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ON THE EQUIVALENCE OF SOME GENERALIZED NETWORK PROBLEMS TO PURE NETWORK PROBLEMS

by

Fred Glover *

D. Klingman

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* Professor of Management Science
The University of Colorado

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CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director
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The University of Texas
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13 ABSTRACT

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1) increasing computational efficiency

2) eliminating roundoff error

3) providing a simple check for infeasibility

4) extending the results in the literature on pure networks to appropriate generalized networks.

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ABSTRACT

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1) increasing computational efficiency
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3) providing a simple check for infeasibility
4) extending the results in the literature on pure networks to appropriate generalized networks.

1.0 Introduction and Conclusion

The purpose of this paper is to show that any generalized network problem whose incidence matrix does not have full row rank is equivalent to a pure network problem. Furthermore, we develop an efficient procedure which may be used simultaneously to determine the rank of the incidence matrix and transform (if possible) the generalized network problem into a pure network problem.

An important generalized network problem that can be transformed into a pure network problem by our procedure is the stock cutting problem of mathematical programming. In general, a number of generalized network problems have a structure that permits them to undergo this transformation. For instance, upon organizing the incidence matrix of a generalized transportation problem in echelon-diagonal form, any generalized transportation problem with repeating echelons and repeating diagonals has the appropriate rank.

Transforming generalized network problems into pure network problems provides obvious computational advantages. While the generalized network problem is next in order after pure networks or distribution problems with respect to ease of computation, the jump in degree of difficulty is such that there exist no algorithms for the former class of problems comparable in speed or efficiency to those for the latter. In fact, no computation times for special purpose algorithms for solving generalized network and

transportation problems [1, 2, 6, 11, 12, 13] have ever been published. Moreover, none of the major computer manufacturers have indicated the availability of special purpose codes for solving generalized network and transportation problems in their libraries or commercial packages. Thus we conclude that the existence of such codes must be rare, and for all practical purposes the only way currently available for solving a generalized network or transportation problem is via a general purpose linear programming code. This situation magnifies the discrepancy in computational effort required to solve pure and generalized network problems. As reported in [9], transportation problems can be solved at least 50 times faster by a special purpose code (utilizing the techniques of [7, 8]) than by general purpose linear programming methods (utilizing sophisticated procedures for exploiting sparse matrices). This difference would undoubtedly be accentuated when the general purpose code is applied to a generalized network structure.

There are, of course, other advantages to transforming a generalized problem into a pure network problem when this is possible. First, by using appropriate specialized algorithms, no round-off error is introduced while solving the pure network problem. Second, if the total supply of the pure network problem is not equal to the total demand, then one can conclude without further effort that the original problem lacks a feasible solution. (Similarly, if the original problem was a generalized transportation problem with

uncapacitated arcs, all admissible, then the check on total supply and total demand of the transformed problem provides a sufficient as well as necessary criterion for determining whether the original problem has a feasible solution.) Finally, our results imply that previous theoretical developments for pure networks can be applied to generalized networks of suitable rank.

1.1 Problem Formulation

Rigorous definitions of networks have appeared many times in the literature, and we will not reiterate them here. Speaking less formally, a pure network may be regarded as consisting of m nodes or junction points, which are connected pairwise by a collection of n directed arcs. Although it is not necessary for all pairs of nodes to be joined, we shall assume for convenience that the network is connected; i.e., there is a path consisting of a sequence of arcs of the network (possibly with some arcs having their orientation reversed) which connects any two given nodes. Any such network may be concisely described by its node-arc incidence matrix E. E is an m x n matrix with entries E_{pq} given as follows:

$$E_{pq} = \begin{cases} -1 \text{ if arc } q \text{ begins at node } p \text{ (node } p \text{ is its tail)} \\ 0 \text{ if arc } q \text{ does not touch node } p \\ +1 \text{ if arc } q \text{ terminates at node } p \text{ (node } p \text{ is its head)} \end{cases}$$

Each column E_k of E has two nonzero entries, -1 and +1, identifying a directed arc (i, j) from node i to node j. For each such admissible (existing) arc (i, j) in the network, we also define the following items:

a) x_{ij} is the flow from node i to node j.

b) c_{ij} is the cost of sending a single unit of flow from node i to node j.

c) b_i is the amount of supply or demand at node i, where supplies are denoted by negative quantities and demands are denoted by positive quantities.

Letting N be the set of all admissible arcs (i, j) in a network, then a capacitated pure network problem can be stated in the form:

Problem 1

$$\text{Minimize} \sum_{(i,j)\in N} c_{ij} x_{ij}$$

subject to:

$$-\sum_{(i,j)\in N} x_{ij} + \sum_{(j,i)\in N} x_{ji} = b_i \quad , \qquad i = 1, 2, \ldots, m$$

$$L_{ij} \le x_{ij} \le U_{ij} \quad , \qquad (i,j) \in N$$

where L_{ij} and U_{ij} are respectively the lower and upper bounds on the amount of flow on the arc (i, j) [U_{ij} need not be finite] and where we assume $\sum_{i=1}^{m} b_i = 0$.

In contrast to a pure network, a generalized network contains nonzero entries in its incidence matrix which are not necessarily ± 1, although each

column of the matrix still contains only two entries of opposite sign. We shall denote the incidence matrix of a generalized network by G. With appropriate scaling of each column of G and its associated cost c_{ij}, the negative element of each column can be set equal to -1. Letting k_{ij} denote the positive component of such a scaled column, the flow along the arc (i, j) may be viewed as being subjected to amplification (or attenuation) by the factor k_{ij} (i.e., the amount flowing into the terminal node j of arc (i, j) is k_{ij} times the amount that leaves the initial node i). The special purpose solution procedures [1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14] assume that this scaling of the columns has been performed before employing their solution techniques. The procedure to be developed here automatically performs this scaling as a byproduct of its other operations. For the case of full row rank, the procedure also creates a list of the arcs which were successfully transformed into "pure arcs"; i.e., those whose columns contain nonzero entries of 1 and -1 in the scaled incidence matrix. Such a list can be used to computational advantage in implementing special purpose solution procedures.

Denoting the nonzero entries of a column of G, corresponding to the directed arc (i, j) from node i to node j, by $-f_{ij}$ and h_{ij} (where $f_{ij} > 0$ and $h_{ij} > 0$), the capacitated generalized network problem can be stated in the form:

Problem 2

$$\text{Minimize} \sum_{(i,j)\in N} \bar{c}_{ij} x_{ij}$$

subject to:

$$-\sum_{(i,j)\in N} f_{ij} x_{ij} + \sum_{(j,i)\in N} h_{ji} x_{ji} = \bar{b}_i \quad , \quad i = 1, 2, \ldots, m$$

$$\bar{L}_{ij} \le x_{ij} \le \bar{U}_{ij} \quad , \quad (i,j) \in N \ .$$

where $\sum_{i=1}^{m} \bar{b}_i$ is not necessarily equal to zero.

Note that if the constraints of Problem 2 are given as inequalities rather than equalities, it is possible to transform the problem into the format of Problem 2 by adding absorbing loops (see [11]), although this may alter the rank of the incidence matrix. In this connection it is important to point out that our procedure can be applied to the inequality form of Problem 2 as well as to the equality form. Thus, if the incidence matrix for the inequality form has rank m-1, one can still obtain an equivalent pure network problem which can be converted into the format of Problem 1 using the standard transformation.

1.2 Scaling Procedure and Equivalence

The procedure developed in this section simultaneously determines the rank of an m x n incidence matrix of a generalized capacitated network and produces an equivalent network problem whose negative entry in each column is -1. If the original incidence matrix has rank m-1, this procedure also yields a pure network problem.

Our procedure can be viewed as a labeling method in which each row and column of the incidence matrix is labeled exactly once. The generation of these labels proceeds by scaling one constraint (row of the incidence matrix) and several variables (columns of the incidence matrix) at each iteration.

The scaling procedure creates two vectors R and K where the i^{th} component, r_i, of R represents the scaling factor for row i and the j^{th} component, k_j, of K represents the scaling factor for column j. Initially these vectors are set to zero, with $r_i = 0$ ($k_j = 0$) indicating that row i (column j) has not been scaled. When the procedure terminates all components of R and K are positive. During the intermediate stages of the procedure, R is a nonnegative vector while the components of K may be positive, zero, or negative. This sign convention on the components of K has the following interpretation:

(a) $k_j = 0$ indicates that neither of the rows containing the nonzero entries in column j have been scaled (hence, the r_i values for these rows are zero).

(b) $k_j < 0$ indicates that exactly one of the rows containing the nonzero entries in column J has been scaled.

(c) $k_j > 0$ indicates that both of the rows containing the nonzero entries in column j have been scaled.

The scaling procedure may then be stated as follows

1. Set R = 0 and K = 0. Set NC = $\emptyset$. (At the termination of the scaling procedure, this set will contain the indexes of those columns whose entries are not ± 1.)
2. Scale row 1 of G by one (i.e., set $G^1 = 1 \cdot G^1$ and set $r_1 = 1$).
3. Scale each column having a nonzero entry in row 1 by the reciprocal of its absolute value in row 1. (That is, set $G_j = \frac{1}{|g_{1j}|} G_j$, and $k_j = -\frac{1}{|g_{1j}|}$, for all j such that $g_{1j} \neq 0$.)
4. If $K > 0$, go to 6 (by connectedness, all rows and columns have been scaled). If $K \not> 0$ select j^* such that $k_{j^*} < 0$ and identify i^* such that $g_{i^*j^*} \neq 0$ and $r_{i^*} = 0$. Scale row i^* by setting $r_{i^*} = \frac{1}{|g_{i^*j^*}|}$, $G^{i^*} = \frac{1}{|g_{i^*j^*}|} G^{i^*}$, and update K by setting $k_{j^*} = -k_{j^*}$.
5. Scale each previously unscaled column with a nonzero entry in row i^* of G by setting $k_j = -\frac{1}{|g_{i^*j}|}$, and $G_j = \frac{1}{|g_{i^*j}|} G_j$ for each j such that $g_{i^*j} \neq 0$ and $k_j = 0$. Then update K and NC by setting $k_j = -k_j$ for each j such that $g_{i^*j} \neq 0$ and $k_j < 0$, and setting $NC = NC \cup \{j\}$ for each of these latter j for which $g_{i^*j} \neq 1$. Then return to Step 4.

6. If NC = $\emptyset$, go to step 7. If not, for each $j \in NC$ identify the index i^* such that $g_{i^*j} < 0$ and if $g_{i^*j} \neq -1$ set

$$k_j = \frac{1}{|g_{i^*j}|} k_j \quad \text{and} \quad G_j = \frac{1}{|g_{i^*j}|} G_j.$$

7. Stop.

Once the procedure terminates with $R > 0$ and $K > 0$, the scaling is completed by multiplying each $\bar{b}_i$ by its corresponding component of R, and by multiplying each $\bar{c}_{ij}$ and dividing each $\bar{L}_{ij}$ and $\bar{U}_{ij}$ by its associated component of K. Otherwise, the problem is unaltered; i.e., uncapacitated problems remain uncapacitated, transportation networks remain transportation networks, cost entries retain their original sign, sources and sinks maintain their identity. (This procedure can be applied directly to network representation rather than to the incidence matrix.)

Theorem: NC = $\emptyset$ when the scaling procedure terminates if and only if the original incidence matrix G has rank m-1.

Proof: First, suppose NC = $\emptyset$ when the scaling procedure terminates. Then every column of the scaled incidence matrix contains two nonzero entries whose values are +1 and -1 as a result of the following observations: each column of the original incidence matrix contained exactly two entries of opposite sign, the scaling weights are all positive, and the scaling procedure does not terminate until there exists no column which has

one of its nonzero elements in a scaled row and the other in an unscaled row (see step 4). By connectedness, this implies all rows have been scaled. Furthermore, each time a row is scaled, each nonzero entry in the row that is also in a scaled column is checked to see if it is ± 1. If not, NC is augmented to include this entry's column index. Also, any nonzero entry in the row that is not in a scaled column is scaled to one (see step 5). This demonstrates that NC = $\emptyset$ implies that all nonzero entries of the scaled incidence matrix are +1 and -1, which, of course, means that the scaling procedure has transformed the generalized network problem into an equivalent pure network problem, and the rank of G is m-1.

Next assume that the rank of G is m-1. Charnes and Raike [4] show that if any row of G is deleted, the remaining rows are linearly independent. Thus if G has rank m-1, any row of G can be formed as a unique linear combination of the other rows in G. The scaling procedure determines this unique combination for row 1. In particular, upon applying Steps 2 and 3 of the procedure to scale each nonzero entry in row 1 to be ± 1, Steps 4 and 5 are designed to determine the unique linear combination of the remaining rows which, when added to row 1, is equal to zero. To accomplish this, each of the ± 1 entries currently in row 1 after applying Step 3 must be

"offset" by scaling the appropriate row so that the unique nonzero entry which is contained in the same columns as the ± 1 entry in row 1 is itself scaled to ± 1. This uniquely determines some r_i, and successive application of the above approach yields Steps 4 and 5 and all r_i values. If NC $\neq \emptyset$ at the termination of the scaling procedure, then at some application of Step 5 there exists a nonzero entry g_{i^*j} in row i^* different from ± 1 whose associated nonzero column entry has already been scaled to ± 1. Moreover, the weight r_{i^*} has been uniquely identified at Step 4 to create a linear combination of the remaining rows equal to the negative of row 1, if this is possible, and the fact that g_{i^*j} cannot "offset" its associated nonzero column entry establishes that such a linear combination does not exist. Consequently, NC must be equal to the null set if G has rank m-1.

The scaling procedure developed in this section is quite simple and easy to implement. Clearly, as a result of the theorem, each column j of the transformed G contains a -1 and some positive number p_j, where $p_j \neq 1$ for at most the cardinality of the set NC. An interesting question arises as to how the cardinality of NC can be minimized. A procedure for accomplishing this would allow one to take increased computational advantage of the pure network structure in a generalized network problem that could not be completely reduced to a pure network. It is reasonable to suppose that "largely pure" networks can be solved more efficiently than those which are saddled with a greater percentage of "generalized" arcs. We defer such considerations to another paper.

1.3 Example

Consider the uncapacitated generalized network problem:

$$\text{Minimize } 3/2\,x_{12} + 4x_{13} + 9x_{23} + 4x_{24} + 1x_{25} + 16x_{36} + 1x_{34} + 2/3\,x_{43} + 16x_{45} + 5x_{46}$$

Subject to

$$\begin{bmatrix} -1/2 & -4 & & & & & & & & & \\ 1/4 & & -3 & -4 & -1/2 & & & & & & \\ & 2 & 3 & & & -2 & -1/2 & 1/3 & & & \\ & & & 4 & & & 1/2 & -1/3 & -4 & -5 & \\ & & & & 3/8 & & & & 3 & & \\ & & & & & 2 & & & & 5 & \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{23} \\ x_{24} \\ x_{25} \\ x_{36} \\ x_{34} \\ x_{43} \\ x_{45} \\ x_{46} \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 0 \\ 0 \\ 6 \\ 4 \end{bmatrix}$$

$$x_{ij} \geq 0$$

The standard scaling of this problem, to introduce a -1 entry in each column (which is required, for example, in the procedures of [1, 2, 6, 11, 12, 13] is accomplished by multiplying the columns by 2, 1/4, 1/3, 1/4, 2, 1/2, 2, 3, 1/4, and 1/5, respectively, to yield the equivalent generalized network problem:

Minimize $3\bar{x}_{12} + 1\bar{x}_{13} + 3\bar{x}_{23} + 1\bar{x}_{24} + 8\bar{x}_{36} + 2\bar{x}_{34} + 2\bar{x}_{43} + 4\bar{x}_{45} + 1\bar{x}_{46}$

Subject to

$$\begin{bmatrix} -1 & -1 & & & & & & & & & \\ 1/2 & & -1 & -1 & -1 & & & & & & \\ & 1/2 & 1 & & & -1 & -1 & 1 & & & \\ & & & 1 & & & 1 & -1 & -1 & -1 & \\ & & & & 3/4 & & & & 3/4 & & \\ & & & & & 1 & & & & 1 & \end{bmatrix} \begin{bmatrix} \bar{x}_{12} \\ \bar{x}_{13} \\ \bar{x}_{23} \\ \bar{x}_{24} \\ \bar{x}_{25} \\ \bar{x}_{36} \\ \bar{x}_{34} \\ \bar{x}_{43} \\ \bar{x}_{45} \\ \bar{x}_{46} \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 0 \\ 0 \\ 6 \\ 4 \end{bmatrix}$$

$\bar{x}_{ij} \geq 0$ where $x_{12} = 2\bar{x}_{12}$, $x_{13} = 1/4\,\bar{x}_{13}$, $x_{23} = 1/3\,\bar{x}_{23}$, $x_{24} = 1/4\,\bar{x}_{24}$, $x_{25} = 2\bar{x}_{25}$, $x_{36} = 1/2\,\bar{x}_{36}$, $x_{34} = 2\bar{x}_{34}$, $x_{43} = 3\bar{x}_{43}$, $x_{45} = 1/4\,\bar{x}_{45}$, and $x_{46} = 1/5\,\bar{x}_{46}$.

This problem is graphically depicted in Figure 1. The number appearing in the rectangle on each arc is the cost per unit of flow along the arc, and the number in the semicircle is the "multiplier value" k_{ij} for the arc.

Figure

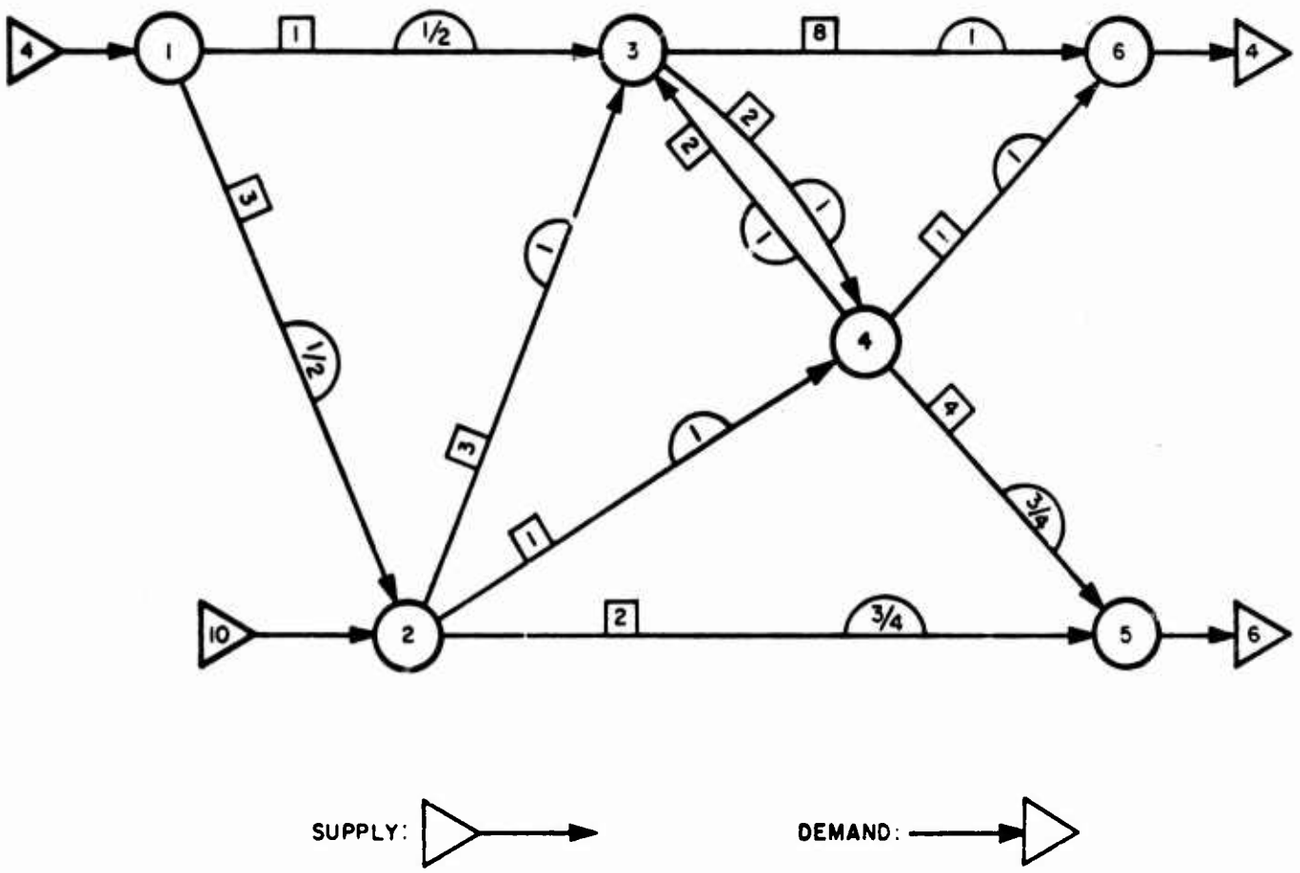

FIGURE 1

Alternatively, applying the scaling procedure of the preceding section to the original incidence matrix, we obtain the following pure network problem:

$$\text{Minimize } 3y_{12} + 1y_{13} + 3/2\,y_{23} + 1/2\,y_{24} + 1y_{25} + 4y_{36} + 1y_{34} + 1y_{43} + 2y_{45} + 1/2\,y_{46}$$

Subject to

$$\begin{bmatrix} -1 & -1 & & & & & & & & & & \\ 1 & & -1 & -1 & -1 & & & & & & & \\ & 1 & 1 & & & -1 & -1 & 1 & & & & \\ & & & 1 & & & 1 & -1 & -1 & -1 & & \\ & & & & 1 & & & & 1 & & & \\ & & & & & 1 & & & & & 1 & \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{13} \\ y_{23} \\ y_{24} \\ y_{25} \\ y_{36} \\ y_{34} \\ y_{43} \\ y_{45} \\ y_{46} \end{bmatrix} = \begin{bmatrix} -4 \\ -20 \\ 0 \\ 0 \\ 16 \\ 8 \end{bmatrix}$$

$y_{ij} \geq 0$ where $R = (1, 2, 2, 2, 8/3, 2)$ and the components of K are given by the coefficients of the y_{ij} in the identities $x_{12} = 2y_{12}$, $x_{13} = 1/4\,y_{13}$, $x_{23} = 1/6\,y_{23}$, $x_{24} = 1/8\,y_{24}$, $x_{25} = 1y_{25}$, $x_{36} = 1/4\,y_{36}$, $x_{34} = 1y_{34}$, $x_{43} = 3/2\,y_{43}$, $x_{45} = 1/8\,y_{45}$, and $x_{46} = 1/10\,y_{46}$.

The fact that total supply equals total demand in this new representation establishes the possible existence of a feasible solution for this problem.

To appreciate the difference in computational effort required to solve these equivalent statements of the original problem the reader is encouraged to apply an appropriate method (e.g., that of [11]) to the generalized network formulation and to apply a standard "minimum cost flow" algorithm (such as the out-of-kilter method [14]) to the pure network formulation.

REFERENCES

[1] Balas, Egon. "The Dual Method for the Generalized Transportation Problem," Management Science 12, 7 (March, 1966), pp. 555-568.

[2] Balas, Egon and P. L. Ivanescu (Hammer). "On the Generalized Transportation Problem," Management Science 11, 1 (September, 1964), pp. 188-202.

[3] Charnes, A. and W. W. Cooper. Management Models and Industrial Applications of Linear Programming, Volumes I and II. New York: John Wiley and Sons, Inc., 1961.

[4] Charnes, A. and W. M. Raike. "One-Pass Algorithms for Some Generalized Network Problems," Operations Research 14, 5 (September-October, 1966), pp. 914-924.

[5] Dantzig, George. Linear Programming and Extensions. Princeton, N.J.: Princeton University Press, 1963.

[6] Eisemann, Kurt. "The Generalized Stepping-Stone Method for the Machine Loading Model," Management Science 11, 1 (September, 1964), pp. 154-177.

[7] Glover, Fred and D. Klingman. "Locating Stepping-Stone Paths in Distribution Problems via the Predecessor Index Method," Transportation Science 4, 2 (May, 1970), pp. 220-225.

[8] Glover, Fred, D. Karney and D. Klingman. "The Augmented Predecessor Index Method for Locating Stepping-Stone Paths and Assigning Dual Prices in Distribution Problems," to appear in Transportation Science.

[9] Glover, Fred, D. Karney, D. Klingman and Al Napier. "A Comparison of Computation Times for Various Starting Procedures, Basis Change Criteria and Solution Algorithms for Distribution Problems," Research Report CS 44, Center for Cybernetic Studies, The University of Texas, Austin, Texas, May 1971.

[10] Glover, Fred, D. Klingman and Al Napier. "A One-Pass Algorithm to Determine a Dual Feasible Basic Solution for A Class of Capacitated Generalized Networks," Research Report CS 42, Center for Cybernetic Studies, The University of Texas, Austin, Texas, October 1970. To appear in Operations Research.

[11] Jewell, W. S. "Optimal Flow Through Networks with Gains," Operations Research 10, 4 (May-June, 1962), pp. 476-499.

[12] Lourie, Janice. "Topology and Computation of the Generalized Transportation Problem," Management Science 11, 1 (September, 1964), 11. 177-187.

[13] Lynn, W. R. and A. Charnes. "The Dyadic Computation of a Generalized Network Model of a Sewage Treatment System," ONR Research Memorandum No. 19, Northwestern University, Evanston, Illinois, October 1959.

[14] Simmonnard, Michel. Linear Programming (translated by William S. Jewell). Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1966.