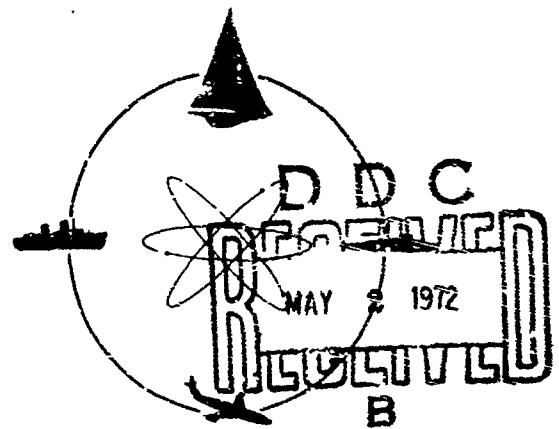


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DAVIDSON LABORATORY

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February 1972

PROPELLER-INDUCED VELOCITY FIELD BY
MEANS OF UNSTEADY LIFTING SURFACE THEORY

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by

W.R. Jacobs and S. Tsakonas

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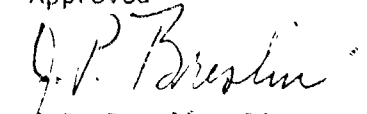
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KEY WORDS

Hydrodynamics

Unsteady Theory for Propeller-Induced Velocity Field

Table of Contents

Abstract	iii
Nomenclature	vii
Introduction	1
Velocity Potential due to Blade Loading	2
Velocity Field	6
Velocity Components	6
Influence Functions	9
Steady and Blade-Frequency Velocity Field for Uniform and Nonuniform Inflows	14
Numerical Results	16
Conclusion	18
References	21
Figures	
Appendix A	
Appendix B	
Appendix C	
Appendix D	

NOMENCLATURE

a	Ω/U
$f(u)$	function defined in Eq (19)
$g_{1,2}(u)$	functions defined in Eq (18a,b)
$h(u)$	function defined in Eq (17)
$I_m(x)$	modified Bessel function of first kind of order m and argument x
$J_n(x)$	Bessel function of order n and argument x
$K_m(x)$	modified Bessel function of second kind
$K^{(m)}$	space function
$\bar{K}_{x,r,\varphi}^{(m,\bar{n})}$	derivatives with respect to x, r, φ , respectively, of modified space function (after chordwise integration)
k	variable of integration
$L(\rho, \theta_\alpha)$	blade loading, lb/ft
l	integer multiple
m	integer
N	number of blades
n	blade index
$\bar{n}$	order of chordwise modes
$\Delta P(\xi, \rho, \theta_o)$	pressure jump across propeller blade, lb/ft ²
q	order of harmonic of velocity field
$\bar{r}$	Descartes distance
r	radial ordinate of field point
r_o	propeller radius
S	lifting surface

t	time, seconds
U	free stream velocity
u	$= k + a\ell N$, variable of integration
x, y, z	Cartesian coordinates of field point
x, r, φ	cylindrical coordinates of field point
$\delta(x)$	Dirac delta function of x
θ_o	angular position of point on propeller blade, in propeller plane, with respect to midchord line
θ_α	angular chordwise location of point on blade
θ_b	projected propeller semichord length, radians
$\bar{\theta}_n$	$= (2\pi/N)(n-1), n=1, 2, \dots, N$
$\Lambda^{(\bar{n})}(x)$	defined in Appendix A
λ	order of harmonic of loading
ξ, ρ, θ_o	cylindrical coordinates of point on propeller blade
ρ	radial ordinate of point on propeller blade
ρ_f	mass density of fluid, slugs/ft ³
σ	angular measure of skewness
τ	variable of integration
Φ	velocity potential
φ	angular position of field point
Ω	angular velocity of propeller

INTRODUCTION

The characteristics of the oscillatory propeller-induced velocity and pressure fields constitute the essentials for knowledge of the vibratory effects of marine propellers on nearby bodies (hull+appendages). Since present technical developments put great emphasis on increased ship speed these vibratory effects have become a very important problem to the naval architect.

The basic tools in the endeavor to evaluate the pressure and velocity field generated by the propeller as a lifting surface operating in nonuniform flow are found in the fundamental Davidson Laboratory studies⁽¹⁻⁵⁾ where the loading problem has been treated by means of the acceleration potential method. The resulting integral equation, which relates the unknown loading distribution to the known velocity distribution normal to the blades, has been solved by using the mode approach and collocation method in conjunction with the so-called "generalized lift operator" technique.

The vibratory pressure field has been the subject of a study⁽⁶⁾ in which the analysis and numerical procedure have been developed and adapted to a high speed digital computer. With the knowledge thus acquired of the vibratory pressure field around an operating propeller, the vibratory effects on a nearby flat boundary can be determined approximately by integrating double the free space pressure over the flat surface.

This information, however, will not be sufficient for the study of vibratory effects in the case of a body of arbitrary shape, if great accuracy is required. The presence of a nearby body or series of bodies should be taken into account by studying the mutual interaction between the marine propeller and the neighboring bodies. The propeller-rudder interaction problem has been the subject of a study⁽⁷⁾ which has considered the complete interaction for any given geometry of the interacting surfaces when both are immersed in nonuniform flow. The hull-propeller-rudder interaction problem can be tackled in similar fashion and thus the long range objective

of evaluating the propeller-induced vibratory effects on the hull is within reach.

An essential requirement of such a study is detailed knowledge of the vibratory velocity field. The present study investigates the propeller-induced vibratory velocity theoretically and on this basis develops a numerical scheme adaptable to a high speed computer for evaluating the velocity field. Thus the existing program, which evaluates the steady and unsteady propeller loadings, the resulting hydrodynamic forces and moments, and the pressure field, is extended to include the velocity field as well. This program should become a highly versatile and useful tool for the ship researcher and ship designer.

The steady and unsteady velocity field around an operating propeller, in a uniform flow field only, has been studied by assuming the propeller either as an actuator disk^(8,9), i.e. a propeller with infinitely many blades, or as a lifting line model⁽¹⁰⁾. The present study, an application of the unsteady propeller lifting-surface theory, takes into account the nonuniformity of the inflow field as well as the exact propeller geometry with its finite number of blades forming helicoidal surfaces, and hence is unique.

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VELOCITY POTENTIAL DUE TO BLADE LOADING

The velocity potential at a point (x, r, φ) due to the loading of the blades of an N-bladed propeller operating in a spatially varying inflow field is given by

$$\begin{aligned} \Phi(x, r, \varphi; q) = & - \frac{1}{4\pi\rho_f U} \sum_{n=1}^N \iint_S \sum_{\lambda=0}^{\infty} \Delta P^{(\lambda)}(\xi, \rho, \theta_0) e^{i\lambda(\Omega t - \bar{\theta}_n)} \\ & \cdot \int_{-\infty}^x e^{i\lambda a(\tau-x)} \frac{\partial}{\partial n} \frac{1}{R} d\tau dS \end{aligned} \quad (1)$$

where q and λ are positive integers defining the desired frequency of the velocity field and the appropriate frequency of the propeller loading, respectively, and

x, r, φ = cylindrical coordinates of point in space referred to axes with origin at the propeller hub

ξ, ρ, θ_0 = cylindrical coordinates of loading point on propeller

U = free stream velocity, ft/sec

ρ_f = fluid density, slugs/ft³

Ω = angular velocity of the propeller, rad/sec

$a = \Omega/U$, 1/ft

$\Delta P^{(\lambda)}(\xi, \rho, \theta_0)$ = propeller loading, or pressure jump across the propeller blade, lb/ft²

$$R = \left\{ (\tau - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos \left[\theta_0 - \Omega t + \bar{\theta}_n - a(\tau - x) - \varphi \right] \right\}^{1/2}, \text{ ft}$$

$\frac{\partial}{\partial n}$ = normal derivative on the helicoidal surface at the loading point
 (ξ, ρ, θ_0) on the propeller

$$\bar{\theta}_n = \frac{2\pi}{N} (n-1) \quad n=1, 2, \dots, N$$

S = blade surface, ft²

The normal derivative to the helicoidal surface (specified by $\xi = \theta_0/a$) is

$$\frac{\partial}{\partial n} = \frac{\rho}{\sqrt{1+a^2\rho^2}} \left(a \frac{\partial}{\partial \xi} - \frac{1}{\rho^2} \frac{\partial}{\partial \theta_0} \right)$$

By making use of the expansion scheme for the reciprocal of the Descartes distance R , viz.:

$$\frac{1}{R} = \frac{1}{r} \sum_{m=-\infty}^{\infty} e^{im(\theta_0 - \Omega t + \bar{\theta}_n - a(\tau - x) - \varphi)} \int_{-\infty}^{\infty} I_m(1k|\rho) K_m(1k|r) e^{i(\tau - \xi)k} dk$$

(when $\rho < r$, otherwise ρ and r are interchanged in the modified Bessel functions)

and the facts that

$$\sum_{n=1}^N e^{-i(\lambda-m)\bar{\theta}_n} = \begin{cases} N, & \lambda - m = \ell N, \ell = 0, \pm 1, \pm 2, \dots \\ 0, & \lambda - m \neq \ell N \end{cases}$$

and from the time-dependent factor

$$\lambda - m = q = \ell N$$

it can be shown that the potential function becomes

$$\Phi(x, r, \varphi; q) = \iint_S \sum_{\lambda=0}^{\infty} \Delta P^{(\lambda)}(\xi, \rho, \theta_0) e^{i q \Omega t} K^{(m)}(x, r, \varphi; \xi, \rho, \theta_0; q) dS \quad (2)$$

where the K-function is given by

$$K^{(m)}(x, r, \varphi; \xi, \rho, \theta_0; q) = \frac{-N\rho}{4\pi^2 \rho_f U \sqrt{1+a^2 \rho^2}} \sum_{\substack{m=-\infty \\ m=\lambda-\ell N}}^{\infty} e^{-im\varphi} \\ \cdot \int_{-\infty}^x e^{ia\ell N(\tau-x)} \left(a \frac{\partial}{\partial \xi} - \frac{1}{\rho^2} \frac{\partial}{\partial \theta_0} \right) \left[e^{i(m)\theta_0} \int_{-\infty}^{\infty} I_m(|k|\rho) K_m(|k|r) e^{i(\tau-\xi)k} dk \right] d\tau \quad (3)$$

It is seen that theoretically an infinite number of frequency constituents of the loading function $\Delta P^{(\lambda)}$ and of the space function $K^{(m)}$ combine to give the velocity potential at blade frequency and multiples thereof, and hence the velocity at any field point. (This was observed by Breslin and Tsakonas in Ref. 11.)

The integration over the blade surface can be converted to integration over its projection in the propeller plane, so that Eq (2) becomes

$$\Phi(x, r, \varphi; q) = \int_{\rho} \int_{-\theta_b}^{\theta_b} \sum_{\lambda=0}^{\infty} \Delta P^{(\lambda)}(\xi, \rho, \theta_0) e^{i q \Omega t} K^{(m)} \frac{\sqrt{1+a^2 \rho^2}}{a\rho} \rho d\theta_0 d\rho$$

where θ_b is the projected semichord at each spanwise location ρ , in radians. Since

$$\theta_0 = \sigma - \theta_b \cos \theta_\alpha = a\xi \quad (4)$$

where σ is angular position of the midchord line of the projected blade

from the generator line through the hub, or skewness, and θ_α is angular chordwise location of the loading point,

$$\Phi(x, r, \varphi; q) = \int_0^\pi \int_0^\infty \sum_{\lambda=0}^\infty L^{(\lambda)}(\rho, \theta_\alpha) e^{i q \Omega t} K^{(m)} \frac{\sqrt{1+a^2 \rho^2}}{a \rho} \sin \theta_\alpha d\theta_\alpha d\rho \quad (5)$$

with $L^{(\lambda)}(\rho, \theta_\alpha) = \Delta P^{(\lambda)}(\xi, \rho, \theta_\alpha) \cdot \rho \theta_b$, lb/ft

On taking the derivatives in (3)

$$K^{(m)} = K^{(m)} \frac{\sqrt{1+a^2 \rho^2}}{a \rho} = \frac{+iN}{4\pi^2 \rho_f U a} \sum_{m=-\infty}^\infty e^{im(\theta_0 - \varphi)} \cdot \int_{-\infty}^x e^{ia\ell N(\tau-x)} \int_{-\infty}^\infty (ak + \frac{m}{2}) I_m(|k|\rho) K_m(|k|r) e^{i(\tau-\xi)k} dk d\tau \quad (6)$$

The τ -integral of (6) involves

$$\int_{-\infty}^x e^{i(a\ell N+k)\tau} d\tau = \pi \delta(a\ell N+k) - \frac{e^{i(k+a\ell N)x}}{k+a\ell N} \quad (7)$$

where $\delta()$ is the Dirac delta function. On substituting Eqs (7) and (4) in (6), it becomes

$$K^{(m)} = + \frac{iN}{4\pi \rho_f U a} \sum_{m=-\infty}^\infty e^{im(\sigma - \varphi)} \cdot \left\{ e^{i\ell N(\sigma - ax)} e^{-i\lambda \theta_b \cos \theta_\alpha} I_m(a|\ell N|\rho) K_m(a|\ell N|r) [-a^2 \ell N + m/\rho^2] \right. \\ \left. - \frac{i}{\pi} \int_{-\infty}^\infty \frac{I_m(|k|\rho) K_m(|k|r) (ak + m/2)}{k + a\ell N} e^{ik(x-\sigma/a)} e^{-i(m-k/a)\theta_b \cos \theta_\alpha} dk \right\} \quad (8)$$

(for $\rho < r$, otherwise ρ and r are interchanged in the modified Bessel functions.) Here only ρ_f and U are dimensional and x, ρ, r, a and k have been nondimensionalized with respect to propeller radius r_0 as has ρ in Eq (5).

The loading function $L^{(\lambda)}(\rho, \theta_\alpha)$ is approximated in the chordwise direction by the Birnbaum mode shapes:

$$L^{(\lambda)}(\rho, \theta_\alpha) = \frac{1}{\pi} \left\{ L^{(\lambda, 1)}(\rho) \cot \frac{\theta_\alpha}{2} + \sum_{\bar{n}=2}^{\infty} L^{(\lambda, \bar{n})}(\rho) \sin(\bar{n}-1)\theta_\alpha \right\}$$

The propeller loading distribution at any frequency is now given in terms of a spanwise (ρ) loading distribution which is available from the existing program^(12,13) and a chordwise (θ_α) distribution of $\bar{n}$ modes which will allow the chordwise integration to be performed analytically.

After the chordwise θ_α -integration, Eq (5) becomes

$$\Phi(x, r, \varphi; q) = \int_{\rho} \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} L^{(\lambda, \bar{n})}(\rho) e^{i\varphi \Omega t} \bar{K}(m, \bar{n}) d\rho \quad (9)$$

where the modified space function is

$$\bar{K}(m, \bar{n}) = \frac{1}{\pi} \int_0^{\pi} K_1^{(m)} \cdot \Theta(\bar{n}) \sin \theta_\alpha d\theta_\alpha$$

$$\Theta(1) = \cot \frac{\theta_\alpha}{2}$$

$$\Theta(\bar{n}) = \sin(\bar{n}-1)\theta_\alpha, \quad \bar{n} > 1$$

With $\lambda - m = q = \ell N, \ell = 0, \pm 1, \pm 2 \dots$, for each ℓ

$$\begin{aligned} \bar{K}(m, \bar{n}) = & + \frac{iNe^{im(\sigma-\varphi)}}{4\pi\rho_f Ua} \left\{ e^{i\ell N(\sigma-ax)} \left(\frac{m}{\rho^2} - a^2 \ell N \right) I_m(a|\ell N|\rho) K_m(a|\ell N|r) \Lambda^{(\bar{n})}(\lambda_{\theta_b}) \right. \\ & - \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ik(x-\sigma/a)} \frac{(m/\rho^2 + ak)}{k + a\ell N} I_m(|k|\rho) K_m(|k|r) \\ & \left. \cdot \Lambda^{(\bar{n})}((m - \frac{k}{a})\theta_b) dk \right\} \quad (10) \end{aligned}$$

for $\rho < r$ (for $\rho > r$, ρ and r are interchanged in the modified Bessel functions). The symbol $\Lambda^{(\bar{n})}(x)$ is defined in Appendix A.

VELOCITY FIELD

The velocities in the x , y and z directions of the Cartesian coordinate system with origin at the propeller hub at a point x, r, φ due to the propeller operating in spatially varying inflow will be the derivatives of the velocity potential Φ with respect to x, y and z , respectively. Since

$$z = r \cos \varphi$$

$$y = r \sin \varphi$$

the derivatives with respect to y and z can be obtained in terms of derivatives with respect to the cylindrical coordinates r and φ as

$$\frac{\Phi}{U} \frac{y}{U} = \frac{\Phi}{U} \frac{r}{U} \sin \varphi + \frac{1}{r} \frac{\Phi}{U} \frac{\varphi}{U} \cos \varphi$$

$$\frac{\Phi}{U} \frac{z}{U} = \frac{\Phi}{U} \frac{r}{U} \cos \varphi - \frac{1}{r} \frac{\Phi}{U} \frac{\varphi}{U} \sin \varphi$$

in non-dimensional form.

Velocity Components

The derivatives with respect to x , r and φ determine the axial, radial and tangential components of the velocity.

Axial component:

$$\frac{\Phi}{U} \frac{x}{U} (x, r, \varphi; q) = \int_0^\infty \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} L(\lambda, \bar{n}) (\rho) e^{i\varphi \Omega t} \bar{K}_x^{(m, \bar{n})} d\rho \quad (11)$$

where $m = \lambda - \ell N = \lambda - q$, and for each ℓ and q

$$\begin{aligned} \bar{K}_x^{(m, \bar{n})} = & - \frac{N e^{im(\sigma - \varphi)}}{4\pi \rho_f U^2 a r_o} \left\{ -a \ell N \left(\frac{m}{\rho^2} - a^2 \ell N \right) e^{i \ell N (\sigma - a x)} I_m(a | \ell N | \rho) \right. \\ & \cdot K_m(a | \ell N | r) \Lambda^{(\bar{n})}(\lambda \theta_b) \\ & \left. - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{k(a k + m/\rho^2)}{k + a \ell N} I_m(|k| \rho) K_m(|k| r) e^{ik(x - \sigma/a)} \right. \\ & \left. \cdot \Lambda^{(\bar{n})}((m - k/a) \theta_b) dk \right\} \quad (12) \end{aligned}$$

for $\rho \leq r$, otherwise ρ and r are interchanged in the modified Bessel functions.

Radial component:

$$\frac{\bar{\Phi}_r}{U}(x, r, \varphi; q) = \int_{\rho} \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} L(\lambda, \bar{n}) (\rho) e^{i q \Omega t} \bar{K}_r^{(m, \bar{n})} d\rho \quad (13)$$

where for $\rho < r$ for each ℓ and q

$$\begin{aligned} \text{a) } \bar{K}_r^{(m, \bar{n})} = & + \frac{i N e^{i m(\sigma - \varphi)}}{4 \pi \rho_f U^2 a r_0} \left\{ \frac{-a |\ell N|}{2} \left(\frac{m}{\rho^2} - a^2 \ell N \right) e^{i \ell N(\sigma - a x)} I_m(a |\ell N| \rho) \right. \\ & \cdot \left[K_{m-1}(a |\ell N| r) + K_{m+1}(a |\ell N| r) \right] \Lambda^{(\bar{n})}(\lambda \theta_b) \\ & + \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{|k| (a k + m/\rho^2)}{k + a \ell N} I_m(|k| \rho) \left[K_{m-1}(|k| r) + K_{m+1}(|k| r) \right] \\ & \cdot e^{i k(x - \sigma/a)} \Lambda^{(\bar{n})}((m - k/a) \theta_b) dk \Big\} \quad (14a) \end{aligned}$$

for $\rho > r$

$$\begin{aligned} \text{b) } \bar{K}_r^{(m, \bar{n})} = & + \frac{i N e^{i m(\sigma - \varphi)}}{4 \pi \rho_f U^2 a r_0} \left\{ \frac{+a |\ell N|}{2} \left(\frac{m}{\rho^2} - a^2 \ell N \right) e^{i \ell N(\sigma - a x)} K_m(a |\ell N| \rho) \cdot \right. \\ & \left[I_{m-1}(a |\ell N| r) + I_{m+1}(a |\ell N| r) \right] \Lambda^{(\bar{n})}(\lambda \theta_0) \\ & - \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{|k| (a k + m/\rho^2)}{k + a \ell N} K_m(|k| \rho) \left[I_{m-1}(|k| r) + I_{m+1}(|k| r) \right] \\ & \cdot e^{i k(x - \sigma/a)} \Lambda^{(\bar{n})}((m - k/a) \theta_b) dk \Big\} \quad (14b) \end{aligned}$$

and for $\rho = r$ it can be shown that

$$\text{c) } \bar{K}_r^{(m, \bar{n})}(\rho=r) = [\text{Eq 14(a)} + \text{Eq 14(b)}] \text{ evaluated at } \rho=r \quad (14c)$$

Tangential component:

$$\frac{\bar{\Phi}_{\varphi}(x, r, \varphi; q)}{r r_0 U} = \frac{1}{r} \int_{\rho} \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} L(\lambda, \bar{n}) (\rho) e^{i q \Omega t} \bar{K}_{\varphi}^{(m, \bar{n})} d\rho \quad (15)$$

where r is nondimensional with respect to r_0 propeller radius and, for $\rho \leq r$ and each ℓ or q

$$\begin{aligned} \bar{K}_{\varphi}^{(m, \bar{n})} = & + \frac{N m e^{i m (\sigma - \varphi)}}{4 \pi \rho_f U^2 a r_o} \left\{ \left(\frac{m}{\rho^2} - a^2 \ell N \right) e^{i \ell N (\sigma - a x)} I_m(a | \ell N | \rho) K_m(a | \ell N | r) \Lambda^{(\bar{n})}(\lambda \theta_b) \right. \\ & - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(a k + m / \rho^2)}{k + a \ell N} I_m(|k| \rho) K_m(|k| r) e^{i k (x - \sigma / a)} \\ & \left. \cdot \Lambda^{(\bar{n})}((m - k/a) \theta_b) dk \right\} \quad (16) \end{aligned}$$

For $\rho > r$, ρ and r are interchanged in the modified Bessel functions.

The integral terms of the modified kernels (Eqs 12, 14, 16) all have Cauchy-type singularities at variable points depending on values of $\ell=0, \pm 1, \pm 2, \dots$. Use of a transformation $u=k+a\ell N$ fixes the singular point at the origin, thus avoiding the complications arising in the numerical integration from singular points at varying values of k .

Influence Functions

Letting $u=k+a\ell N$, Eq (12) becomes

$$\begin{aligned} \bar{K}_x^{(m, \bar{n})} = & - \frac{N e^{i (\lambda \sigma - m \varphi - a \ell N x)}}{4 \pi \rho_f U^2 a r_o} \\ & \cdot \left\{ a \ell N (a^2 \ell N - m / \rho^2) I_m(a | \ell N | \rho) K_m(a | \ell N | r) \Lambda^{(\bar{n})}(\lambda \theta_b) \right. \\ & \left. - \frac{i}{\pi} \int_0^{\infty} \frac{h(u) - h(-u)}{u} du \right\} \quad \text{for } \rho \leq r \quad (17) \end{aligned}$$

$$\begin{aligned} \text{where } h(u) = & (u - a \ell N) (a u - a^2 \ell N + m / \rho^2) I_m(|u - a \ell N| \rho) K_m(|u - a \ell N| r) e^{i u (x - \sigma / a)} \\ & \cdot \Lambda^{(\bar{n})}((\lambda - u/a) \theta_b) \end{aligned}$$

The integrand at $u=0$ is evaluated in Appendix B as

a) when $q=\ell=0$, $m=\lambda=0$

$$\lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] = 0$$

b) When $q=l=0$, $m=\lambda \neq 0$

$$\lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] = \begin{cases} (1/\rho^2)(\rho/r)^m \Lambda^{(\bar{n})}(\lambda \theta_b) & \text{for } \rho \leq r \\ (1/\rho^2)(r/\rho)^m \Lambda^{(\bar{n})}(\lambda \theta_b) & \text{for } \rho \geq r \end{cases}$$

c) When $q=lN \neq 0$

$$\begin{aligned} \lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] = & -2 \left\{ I_m(a|lN|\rho) K_m(a|lN|r) \right\} \\ & \cdot \left[2a^2 lN - m/\rho^2 - i(x - \sigma/a) a lN (a^2 lN - m/\rho^2) \right] \Lambda^{(\bar{n})}(\lambda \theta_b) \\ & - i \theta_b lN (a^2 lN - m/\rho^2) \Lambda_1^{(\bar{n})}(\lambda \theta_b) \Big] \\ & + \left\{ \rho K_m(a|lN|r) \left[I_{m-1}(a|lN|\rho) + I_{m+1}(a|lN|\rho) \right] - r I_m(a|lN|\rho) \right. \\ & \cdot \left. \left[K_{m-1}(a|lN|r) + K_{m+1}(a|lN|r) \right] \right\} \cdot a lN (a^2 lN - m/\rho^2) \Lambda^{(\bar{n})}(\lambda \theta_b), \text{ for } \rho \leq r \end{aligned}$$

Where the upper sign (-) is used when $l > 0$ and the lower sign (+) when $l < 0$. When $\rho > r$, ρ and r are interchanged in the factors in braces (involving modified Bessel functions and their derivatives).

The closed term of (17) can be shown to be equal to zero when $q=l=0$.

Equation (14a) for $\rho < r$ becomes with $u=k+a lN$

$$\begin{aligned} \bar{K}_r^{(m, \bar{n})}(\rho < r) = & \frac{-N e^{i(\lambda \sigma - m \varphi - a l N x)}}{4 \pi \rho_f U^2 a r_0} \\ & \cdot \left\{ -\frac{1}{2} a |lN| (a^2 lN - m/\rho^2) I_m(a|lN|\rho) \left[K_{m-1}(a|lN|r) + K_{m+1}(a|lN|r) \right] \Lambda^{(\bar{n})}(\lambda \theta_b) \right. \\ & \left. + \frac{1}{2\pi} \int_0^\infty \frac{g_1(u) - g_1(-u)}{u} du \right\} \end{aligned} \quad (18a)$$

$$\begin{aligned} \text{where } g_1(u) = & |u - a lN| (a u - a^2 lN - m/\rho^2) I_m(|u - a lN|\rho) \left[K_{m-1}(|u - a lN|r) + K_{m+1}(|u - a lN|r) \right] \\ & \cdot e^{i u (x - \sigma/a)} \Lambda^{(\bar{n})}((\lambda - u/a) \theta_b) \end{aligned}$$

It can be easily shown that when $q=l=0$ the closed term of (18a) becomes

$$(1/2)(\rho/r)^m (m/r\rho^2) \Lambda^{(\bar{n})}(\lambda\theta_b)$$

where $m=\lambda$ since $q=0$.

The integrand at $u=0$ is evaluated in Appendix C as

a) When $q=\ell=0$, $m=\lambda=0$

$$\lim_{u \rightarrow 0} \left[\frac{g_1(u) - g_1(-u)}{u} \right] = (4a/r) \Lambda^{(\bar{n})}(0)$$

b) When $q=\ell=0$, $m=\lambda \neq 0$

$$\begin{aligned} \lim_{u \rightarrow 0} \left[\frac{g_1(u) - g_1(-u)}{u} \right] = & (2/r)(\rho/r)^m \left\{ [a + (im/\rho^2)(x - \sigma/a)] \Lambda^{(\bar{n})}(\lambda\theta_b) \right. \\ & \left. + (im/\rho^2)(\theta_b/a) \Lambda_1^{(\bar{n})}(\lambda\theta_b) \right\} \end{aligned}$$

c) When $q=\ell N \neq 0$

$$\begin{aligned} \lim_{u \rightarrow 0} \left[\frac{g_1(u) - g_1(-u)}{u} \right] = & \pm 2 \left\{ I_m(a|\ell N|\rho) [K_{m-1}(a|\ell N|r) + K_{m+1}(a|\ell N|r)] \right. \\ & \cdot \left\{ 2a^2 \ell N - m/\rho^2 - i(x - \sigma/a)(a\ell N)(a^2 \ell N - m/\rho^2) \right\} \Lambda^{(\bar{n})}(\lambda\theta_b) \\ & - i\theta_b \ell N (a^2 \ell N - m/\rho^2) \Lambda_1^{(\bar{n})}(\lambda\theta_b) \Big\} \\ & + a\ell N (a^2 \ell N - m/\rho^2) \Lambda^{(\bar{n})}(\lambda\theta_b) \\ & \cdot \left\{ \rho [I_{m-1}(a|\ell N|\rho) + I_{m+1}(a|\ell N|\rho)] [K_{m-1}(a|\ell N|r) + K_{m+1}(a|\ell N|r)] \right. \\ & \left. - r I_m(a|\ell N|\rho) [K_{m-2}(a|\ell N|r) + 2K_{m-1}(a|\ell N|r) + K_{m+2}(a|\ell N|r)] \right\} \end{aligned}$$

where the upper sign (+) is used when $\ell > 0$ and the lower sign (-) when $\ell < 0$.

Equation (14b) for $\rho > \cdot$ becomes with $u=k+a\ell N$

$$\bar{K}_r^{(m, \bar{n})}(\rho > r) = - \frac{Ne^{i(\lambda\sigma - m\phi - a\ell N x)}}{4\pi\rho_f U^2 a r_0} \cdot \left\{ (1/2)a|\ell N|(a^2\ell N - m/\rho^2)K_m(a|\ell N|\rho) \left[I_{m-1}(a|\ell N|r) + I_{m+1}(a|\ell N|r) \right] \Lambda^{(\bar{n})}(\lambda\theta_b) \right. \\ \left. + (1/2\pi) \int_{-\infty}^{\infty} \frac{g_2(u) - g_2(-u)}{u} du \right\} \quad (18b)$$

$$\text{where } g_2(u) = -|u - a\ell N|(au - a^2\ell N + m/\rho^2)K_m(|u - a\ell N|\rho) \left[I_{m-1}(|u - a\ell N|r) + I_{m+1}(|u - a\ell N|r) \right] \\ \cdot e^{iu(x - \sigma/a)} \Lambda^{(\bar{n})}((\lambda - u/a)\theta_b)$$

In this case when $q=\ell=0$ the closed term is

$$-(i/2)(r/\rho)^m (m/r\rho^2) \Lambda^{(\bar{n})}(\lambda\theta_b)$$

The integrand at $u=0$ is evaluated in Appendix C as

a) when $q=\ell=0$, $m=\lambda=0$

$$\lim_{u \rightarrow 0} \left[\frac{g_2(u) - g_2(-u)}{u} \right] = 0$$

b) when $q=\ell=0$, $m=\lambda \neq 0$

$$\lim_{u \rightarrow 0} \left[\frac{g_2(u) - g_2(-u)}{u} \right] = (-2/r)(r/\rho)^m \left\{ [a + (im/\rho^2)(x - \sigma/a)] \Lambda^{(\bar{n})}(\lambda\theta_b) \right. \\ \left. + (im/\rho^2)(\theta_b/a) \Lambda_1^{(\bar{n})}(\lambda\theta_b) \right\}$$

c) when $q=\ell N \neq 0$

$$\lim_{u \rightarrow 0} \left[\frac{g_2(u) - g_2(-u)}{u} \right] = -2K_m(a|\ell N|\rho) \left[I_{m-1}(a|\ell N|r) + I_{m+1}(a|\ell N|r) \right] \\ \cdot \left\{ \left[2a^2\ell N - m/\rho^2 - i(x - \sigma/a)a\ell N(a^2\ell N - m/\rho^2) \right] \Lambda^{(\bar{n})}(\lambda\theta_b) \right. \\ \left. - i\theta_b\ell N(a^2\ell N - m/\rho^2) \Lambda_1^{(\bar{n})}(\lambda\theta_b) \right\}$$

$$\begin{aligned}
& + (a\ell N) (a^2 \ell N - m/\rho^2) \Lambda^{(\bar{n})}(\lambda \theta_b) \\
& \cdot \left\{ \rho \left[K_{m-1}(a|\ell N|\rho) + K_{m+1}(a|\ell N|\rho) \right] \left[I_{m-1}(a|\ell N|r) + I_{m+1}(a|\ell N|r) \right] \right. \\
& \quad \left. - r K_m(a|\ell N|\rho) \left[I_{m-2}(a|\ell N|r) + 2I_m(a|\ell N|r) + I_{m+2}(a|\ell N|r) \right] \right\}
\end{aligned}$$

where the upper sign (-) is used when $\ell > 0$ and the lower sign (+) when $\ell < 0$.

With $u = k + a\ell N$, Eq (14c) for $\rho = r$ becomes

$$\bar{K}_r^{(m, \bar{n})}(\rho=r) \left[\text{Eq (18a)} + \text{Eq (18b)} \right] \text{ evaluated at } \rho=r \quad (18c)$$

On substituting $u = k + a\ell N$ Eq (16) becomes

$$\begin{aligned}
\bar{K}_\varphi^{(m, \bar{n})} &= - \frac{Ne^{i(\lambda\sigma - m\varphi - a\ell N x)}}{4\pi\rho_f U^2 a r_0} \\
&\cdot \left\{ m(a^2 \ell N - m/\rho^2) I_m(a|\ell N|\rho) K_m(a|\ell N|r) \Lambda^{(\bar{n})}(\lambda \theta_b) \right. \\
&\quad \left. + (1/\pi) \int_0^\infty \left[\frac{f(u) - f(-u)}{u} \right] du \right\} \quad (19)
\end{aligned}$$

where $f(u) = (au - a^2 \ell N + m/\rho^2) I_m(|u - a\ell N|\rho) K_m(|u - a\ell N|r) e^{iu(x - \sigma/a)} \Lambda^{(\bar{n})}((\lambda - u/a)\theta_b)$

it can be shown that when $q = \ell = 0$ the closed term of (19) becomes

$$(-m/2\rho^2)(\rho/r)^m \Lambda^{(\bar{n})}(\lambda \theta_b) \quad \text{for } \rho \leq r$$

$$(-m/2\rho^2)(r/\rho)^m \Lambda^{(\bar{n})}(\lambda \theta_b) \quad \text{for } \rho \geq r$$

The integrand of (19) is evaluated for $u=0$ in Appendix D as

a) when $q = \ell = 0$, $m = \lambda = 0$

$$\lim_{u \rightarrow 0} m \left[\frac{f(u) - f(-u)}{u} \right] = 0$$

b) when $q=\ell=0$, $m=\lambda \neq 0$

$$\lim_{u \rightarrow 0} m \left[\frac{f(u) - f(-u)}{u} \right] = \left\{ \left[a + i(x - \sigma/a)m/\rho^2 \right] \Lambda^{(\bar{n})}(\lambda\theta_b) + (i\theta_b/a)(m/\rho^2) \Lambda_1^{(\bar{n})}(\lambda\theta_b) \right\} \\ \cdot \begin{cases} (\rho/r)^m & \text{for } \rho \leq r \\ (r/\rho)^m & \text{for } \rho \geq r \end{cases}$$

c) when $q=\ell N \neq 0$, $m=\lambda - \ell N$

$$\lim_{u \rightarrow 0} m \left[\frac{f(u) - f(-u)}{u} \right] = +2m \left\{ \left[a - i(x - \sigma/a)(a^2 \ell N - m/\rho^2) \right] \Lambda^{(\bar{n})}(\lambda\theta_b) \right. \\ \left. - (i\theta_b/a)(a^2 \ell N - m/\rho^2) \Lambda_1^{(\bar{n})}(\lambda\theta_b) \right\} I_m(a|\ell N|\rho K_m(a|\ell N|r)) \\ +_{-m}(a^2 \ell N - m/\rho^2) \Lambda^{(\bar{n})}(\lambda\theta_b) \\ \cdot \left\{ \rho K_m(a|\ell N|r) \left[I_{m-1}(a|\ell N|\rho) + I_{m+1}(a|\ell N|\rho) \right] \right. \\ \left. - r I_m(a|\ell N|\rho) \left[K_{m-1}(a|\ell N|r) + K_{m+1}(a|\ell N|r) \right] \right\}$$

for $\rho \leq r$, otherwise ρ and r are interchanged in the product of the modified Bessel functions and its derivative (the factor in braces in the second term). Here the upper sign (+) is used when $\ell > 0$ and the lower sign (-) when $\ell < 0$

When $m=0$, i.e. $\lambda=\ell N$ whatever the value of ℓ ,

$$\bar{K}_{\varphi}^{(m, \bar{n})} = 0$$

STEADY-STATE AND BLADE-FREQUENCY VELOCITY FIELD FOR NONUNIFORM AND UNIFORM INFLOWS

It is easily seen from Eq (9) that in the steady-state, nonuniform inflow case, when $q=\ell=0$, the velocity potential is given by

$$\Phi(x, r, \varphi; 0) = \text{R.P.} \int_{\rho} \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} L^{(\lambda, \bar{n})}(\rho) \bar{K}^{(\lambda, \bar{n})} d\rho \quad (20)$$

whereas in the unsteady blade-frequency case where $|\ell|=1$ the velocity potential becomes

$$\Phi(x, r, \varphi; N) = \text{R.P.} \left\{ e^{iN\Omega t} \int_{\rho} \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} \left\{ L^{(\lambda, \bar{n})}(\rho) \bar{K}^{(\lambda-N, \bar{n})} + \text{conj.} \left[L^{(\lambda, \bar{n})}(\rho) \bar{K}^{(\lambda+N, \bar{n})} \right] \right\} d\rho \right\}$$

where in the first term on the R.H., $\bar{K}^{(m, \bar{n})}$ is evaluated at $m=\lambda-N, \ell=1$, and in the second term at $m=\lambda+N, \ell=-1$. Therefore the blade-frequency velocity potential is

$$\Phi(x, r, \varphi; N) = \int_{\rho} \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} \left\{ L^{(\lambda, \bar{n})}(\rho) \bar{K}^{(\lambda-N, \bar{n})} + \text{conj.} \left[L^{(\lambda, \bar{n})}(\rho) \bar{K}^{(\lambda+N, \bar{n})} \right] \right\} d\rho \quad (21)$$

In the uniform inflow case, on the other hand, with $\lambda=0$, the corresponding values are:

$$\Phi(x, r, \varphi; 0) = \int_{\rho} \sum_{\bar{n}=1}^{\infty} L^{(0, \bar{n})}(\rho) \bar{K}^{(0, \bar{n})} d\rho \quad (22)$$

and

$$\begin{aligned} \Phi(x, r, \varphi; N) &= \int_{\rho} \sum_{\bar{n}=1}^{\infty} L^{(0, \bar{n})}(\rho) \left[\bar{K}^{(-N, \bar{n})} + \text{conj.} \bar{K}^{(N, \bar{n})} \right] d\rho \\ &= 2 \int_{\rho} \sum_{\bar{n}=1}^{\infty} L^{(0, \bar{n})}(\rho) \bar{K}^{(-N, \bar{n})} d\rho \end{aligned} \quad (23)$$

The velocity field is given by the derivatives with respect to x, y, z of these velocity potentials. In non-dimensional form in the steady-state, nonuniform inflow case, $q=\ell=0$, $m=\lambda$

$$\frac{v_x^{(0)}}{U} = \frac{\Phi_x(0)}{U} = \int_{\rho} \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} L^{(\lambda, \bar{n})}(\rho) \bar{K}_x^{(\lambda, \bar{n})} d\rho \quad (24a)$$

$$\frac{v_r^{(0)}}{U} = \frac{\Phi_r(0)}{U} = \int_{\rho} \sum_{\bar{n}=1}^{\infty} \sum_{\lambda=0}^{\infty} L^{(\lambda, \bar{n})}(\rho) \bar{K}_r^{(\lambda, \bar{n})} d\rho \quad (24b)$$

$$\frac{v_{\phi}^{(0)}}{U} = \frac{\phi_{\phi}(0)}{r r_o U} = \frac{1}{r} \int_{\rho} \sum_{\bar{n}=1} \sum_{\lambda=0} L(\lambda, \bar{n}) (\rho) \bar{K}_{\phi}(\lambda, \bar{n}) d\rho \quad (24c)$$

where $\bar{K}_x^{(m, \bar{n})}$, $\bar{K}_r^{(m, \bar{n})}$ and $\bar{K}_{\phi}^{(m, \bar{n})}$ are given by Eqs (17), (18a, b or c) and (19), respectively. Note that the argument of the potential refers to the value of the frequency.

In the unsteady, blade-frequency case $q=N$, $|l|=1$, $m=\lambda-\ell N$

$$\begin{aligned} \frac{v_x^{(N)}}{U} = \frac{\phi_x(N)}{U} &= \int_{\rho} \sum_{\bar{n}=1} \sum_{\lambda=0} \left\{ L(\lambda, \bar{n}) (\rho) \bar{K}_x^{(\lambda-N, \bar{n})} \right. \\ &\quad \left. + \text{conj.} \left[L(\lambda, \bar{n}) (\rho) \bar{K}_x^{(\lambda+N, \bar{n})} \right] \right\} d\rho \end{aligned} \quad (25a)$$

$$\begin{aligned} \frac{v_r^{(N)}}{U} = \frac{\phi_r(N)}{U} &= \int_{\rho} \sum_{\bar{n}=1} \sum_{\lambda=0} \left\{ L(\lambda, \bar{n}) (\rho) \bar{K}_r^{(\lambda-N, \bar{n})} \right. \\ &\quad \left. + \text{conj.} \left[L(\lambda, \bar{n}) (\rho) \bar{K}_r^{(\lambda+N, \bar{n})} \right] \right\} d\rho \end{aligned} \quad (25b)$$

$$\begin{aligned} \frac{v_{\phi}^{(N)}}{U} = \frac{\phi_{\phi}(N)}{r r_o U} &= \frac{1}{r} \int_{\rho} \sum_{\bar{n}=1} \sum_{\lambda=0} \left\{ L(\lambda, \bar{n}) (\rho) \bar{K}_{\phi}^{(\lambda-N, \bar{n})} \right. \\ &\quad \left. + \text{conj.} \left[L(\lambda, \bar{n}) (\rho) \bar{K}_{\phi}^{(\lambda+N, \bar{n})} \right] \right\} d\rho \end{aligned} \quad (25c)$$

NUMERICAL RESULTS

A numerical procedure for the evaluation of the velocity field induced by a propeller operating in spatially uniform or nonuniform inflow has been developed and incorporated in the existing program which computes the loading on the interacting blades, the resulting steady and unsteady hydrodynamic forces and moments and the pressure field.

The enlarged program has been used to calculate the steady-state and blade-frequency theoretical velocities in the x, y, and z directions in the neighborhood of the NSRDC 3-blade, 12-inch diameter, marine propeller

No. 4113 of blade area ratio $BAR = 0.6$, which had been treated in earlier studies. The blade loading distributions and resulting forces and moments had been presented in Refs 3, 5 and 12, the blade-frequency pressures in the vicinity of the propeller in Ref 6.

In Ref 6 the theoretical pressure calculations were shown to be in satisfactory agreement with the available experimental results⁽¹⁴⁾ for both uniform flow and the nonuniform flow generated by a wake screen. The present computations of the velocity field have been performed for the same conditions, viz.,

Advance ratio, J	0.833 (Design)
Tip Clearance	5% of diameter ($r = 1.1$)
Angle ϕ (clockwise from 12 M position looking aft)	0

over a range of axial distance x from the propeller plane, from -0.3 of radius (upstream) to $+0.3$ radius (downstream). The harmonic analysis of the screen wake survey was that supplied by NSRDC.

The results of the calculations are shown in the figures. Figures 1 and 2 present the real and imaginary parts of the unsteady, blade-frequency x , y and z components of the velocity, in nonuniform and uniform flow, respectively. Figures 3 and 4 depict the steady-state components.

Figures 5-7 are comparisons of the unsteady, blade-frequency velocities for uniform and nonuniform flow conditions. These figures show that nonuniformity in the flow reduces the amplitudes of the velocities in the neighborhood of the propeller. This was to be expected from the comparisons of the experimental blade-rate pressures for the same tip clearance, angle ϕ and J in nonuniform and uniform flow given in Figs 33-35 of Ref 14, where it is seen that nonuniformity in the flow increases the pressure magnitudes.*

The pressure can be computed from the velocity components by using Bernoulli's linearized equation for free-space pressure

*It is to be noted that the peak in blade-rate pressure in uniform flow in Fig 34 of Ref 14 is in error, as comparison with Figs 15 and 16 of that reference will show.

$$\begin{aligned}
 P(x, r, \varphi; t) &= -\rho_f \left(\frac{\partial \Phi}{\partial t} + U \frac{\partial \Phi}{\partial x} \right) \\
 &= -\rho_f (\Omega r r_o V_\varphi + UV_x)
 \end{aligned}$$

(The pressure on a flat plate is twice the free-space value.) Calculations of the blade-frequency pressure in a uniform flow field for the small axial range $-0.3 < x < 0.3$, using the velocity components shown in Figure 2, check quite well with the corresponding pressures on a flat plate arising from blade loading in Figure 13 of Reference 6, which compared well with experimentally obtained values.

CONCLUSION

The vibratory velocity field induced by an operating propeller in uniform and nonuniform inflow fields has been determined on the basis of lifting surface theory, and the developed numerical procedure has been incorporated in the existing program, adapted to the CDC 6600 high-speed digital computer, which evaluates the steady and unsteady propeller loadings, the resulting hydrodynamic forces and moments, and the pressure field.

The analysis demonstrates that, in contrast to the case of uniform flow, in nonuniform flow the velocity at any field point is made up of a large number of combinations of the frequency constituents of the loading function with those of the space function.

The results of calculations for a 3-blade propeller in uniform flow and in a 3-cycle screen-generated wake show that the effect of the nonuniformity in the flow is to reduce the amplitudes of the velocities in the neighborhood of the propeller. Although no experimental velocity data are available for this case, this effect of the nonuniformity is confirmed by the experimental data on pressure fields induced by the same propeller under the same conditions which show that the nonuniformity increases the pressure amplitudes. Furthermore, pressures computed from the theoretical velocity components in a uniform flow field show good agreement with both earlier calculations of the pressures due to blade loading and experimental pressure measurements.

The particular case of a propeller in a screen wake was selected for purposes of comparison and because, the wake being pure third harmonic and strong, wake measurements were accurate. Unfortunately these wake characteristics are not those of a typical ship.

It is to be noted, furthermore, that the present report has considered only the effect of blade loading on the velocity field. Blade thickness is also important. An expression for the velocity field due to this effect has been developed analytically, and will be incorporated in the present program and applied to the case of a propeller in an actual ship wake, in an extension of this investigation.

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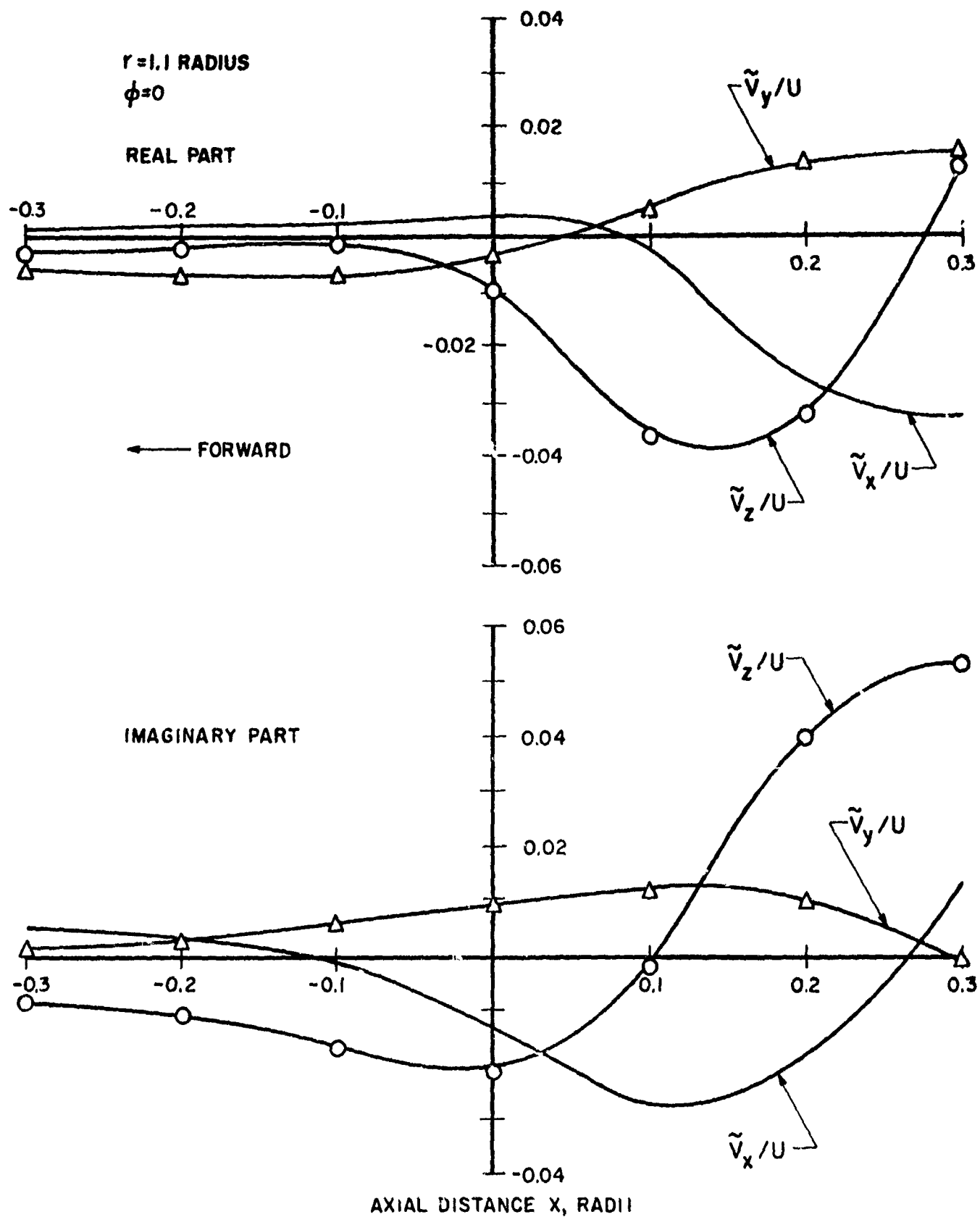


FIG. 1. BLADE-FREQUENCY VELOCITY FIELD AROUND PROPELLER 4118 IN NONUNIFORM INFLOW (3-CYCLE SCREEN WAKE)

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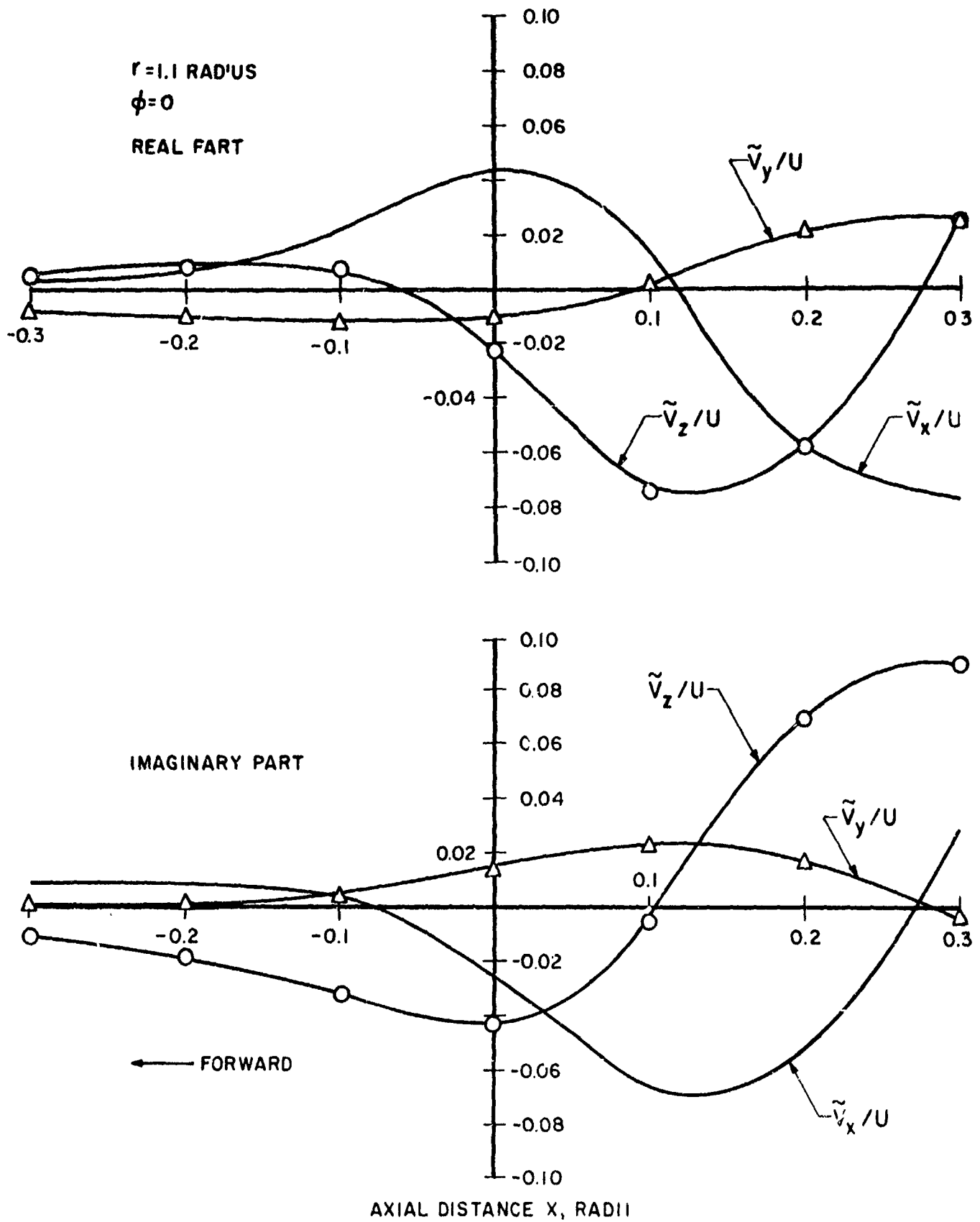


FIG. 2. BLADE-FREQUENCY VELOCITY FIELD AROUND PROPELLER 4118 IN UNIFORM INFLOW

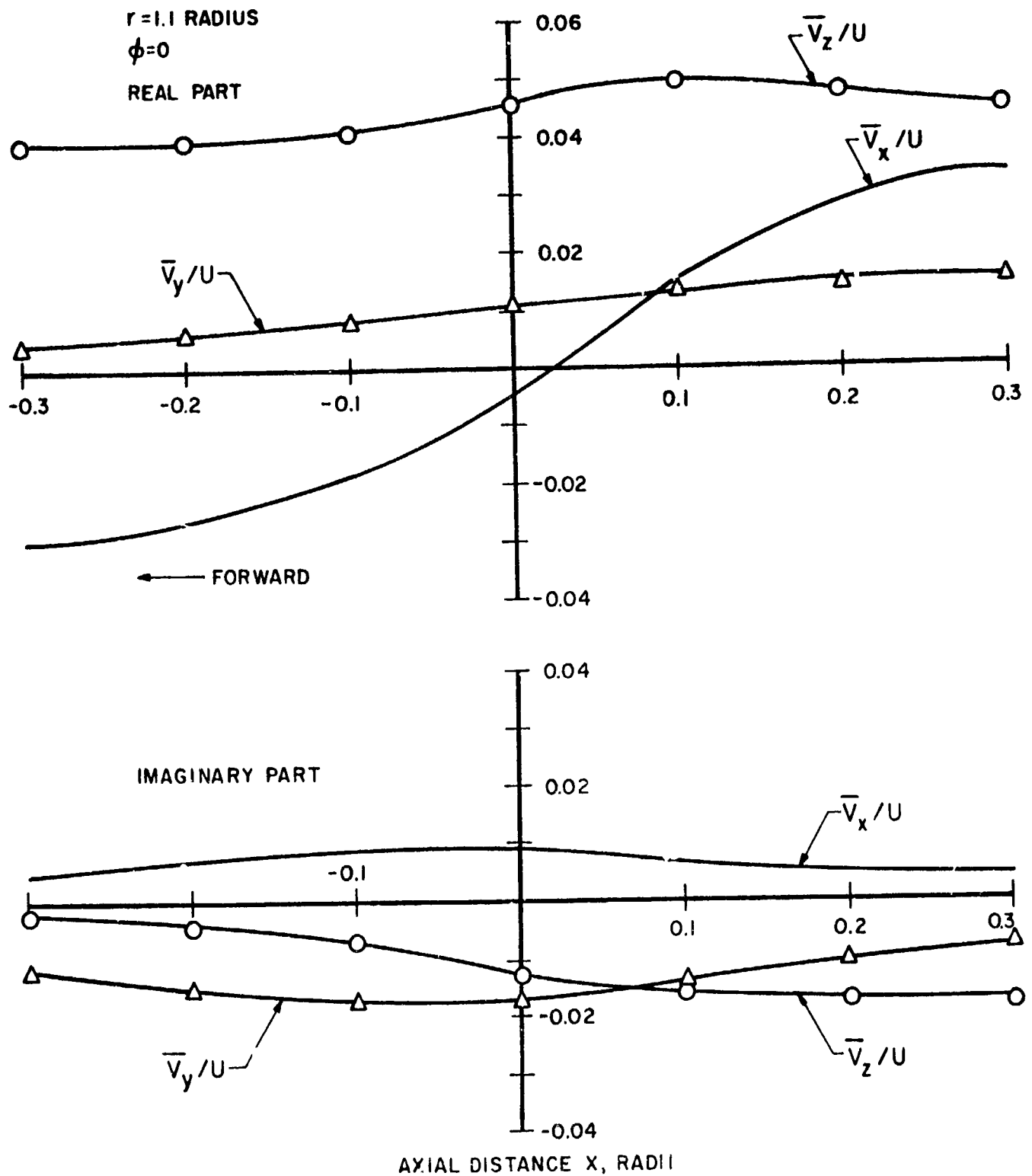


FIG. 3. STEADY-STATE VELOCITY FIELD AROUND PROPELLER 4118 IN NON-UNIFORM INFLOW (3-CYCLE SCREEN WAKE)

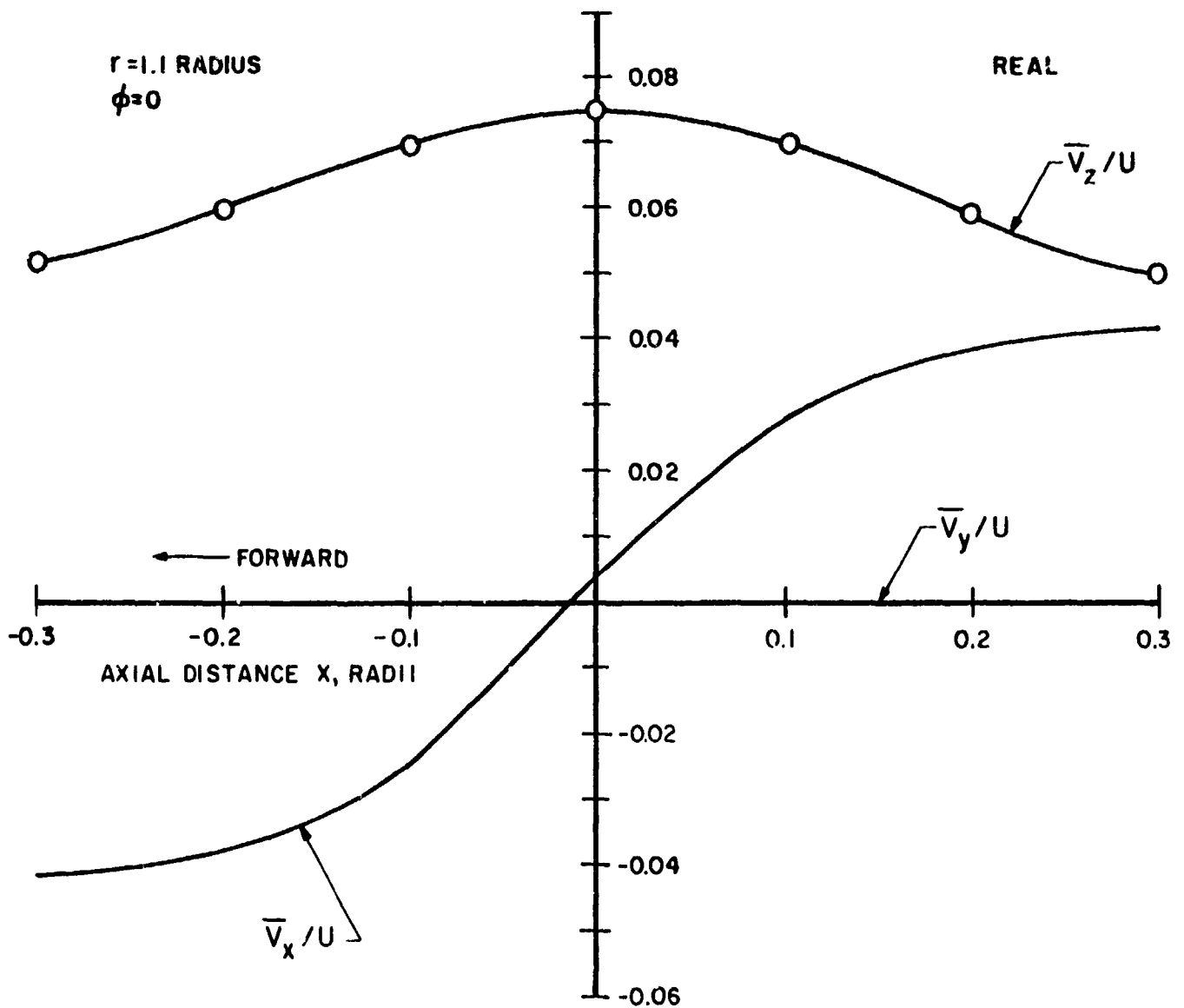


FIG. 4. STEADY-STATE VELOCITY FIELD AROUND PROPELLER 4118 IN UNIFORM FLOW

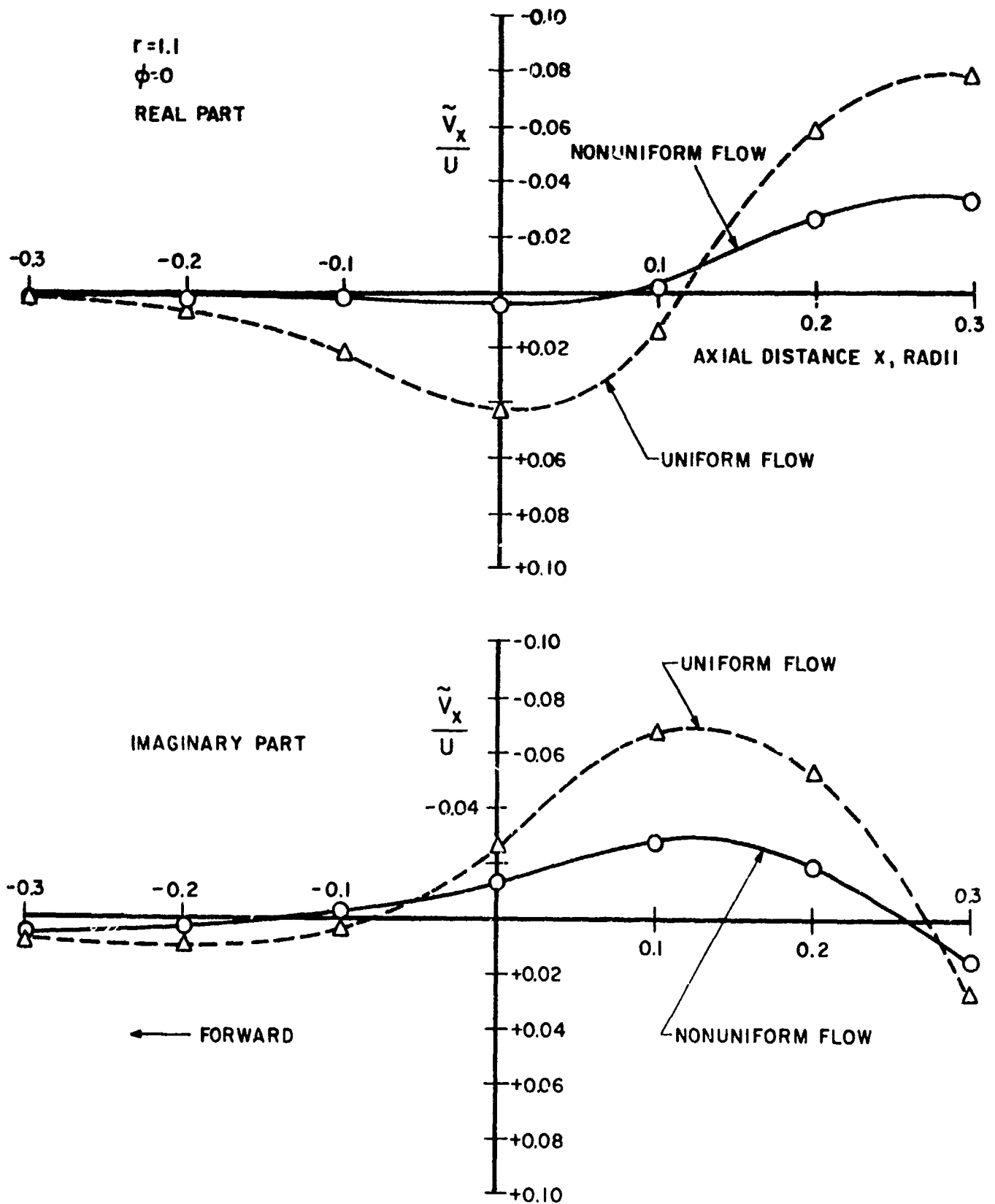


FIG. 5. UNSTEADY BLADE-FREQUENCY $\tilde{v}_x/U$ VERSUS AXIAL DISTANCE x/r_0 FOR UNIFORM AND NONUNIFORM FLOW CASES

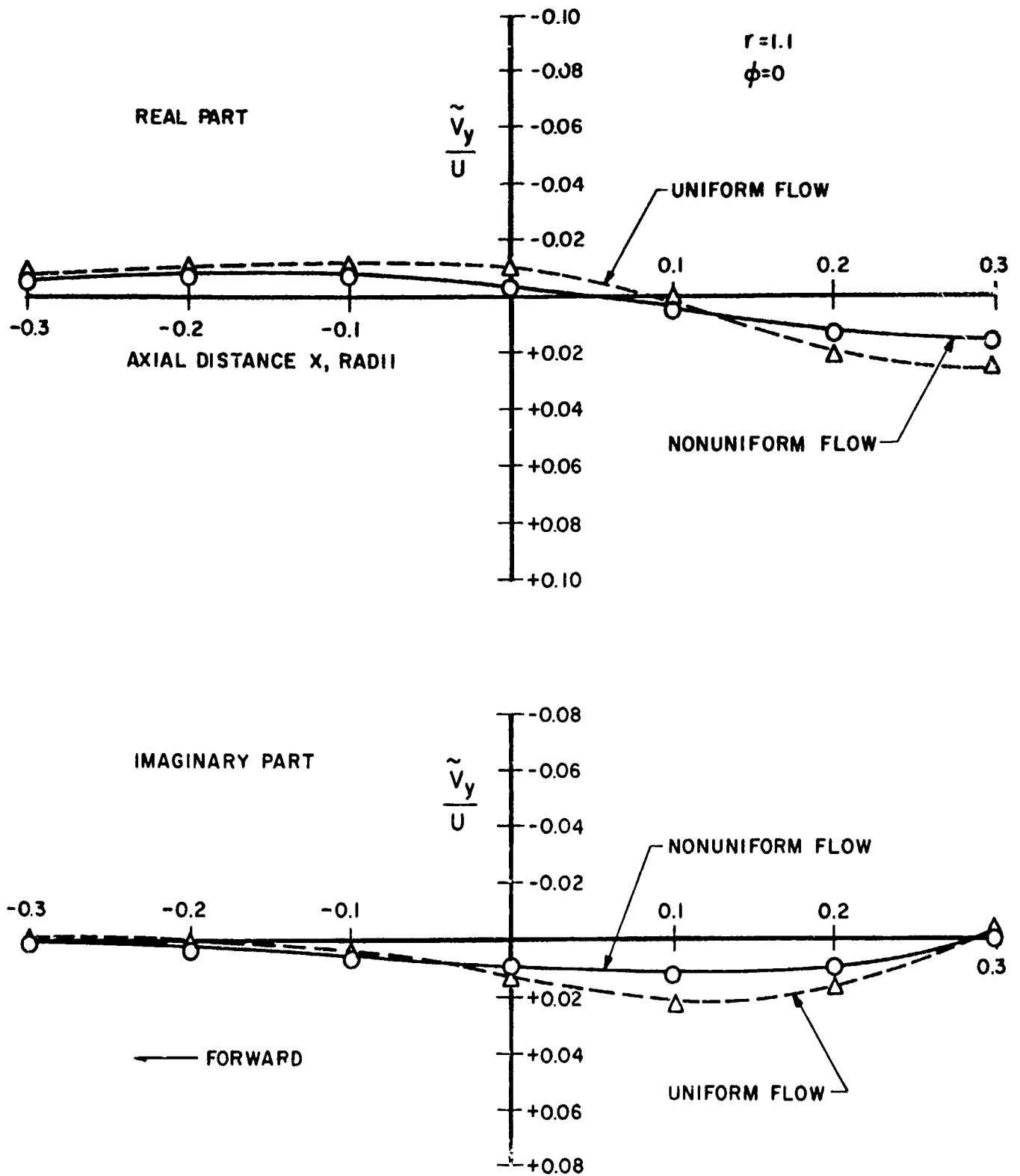


FIG. 6. UNSTEADY BLADE FREQUENCY $\tilde{V}_y / U$ VERSUS AXIAL DISTANCE X/r_0 FOR UNIFORM AND NONUNIFORM FLOW CASES

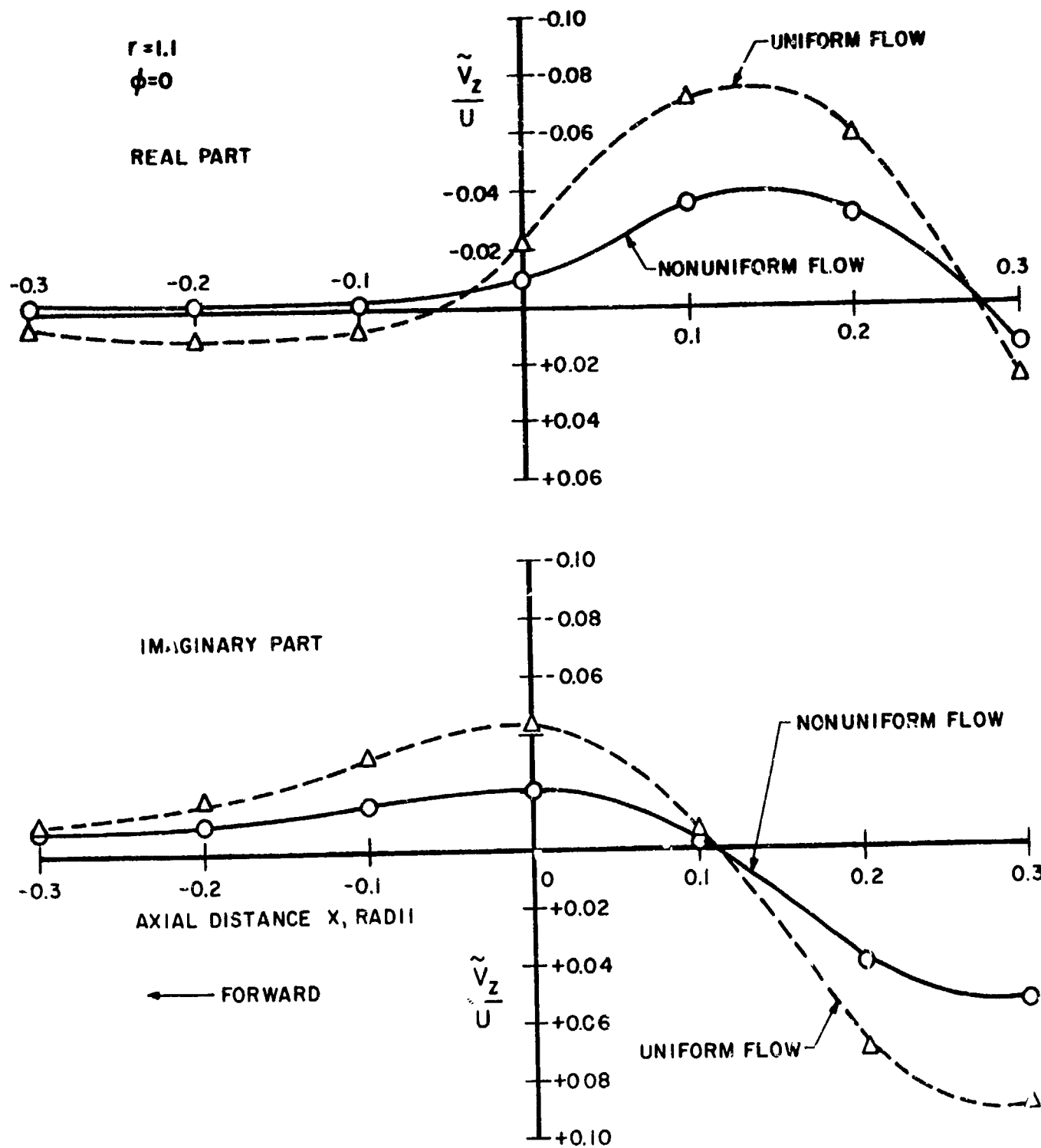


FIG. 7. UNSTEADY BLADE-FREQUENCY $\tilde{V}_z/U$ VERSUS AXIAL DISTANCE x/r_0 FOR UNIFORM AND NONUNIFORM FLOW CASES

APPENDIX A

$$\Lambda^{(\bar{n})}(x) \text{ and } \Lambda_1^{(\bar{n})}(x)$$

$$I \quad \Lambda^{(\bar{n})}(x) = \frac{1}{\pi} \int_0^\pi \Theta(\bar{n}) \exp(-ix \cos \theta_\alpha) \sin \theta_\alpha \, d\theta_\alpha \quad (A-1)$$

$$\text{where } \Theta(1) = \cot(\theta_\alpha/2)$$

$$\Theta(\bar{n}) = \sin(\bar{n}-1)\theta_\alpha, \quad \bar{n} > 1$$

Then

$$\Lambda^{(1)}(x) = J_0(x) - iJ_1(x)$$

$$\Lambda^{(\bar{n})}(x) = \frac{(-i)^{\bar{n}-2}}{2} [J_{\bar{n}-2}(x) + J_{\bar{n}}(x)], \quad \bar{n} > 1$$

where $J_n(x)$ is the Bessel function of the first kind of order n and argument x

$$II \quad \Lambda_1^{(\bar{n})}(x) = \frac{1}{\pi} \int_0^\pi \Theta(\bar{n}) \exp(-ix \cos \theta_\alpha) \cos \theta_\alpha \sin \theta_\alpha \, d\theta_\alpha \quad (A-2)$$

$$\Lambda_1^{(1)}(x) = \frac{1}{2} [J_0(x) - J_2(x)] - iJ_1(x)$$

$$\Lambda_1^{(\bar{n})}(x) = \frac{(-i)^{\bar{n}+1}}{4} [J_{\bar{n}-3}(x) - J_{\bar{n}+1}(x)], \quad \bar{n} > 1$$

It is to be noted that

$$\begin{aligned} \frac{\partial \Lambda^{(\bar{n})}(x)}{\partial x} &= -\frac{1}{\pi} \int_0^\pi \Theta(\bar{n}) \exp(-ix \cos \theta_\alpha) \cos \theta_\alpha \sin \theta_\alpha \, d\theta_\alpha \\ &= -i\Lambda_1^{(\bar{n})}(x) \end{aligned}$$

APPENDIX B

Evaluation of the integrand of $\bar{K}_x$ at $u = 0$ The integrand of $\bar{K}_x^{(m, \bar{n})}$ is

$$\left[\frac{h(u) - h(-u)}{u} \right] \quad (B-1)$$

$$\text{where } h(u) = (u - a\ell N) \left(au - \varepsilon^2 \ell N + \frac{m}{\rho^2} \right) \left[I_m(1u - a\ell N | \rho) K_m(1u - a\ell N | r) \right] \\ \cdot e^{iu(x - \frac{\sigma}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right)$$

and $q = \ell N$, $\ell = 0, \pm 1, \pm 2, \dots$ 1. $q = \ell = 0$, $m = \lambda$

$$h(u) = u \left(au + \frac{m}{\rho^2} \right) \left[I_m(u\rho) K_m(ur) \right] e^{iu(x - \frac{\sigma}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right)$$

$$h(-u) = -u \left(-au + \frac{m}{\rho^2} \right) \left[I_m(u\rho) K_m(ur) \right] e^{-iu(x - \frac{\sigma}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right)$$

Then

$$\frac{h(u) - h(-u)}{u} = \left[I_m(u\rho) K_m(ur) \right] \left\{ au \left[e^{iu(x - \frac{\sigma}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right) \right. \right. \\ \left. \left. - e^{-iu(x - \frac{\sigma}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right) \right] \right. \\ \left. + \frac{m}{\rho^2} \left[e^{iu(x - \frac{\sigma}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right) + e^{-iu(x - \frac{\sigma}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right) \right] \right\}$$

a) when $m = \lambda \neq 0$

$$\lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] = \begin{cases} \frac{1}{\rho^2} \left(\frac{\rho}{r} \right)^m \Lambda^{(\bar{n})}(\lambda \theta_b) & \text{for } \rho < r \\ \frac{1}{\rho^2} \left(\frac{r}{\rho} \right)^m \Lambda^{(\bar{n})}(\lambda \theta_b) & \text{for } \rho > r \end{cases} \quad (B-1a)$$

b) when $m = \lambda = 0$

$$\lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] = 0 \quad (B-1b)$$

11. $q = \ell N > 0$

$$\begin{aligned} \lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] &= \lim_{u \rightarrow 0} \frac{1}{u} \left\{ (u - a\ell N) \left(au - a^2 \ell N + \frac{m}{2} \right) \left[I_m((a\ell N - u)\rho) K_m((a\ell N - u)r) \right] \right. \\ &\quad \cdot e^{iu(x - \frac{\sigma}{a})} \Lambda(\bar{n}) \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right) \\ &\quad - (-u - a\ell N) \left(-au - a^2 \ell N + \frac{m}{2} \right) \left[I_m((a\ell N + u)\rho) K_m((a\ell N + u)r) \right] \\ &\quad \cdot e^{-iu(x - \frac{\sigma}{a})} \Lambda(\bar{n}) \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right) \left. \right\} \quad \text{for } \rho < r \end{aligned}$$

By L'Hospital's rule

$$\begin{aligned} \lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] &= \left. \frac{\partial [h(u) - h(-u)]}{\partial u} \right|_{u=0} \\ &= \lim_{u \rightarrow 0} \left\{ \left[au - a^2 \ell N + \frac{m}{2} + au - a^2 \ell N + i \left(x - \frac{\sigma}{a} \right) (u - a\ell N) \left(au - a^2 \ell N + \frac{m}{2} \right) \right] \right. \\ &\quad \cdot \left[I_m((a\ell N - u)\rho) K_m((a\ell N - u)r) \right] e^{iu(x - \frac{\sigma}{a})} \Lambda(\bar{n}) \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right) \\ &\quad + \left[-au - a^2 \ell N + \frac{m}{2} - au - a^2 \ell N + i \left(x - \frac{\sigma}{a} \right) (-u - a\ell N) \left(-au - a^2 \ell N + \frac{m}{2} \right) \right] \\ &\quad \cdot \left[I_m((a\ell N + u)\rho) K_m((a\ell N + u)r) \right] e^{-iu(x - \frac{\sigma}{a})} \Lambda(\bar{n}) \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right) \\ &\quad + (u - a\ell N) \left(au - a^2 \ell N + \frac{m}{2} \right) e^{iu(x - \frac{\sigma}{a})} \Lambda(\bar{n}) \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right) \left[-\rho I'_m((a\ell N - u)\rho) K_m((a\ell N - u)r) \right. \\ &\quad \left. - r I_m((a\ell N - u)\rho) K'_m((a\ell N - u)r) \right] \\ &\quad - (-u - a\ell N) \left(-au - a^2 \ell N + \frac{m}{2} \right) e^{-iu(x - \frac{\sigma}{a})} \Lambda(\bar{n}) \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right) \left[\rho I'_m((a\ell N + u)\rho) K_m((a\ell N + u)r) \right. \\ &\quad \left. + r I_m((a\ell N + u)\rho) K'_m((a\ell N + u)r) \right] \left. \right\} \end{aligned}$$

$$\begin{aligned}
& + (u - a \ell N) (a u - a^2 \ell N + \frac{m}{2}) \left[I_m((a \ell N - u) \rho) K_m((a \ell N - u) r) \right] e^{i u (x - \frac{\sigma}{a})} \frac{\partial \Lambda^{(\bar{n})}((\lambda - \frac{u}{a}) \theta_b)}{\partial u} \\
& - (-u - a \ell N) (-a u - a^2 \ell N + \frac{m}{2}) \left[I_m((a \ell N + u) \rho) K_m((a \ell N + u) r) \right] e^{-i u (x - \frac{\sigma}{a})} \frac{\partial \Lambda^{(\bar{n})}((\lambda + \frac{u}{a}) \theta_b)}{\partial u}
\end{aligned}$$

where

$$\left. \frac{\partial \Lambda^{(\bar{n})}((\lambda - \frac{u}{a}) \theta_b)}{\partial u} \right|_{u=0} = + i \frac{\theta_b}{a} \Lambda_1^{(\bar{n})}(\lambda \theta_b)$$

$$\left. \frac{\partial \Lambda^{(\bar{n})}((\lambda + \frac{u}{a}) \theta_b)}{\partial u} \right|_{u=0} = - i \frac{\theta_b}{a} \Lambda_1^{(\bar{n})}(\lambda \theta_b)$$

and $\Lambda_1^{(\bar{n})}(\lambda \theta_b)$ is defined in Appendix A.

Then, whatever m ,

$$\begin{aligned}
\lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] &= -2 \left[I_m(a \ell N \rho) K_m(a \ell N r) \right] \\
&\cdot \left\{ \left[+2a^2 \ell N - \frac{m}{2} - i(x - \frac{\sigma}{a})(a \ell N)(a^2 \ell N - \frac{m}{2}) \right] \Lambda^{(\bar{n})}(\lambda \theta_b) \right. \\
&- i \theta_b \ell N (a^2 \ell N - \frac{m}{2}) \Lambda_1^{(n)}(\lambda \theta_b) \} \\
&- 2(a \ell N)(a^2 \ell N - \frac{m}{2}) \Lambda^{(\bar{n})}(\lambda \theta_b) \\
&\cdot \left[\rho I_m'(a \ell N \rho) K_m(a \ell N r) + r I_m(a \ell N \rho) K_m'(a \ell N r) \right]
\end{aligned} \tag{B-2}$$

where

$$\left. \begin{aligned}
I_m'(a \ell N \rho) &= \frac{1}{2} \left[I_{m-1}(a \ell N \rho) + I_{m+1}(a \ell N \rho) \right] \\
K_m'(a \ell N r) &= -\frac{1}{2} \left[K_{m-1}(a \ell N r) + K_{m+1}(a \ell N r) \right]
\end{aligned} \right\} \tag{B-2a}$$

Here $\rho < r$ otherwise ρ and r are interchanged in the modified Bessel functions and the last term in brackets of (B-2).

III. $q = \ell N < 0$, whatever m ,

$$\begin{aligned} \lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] &= \lim_{u \rightarrow 0} \frac{1}{u} \left\{ (u - a\ell N) \left(au - a^2 \ell N + \frac{m}{2} \right) \left[I_m((u + a|\ell N|)\rho) K_m((u + a|\ell N|)r) \right] \right. \\ &\quad \cdot e^{iu(x - \frac{q}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right) \\ &\quad - (-u - a\ell N) \left(-au - a^2 \ell N + \frac{m}{2} \right) \left[I_m((-u + a|\ell N|)\rho) K_m((-u + a|\ell N|)r) \right] \\ &\quad \cdot e^{-iu(x - \frac{q}{a})} \Lambda^{(\bar{n})} \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right) \left. \right\} \end{aligned}$$

$$\begin{aligned} \lim_{u \rightarrow 0} \left[\frac{h(u) - h(-u)}{u} \right] &= -2 \left[I_m(a|\ell N|\rho) K_m(a|\ell N|r) \right] \\ &\quad \cdot \left\{ \left[2a^2 \ell N - \frac{m}{2} - 1 \left(x - \frac{q}{a} \right) (a\ell N) \left(a^2 \ell N - \frac{m}{2} \right) \right] \Lambda^{(n)}(\lambda \theta_b) \right. \\ &\quad - I_{\theta_b \ell N} \left(a^2 \ell N - \frac{m}{2} \right) \Lambda_1^{(\bar{n})}(\lambda \theta_b) \left. \right\} \\ &\quad + 2(a\ell N) \left(a^2 \ell N - \frac{m}{2} \right) \Lambda^{(\bar{n})}(\lambda \theta_b) \\ &\quad \cdot \left[\rho I_m'(a|\ell N|\rho) K_m(a|\ell N|r) + r I_m(a|\ell N|\rho) K_m'(a|\ell N|r) \right] \quad (B-3) \end{aligned}$$

for $\rho < r$. Again if $\rho > r$, ρ and r are interchanged in the modified Bessel functions and the last term in brackets in (B-3).

.. PENDIX C

Evaluation of the Integrand of $\bar{K}_r$ at $u=0$

A. For the case: $\rho < r$

The integrand of $\bar{K}_r^{(m, \bar{n})} (\rho < r)$ is

$$\left[\frac{g_1(u) - g_1(-u)}{u} \right] \quad (C-1)$$

where $g_1(u) = |u - a\ell N| (au - a^2\ell N + m/\rho^2) \{ I_m(|u - a\ell N| \rho) \}$

$$\cdot \{ K_{m-1}(|u - a\ell N| r) + K_{m+1}(|u - a\ell N| r) \} e^{iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda - u/a) \theta_b)$$

and $q = \ell N$, $\ell = 0, \pm 1, \pm 2, \dots$

I $q = \ell = 0$, $m = \lambda$

$$g_1(u) = u(au + m/\rho^2) \{ I_m(u\rho) [K_{m-1}(ur) + K_{m+1}(ur)] \} e^{iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda - u/a) \theta_b)$$

$$\text{and } g_1(-u) = u(-au + m/\rho^2) \{ I_m(u\rho) [K_{m-1}(ur) + K_{m+1}(ur)] \} e^{-iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda + u/a) \theta_b)$$

(B-2)

Then

$$\begin{aligned} \left[\frac{g_1(u) - g_1(-u)}{u} \right] = & u \{ I_m(u\rho) [K_{m-1}(ur) + K_{m+1}(ur)] \} \left\{ a \left[e^{iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda - u/a) \theta_b) \right. \right. \\ & \left. \left. + e^{-iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda + u/a) \theta_b) \right] \right. \\ & \left. + \frac{m}{\rho^2 u} \left[e^{iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda - u/a) \theta_b) \right. \right. \\ & \left. \left. - e^{-iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda + u/a) \theta_b) \right] \right\} \end{aligned}$$

a) when $m \neq \lambda \neq 0$

$$\lim_{u \rightarrow 0} \frac{g_1(u) - g_1(-u)}{u} = \left[\frac{\rho}{r} \right]^m \frac{a}{r} \{ 2\Lambda^{(\bar{n})}(\lambda \theta_b) \} \\ + \left[\frac{\rho}{r} \right]^m \frac{1}{r} (m/\rho^2) \lim_{u \rightarrow 0} \left\{ \frac{e^{iu(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda-u/a)\theta_b) - e^{-iu(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda+u/a)\theta_b)}{u} \right\}$$

But

$$\lim_{u \rightarrow 0} \left[\frac{e^{iu(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda-u/a)\theta_b) - e^{-iu(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda+u/a)\theta_b)}{u} \right] = \\ i(x-\sigma/a) [2\Lambda^{(\bar{n})}(\lambda \theta_b)] + \left[\frac{\partial \Lambda^{(\bar{n})}((\lambda-u/a)\theta_b)}{\partial u} - \frac{\partial \Lambda^{(\bar{n})}((\lambda+u/a)\theta_b)}{\partial u} \right]_{u=0} \\ = 2i(x-\sigma/a) \Lambda^{(\bar{n})}(\lambda \theta_b) + 2i(\theta_b/a) \Lambda_1^{(\bar{n})}(\lambda \theta_b) \\ \text{since } \frac{\partial \Lambda^{(\bar{n})}((\lambda-u/a)\theta_b)}{\partial u} \Big|_{u=0} = -\frac{\partial \Lambda^{(\bar{n})}((\lambda+u/a)\theta_b)}{\partial u} \Big|_{u=0} = i(\theta_b/a) \Lambda_1^{(\bar{n})}(\lambda \theta_b)$$

with $\Lambda_1^{(\bar{n})}(x)$ as defined in Appendix A

Therefore the integrand at $u=0$ is

$$\frac{2}{r} \left[\frac{\rho}{r} \right]^m \left\{ \left[a + (m/\rho^2)(x-\sigma/a) \right] \Lambda^{(\bar{n})}(\lambda \theta_b) + (im/\rho^2)(\theta_b/a) \Lambda_1^{(\bar{n})}(\lambda \theta_b) \right\} \quad (C-1a)$$

b) when $m=\lambda=0$ the integrand at $u=0$ is easily seen to be

$$\frac{4a}{r} \Lambda^{(\bar{n})}(0) \quad (C-1b)$$

II $q = \ell N > 0$

$$\lim_{u \rightarrow 0} \left[\frac{g_1(u) - g_1(-u)}{u} \right] = \lim_{u \rightarrow 0} \frac{1}{u} \left\{ (a\ell N - u)(au - a^2\ell N + m/\rho^2) \left\{ I_m((a\ell N - u)\rho) \right. \right. \\ \left. \left. \cdot \left[K_{m-1}((a\ell N - u)r) + K_{m+1}((a\ell N - u)r) \right] \right\} e^{iu(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda-u/a)\theta_b) \right\}$$

$$\begin{aligned}
& -(a\ell N+u) \{-au-a^2\ell N+m/\rho^2\} \left\{ I_m((a\ell N+u)\rho) \right. \\
& \cdot \left[K_{m-1}((a\ell N+u)r) + K_{m+1}((a\ell N+u)r) \right] \left. \right\} e^{-i u(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda+u/a)\theta_b) \} \\
& = \frac{0}{0}
\end{aligned}$$

By L'Hospital's rule

$$\lim_{u \rightarrow 0} \left[\frac{g_1(u) - g_1(-u)}{u} \right] = \left[\frac{\partial [g_1(u) - g_1(-u)]}{\partial u} \right]_{u=0}$$

Then, whatever m ,

$$\begin{aligned}
\lim_{u \rightarrow 0} \left[\frac{g_1(u) - g_1(-u)}{u} \right] &= 2 \left\{ I_m(a|\ell N|\rho) \left[K_{m-1}(a|\ell N|r) + K_{m+1}(a|\ell N|r) \right] \right. \\
&\cdot \left\{ [2a^2\ell N-m/\rho^2 - i(x-\sigma/a)(a\ell N)(a^2\ell N-m/\rho^2)] \Lambda^{(\bar{n})}(\lambda\theta_b) \right. \\
&\quad \left. - i e_b \ell N (a^2\ell N-m/\rho^2) \Lambda_1^{(\bar{n})}(\lambda\theta_b) \right\} \\
&+ 2(a\ell N)(a^2\ell N-m/\rho^2) \Lambda^{(\bar{n})}(\lambda\theta_b) \\
&\cdot \left\{ \rho I'_m(a|\ell N|\rho) [K_{m-1}(a|\ell N|r) + K_{m+1}(a|\ell N|r)] \right. \\
&\quad \left. + r I'_m(a|\ell N|\rho) [K'_{m-1}(a|\ell N|r) + K'_{m+1}(a|\ell N|r)] \right\} \quad (C-2)
\end{aligned}$$

$$\text{where } I'_m(a|\ell N|\rho) = \frac{1}{2} [I_{m-1}(a|\ell N|\rho) + I_{m+1}(a|\ell N|\rho)] \quad (C-2a)$$

$$K'_{m-1}(a|\ell N|r) + K'_{m+1}(a|\ell N|r) = \frac{-1}{2} [K_{m-2}(a|\ell N|r) + 2K_m(a|\ell N|r) + K_{m+2}(a|\ell N|r)]$$

III $q = \ell N < 0$, whatever m ,

$$\begin{aligned}
& \lim_{u \rightarrow 0} \left[\frac{g_1(u) - g_1(-u)}{u} \right] \\
&= \lim_{u \rightarrow 0} \frac{1}{u} \left\{ (u-a\ell N) (au-a^2\ell N+m/\rho^2) \left\{ I_m((u+a|\ell N|\rho) \right. \right. \\
&\quad \cdot \left. \left. [K_{m-1}((u+a|\ell N|r) + K_{m+1}((u+a|\ell N|r))] \right\} \right. \\
&\quad \cdot e^{i u(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda-u/a)\theta_b)
\end{aligned}$$

$$\begin{aligned}
& -(-u-a\ell N)(-au-a^2\ell N+m/\rho^2)\left\{I_m((-u+a|\ell N|)\rho)\left[K_{m-1}((-u+a|\ell N|)r)+K_{m+1}((-u+a|\ell N|)r)\right]\right\} \\
& \quad e^{-iu(x-\sigma/a)}\Lambda^{(\bar{n})}((\lambda+u/a)\theta_b)\} \\
& = -2\left\{I_m(a|\ell N|\rho)\left[K_{m-1}(a|\ell N|r)+K_{m+1}(a|\ell N|r)\right]\right\} \\
& \quad \cdot \left\{[2a^2\ell N-m/\rho^2-1(x-\sigma/a)(a\ell N)(a^2\ell N-m/\rho^2)]\Lambda^{(\bar{n})}(\lambda\theta_b)\right. \\
& \quad \left.-i\theta_b\ell N(a^2\ell N-m/\rho^2)\Lambda_1^{(\bar{n})}(\lambda\theta_b)\right\} \\
& \quad +2(a\ell N)(a^2\ell N-m/\rho^2)\Lambda^{(\bar{n})}(\lambda\theta_b)\left\{\rho I'_m(a|\ell N|\rho)\left[K_{m-1}(a|\ell N|r)+K_{m+1}(a|\ell N|r)\right]\right. \\
& \quad \left.+r I'_m(a|\ell N|\rho)\left[K'_{m-1}(a|\ell N|r)+K'_{m+1}(a|\ell N|r)\right]\right\} \quad (C-3)
\end{aligned}$$

B. For the case: $\rho > r$

The integrand of $\bar{K}_r^{(m,\bar{n})}(\rho > r)$ is

$$\left[\frac{g_2(u)-g_2(-u)}{u}\right] \quad (C-4)$$

where $g_2(u)$ is given in Eq (18b)

1. $q=\ell=0$, $m=\lambda$

$$\begin{aligned}
\left[\frac{g_2(u)-g_2(-u)}{u}\right] & = -uK_m(u\rho)\left[I_{m-1}(ur)+I_{m+1}(ur)\right] \\
& \cdot \left\{a\left[e^{iu(x-\sigma/a)}\Lambda^{(\bar{n})}((\lambda-u/a)\theta_b)+e^{-iu(x-\sigma/a)}\Lambda^{(\bar{n})}((\lambda+u/a)\theta_b)\right]\right. \\
& \left.+\frac{m}{\rho^2u}\left[e^{iu(x-\sigma/a)}\Lambda^{(\bar{n})}((\lambda-u/a)\theta_b)-e^{-iu(x-\sigma/a)}\Lambda^{(\bar{n})}((\lambda+u/a)\theta_b)\right]\right\}
\end{aligned}$$

a) when $m=\lambda \neq 0$ it can be shown that

$$\begin{aligned}
\lim_{u \rightarrow 0} \left[\frac{g_2(u)-g_2(-u)}{u}\right] & = -(r/\rho)^m(2/r)\left\{[a+(im/\rho^2)(x-\sigma/a)]\Lambda^{(\bar{n})}(\lambda\theta_b)\right. \\
& \left.+ (im/\rho^2)(\theta_b/a)\Lambda_1^{(\bar{n})}(\lambda\theta_b)\right\} \quad (C-4a)
\end{aligned}$$

b) when $m=\lambda=0$

$$\lim_{u \rightarrow 0} \left[\frac{g_2(u) - g_2(-u)}{u} \right] = 0 \quad (C-4b)$$

II $q = \ell N > 0$

$$\begin{aligned} & \lim_{u \rightarrow 0} \left[\frac{g_2(u) - g_2(-u)}{u} \right] \lim_{u \rightarrow 0} \frac{1}{u} \left\{ -a\ell N - u \right\} (au - a^2 \ell N + m/\rho^2) K_m((a\ell N - u)\rho) \\ & \quad \cdot \left[I_{m-1}((a\ell N - u)r) + I_{m+1}((a\ell N - u)r) \right] e^{iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda - u/a)\theta_b) \\ & \quad + (a\ell N + u) (-au - a^2 \ell N + m/\rho^2) K_m((a\ell N + u)\rho) \\ & \quad \left[I_{m-1}((a\ell N + u)r) + I_{m+1}((a\ell N + u)r) \right] e^{-iu(x-\sigma/a)} \Lambda(\bar{n}) ((\lambda + u/a)\theta_b) \} \\ & = -2K_m(a\ell N\rho) \left[I_{m-1}(a\ell Nr) + I_{m+1}(a\ell Nr) \right] \\ & \quad \left\{ \left[2a^2 \ell N - m/\rho^2 - 1(x-\sigma/a)a\ell N(a^2 \ell N - m/\rho^2) \right] \Lambda(\bar{n}) (\lambda\theta_b) \right. \\ & \quad \left. - 1\theta_b \ell N(a^2 \ell N - m/\rho^2) \Lambda_1(\bar{n}) (\lambda\theta_b) \right\} \\ & \quad - 2a\ell N(a^2 \ell N - m/\rho^2) \Lambda(\bar{n}) (\lambda\theta_b) \left\{ \rho K'_m(a\ell N\rho) \left[I_{m-1}(a\ell Nr) + I_{m+1}(a\ell Nr) \right] \right. \\ & \quad \left. + rK_m(a\ell N\rho) \left[I'_{m-1}(a\ell Nr) + I'_{m+1}(a\ell Nr) \right] \right\} \quad (C-4c) \end{aligned}$$

where $K'_m(a\ell N\rho) = -\frac{1}{2} \left[K_{m-1}(a\ell N\rho) + K_{m+1}(a\ell N\rho) \right]$

and

$$I'_{m-1}(a\ell Nr) + I'_{m+1}(a\ell Nr) = \frac{1}{2} \left[I_{m-2}(a\ell Nr) + 2I_m(a\ell Nr) + I_{m+2}(a\ell Nr) \right]$$

III $q = \ell N < 0$

$$\begin{aligned}
 \lim_{u \rightarrow 0} \left[\frac{g_2(u) - g_2(-u)}{u} \right] &= \lim_{u \rightarrow 0} \frac{1}{u} \left\{ -(u - a\ell N) (au - a^2 \ell N + m/\rho^2) K_m((u+a|\ell|N)\rho) \right. \\
 &\quad \cdot e^{iu(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda-u/a)\theta_b) \\
 &\quad \cdot [I_{m-1}((u+a|\ell|N)r) + I_{m+1}((u+a|\ell|N)r)] \\
 &\quad + (-u - a\ell N) (-au - a^2 \ell N + m/\rho^2) K_m((-u+a|\ell|N)\rho) e^{-iu(x-\sigma/a)} \Lambda^{(\bar{n})}((\lambda+u/a)\theta_b) \\
 &\quad \cdot [I_{m-1}((-u+a|\ell|N)r) + I_{m+1}((-u+a|\ell|N)r)] \Big\} \\
 &= 2K_m(a|\ell N|\rho) [I_{m-1}(a|\ell N|r) + I_{m+1}(a|\ell N|r)] \\
 &\quad \cdot \left\{ [2a^2 \ell N - m/\rho^2 - i(x-\sigma/a)a\ell N(a^2 \ell N - m/\rho^2)] \Lambda^{(\bar{n})}(\lambda\theta_b) \right. \\
 &\quad \left. - i\theta_b \ell N(a^2 \ell N - m/\rho^2) \Lambda_1^{(\bar{n})}(\lambda\theta_b) \right\} \\
 &\quad - 2a\ell N(a^2 \ell N - m/\rho^2) \Lambda^{(\bar{n})}(\lambda\theta_b) \left\{ \rho K_m'(a|\ell N|\rho) [I_{m-1}(a|\ell N|r) + I_{m+1}(a|\ell N|r)] \right. \\
 &\quad \left. + rK_m(a|\ell N|\rho) [I_{m-1}'(a|\ell N|r) + I_{m+1}'(a|\ell N|r)] \right\} \quad (C-4d)
 \end{aligned}$$

APPENDIX D

Evaluation of the Integrand of $\tilde{K}_\varphi$ at $u = 0$ The Integrand of $\tilde{K}_\varphi(m, \bar{n})$ is

$$m \left[\frac{f(u) - f(-u)}{u} \right] \quad (D-1)$$

$$\text{where } f(u) = (au - a^2 \ell N + \frac{m}{2}) I_m(u\rho) K_m(ur) e^{iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda - \frac{u}{a}) \theta_b)$$

$$q = \ell N, \ell = 0, \pm 1, \pm 2, \dots$$

$$1) q = \ell = 0, m = \lambda$$

$$f(u) = (au + \frac{m}{2}) I_m(u\rho) K_m(ur) e^{iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda - \frac{u}{a}) \theta_b)$$

$$\bar{f}(-u) = (-au + \frac{m}{2}) I_m(u\rho) K_m(ur) e^{-iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda + \frac{u}{a}) \theta_b)$$

Then

$$\begin{aligned} \lim_{u \rightarrow 0} \frac{m[f(u) - f(-u)]}{u} &= \lim_{u \rightarrow 0} \left\{ \frac{m}{u} I_m(u\rho) K_m(ur) \left[au \left[e^{iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda - \frac{u}{a}) \theta_b) \right. \right. \right. \\ &\quad \left. \left. + e^{-iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda + \frac{u}{a}) \theta_b) \right] \right. \\ &\quad \left. + \frac{m}{2} \left[e^{iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda - \frac{u}{a}) \theta_b) \right. \right. \\ &\quad \left. \left. - e^{-iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda + \frac{u}{a}) \theta_b) \right] \right] \Big\} \\ &= \lim_{u \rightarrow 0} \left\{ am I_m(u\rho) K_m(ur) \left[e^{iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda - \frac{u}{a}) \theta_b) \right. \right. \\ &\quad \left. \left. + e^{-iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda + \frac{u}{a}) \theta_b) \right] \right. \\ &\quad \left. + \frac{m^2}{2} I_m(u\rho) K_m(ur) \left[\frac{e^{iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda - \frac{u}{a}) \theta_b) - e^{-iu(x - \frac{\sigma}{a}) \Lambda(\bar{n})} ((\lambda + \frac{u}{a}) \theta_b)}{u} \right] \right\} \end{aligned}$$

a) when $m = \lambda \neq 0$

$$\begin{aligned}
 \lim_{u \rightarrow 0} \frac{m[f(u) - f(-u)]}{u} &= a\left(\frac{\rho}{r}\right)^m \Lambda^{\bar{n}}(\lambda \theta_b) \\
 &+ \frac{m^2}{2} \frac{1}{2m} \left(\frac{\rho}{r}\right)^m \lim_{u \rightarrow 0} \frac{\left[e^{iu(x - \frac{\sigma}{a})} \Lambda^{\bar{n}}((\lambda - \frac{u}{a})\theta_b) - e^{-iu(x - \frac{\sigma}{a})} \Lambda^{\bar{n}}((\lambda + \frac{u}{a})\theta_b) \right]}{u} \\
 &= a\left(\frac{\rho}{r}\right)^m \Lambda^{\bar{n}}(\lambda \theta_b) \\
 &+ \frac{m^2}{2} \frac{1}{2m} \left(\frac{\rho}{r}\right)^m \left[i(x - \frac{\sigma}{a}) \Lambda^{\bar{n}}(\lambda \theta_b) + i(x - \frac{\sigma}{a}) \Lambda^{\bar{n}}(\lambda \theta_b) \right. \\
 &+ \left. \frac{m^2}{2} \frac{1}{2m} \left(\frac{\rho}{r}\right)^m \left[\frac{\partial \Lambda^{\bar{n}}((\lambda - \frac{u}{a})\theta_b)}{\partial u} - \frac{\partial \Lambda^{\bar{n}}((\lambda + \frac{u}{a})\theta_b)}{\partial u} \right]_{u=0} \right] \\
 &= \left(\frac{\rho}{r}\right)^m \left\{ \left[a + i(x - \frac{\sigma}{a}) \frac{m}{2} \right] \Lambda^{\bar{n}}(\lambda \theta_b) + i \frac{\theta_b}{a} \frac{m}{2} \Lambda_1^{\bar{n}}(\lambda \theta_b) \right\} \quad (D-1a)
 \end{aligned}$$

b) when $m = \lambda = 0$, the integrand at $u = 0$ is

$$\lim_{u \rightarrow 0} \frac{m[f(u) - f(-u)]}{u} = 0 \quad (D-1b)$$

In fact, the entire integral is zero.

2) $q = \ell N > 0$

$$\begin{aligned}
 \lim_{u \rightarrow 0} \frac{m[f(u) - f(-u)]}{u} &= \lim_{u \rightarrow 0} \frac{m}{u} \left\{ (au - a^2 \ell N + \frac{m}{2}) I_m((a \ell N - u)\rho) K_m((a \ell N - u)r) \right. \\
 &\quad \cdot e^{iu(x - \frac{\sigma}{a})} \Lambda^{\bar{n}}((\lambda - \frac{u}{a})\theta_b) \\
 &\quad - (-au - a^2 \ell N + \frac{m}{2}) I_m((a \ell N + u)\rho) K_m(a \ell N + u)r) \\
 &\quad \cdot e^{-iu(x - \frac{\sigma}{a})} \Lambda^{\bar{n}}((\lambda + \frac{u}{a})\theta_b) \left. \right\}
 \end{aligned}$$

By L'Hospital's rule, the integrand at $u = 0$ is

$$\begin{aligned}
& \lim_{u \rightarrow 0} e^{iu(x - \frac{g}{a})} \left\{ \left[a + i \left(x - \frac{g}{a} \right) \left(au - a^2 \ell N + \frac{m}{2} \right) \right] I_m((a \ell N - u) \rho) K_m((a \ell N - u) r) \right. \\
& \quad + \left(au - a^2 \ell N + \frac{m}{2} \right) \left\{ -\rho I_m'((a \ell N - u) \rho) K_m((a \ell N - u) r) - r I_m((a \ell N - u) \rho) K_m'((a \ell N - u) r) \right\} \Big] \\
& \quad \cdot \Lambda^{(\bar{n})} \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right) \\
& \quad + \left(au - a^2 \ell N + \frac{m}{2} \right) I_m((a \ell N - u) \rho) K_m((a \ell N - u) r) \frac{\partial \Lambda^{(\bar{n})} \left(\left(\lambda - \frac{u}{a} \right) \theta_b \right)}{\partial u} \Big\} \\
& - e^{-iu(x - \frac{g}{a})} \left\{ \left[-a - i \left(x - \frac{g}{a} \right) \left(-au - a^2 \ell N + \frac{m}{2} \right) \right] I_m((a \ell N + u) \rho) K_m((a \ell N + u) r) \right. \\
& \quad + \left(-au - a^2 \ell N + \frac{m}{2} \right) \left[\rho I_m'((a \ell N + u) \rho) K_m((a \ell N + u) r) + r I_m((a \ell N + u) \rho) K_m'((a \ell N + u) r) \right] \\
& \quad \cdot \Lambda^{(\bar{n})} \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right) \\
& \quad + \left(-au - a^2 \ell N + \frac{m}{2} \right) I_m((a \ell N + u) \rho) K_m((a \ell N + u) r) \frac{\partial \Lambda^{(\bar{n})} \left(\left(\lambda + \frac{u}{a} \right) \theta_b \right)}{\partial u} \Big\} \\
& = 2m \left\{ \left[a + i \left(x - \frac{g}{a} \right) \left(-a^2 \ell N + \frac{m}{2} \right) \right] I_m(a \ell N | \rho) K_m(a \ell N | r) \Lambda^{(\bar{n})}(\lambda \theta_b) \right. \\
& \quad - \left(-a^2 \ell N + \frac{m}{2} \right) \Lambda^{(\bar{n})}(\lambda \theta_b) \left[\rho I_m'(a \ell N | \rho) K_m(a \ell N | r) + r I_m(a \ell N | \rho) K_m'(a \ell N | r) \right] \\
& \quad + \left(-a^2 \ell N + \frac{m}{2} \right) I_m(a \ell N | \rho) K_m(a \ell N | r) \left[i \frac{\theta_b}{a} \Lambda_1^{(\bar{n})}(\lambda \theta_b) \right] \Big\} \tag{D-2}
\end{aligned}$$

where

$$\begin{aligned}
I_m'(a \ell N | \rho) &= \frac{1}{2} \left[I_{m-1}(a \ell N | \rho) + I_{m+1}(a \ell N | \rho) \right] \\
K_m'(a \ell N | r) &= -\frac{1}{2} \left[K_{m-1}(a \ell N | r) + K_{m+1}(a \ell N | r) \right] \tag{D-2a}
\end{aligned}$$

When $m = 0$ the integrand is zero.

3) $q = \ell N < 0$

$$\lim_{u \rightarrow 0} \frac{m[f(u) - f(-u)]}{u} =$$

$$\lim_{u \rightarrow 0} \frac{m}{u} \left\{ (au - a^2 \ell N + \frac{m}{2}) I_m((u - a \ell N) \rho) K_m((u - a \ell N) r) e^{i u (x - \frac{\sigma}{a})} \Lambda^{(\bar{n})}((\lambda - \frac{u}{a}) \theta_b) \right. \\ \left. - (-au - a^2 \ell N + \frac{m}{2}) I_m((-u - a \ell N) \rho) K_m((-u - a \ell N) r) e^{-i u (x - \frac{\sigma}{a})} \Lambda^{(\bar{n})}((\lambda + \frac{u}{a}) \theta_b) \right\}$$

It can be easily shown that

$$\lim_{u \rightarrow 0} \frac{m[f(u) - f(-u)]}{u} =$$

$$2m \left\{ [a + i(x - \frac{\sigma}{a}) (-a^2 \ell N + \frac{m}{2})] I_m(a \ell N | \rho) K_m(a \ell N | r) \Lambda^{(\bar{n})}(\lambda \theta_b) \right. \\ \left. + (-a^2 \ell N + \frac{m}{2}) \Lambda^{(\bar{n})}(\lambda \theta_b) \left[\rho I'_m(a \ell N | \rho) K_m(a \ell N | r) + r I_m(a \ell N | \rho) K'_m(a \ell N | r) \right] \right. \\ \left. + (-a^2 \ell N + \frac{m}{2}) I_m(a \ell N | \rho) K_m(a \ell N | r) \left[i \frac{\theta_b}{a} \Lambda^{(\bar{n})}_1(\lambda \theta_b) \right] \right\} \quad (D-3)$$

When $m = 0$ the integrand is zero.