ 100K
R₁ 20K
R₄ 51K
R₃ 20K
10 pf
0.05μf
0.05μf
0.01
μA739

C = 0.01μfd
R = 75K + 20K VARIABLE
f_o = 187 Hz

e_{in}
e_{out}
e_{out}

Fig. 3 – Completed dual-notch filter with notch frequencies at 96 and 187 Hz respectively.

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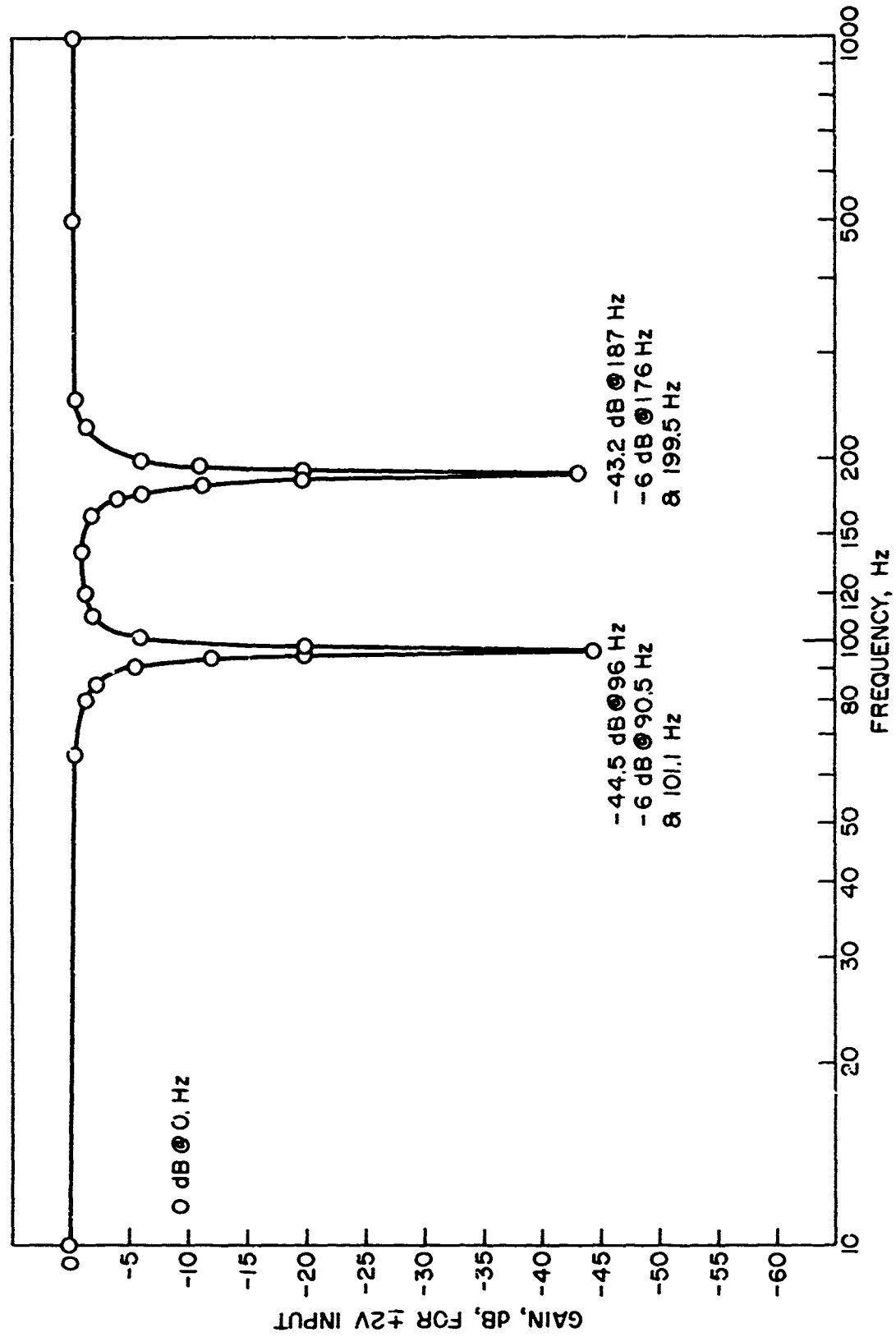
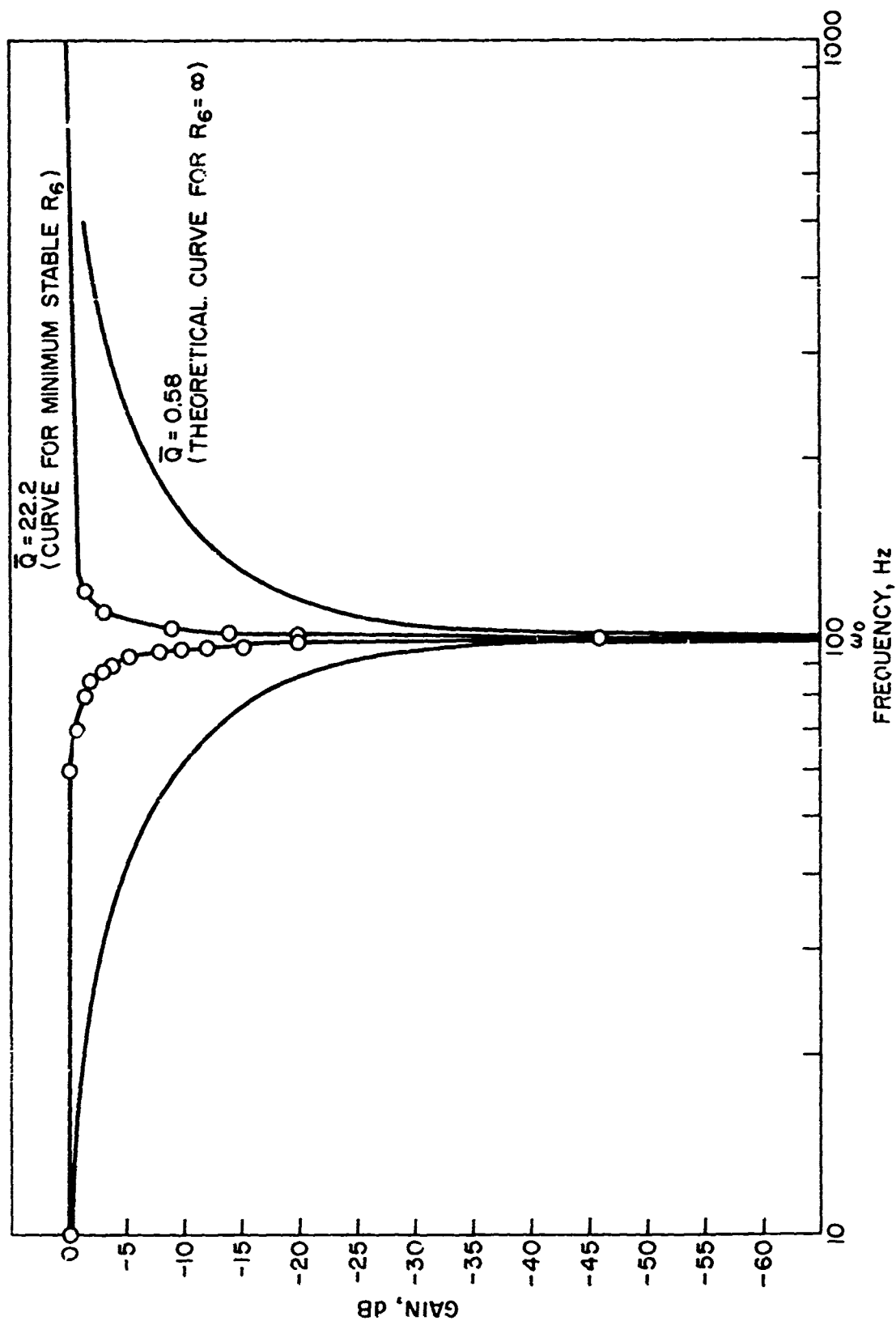Fig. 4 — Typical gain curve of dual notch filter.

Fig. 5 - Maximum $\bar{Q}$ of design example vs a bridged-T with no feedback loop.
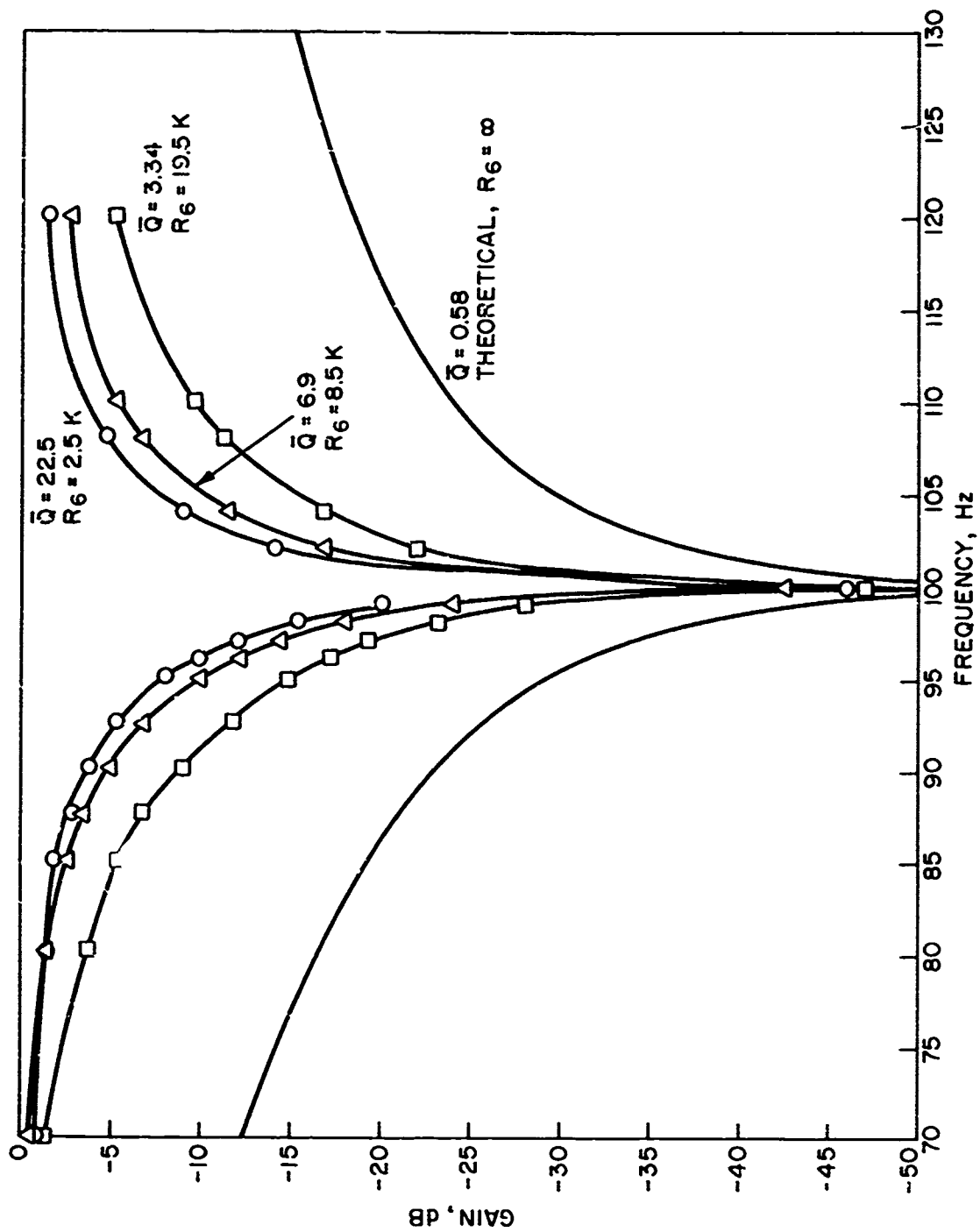
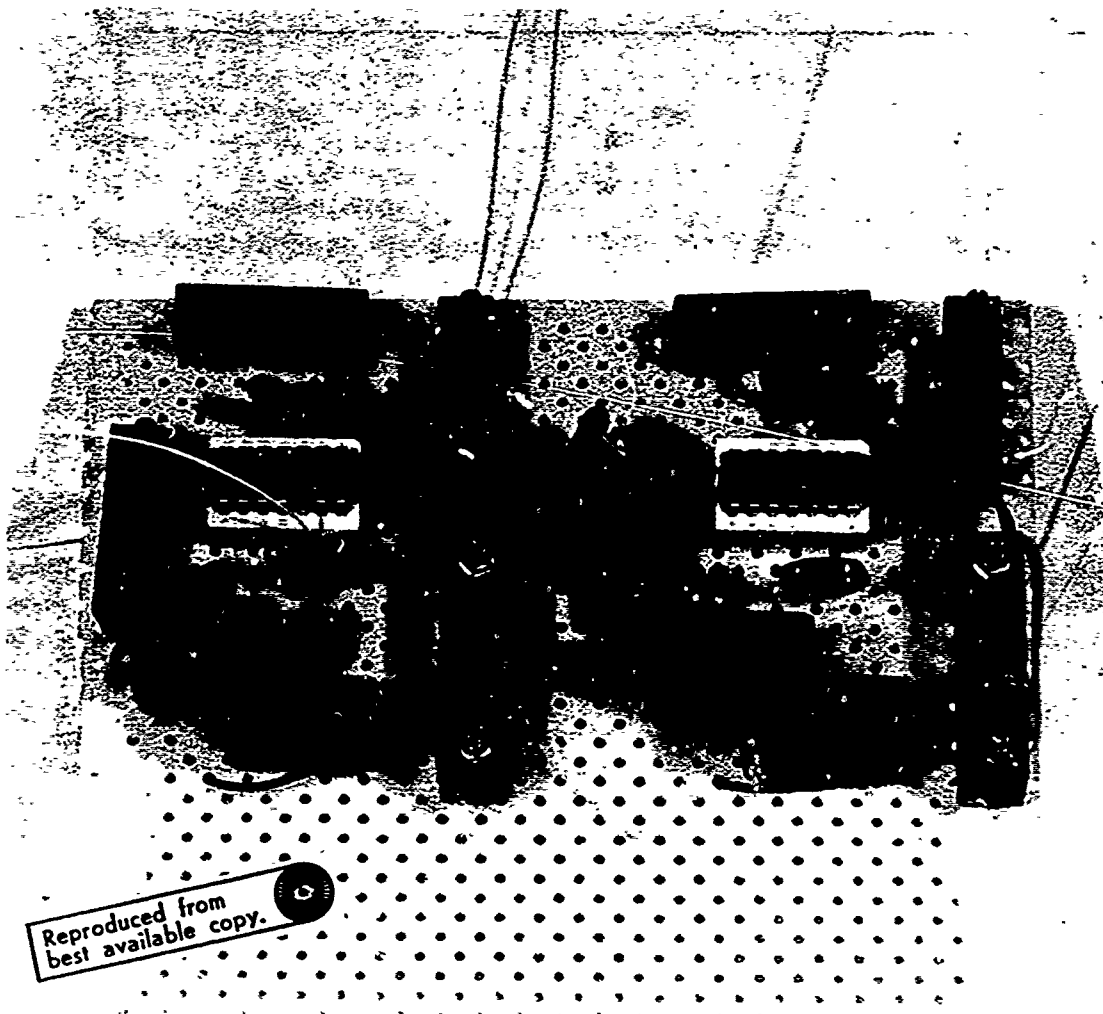Fig. 6 – Typical $\bar{Q}$ values available from design example.

Fig. 7 – Picture, completed dual filter.

APPENDIX

I. Output of a bridged T-Network (see Figure A) for low driving impedance and high load impedance:

1. Assuming $Z_L = \infty$

$$Z_S = 0$$

and let $\omega_o = 1/RC$

$$A = -i\,\frac{\omega_o}{\omega}\,.$$

Then it can be shown that

$$\frac{e_2}{e_{in}} = \frac{A^2 + 2A}{A^2 + 3A + 1}$$

$$\frac{e_1}{e_{in}} = \frac{1}{A^2 + 3A + 1}$$

$$\frac{e_{out}}{e_{in}} = \frac{e_1 + e_2}{e_{in}} = \frac{A^2 + 2A + 1}{A^2 + 3A + 1}$$

2. Thus

$$\frac{e_{out}}{e_{in}} = \frac{-\left(\frac{\omega_o}{\omega}\right)^2 - 2i\left(\frac{\omega_o}{\omega}\right) + 1}{-\left(\frac{\omega_o}{\omega}\right)^2 - 3i\left(\frac{\omega_o}{\omega}\right) + 1}$$

$$= \frac{\left[\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} - 2i\right]\left[\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} + 3i\right]}{\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2 + 9}$$

$$= \frac{\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2 + 6}{\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2 + 9} + i\,\frac{\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)}{\left(\frac{\omega}{\omega_o} + \frac{\omega_o}{\omega}\right)^2 + 9}$$

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3. Letting $X = \left(\dfrac{\omega}{\omega_o} - \dfrac{\omega_o}{\omega}\right)$

$$\frac{e_{out}}{e_{in}} = \frac{X^2 + 6}{X^2 + 9} + i\,\frac{X}{X^2 + 9}$$

$$= \frac{5}{6} + \frac{1}{6}\left[\frac{X^2 - 9}{X^2 + 9} + i\,\frac{6X}{X^2 + 9}\right]$$

or

$$\frac{e_{out}}{e_{in}} = \frac{5}{6} + \frac{1}{6}\,e^{i\alpha}$$
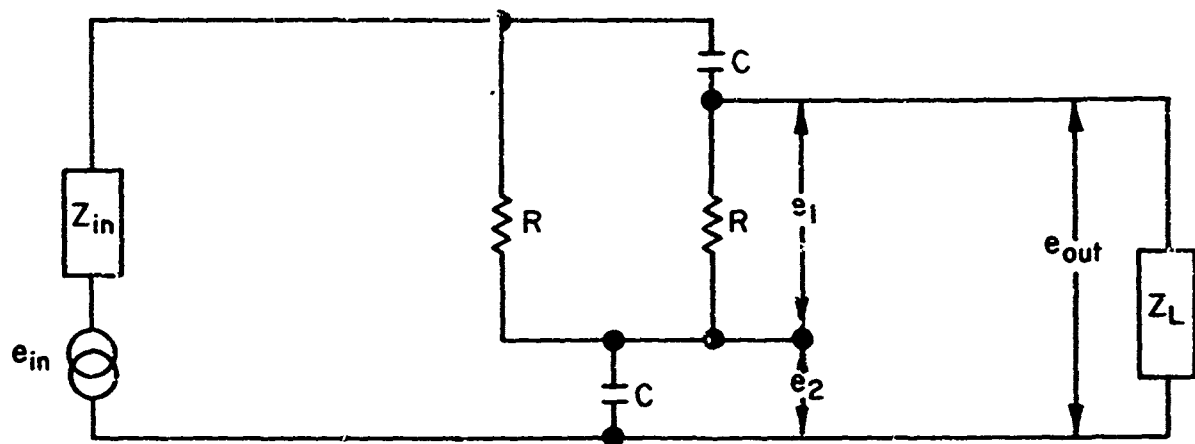
where

$$\cos\alpha + i\sin\alpha = \frac{X^2 - 9}{X^2 + 9} + i\,\frac{6X}{X^2 + 9} \quad .$$

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Fig. A - Bridged T-network.