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INFINITE COMPOSITION OF MÖBIUS TRANSFORMATIONS

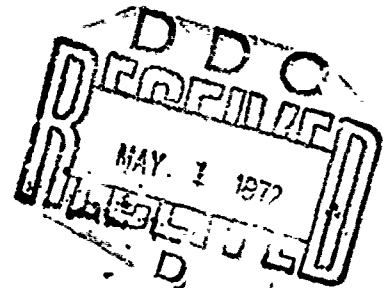
John Gill

Abstract. A sequence of Möbius transformations $\{t_n\}_{n=1}^{\infty}$, which converges to a parabolic or elliptic transformation t , may be employed to generate a second sequence $\{T_n\}_{n=1}^{\infty}$ by setting $T_n = t_1 \circ \dots \circ t_n$. The convergence behavior of $\{T_n\}$ is investigated and the ensuing results are shown to apply to continued fractions which are periodic in the limit.

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INFINITE COMPOSITION OF MÖBIUS TRANSFORMATIONS

John Gill

This paper treats the convergence behavior of sequences of Möbius transformations $\{T_n(z)\}$ which are generated in the following way:

Let $t_n(z) = (a_n z + b_n)/(c_n z + d_n)$, where $t = \lim_{n \rightarrow \infty} t_n$ is either parabolic or elliptic.

Set $T_1(z) = t_1(z)$, $T_n(z) = T_{n-1}(t_n(z))$, $n = 2, 3, \dots$.

Our approach is essentially the same as that of Magnus and Mandell [1], who investigated the cases in which the t_n and t are hyperbolic or loxodromic, and in which the t_n and t are all elliptic. They established conditions on the fixed points $\{u_n\}$ and $\{v_n\}$ of $\{t_n\}$ that insure behavior of $\{T_n(z)\}$ very much like that observed in the special case $t_n = t$ for all n [2]. Convergence is in the extended plane, so that divergence is of an oscillatory nature only.

The present paper consists of results concerning the two remaining possible combinations of t_n and t :

1. t_n any type and t parabolic, and 2. t_n elliptic or loxodromic and t elliptic. The principle result obtained

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in the investigation of case (2) is an extension and sharpening of the main theorem in [1].

The Parabolic Case. We first consider the case in which

$t = \lim t_n$ is parabolic, with a finite fixed point v . Some conditions on the rates at which u_n and v_n approach v are necessary, as the following example illustrates.

Example 1. Let $t_n = [n/(n+1)]^s z + 1$, where $s = 1 + iy$, $y \neq 0$. Then $t = z + 1$, which is parabolic with fixed point $v = \infty$. We have

$$T_n(z) = z/(n+1)^s + \zeta_n(s),$$

where $\zeta_n(s)$ is the truncated Riemann-Zeta function.

It can be shown, [3], p. 235, that $\zeta_n(s)$ oscillates finitely as $n \rightarrow \infty$ for the prescribed values of s .

Set $X(z) = z/(z-1)$. Then $X^{-1} \circ t_n \circ X(z) = t_n^*(z)$ and $t_n^*(z)$ are the same type of transformation, [1], and $t^* = X^{-1} \circ t \circ X$ has the fixed point $v^* = 1$. Obviously

$$T_n^*(z) = t_1^* \circ \dots \circ t_n^*(z) = X^{-1} \circ T_n \circ X(z)$$

has the same convergence behavior as $T_n(z)$.

→ Theorem 1. Let $\{t_n\}$ be a sequence of Möbius transformations converging to a parabolic transformation t , having a finite fixed point v . If there exists an ordering of u_n and v_n ,

the fixed points of t_n , such that $\sum |u_n - v_n|$ and $\sum |v_{n+1} - v_n|$ both converge, then the sequence $\{T_n(z)\}$ converges in the extended plane for every z .

Proof. Assume the t_n 's and c have been normalized so that $a_n d_n - b_n c_n = ad - bc = 1$, and that $a + d = 2$.

We first observe that any t_n may be written implicitly

$$(1) \quad \frac{1}{t_n(z) - v_n} = \frac{k_n}{z - v_n} + q_n,$$

where

$$k_n = \begin{cases} 1 & \text{if } t_n \text{ is parabolic} \\ \frac{a_n - c_n u_n}{a_n - c_n v_n} & \text{if } t_n \text{ is non-parabolic} \end{cases}$$

and

$$q_n = \begin{cases} c_n & \text{if } t_n \text{ is parabolic} \\ \frac{k_n - 1}{v_n - u_n} & \text{if } t_n \text{ is non-parabolic} \end{cases}.$$

It may easily be shown that $\lim k_n = 1$ and $\lim q_n = c \neq 0$.

Next, we set

$$Y_n(z) = 1/(z - v_n), \quad K_n(z) = k_n \cdot z, \quad Q_n(z) = q_n - c.$$

Then

$$t_n(z) = Y_n^{-1} \circ Q_n \circ K_n \circ Y_n(z).$$

Set

$$w_n(z) = Q_n \circ K_n \circ Y_n \circ Y_{n+1}^{-1}(z),$$

$$S_n(z) = Q_n \circ K_n \circ Y_n(z), \quad n = h, h+1, \dots,$$

where h will be chosen later.

Thus

$$T_n(z) = T_{h-1} \circ Y_h^{-1} \circ w_h \circ \dots \circ w_n \circ S_n(z).$$

Direct computation shows that $w_n(z) = (p_n z + q_n) / (r_n z + 1)$,

where: $r_n = v_{n+1} - v_n$ and $p_n = k_n + q_n r_n$.

We set $W_n^h(z) = w_h \circ \dots \circ w_n(z)$, and consider the convergence behavior of $\{W_n^h \circ S_n(z)\}_{n=h+1}^\infty$ for a fixed value of h .

Let

$$W_n^h(z) = \frac{A_n^h z + B_n^h}{C_n^h z + D_n^h},$$

where

$$(2) \quad A_n^h = p_n A_{n-1}^h + r_n B_{n-1}^h$$

$$(3) \quad B_n^h = q_n A_{n-1}^h + B_{n-1}^h$$

$$(4) \quad C_n^h = p_n C_{n-1}^h + r_n D_{n-1}^h$$

$$(5) \quad D_n^h = q_n C_{n-1}^h + D_{n-1}^h.$$

It follows from (2) and (3) that

$$\begin{aligned}
 A_n^h &= \prod_h p_i + \sum (\prod p_i) q_{k_1} r_{k_2} + \sum (\prod p_i) q_{k_1} r_{k_2} q_{k_3} r_{k_4} + \dots \\
 (6) \quad &+ \sum (\prod p_i) q_{k_1} r_{k_2} \dots q_{k_{2j-1}} r_{k_{2j}} ,
 \end{aligned}$$

where $h < k_1 < \dots < k_{2j} \leq h+m = n$, $1 < l \leq 2j$. The q and r -factors alternate, and $(\prod p_i)$ designates finite p -products, with $i \geq h$.

Lemma 1. Suppose $\{r_{h+k_j}\}_{j=1}^l$ are the r -factors in a term of A_n^h . Then there are no more than s terms having this specific set of r -factors in A_n^h , where $s \leq \prod_{i=1}^l k_i$.

Proof. The proof is by induction on the auxilliary recurrence relations:

$$A_{h+m}^h = A_{h+m-1}^h + r_{h+m} B_{h+m-1}^h \quad \text{and} \quad B_{h+m}^h = A_{h+m-1}^h + B_{h+m-1}^h .$$

We observe that

$$p_i = k_i + q_i r_i = 1 + (v_i - u_i) q_i + q_i r_i ,$$

so that, by hypothesis, $\prod p_i$ converges, and there exists a positive number M such that both $|\prod p_i|$ and $|q_i|$ are less than M for i greater than some h .

Fix $\epsilon > 0$ and choose h so large that the following conditions are met, in addition to those described above:

$|\prod_{i=1}^n p_i - 1| < \epsilon/2$, for $n \geq h$, and $\sum_{m=1}^{\infty} m|r_{h+m}| < \epsilon/M$, where

$$M = \min\{1, M, \epsilon/(2M+\epsilon)\}.$$

Consequently, by the preceding remarks and lemma 1,

$$\begin{aligned} |A_n^h - \prod_{i=1}^n p_i| &\leq \sum |(\prod p_i) q_{k_1} r_{k_2}| + \dots + \sum |(\prod p_i) q_{k_1} \dots r_{k_{2j}}| \\ &< M^2 (\epsilon/M) + \dots + M^{j+1} (\epsilon/M)^j \\ &< \epsilon/2. \end{aligned}$$

Hence

$$|A_n^h - 1| \leq |\prod_{i=1}^n p_i - 1| + \epsilon/2 < \epsilon$$

In an entirely similar manner it may be shown that

$$|C_n^h| < \epsilon, \text{ for a sufficiently large } h.$$

(2) and (3) give

$$A_{h+m}^h - k_{h+m}^h A_{h+m-1}^h = q_{h+m}^h r_{h+m}^h A_{h+m-1}^h + r_{h+m}^h B_{h+m-1}^h,$$

from which we obtain

$$(7) \quad A_{h+m}^h - A_{h+m-1}^h = (k_{h+m}^h - 1) A_{h+m-1}^h + r_{h+m}^h B_{h+m}^h$$

add both sides of (7).

$$(8) \quad A_{h+m}^h - p_h = \sum_{j=1}^m (k_{h+j}^h - 1) A_{h+j-1}^h + \sum_{j=1}^m r_{h+j}^h B_{h+j}^h.$$

(3) gives, upon summing,

$$(9) \quad B_{h+m}^h = q_h + \sum_{j=1}^m q_{h+j} A_{h+j-1}^h.$$

We combine (8) and (9) to obtain

$$(10) \quad A_{h+n}^h = p_h + \sum_{j=1}^m (k_{h+j} - 1) A_{h+j-1}^h + \sum_{j=1}^m r_{h+j} (q_h + \sum_{i=1}^j q_{h+i} A_{h+i-1}^h)$$

Thus, from (10), if $|q_{h+n}| < M$ and $|A_m^h| < 3$,

$$\begin{aligned} |A_{h+m+1}^h - A_{h+m}^h| &< 3|k_{h+m+1} - 1| + M|r_{h+m+1}|[1 + 3(m+2)] \\ &< 3[|k_{h+m+1} - 1| + M(m+3)|r_{h+m+1}|]. \end{aligned}$$

Therefore

$$\begin{aligned} |A_{h+m+n}^h - A_{h+m}^h| &\leq \sum_{j=1}^n |A_{h+m+j}^h - A_{h+m+j-1}^h| \\ &\leq 3M \left[\sum_{j=1}^n |v_{h+m+j} - u_{h+m+j}| + \sum_{j=1}^n (m+j+2) |r_{h+m+j}| \right]. \end{aligned}$$

The last expression on the right may be made arbitrarily small by choosing m sufficiently large and n a positive integer. The Cauchy criterion is satisfied and we have

$$(11) \quad \lim_{n \rightarrow \infty} A_n^h = i(A, h) \approx 1.$$

Similarly,

$$(12) \quad \lim_{n \rightarrow \infty} C_n^h = i(C, h) \approx 0.$$

It is obvious, from (9), that

$$(13) \quad \lim_{n \rightarrow \infty} B_n^h = \infty.$$

Also,

$$\begin{aligned} A_n^h D_n^h - B_n^h C_n^h &= \det W_n^h = \prod_h^{n-2} (\det w_j) = \prod_h^{n-2} k_j \\ &= \prod_h^{n-2} [1 + q_j(v_j - u_j)]. \end{aligned}$$

The hypothesis implies the convergence of this product to some number close to one, as $n \rightarrow \infty$.

Hence

$$(14) \quad \lim_{n \rightarrow \infty} (D_n^h / B_n^h) = \lambda_h \approx 0.$$

It is now possible to complete the proof of Theorem 1 for $z \neq v$. We have, from (11), (12), (13), and (14),

$$\lim_{n \rightarrow \infty} [W_n^h \circ S_n(z)] = \lim_{n \rightarrow \infty} \frac{(A_n^h / B_n^h) S_n(z) + 1}{(C_n^h / B_n^h) S_n(z) + (D_n^h / B_n^h)} = 1 / \lambda_h.$$

$$\lim_{n \rightarrow \infty} T_n(z) = T_{h-1} \circ Y_h(1 / \lambda_h), \quad z \neq v.$$

We divide numerator and denominator of $W_n^h \circ S_n(v)$ by $S_n(v)$ and find, after some computation, that

$$\lim_{n \rightarrow \infty} T_n(v) = T_{h-1} \circ Y_h(1/l_h) .$$

Corollary 1. Let $\{t_n\}$ be a sequence of normalized Möbius transformations converging to t , which is parabolic and has a finite fixed point. If $t_n(z) = (a_n z + b_n)/(c_n z + d_n)$, then the convergence of the following four series imply the convergence of $\{T_n(z)\}$ for every z : $\sum n|\sqrt{[(a_{n+1} + d_{n+1})^2 - 4]}|$, $\sum n|a_{n+1} - a_n|$, $\sum n|c_{n+1} - c_n|$, $\sum n|d_{n+1} - d_n|$.

The following example shows that the hypotheses of theorem 1, although sufficient, are not necessary.

Example 2. Let

$$t_n(z) = [(v_n + 1)z - v_n^2]/[z + (1 - v_n)] ,$$

where $v_1 = 0$ and $v_n = \sum_{k=1}^{n-1} (-1)^k/k$ for $n \geq 2$. Then

$\lim v_n = v = -\log 2$, and both t_n and t are parabolic. An intricate investigation, somewhat similar to the proof of theorem 1, shows that $\{T_n(z)\}$ converges for every $z \neq v$.

Case 2. We next consider the case in which

$t = \lim t_n$ is elliptic.

Theorem 2. Let $\{t_n\}$ be a sequence of Möbius transformations having fixed points $\{u_n\}$ and $\{v_n\}$, chosen so that

$|k_n| \leq 1$. Let $t = \lim t_n$ be an elliptic transformation having finite fixed points u and v .

(i) If $\sum |u_n - u_{n-1}| < \infty$, $\sum |v_n - v_{n-1}| < \infty$, and $\Pi k_n \rightarrow 0$, then $\{T_n(z)\}$ converges for every z except perhaps $z = v$.

(ii) If $\sum |u_n - u_{n-1}| < \infty$, $\sum |v_n - v_{n-1}| < \infty$, and $\Pi |k_n|$ converges, then $\{T_n(z)\}$ diverges by oscillation for $z \neq u, v$ and converges to distinct values for $z = u$ and $z = v$.

Proof. Set $Y_n(z) = (z - u_n)/(z - v_n)$, $K_n(z) = k_n z$, $w_n(z) = K_{n-1} \circ Y_{n-1} \circ Y_n^{-1}(z)$, $S_n(z) = K_n \circ Y_n(z)$, and $W_n^h(z) = w_n \circ \dots \circ w_{n-1}(z)$

$$= \frac{A_n^h z + B_n^h}{C_n^h z + D_n^h} . \text{ Then}$$

$$t_n(z) = Y_n^{-1} \circ K_n \circ Y_n(z) ,$$

and

$$T_n(z) = T_{n-1} \circ Y_{n-1}^{-1} \circ W_n^h \circ S_n(z) .$$

As before, $w_n(z) = (p_n z + q_n)/(r_n z + 1)$, where $p_n =$

$k_n(v_{n+1} - \bar{v}_n)/(v_n - u_{n+1})$, etc.

We choose a positive ϵ and find an h such that

$|\sum_{j=h}^n p_j| < \epsilon$ and $|C_n^h| < \epsilon$ for $n > h$. Thus $\lim_{n \rightarrow \infty} D_n^h =$

$\lambda(B, h) \approx 0$ and $\lim_{n \rightarrow \infty} D_n^h = \lambda(D, h) \approx 1$.

The following formula is established by induction:

$$(15) \quad A_n^h = \prod_{j=h}^n p_j + \sum_{m=n}^{n-2} \left(\prod_{j=m+1}^n p_j \right) r_{m+1} B_m^h + r_n B_{n-1}^h .$$

We observe that $\prod_h^n |p_j| = \prod_h^n |k_j| \cdot \prod_h^n (1+s_j)$, where $\sum |s_j| < \infty$.

Therefore, in case (i), $\prod_h^n |p_j| \rightarrow 0$, as $n \rightarrow \infty$. The three terms in (15) tend to zero, as $n \rightarrow \infty$. Hence, $\lim_{n \rightarrow \infty} A_n^h = 0$. In

similar fashion, $\lim_{n \rightarrow \infty} C_n^h = 0$.

Consequently,

$$\lim_{n \rightarrow \infty} T_n(z) = T_{h-1} \circ y_h^{-1} \circ \lim_{n \rightarrow \infty} W_n^h(S_n(z)) = T_{h-1} \circ y_h^{-1} \circ \frac{L(B, h)}{L(D, h)}$$

for $z \neq v$.

The hypotheses of case (ii), and the observed behavior of the coefficients of W_n^h provide a straightforward proof of the next lemma.

Lemma 2. For a fixed $z \neq v$, there exist finite numbers M and h_0 such that $h > h_0$, $n \geq h$, $m \geq h-1$ imply

$$|S_n(z)| < M \text{ and } |T_n^h(z) - v_m| > |u-v|/4(1+M).$$

The following formula may be established by induction on n , using (1) and the fact that $\frac{1}{t_{n+1}(z) - v_n} = \frac{1}{t_{n+1}(z) - v_{n+1}} +$

$$\frac{v_n - v_{n+1}}{(t_{n+1}(z) - v_n)(t_{n+1}(z) - v_{n+1})} :$$

$$(16) \quad \frac{1}{T_n^h(z) - v_h} = \frac{\prod_{j=1}^n k_j}{z - v_n} + \sum_{m=h}^{n-1} \sum_{j=1}^m (\prod k_j) \frac{v_m - v_{m+1}}{(T_n^{m+1}(z) - v_m)(T_n^{m+1}(z) - v_{m+1})} \\ + \sum_{m=h-1}^{n-1} \sum_{j=1}^m (\prod k_j) \frac{k_{m+1} - 1}{v_{m+1} - u_{m+1}},$$

where $\prod_{j=1}^{h-1} k_j \equiv 1$.

We may rewrite (16) in the form

$$(17) \quad \frac{1}{T_n^h(z) - v_h} = \frac{\prod_{j=1}^n k_j (z - u_n)}{(z - v_n)(v_n - u_n)} \\ + \sum_{h=1}^{n-1} \sum_{j=1}^m (\prod k_j) \frac{v_m - v_{m+1}}{(T_n^{m+1}(z) - v_m)(T_n^{m+1}(z) - v_{m+1})} \\ + \sum_{h+1}^{n-1} \sum_{j=1}^m (\prod k_j) \frac{v_{m+1} - v_m + u_m - u_{m+1}}{(v_m - u_m)(v_{m+1} - u_{m+1})} \\ + \frac{k_h - 1}{v_h - u_h} - \frac{k_h}{v_{h+1} - u_{h+1}}.$$

Set

$$\prod_{j=1}^n k_j = \exp(i \sum_{j=1}^n \theta_j) \prod_{j=1}^n |k_j|,$$

$$F = F(z) = \frac{z - u}{(z - v)(v - u)}, \quad R = |F| \sin(|\theta'|/4), \quad \text{where}$$

$$\arg k = \theta = \theta' \pmod{2\pi}, \quad |\theta'| \leq \pi. \quad \dots$$

We choose h so large that the following conditions are satisfied, in addition to previous stipulations:

$$(18) \quad |f_1| < R/6, \quad \text{where} \quad F + f_1 = \frac{z-u_n}{(z-v_n)(v_n-u_n)}$$

$$(19) \quad |f_2| < R/6, \quad \text{where} \quad \frac{k_h-1}{v_h-u_h} - \frac{k_h}{v_{h+1}-u_{h+1}} = f_2 + \frac{1}{u-v}$$

$$(20) \quad |f_3| < \min\{1, R/6|F|\}, \quad \text{where} \quad \prod_{h=1}^n |k_j| = 1 + f_3$$

$$(21) \quad \sum_h |v_{m+1}-v_m| < \frac{R|v-u|^2}{96(i+M)^2}$$

$$(22) \quad \sum_h |u_{m+1}-u_m| < \frac{R|v-u|^2}{48}$$

$$(23) \quad |v_m-u_m| > \frac{|v-u|}{2}, \quad m \geq h-1.$$

Then, from (17), we obtain

$$(24) \quad \frac{1}{T_n^h(z)-v_h} = |F| \exp\left[i\left(\arg F + \sum_h^n \theta_j\right)\right] + \frac{1}{u-v} + H(h,n),$$

where $|H(h,n)| < R$.

The sum of the first two terms of (24) is a point on a circle C with center $\frac{1}{u-v}$ and radius $|f|$. Hence $\frac{1}{T_n^h(z)-v_h}$

lies in a disc $U(h,m)$ of radius R with center g_n on C . R has been chosen so that three tangent discs of radius R with centers on C can be constructed if the centers of the two end discs are separated by a central angle of θ' .

Clearly, the sequence $\left\{ \frac{1}{T_n^h(z) - v_h} \right\}_{n=h}^{\infty}$ diverges by oscillation, so that $\left\{ T_n^h(z) \right\}_{n=h}^{\infty}$ must do likewise. The pattern of divergence bears a close resemblance to that observed when $t_n = t$ for all n . In this special case

$$\frac{1}{T_n(z) - v} = |F| \exp[i(\arg F + n\theta)] + \frac{1}{u-v}.$$

Convergence at $z = u$ is easily established, since $S_n(u) \rightarrow 0$. We return to the beginning of the proof of case (ii) and interchange the u_n 's and v_n 's, in order to show convergence at $z = v$. The development in [1] can be paraphrased to show that $\lim T_n(u) \neq \lim T_n(v)$.

Corollary 2. If the transformations t_n converge to the elliptic transformation t , where $a_n d_n - b_n c_n = ad - bc = 1$ and $\sum |a_n - a_{n-1}|$, $\sum |b_n - b_{n-1}|$, $\sum |c_n - c_{n-1}|$, and $\sum |d_n - d_{n-1}|$ all converge, then $\{T_n(z)\}$

- (i) converges for $z \neq v$, if $\prod k_n \rightarrow 0$
- (ii) diverges for $z \neq u, v$, and converges to distinct values at u and v , if $\prod |k_n|$ converges.

Continued fractions may be interpreted as compositions of Möbius transformations, and may be written so as to display the fixed points. Set $t_n(z) = \frac{-u_n v_n}{-(u_n + v_n) + z}$, to obtain

$$(25) \quad \frac{-u_1 v_1}{-(u_1 + v_1)} + \frac{-u_2 v_2}{-(u_2 + v_2)} + \dots,$$

whose n^{th} approximant is $T_n(0)$.

The following two examples are applications of theorems 1 and 2 to continued fractions which are periodic in the limit.

Example 3. Let $u_n = |u_n| \exp(i\theta_n)$, $v_n = |v_n| \exp(i\phi_n)$, where $\lim |u_n| = \lim |v_n| = c \neq 0$, $\lim \theta_n = \theta$, $\lim \phi_n = \phi$, $\theta \neq \phi \pmod{2\pi}$. Then $\lim k_n = \lim \frac{u_n}{v_n} \exp[i(\theta_n - \phi_n)] = k = \exp[i(\theta - \phi)]$, so that t is elliptic. Theorem 2, case (i) guarantees the convergence of (25), provided $|u_n|$ and $|v_n|$ are chosen so that $\lim \left| \frac{u_n}{v_n} \right| \rightarrow 0$, (e.g., $|u_n| = 1 - \frac{1}{n^2}$, $|v_n| = 1 + \frac{1}{n}$).

Example 4. Let $u_n = c + \epsilon_n$, $v_n = c + \delta_n$, where $\lim \epsilon_n = \lim \delta_n = 0$, $c \neq 0$, $\sum |\epsilon_n - \delta_n| < \infty$, $\sum |\delta_{n+1} - \delta_n| < \infty$. e.g., $u_n = -\frac{1}{2} - \frac{i}{n^2}$, $v_n = -\frac{1}{2} + \frac{i}{n^2}$. Then t is parabolic, and theorem 1 insures the convergence of (25).

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