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APPLIED PHYSICS LABORATORY

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ON THE SOLUTION OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS  
BY MEANS OF NUMERICAL INTEGRATION OPERATORS

by

A. V. Nesterchuk [Nesterčuk]

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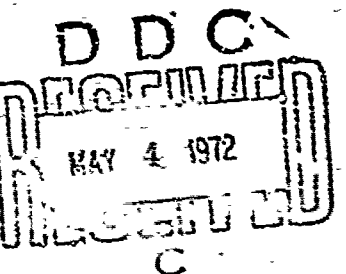
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SUMMARY

Operators of numerical integration of functions are constructed. Also considered is the question of estimating the total error. The operators are applied to the solution of a problem with initial conditions for ordinary linear differential equations with constant coefficients. It was determined that the process of finding the solution of a differential equation of the type being examined by means of numerical integration operators reduces to arithmetic operations with numerical matrices and, consequently, is highly suitable for computer realization. An example is given, illustrating the advantages of the proposed method.

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ON THE SOLUTION OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS  
BY MEANS OF NUMERICAL INTEGRATION OPERATORS<sup>\*)</sup>

by

A. V. Nesterchuk [Nesterčuk]

Considered in [1,2] are several aspects of the solution of differential equations by means of numerical integration operators. Considered in the present paper are new numerical integration operators and their application to the computer solution of ordinary linear differential equations. Also examined is the problem of estimating the total error.

Given a function  $f(x)$  continuous on the segment  $[0,1]$ . We divide the segment  $[0,1]$  into  $n$  equal parts,  $h = \frac{1}{n}$ . We introduce the notation

$$x_0 = 0, x_i = ih, f_i = f(x_i), [f_i] = [f_0, f_1, \dots, f_n], \quad i = 0, 1, 2, \dots, n.$$

We shall calculate  $\int_0^x f(t) dt$ ,  $0 \leq x \leq 1$  from one of the quadrature formulas, e.g., by the trapezoidal formula. Then

$$\begin{aligned} \int_0^1 f(t) dt &\approx h \cdot \frac{1}{2} (f_0 + f_1) = h [f_0, f_1] \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \int_0^{2h} f(t) dt &\approx h \cdot \frac{1}{2} (f_0 + 2f_1 + f_2) = h [f_0, f_1, f_2] \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \\ &\dots \end{aligned}$$

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<sup>\*)</sup> Translated from Ukr. Mat. Zh. [Ukr. Math. J.], Vol. 17, No. 4, pp. 112-119 (1965).

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error and the computational error. Apropos the total error, the following theorem holds.

Theorem 1. If the function  $f(x)$ , defined and continuous on the segment  $[0,1]$ , has on this segment a second derivative for which the following condition is fulfilled:

$$|f''(\xi)| \leq M, \quad 0 \leq \xi \leq 1,$$

then in calculating  $\int_0^x f(t) dt$ ,  $0 \leq x \leq 1$ , by means of the operator  $A_1$ , when the errors in the calculated values of the function  $f^*(ih)$  ( $i = 0, 1, \dots, n$ ) do not exceed the quantity  $\frac{Mh^2}{12}$ , the vector of the total errors  $R_1^*$  has the form

$$R_1^* = \frac{Mh^3}{12} [0, 2, 3, \dots, n+1].$$

Proof. Let  $f(x)$  be the exact and  $f^*(x)$  the calculated values of the function. The total-error vector for the approximate vector  $h[f_1^*]A_1$  is determined in the form of the difference

$$\left[ \int_0^{kh} f(t) dt \right] - h[f_1^*]A_1, \quad k = 0, 1, 2, \dots, n. \quad (4)$$

We calculate the  $i$ -th component of vector (4). Taking (2) into account, we get

$$\begin{aligned} & \frac{1}{2}h(f_0 + 2f_1 + \dots + 2f_{i-1} + f_i) - \frac{1}{12}h^3 f''(\xi) - h[f_0^*, f_1^*, \dots, f_i^*] \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2 \\ 1 \end{bmatrix} = \\ & = \frac{1}{2}h(f_0 + 2f_1 + \dots + 2f_{i-1} + f_i) - \frac{1}{12}h^3 f''(\xi) - \frac{1}{2}h(f_0^* + 2f_1^* + \dots \\ & \dots + 2f_{i-1}^* + f_i^*) = \frac{1}{2}h((f_0 - f_0^*) + (2f_1 - 2f_1^*) + \dots \\ & \dots + 2(f_{i-1} - f_{i-1}^*) + (f_i - f_i^*)) - \frac{1}{12}h^3 f''(\xi), \\ & i = 0, 1, 2, \dots, n; \quad 0 \leq \xi \leq ih. \end{aligned} \quad (5)$$

By the condition of the theorem  $|f_i - f_i^*| \leq \frac{Mh^2}{12}$ ; then, from (5)

$$\left| \left( \int_0^M f(t) dt \right) - h [f_k] A_1 \right| < \frac{1}{2} h \cdot 2 \cdot \frac{Mh^2}{12} i + \frac{1}{12} h^3 M =$$

$$= \frac{Mh^3}{12} (i+1), \quad i = 1, 2, 3, \dots, n. \quad (6)$$

Taking the obvious equality  $\left| \left( \int_0^M f(t) dt \right) - h [f_k] A_1 \right|_0 = 0$  and (6) into account, we can write

$$R_1^* = \frac{Mh^3}{12} [0, 2, 3, \dots, n+1].$$

More accurate will be an operator constructed by means of second- and third degree parabolas. When  $x = h$  we use the trapezoidal formula; then

$$\int_0^h f(t) dt \approx \frac{1}{12} h (f_0 + f_1). \quad (7)$$

When  $x = 2h$ , by the Simpson formula we have

$$\int_0^{2h} f(t) dt \approx \frac{1}{3} h (f_0 + 4f_1 + f_2), \quad (8)$$

and when  $x = 3h$ , we have by the "3/8" Simpson formula

$$\int_0^{3h} f(t) dt \approx \frac{3}{8} h (f_0 + 3f_1 + 3f_2 + f_3). \quad (9)$$

But if  $x = 4h$ , we apply the "one-third" Simpson rule twice, and when  $x = 5h$  this same rule is applied from 0 to  $2h$  and the "three-eighths" Simpson rule from  $2h$  to  $5h$ . As a result, by analogy with (1) we get

$$\left[ \int_0^{ih} f(t) dt \right] \approx h [f_k] A_2, \quad k = 0, 1, 2, \dots, i; \quad i = 0, 1, 2, \dots, n, \quad (10)$$

where

$$A_n = \frac{1}{24} \begin{bmatrix} 0 & 12 & 8 & 9 & 8 & 8 & \dots & 8 \\ 0 & 12 & 32 & 27 & 32 & 32 & \dots & 32 \\ 0 & 0 & 8 & 27 & 16 & 17 & \dots & 16 \\ 0 & 0 & 0 & 9 & 32 & 27 & \dots & 32 \\ 0 & 0 & 0 & 0 & 8 & 27 & \dots & 16 \\ 0 & 0 & 0 & 0 & 0 & 9 & \dots & 32 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 8(9) \end{bmatrix}.$$

The last diagonal element equals eight if  $n$  is even and 9 if  $n$  is odd. The error in this case can be found on the basis of the equalities

$$\begin{aligned} y_1 &= \frac{1}{2} h(f_0 + f_1) - \frac{1}{12} h^3 f''(\xi_1), & 0 < \xi_1 < h, \\ y_2 &= \frac{1}{3} h(f_0 + 4f_1 + f_2) - \frac{1}{90} h^5 f^{(iv)}(\xi_2), & 0 < \xi_2 < 2h, \\ y_3 &= \frac{3}{8} h(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3}{80} h^5 f^{(iv)}(\xi_3), & 0 < \xi_3 < 3h, \\ y_4 &= \frac{1}{3} h(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) - \frac{1}{90} (f^{(iv)}(\xi_4) + f^{(iv)}(\xi_5)), \\ & 0 < \xi_4 < 2h, \quad 2h < \xi_5 < 4h, \\ y_5 &= \frac{1}{24} h(8f_0 + 32f_1 + 17f_2 + 27f_3 + 27f_4 + 9f_5) - \\ & - \frac{h^5}{720} (3f^{(iv)}(\xi_6) + 27f^{(iv)}(\xi_7)), \\ & 0 < \xi_6 < 2h, \quad 2h < \xi_7 < 5h, \\ & \dots \end{aligned} \quad (11)$$

Thus, it follows from (11) that

$$|y_i| - h |f_i| A_i \leq R_i, \quad (12)$$

where

$$R_i = \frac{Mh^5}{720} [0, 1, 8, 27, 16, 35, 24, \dots],$$

$$M = \max_{0 \leq i \leq n} (|60h^{-2} f''(\xi)|, |f^{(iv)}(\xi)|),$$

$$k = 0, 1, 2, \dots, i; \quad i = 0, 1, 2, \dots, n.$$

Insofar as the total error is concerned, a theorem analogous to theorem 1



is fulfilled in the process of applying the operator  $A_2$ .

Theorem 2. If a function  $f(x)$ , given and continuous on the segment  $[0,1]$ , has on this segment derivatives of up to and including the fourth order and if the conditions  $|60h^{-2}f''(\xi)| \leq M$ ,  $|f^{(iv)}(\xi)| \leq M$ ,  $0 \leq \xi \leq 1$ , are fulfilled, in calculating  $\int_0^x f(t)dt$  by means of the operator  $A_2$ , when the errors of the calculated values of the function  $f^*(x)$  in each integration step  $h$  do not exceed the value  $\frac{Mh^4}{720}$ , the total-error vector  $R_2^*$  has the form

$$R_2^* = \frac{Mh^4}{720} [0, 2, 10, 30, 20, 40, 30, 50, \dots].$$

The proof of theorem 2 is analogous to the proof of theorem 1, and therefore we will not give it here.

Upon repeated integration by means of numerical operators, the matrices representing these operators multiply out. In fact, if we calculate  $\int_0^x dt \int_0^t f(\tau) d\tau$ ,

then, denoting  $\int_0^t f(\tau) d\tau$  by  $\Phi(t)$ , we get from (1)

$$\left[ \int_0^x \Phi(t) dt \right] \approx h [\Phi(t)] A_i = h^2 [f] A_i^2, \quad i = 1, 2. \quad (13)$$

It is easy to see that

$$\left[ \underbrace{\int_0^x dt \int_0^t d\tau \dots \int_0^{\tau} f(u) du}_m \right] \approx h^m [f] A_i^m, \quad i = 1, 2. \quad (14)$$

It should be noted, however, that it is not always expedient to use formula (14), since the calculations can be made more successfully if the left-hand part of formula (14) is replaced by the Cauchy formula:

$$\int_0^{kh} d\eta_1 \int_0^{\eta_1} d\eta_2 \int_0^{\eta_2} d\eta_3 \dots \int_0^{\eta_{m-1}} f(\eta_m) d\eta_m = \frac{1}{(m-1)!} \int_0^{kh} (kh-t)^{m-1} f(t) dt, \quad (15)$$

and then formula (1) is applied.

Now let us consider the solution of ordinary linear differential equations with constant coefficients by means of numerical integration operators.

Given the differential equation

$$ay' + by = f(x), \quad 0 \leq x \leq 1, \quad (16)$$

where  $a$  and  $b$  are constants, with the initial condition

$$y(0) = y_0. \quad (17)$$

By numerical integration of Eq. (16) taking (1) and (17) into account we get

$$a[y_A] - ay_0[1, 1, \dots, 1] + bh[y_A]A_1 = h[f_A]A_1$$

or

$$[y_A] \{aI + bhA_1\} = h[f_A]A_1 + ay_0[1, 1, \dots, 1], \quad i = 1, 2, \quad (18)$$

where  $I$  is the unit matrix. The matrix  $aI + bhA_1$  is invertible, since it is triangular and its diagonal elements are different from zero; and when  $h \neq -\frac{a}{ba_{jj}}$

( $a_{jj}$  are the diagonal elements of the matrix  $A_1$ )  $\det \{aI + bhA_1\} \neq 0$ . From (18) we get

$$[y_A] = \{h[f_A]A_1 + ay_0[1, 1, \dots, 1]\} B^{-1}, \quad (19)$$

where  $B = aI + bhA_1$ , ( $i = 1, 2, \dots$ ), is a matrix which we will call "resolving".

Equality (19) is the computational formula for finding the solution to the problem (16)-(17).

Let us now consider the second-order differential equation

$$ay'' + by' + cy = f(x), \quad x \in [0, 1], \quad (20)$$

$$y(0) = y_0, \quad y'(0) = y'_0,$$

where  $a$ ,  $b$ ,  $c$  are constants.

Integrating the first time we get

$$a(y' - y'_0) + b(y - y_0) + c \int_0^x y dt = \int_0^x f(t) dt, \quad (21)$$

$$ay' + by + c \int_0^x y dt = \int_0^x f(t) dt + ay'_0 + by_0.$$

Now we integrate (21) taking (13) into account and get

$$ay + b \int_0^x y dt + c \int_0^x dt \int_0^t y d\tau = \int_0^x dt \int_0^t f(\tau) d\tau + \int_0^x (ay'_0 + by_0) dt + ay_0,$$

or

$$\begin{aligned}
 a[y_k] + bh[y_k]A_i + ch^2[y_k]A_i^2 = h^2[f_k]A_i^2 + \\
 + (ay'_0 + by_0)[0, 1, 2, \dots, n] + ay_0[1, 1, \dots, 1], \quad i = 1, 2; \\
 k = 0, 1, 2, \dots, n.
 \end{aligned} \tag{22}$$

For the given equation we call the matrix

$$B = aI + bhA_i + ch^2A_i^2, \quad i = 1, 2.$$

the "resolving" matrix.

In view of the fact that the matrix  $A_j$  is triangular, for its invertibility it is necessary to choose  $h$  such that

$$a + bha_{ii} + ch^2a_{ii}^2 \neq 0,$$

then  $\det B \neq 0$  and formula (22) can be written in the form

$$\begin{aligned}
 [y_k] = \{h^2[f_k]A_i^2 + (ay'_0 + by_0)[0, 1, 2, \dots, n] + ay_0[1, 1, \dots, 1]\} B^{-1}, \\
 i = 1, 2; \quad k = 0, 1, 2, \dots, n.
 \end{aligned} \tag{23}$$

Formula (23) will be the computational formula for finding the solution to problem (20). When integrating Eq. (21) it is also possible to use equality (15) instead of (13). We then get

$$\begin{aligned}
 [y_k] \{aI + bhA_i + ch^2[k - j]_d A_i\} = \\
 = h^2[f_k][k - j]_d A_i + (ay'_0 + by_0)[0, 1, 2, \dots, n] + ay_0[1, 1, \dots, 1],
 \end{aligned}$$

where  $i = 1, 2$ ;  $j = 0, 1, 2, \dots, k$ ;  $k = 0, 1, 2, \dots, n$ ;  $[k - j]_d$  is a diagonal matrix.

We proceed analogously in the solution of an  $m$ -order linear differential equation, applying  $m$ -fold integration using equality (14) or replacing by formula (15).

From (19) and (23) it is clear that the process of finding the solution of a differential equation reduces to arithmetic operations with numerical matrices, which makes this method suitable for use in the computer solution of differential equations of the indicated type, since standard programs are used to multiply

and invert matrices.

Here the process of inverting the matrix B has an appreciable effect on the accuracy of calculation. This must be done by the method indicated in [3], i.e., by introducing a correction to the elements of the matrix  $B^{-1}$ .

Example. Let us consider the simplest differential equation

$$y' + 2y = 8x^2 - 4x, \quad x \in [0, 1] \quad (24)$$

with initial condition  $y(0) = 1$ , having the obvious solution  $y = 4x^2 - 6x + 3 - 2e^{-2x}$ . For the solution we use the operator  $A_2$  with  $h = 0.2$ . The computational formula for finding the solution of Eq.(24) has the form

$$[y_k] = [0.2 f_k] A_2 + [1, 1, \dots, 1] B^{-1},$$

where  $f(x) = 8x^2 - 4x$ ,  $B = I + 0.4A_2$ ,  $k = 0, 1, 2, 3, 4, 5$ .

The solution matrix, obtained analytically, is written as

$$[y_k]_{\text{anal}} = [1, 0.6194, 0.3410, 0.2376, 0.3562, 0.8343];$$

obtained by the proposed method it is written

$$[y_k]_{A_2} = [1, 0.6205, 0.3408, 0.2365, 0.3552, 0.8331];$$

and by the Runge-Kutta method (with computational formula

$$\Delta y_i \approx \frac{1}{6} (k_1 + 4k_2 + k_3), \quad k_1 = hf(x_i, y_i), \quad k_2 = hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right), \quad k_3 = hf(x_i + h, y_i + 2k_2 - k_1)$$

$$[y_k]_{\text{RK}} = [1, 0.6139, 0.3369, 0.2334, 0.3512, 0.7246].$$

The relative errors at selected nodes are, respectively:

by the proposed method:

$$\delta_1 = 0.16\%, \quad \delta_2 = 0.07\%, \quad \delta_3 = 0.46\%, \quad \delta_4 = 0.28\%, \quad \delta_5 = 0.14\%;$$

by the Runge-Kutta method:

$$\eta_1 = 0.89\%, \quad \eta_2 = 0.67\%, \quad \eta_3 = 1.77\%, \quad \eta_4 = 1.57\%, \quad \eta_5 = 13.2\%.$$

Setting  $\xi = z$  in (13), we get

$$\frac{M(z) - M^*(\bar{z})}{z - \bar{z}} = \sum_{j=0}^{\infty} [Q_j(z) + P_j(z) M(z)]^* [Q_j(z) + P_j(z) M(z)]. \quad (14)$$

Starting with relations (9) and (14), and then completely repeating the proof of lemma 4.1 from [1], we arrive at the following statement.

Lemma 3. In the completely indefinite case, for every finite domain G of the complex plane there exists a constant  $q_G$  such that

$$\left\| \sum_{j=0}^{\infty} Q_j(z) Q_j(z) \right\| \leq q_G. \quad (15)$$

Let us give without proof one more statement which describes the  $M(z)$  at any fixed point  $z$  ( $\text{Im } z \neq 0$ ) that correspond to all orthogonal operator distribution functions  $\tau(\lambda)$  of the positive-definite function  $F(t)$ .

Theorem 2. In the completely indefinite case, for fixed  $z$  ( $\text{Im } z \neq 0$ ) the set of  $M(z)$

$$M(z) = O(z) + \frac{1}{z - \bar{z}} \Gamma^{1/2}(\bar{z}) U \Gamma^{1/2}(z) \quad (16)$$

corresponding to all possible orthogonal distribution functions  $\tau(\lambda)$  of the function  $F(t)$  ranges over the "operator circle"  $K_z$  [1] with center

$$O(z) = \sum_{j=0}^{\infty} P_j^*(z) P_j(z) \left[ \frac{1}{z - \bar{z}} E + \sum_{j=0}^{\infty} P_j^*(z) Q_j(z) \right]$$

and radii (left and right)  $r_{\eta} = \frac{1}{|z - \bar{z}|} \Gamma^{1/2}(\bar{z})$ ,  $r_{\zeta} = \frac{1}{|z - \bar{z}|} \Gamma^{1/2}(z)$ ,  $U$  ranges over all unitary operators in  $H$ .

Replacing  $z$  by  $\xi$  and  $\bar{z}$  by  $\bar{\xi}$  in (13) and multiplying we get

$$M(z) E_0(\xi, z) + E_1(\xi, z) = [M(z) D_0(\xi, z) + D_1(\xi, z)] M(\xi), \quad (17)$$

where

$$\begin{aligned} D_0(\xi, z) &= -(\xi - z) \sum_{j=0}^{\infty} P_j^*(\bar{z}) P_j(\xi), & D_1(\xi, z) &= E - (\xi - z) \sum_{j=0}^{\infty} Q_j^*(\bar{z}) P_j(\xi), \\ E_0(\xi, z) &= \xi + (\xi - z) \sum_{j=0}^{\infty} P_j^*(\bar{z}) Q_j(\xi), & E_1(\xi, z) &= (\xi - z) \sum_{j=0}^{\infty} Q_j^*(\bar{z}) Q_j(\xi). \end{aligned} \quad (18)$$

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