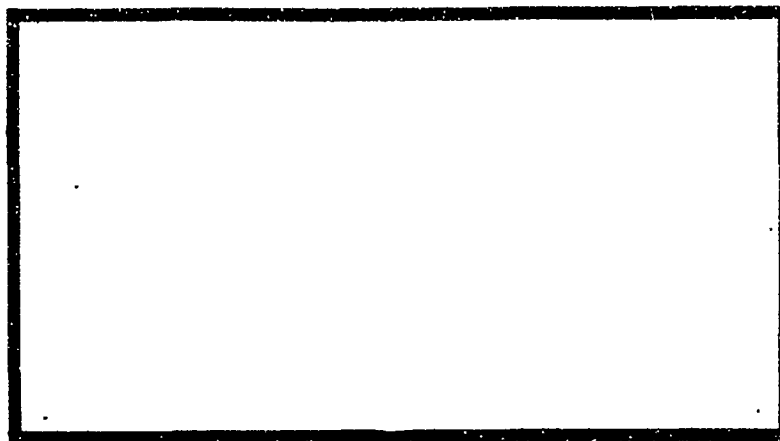


AD743611



UNITED STATES AIR FORCE
AIR UNIVERSITY
AIR FORCE INSTITUTE OF TECHNOLOGY
Wright-Patterson Air Force Base, Ohio

Reproduced by
NATIONAL TECHNICAL
INFORMATION SERVICE
Springfield, Va. 22151

2 D C
JUN 28 1972
RECEIVED
L3 D

R
12

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Air Force Institute of Technology (AFIT-SE) Wright-Patterson AFB, Ohio 45433		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		3b. GROUP	
3. REPORT TITLE Bayesian Reliability Assessment for Systems Program Decisions			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) AFIT Thesis			
5. AUTHOR(S) (First name, middle initial, last name) Lewis R. White Capt USAF			
6. REPORT DATE December 1971		7a. TOTAL NO. OF PAGES 123	7b. NO. OF REFS 486
8a. CONTRACT OR GRANT NO. b. PROJECT NO. N/A		9a. ORIGINATOR'S REPORT NUMBER(S) GRE/MATH/66-11	
c. d.		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
10. DISTRIBUTION STATEMENT Approved for public release; distribution unlimited			
Approved by Public release; LAW AFR 190-127 Keith A. Williams, 1st Lt., USAF Acting Director of Information		SPONSORING MILITARY ACTIVITY	
13. ABSTRACT Bayesian statistics provide the necessary mathematical techniques to pool all available subjective and experimental information when estimating reliability. The uncertainties associated with analytical predictions or limited test data considered separately are significantly reduced when these two sources of information are combined. The introduction of judgment and pertinent engineering theory and experience to qualify point estimates is the key to realistic and practical solutions to decision problems in which reliability is a primary consideration. A method for periodic reliability assessment is presented. A hypothetical example is used to show how iterative inference on system reliability can be drawn from initial estimates of unit/subsystem reliability and heterogeneous time and failure data accumulated during various stages of design verification, electrical performance, environmental, etc. testing. Sample worksheets for recording inputs required for the assessment technique are provided as an appendix.			

DD FORM 1473
1 NOV 65

Unclassified

Security Classification

BAYESIAN RELIABILITY ASSESSMENT
FOR SYSTEM PROGRAM DECISIONS

THESIS

GRE/MATH/66-11 Lewis Fay White
Captain USAF

Approved for public release; distribution unlimited

14.	KEY WORDS	LINK A		LINK B		LINK C	
		ROLE	WT	ROLE	WT	ROLE	WT
	System reliability Bayesian Statistics Decision Theory						

BAYESIAN RELIABILITY ASSESSMENT
FOR SYSTEM PROGRAM DECISIONS

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the
Requirements for the Degree of

Master of Science

by

Lewis Ray White, B.S.M.E.

Capt

USAF

Graduate Reliability Engineering

December 1971

Approved for public release; distribution unlimited

Preface

This thesis is the presentation of the results of an extensive literature search to discover applications of Bayes formula for the solution of typical decision problems encountered by reliability and maintainability, R/M, engineers.

The purpose of this study is to present an overview of the basic elements and fundamental concepts of general decision theory and show how they might be useful to the R/M practitioner. It is hoped that this brief exposure to some of the organized and systematic techniques to decision-making will create a thirst for more detailed individual investigation for ways to treat similar situations arising in both personal and professional life.

My most sincere thanks are extended to my Thesis Advisor, Professor A. H. Moore for his inspiration, motivation, and consultation. I am also deeply indebted to my wife, [REDACTED] whose degree of belief in me was great enough to assign a prior subjective probability of thesis completion equal to one. Bayes rule always works with a driving force like that.

Lewis R. White

Contents

	Page
Preface	ii
List of Figures	v
List of Tables	vi
Abstract	vii
I. Introduction	1
II. Decision Theory Review	4
Basic Elements	6
Primary Sets	6
Supplementary Sets	9
Fundamental Concepts	11
Definition of the Problem	12
Assessment of the Uncertainties	12
Comparison of Consequences	14
Characteristics of Utility	15
Calculation of Expected Value	16
III. Bayes Theorem	17
Development	17
Interpretation	18
Controversy of Usage	19
Application to Reliability Problems	20
IV. Prior Distribution	23
Assigning Gamma Parameters	25
Upper Bound Method	28
Coefficient of Variation Method	28
Equivalent Test Time Method	31
Assigning Beta Parameters	32
V. Bayesian Reliability Assessment Example	34
Definition of the Problem	35
Initial Assessment of Uncertainties	36
Comparison of Consequences	40
Calculation of Expected Value	47
Second Assessment of Uncertainties	48
Final Assessment of Uncertainties	49

Contents

	Page
VI Assessment Technique Analysis and Refinement	53
Examination of Input Contributions	53
Gamma Vs Normal System Distributions	56
Other Practical Considerations	61
Weighting Prior Assignments	61
Weighting Heterogeneous Test Data	62
Revised General Approach	63
VII Conclusions and Recommendations	66
Conclusions	66
Recommendations	67
Bibliography	68
Appendix A: Supplemental Bibliography	87
Appendix B: Sample Worksheets	108
Vita	116

List of Figures

<u>Figure</u>		<u>Page</u>
1	Probability Diagram	17
2	Reliability Growth Curves	45
3	Effect of Test Data on Antenna Array Gamma Distribution	54
4	Effect of Test Data on Emitter/Detector Gamma Distribution	55

List of Tables

<u>Table</u>		<u>Page</u>
I	Values for a if λ_p Assigned Median	29
II	Values for a if λ_p Assigned Mean	29
III	Equivalent Test Time for Priors	32
IV	NEW System Reliability Predictions	37
V	Parameters for a System Prior Distribution	37
VI	Values for $P(\lambda_j)$ Based on Predictions	40
VII	Life Cycle Cost Factors	42
VIII	Monthly Maintenance Expense, $R(N, \lambda)$	43
IX	Production Release Decision Matrix	47
X	Cumulative Unit/Subsystem Test Data	48
XI	Values for $P(\lambda_j)$ Based on Test Data	49
XII	Parameters for a System Posterior Distribution	52
XIII	Values for $P(\lambda_j)$ Based on Combined Inputs	52
XIV	Sample Failure Information	58
XV	$E(a_j)$ Values Assuming Gamma System Distribution	59
XVI	$E(a_j)$ Values Assuming Normal System Distribution	60

Abstract

Bayesian statistics provide the necessary mathematical techniques to pool all available subjective and experimental information when estimating reliability. The uncertainties associated with analytical predictions or limited test data considered separately are significantly reduced when these two sources of information are combined. The introduction of judgement and pertinent engineering theory and experience to qualify point estimates is the key to realistic and practical solutions to decision problems in which reliability is a primary consideration.

A method for periodic reliability assessment is presented. A hypothetical example is used to show how iterative inferences on system reliability can be drawn from initial estimates of unit/subsystem reliability and heterogeneous time and failure data accumulated during various stages of design verification, electrical performance, environmental, etc., testing. Sample worksheets for recording inputs required for the assessment technique are provided as an appendix.

BAYESIAN RELIABILITY ASSESSMENT
FOR SYSTEM PROGRAM DECISIONS

I. Introduction

The number of cost-plus type government contracts which permit generous budgets for extensive reliability/maintainability studies and tests to minimize risk is dwindling. The time has come when both government and industry have been forced to tighten their money belts. The resulting squeeze has made backbones stiffer and eyes more pointed. In today's austere atmosphere, what used to be accepted techniques and practices are challenged as potential candidates for modification and revision in an attempt to reduce costs.

But although there is less money to spend, the requirements are just as stringent, if not more so. The impact of this current climate is felt throughout all organizational levels in the military-industrial complex. Essentially, practicality and utility are the keynotes that are replacing desirability, feasibility and availability. The reliability/maintainability practitioner must adjust to this new environment.

The theme of getting the most for the money is really nothing new. However, the recent reduction in resources has resulted in more scrutiny of actions taken to achieve this end. No longer is a sole recommendation on a particular issue accepted at face value. The following simple, straightforward and sensible questions are being asked more frequently:

- a. Is this the only way to solve the problem?
- b. Were other solutions considered?
- c. Why is the proposed solution considered the best?

Decision theory provides the framework for systematically addressing these questions and quantifying the uncertainties associated with their answers. The primary purpose of this thesis is to present a method for periodic assessment of system reliability risk based only on analytical predictions and limited usage experience. Particular emphasis is placed on the use of subjective probability and Bayes' formula. These techniques have received much attention in recent years and considerable literature is available for those who wish to pursue more advanced analyses than the one presented. As an aid to those who desire more detail, an additional study objective is to provide a selective listing of references in which general decision theory and Bayesian analyses are used to solve problems of choice usually encountered in the field of reliability and maintainability. These techniques are considered to be valuable analytical tools for assessing alternatives during the establishment, specification, verification and demonstration of quantitative reliability/maintainability requirements throughout a system's life.

To achieve study objectives, a comprehensive survey of recent literature was conducted with the aid of abstracting, indexing, and search services. The results of this survey and the subsequent study are presented as follows: Section II summarizes the elements, concepts and notation associated with general decision theory; Section III introduces Bayes' formula and outlines how it may be used to combine

all available relevant information, both subjective judgements and objective data, in the decision process; Section IV addresses the selection of prior distributions applicable to the reliability assessment problem; Section V outlines a proposed method for applying Bayesian techniques to assess the reliability of a system prior to formal testing; Section VI provides a discussion of the assessment technique with a few refinements; and Section VII contains the conclusions and recommendations of the study. A supplementary bibliography is provided as an appendix. Listed are both references which are probably directly applicable to the thesis but were not acquired, and also those which are indirectly or remotely pertinent to the investigation but perhaps of interest to persons working in other functional areas in which the generalized ideas presented are applicable. Also appended are sample worksheets which might be used in situations similar to the example presented.

II. Decision Theory Review

Since man was created, he has been making decisions. As the species expanded, other men (and oftentimes women) started telling him how he should make (or should have made) decisions. In recent years, startling new insights have evolved in the recommended processes for decision-making. Today decision theory is considered almost a distinct scientific discipline. Does this mean that, because we all make decisions (or suffer the consequences from them), we should rush out and obtain a rash of textbooks and read up on the subject? The answer to this question depends on the nature of decisions involved. Most decision situations are trivial and require very little thought; therefore, to perform extensive analyses before making simple choices would be ridiculous to say the least.

Nevertheless, there are occasions when there is much at stake and thorough examination and evaluation of various options and their implications are definitely in order. In these important cases, some knowledge of the precepts and principles of decision theory can be extremely helpful. However, it is stressed from the outset that the study and application of decision theory will not add to the information available to the decision maker, but will merely help him organize the relevant facts and opinions in a manner which will assist him in making a rational choice. Decision theory is essentially a logically consistent systematic approach to the selection of one of several alternative courses of action available to a decision maker.

The methodical approach which is outlined in this section should be recognized as a basic conceptual framework for the analytical treatment of decision situations. It should only be used as a guide and not be considered as a dogmatic delineation of how to treat all problems. Each decision situation is unique and must be handled in light of its particular characteristics. A primary principle in decision theory is that selection problems can be broken down into their constituent elements which are usually easier to analyze. The methodology also puts these individual elements in proper perspective with each other so, although they can be treated separately, the "big picture" is not lost. The detailed analysis of these interrelated portions of the problem are accomplished in various ways but each has a common prerequisite - the willingness to think quantitatively.

This requirement is established so the powerful mathematical techniques associated with decision theory can be employed to formulate and solve decision models. No longer does the decision maker have to rely solely on emotion and intuition. However, the use of analytical tools by no means eliminates or diminishes the importance of a decision maker's experience and his instincts and insights derived from similar situations. Rather than reducing the need for human judgement, the framework actually provides a mechanism for the explicit consideration of personal experience and opinion. As a matter of fact, the additional clarity resulting from the distinct differentiation between judgements involving the likelihood of an event and those concerning the worth of an alternative is a definite advantage of the process.

There are really several processes associated with the general study of decision theory. There is no agreed standard procedure on how to structure and solve decision problems. The synopsis which follows is a presentation of a process that is typical to the solution of decision problems dealing with uncertainty. The following parts, principles and procedures are considered of the most value for the decision maker who works primarily in the reliability and maintainability arena.

Basic Elements

As previously mentioned, decision situations can be broken down or decomposed into sets of factors which can be analyzed individually at first and collectively later. Of course it is more difficult to dissect and recombine partial analyses of problems which are more complex or have greater uncertainty surrounding them. Although the decomposition effort may be minimal or extensive, there are several sets of factors that are common to all decisions made under uncertainty. Depending on the preliminary results of a partial analyses, this common or primary group may be augmented by an additional or supplementary group.

Primary Sets. The primary group of elements consists of: a set of the available alternative courses of action; a set of the possible states of nature; and a set of consequences of each alternative and state of nature combination.

The action set will be denoted by A ,

where $A = \{a_i\}$ for $i = 1, 2, \dots, m$

and a_i = a specific course-of action

m = the total number of alternatives which
are feasible and practical

There is no prescribed procedure on how to develop a list of possible actions that might be available. Obviously, the compilation of this list deserves considerable thought to assure all reasonable options are included. The decision maker is encouraged to apply his ingenuity, imagination, and initiative to the fullest extent. The methodology does require that members of the action set be mutually exclusive and collectively exhaustive. The mutually exclusive property limits the selection to only one member from the set - combinations are not permitted. The collectively exhaustive property merely means that the list should be complete in the sense that one of the members must be chosen. Thus, the solution to the decision situation is the selection of a single item from this action set. The primary difficulty in making this choice is usually due to the uncertainties of the situation, in other words, not knowing exactly what will happen should a particular alternative be selected. These uncertainties usually stem from unknown states of nature which constitute the second primary set. The nature state set will be denoted by Θ ,

where $\Theta = \{\theta_j\}$ for $j = 1, 2, \dots, n$

and θ_j = a possible event that can occur or happen
which is relevant to one or more of the
actions, a_i , under consideration. Nature
exists in exactly one of these unknown states.

n = the total number of states that have a
potential impact on the problem.

As in the action set, there is no suggested way to enumerate all the uncertainties involved in a particular decision situation. Again, the skills of the individual decision maker play an important role in judging the relevance of the many unknown factors bearing on a particular problem. The decision maker usually has considerable latitude in assigning members to the nature state set. However, he must obey the exclusive and exhaustive rules previously mentioned. Therefore, he should clearly and concisely describe each event in the set in a manner which prevents overlapping and assures completeness. Difficulties encountered in following the exclusive rule can be alleviated by judicious definition and careful grouping of each member. In many situations, it may be necessary to construct the set by taking pairs, triplets and higher order combinations from other lists of more obvious factors. This technique is also sometimes helpful in adhering to the exhaustive rule. Of course, it is highly unlikely that each possible uncertain event can be identified in each and every case. The time and effort devoted to the composition of the nature state should be tailored to the importance of the decision required. Essentially, the practical significance of the exhaustive rule is that the list of uncertain events should cover all the known contingencies likely to affect the selection of a particular course of action. This is not to say that there might not be unknown factors that bear on the decision problem. Thus, there will be a few cases when the consequence of a particular action will depend not only on factors considered by the decision maker but also unknown unknowns.

However, in general, consequences can be foreseen and it is convenient to designate them as a set.

The third and final primary set, the consequent set, will be denoted by C ,

$$\text{where } C = \{C_{ij}\} \text{ for } i = 1, 2, \dots, m \\ j = 1, 2, \dots, n$$

and C_{ij} = the result of a particular available course of action, a_i , if a specific anticipated state of nature, θ_j , occurs.

A consequence is essentially the position, posture, predicament or state of affairs associated with an individual optional act and uncertain event combination. Thus, conceptually, there are $m \cdot n$ consequences possible in every decision situation. A particular course of action, if selected, could lead to one of n consequences depending on which state of nature exists. Within this group, there are consequences which are good, desirable, beneficial, or profitable as well as those which are bad, unwanted, detrimental or costly. The final choice should represent an optimum balance among these positive and negative features. Sometimes this choice is obvious. Many times the selection requires no more than just considerable deliberation of the facts on hand. But there are some instances when assessed risks are too great and additional data must be obtained and analyzed before an effective discrimination among the alternatives can be made. When these situations occur it is necessary to extend the basic approach to include the formulation and evaluation of supplementary sets.

Supplementary Sets. The supplementary group consists of two sets - one which includes the various ways to obtain additional information

and one which includes the supplementary data accumulated.

The first supplementary set is really a family of possible experiments. This set will be noted by E ,

where $E = \{e_k\}$ for $k = 1, 2, \dots, r$

and e_k = a specific experiment

r = the total number of experiments which are applicable and appropriate.

Members of this set consist of any data collection methods, techniques, plans, etc., which can be used to discover more about the true state of nature. A working knowledge of statistical theory is a definite asset in the formulation and application of this set. The manner and extent to which sampling is accomplished directly affect the validity and creditability of the results. Usually, both the type of additional information needed and the methods of acquisition are obvious. The need for the data collected to be both representative and sufficient is intuitive to most people. However, if experimentation is to be an important aspect of a complex decision problem, then the advice of a competent statistician is almost mandatory. His role will be to assure that the experiments being considered can generate useful outcomes. Whether personally observed or generated by a sophisticated experiment under someone else's direction, the resultant sample data constitutes the other supplementary set.

The outcome set will be denoted by X ,

where $X = \{x_1\}$ for $1 = 1, 2, \dots, s$

and x_1 = a possible outcome of the experiment set E

s = the total number of potential observations of all e_k in E .

The size of this set can vary from a few denumerable elements for simple decision situations involving discrete variables to an infinite list of items for complex problems involving continuous variables. In most cases, the enumeration of all possible outcomes would probably require an inordinate amount of time and effort. Generally, the primary interest is to obtain an appreciation for the range of outcomes. For routine, uncomplicated decision situations, the lowest and highest values anticipated for the results of a small number of experiments are usually either apparent or easily obtained. However, in more complex situations, the assistance of a statistician is highly desirable. Both the range and other meaningful characteristics of the expected results can usually be determined from mathematical equations, tables, charts, and other statistical tools not understood by all decision makers. One such tool is Bayes Theorem which allows the pooling of all available information when making inferences about unknown quantity (i.e., the true state of nature θ). This Theorem, and the related prior distribution (which reflects knowledge before experimentation) will be addressed in more detail in the next two sections. The importance of the roles that these topics have in the overall decision process will become apparent shortly.

Fundamental Concepts

In the preceding discussion, a general description of the essential and auxiliary ingredients was provided along with some of the conditions and constraints for their formulation. Now, general guidelines will be outlined on how to shape these basic building blocks and construct a fundamental framework within which a variety of decision problems can be solved.

Definition of the Problem. First and foremost in any problem solution process should be a careful examination of the issue or question raised. Odiorne defines a problem as "the difference between present condition and desired condition" (Ref 154:15). He also states that commitment is necessary for problem solving for "the committed man has to choose and decide among alternate solutions and moves. The uncommitted man can delay, put it off, and not get things done." (Ref 154:16). These "things to do" constitute the desired condition or objectives. Objectives should be stated in clear concise terms which collectively can be a guiding light providing illuminating direction toward their achievement. The precise and complete delineation of the difference between what is wanted and what is available is a primary prerequisite for future efforts to bridge the gap by enumerating and evaluating the alternate avenues available.

Assessment of the Uncertainties. The impact of each of these options is dependent on an unknown state of nature existing at the time action is taken. Although the exact state is unknown, it is assumed to be a member of the set, Θ , of possible states. In most situations, the members of Θ are not all equally likely to occur. In fact, the decision maker may have encountered similar situations and thus has a basis for weighing a particular θ_j more heavily than others.

The English language has several words to describe aspects of the uncertainty that is felt on such occasions: one of them, likely, has been used in the previous sentence. Others are probable, credible, plausible and expressions derived from words like chance or odds. Our aim is to describe this uncertainty numerically; for number is the essence of the scientific method and it is by measuring things that we know them. (Ref 137:13).

The term which will be used throughout this thesis is probability. It is intended that a generic definition apply to this yardstick for measuring uncertainty. Subsequent to the following discussion on the various kinds of probability, the word will be used without qualifying adjectives.

The word "probability" is used by practicing mathematicians and statisticians in several different ways and means different things to different people. To circumvent these semantic obstacles, various descriptors have been used to provide more explicit definitions.

Typical of the phrases often used are those discussed by Good:

...a physical probability (also called "material probability," "intrinsic probability," "propensity," or "chance") is a probability that is regarded as an intrinsic property of the material world, existing irrespective of minds and logic... A psychological probability is a degree of belief or intensity of conviction that is used for betting purposes, for making decisions, or for any other purpose, not necessarily after mature consideration and not necessarily with any attempt at "consistency" with one's other opinions... When a person or persons, called "you," uses a fairly consistent set of probabilities, they are called subjective ("personal") or multi subjective ("multi personal") probabilities. (Ref 94:6).

Thus, physical probability corresponds to the relative frequency interpretation, and is measured by observing how often a particular event occurs in relation to the total number of attempts made. Subjective probability (the special case of psychological probabilities to which future discussion will be limited) applies when "one is quantifying his personal judgements based on his experience and knowledge, insight and information." (Ref 116: 28). The mathematical properties of both these general classes must obey the postulates and laws of probability theory. Using the event set notation and defining $P()$ as "the probability of the event (), the three basic requirements for

mutually exclusive (or disjoint) events are as follows:

$$(1) \quad 0 \leq P(\theta_j) \leq 1$$

$$(2) \quad P(\Theta) = 1$$

$$(3a) \quad P(\theta_1 \text{ or } \theta_2 \text{ or } \dots) = P(\theta_1) + P(\theta_2) + \dots$$

$$(3b) \quad P(\theta_j \text{ and any other } \theta_j) = 0$$

Rule (1) represents the only new requirement since rules (2) and (3a, 3b) are merely reexpressions of the exhaustive and exclusive rules, respectively. Rule (1) can also be satisfied if odds are quoted for uncertainty. For example, if a particular event is favored 4 to 1, the equivalent statement is that it is likely to occur 4 out of 5 times or with .80 probability. Another aid in establishing a number between 0 and 1 as a measure of likelihood is the comparison of the chances of a particular event with those of a random point falling within a designated area of a unit square. (Ref 137:19).

Comparison of Consequences. In addition to evaluating the data (either objective or subjective) relating to the likelihood of the state of nature, $P(\theta_j)$, the decision maker must also examine the potential impacts of the individual choices, a_i . As previously stated, not all consequences are equally favorable. Thus, the decision maker should list the possible consequences in a relative ranking sequence which reflects the degrees of achievement toward the objectives. Obviously, the particular consequence which represents the greatest step toward the goal is to be the most preferred. Also, great gains should rank higher than shorter ones. Therefore, as Lindley explains:

It follows that the next task is to provide something more than just a ranking of the consequences. In order to do this a standard is introduced and coherent comparison with it provides a numerical assessment, just as with the uncertain events. In the case of probability, the standard was a random point in a unit square. For the consequences, two reference consequences are used; one of these is better than, or at rate not worse than, any of the consequences in the relevant table; the other is similarly worse than, or at most not better than, all the C_{ij} . (Ref 137:52).

He goes on to develop a separate probability measure relating to the attractiveness of a particular C_{ij} in comparison with the most preferred consequence. He calls this numerical measure "a utility of the consequence." (Ref 137:53).

Characteristics of Utility. The numerical measure for consequences does not have to be a probability function. In fact, a quantitative yardstick, although highly desirable, is not mandatory. Qualitative expressions may be used if more appropriate. Miller states that "in addition to the common practice of measuring the utility of consequences on an objective scale such as dollars, gain may be a more subjective concept including factors such as reputation, happiness, security, or any other characteristic associated with well-being." (Ref 148:3-6). In addition, Schaifer indicates that "consequences might also involve "physical assets, technological know-how,...the behavior of various people,...and the decision maker's own personal position..." (Ref 186:40). The point really is that there is no common denominator or standard dimension against which the worth of a consequence can be gauged. However, there is a definite advantage to using numerical measures because they are easier to handle mathematically. The basic requirement in assigning a quantitative index to a value judgement is

coherence. A decision maker is considered coherent if he uses a utility function which assigns a higher utility number to a most preferred consequence and equal numbers to consequences for which he is indifferent. A particular quantity for a utility measure may be designated $U(C_{ij})$ and may fall within any arbitrarily selected range that is appropriate to the situation.

Calculation of Expected Value. Once the numerical measures $P(\theta_j)$ and $U(C_{ij})$ have been determined, the next step is to combine them in a manner that the relative merits of the various a_i can be assessed. To do this, the quantity known as expected value will be used. The expected value, or mathematical expectation, of a particular a_i , written $E(a_i)$ can be computed from the following relationship.

$$E(a_i) = \sum_{j=1}^n U(C_{ij})P(\theta_j) \quad (2-1)$$

Thus, $E(a_i)$ is a measure of the extent that a_i can solve the problem considering the circumstances known to the decision maker. The logical choice is to select the alternative from A which has the greatest $E(a_i)$. Since this choice is highly dependent on the $P(\theta_j)$, these quantities must reflect all known information about Θ - not necessarily just the recorded results of sampling. Bayes Theorem serves as the basis for the current practice of combining subjective and test data.

III. Bayes Theorem

Bayes Theorem is essentially a simple relation between probabilities that two different events will occur. The basic expression which describes the relationship is

$$P(A/B) = \frac{P(A) \cdot P(B/A)}{P(B)} \quad (3-1)$$

where the / is to be read "given."

Development

There is really nothing special about the formula per se - its derivation is quite straightforward.

Consider the "probability diagram" depicted in Figure 1, where A and B represent two events (not disjoint as were the θ_j).

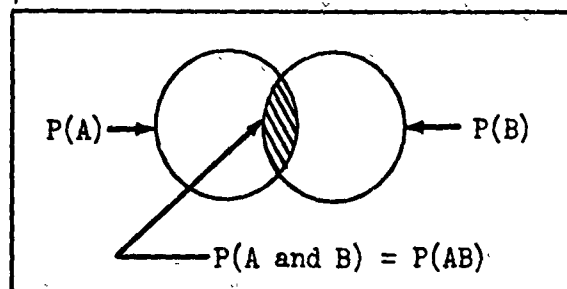


Figure 1

Probability Diagram
(from Ref 213:344)

The probability of event B given that event A has occurred, $P(B/A)$ is the portion that common shaded area, $P(AB)$, is to the total area $P(A)$.

In equation form,

$$P(B/A) = \frac{P(AB)}{P(A)} \quad (3-2)$$

Likewise, $P(A/B)$ can be determined by

$$P(A/B) = \frac{P(AB)}{P(B)} \quad (3-3)$$

Note $P(AB)$ is common to both equations and thus

$$P(AB) = P(A/B) P(B) = P(B/A) P(A)$$

When each term of the latter equality is divided by $P(B)$, Bayes Theorem results.

$$P(A/B) = \frac{P(B/A) P(A)}{P(B)} \quad (3-4)$$

Interpretation

So what is all the fuss about an easy rearrangement of terms? The answer is in how these terms may be interpreted. Addressing each term individually and replacing A with θ_j ; and B with x_1 :

- $P(\theta_j)$ = Prior probability (what is known about θ_j before additional data x_1 is available)
- $P(x_1/\theta_j)$ = Likelihood of observing x_1 (how probable is the sampling results, assuming the true state of nature is a particular θ_j)
- $P(x_1)$ = Probability of test observations for any θ_j in Θ
- $P(\theta_j/x)$ = Posterior probability (what is known about θ_j after x_1 has been obtained and analyzed)

Thus, Bayes Theorem provides the framework which allows redetermination of $P(\theta_j)$ based on additional information. Decisions dependent on knowledge of $P(\theta_j)$ are better when based on all available information. But there is some controversy between the Bayes approach and classical statistical methods when some or all of the prior knowledge consists

of theoretical considerations, design analyses, engineering judgement, etc.

Controversy of Usage

The key to the question concerning the applicability of Bayes Theorem for updating $P(\theta_j)$ is the use of subjective rather than objective information to establish the prior probability. The argument advanced by the purists is that the introduction of intuition and guesswork into statistics constitutes an unnecessary bias and the resultant inferences made are not valid. The reason for these doubts is centered around the various definitions of probability discussed in the previous section. Essentially, there are two schools of thought which are prevalent today. For future discussions, the terms designated by Weir for the advocates of these two philosophies will be used:

(1) "Classicist" which will refer to the group which adopts the frequency interpretation of probability. This faction believes in the concept of unknown parameters located at one unknown point, which can be estimated with increasing precision as test data builds up; information of a non frequency nature cannot be directly used in this estimate. Probabilities based upon outcomes of games of chance (e.g., flipping coins, tossing dice, dealing cards) are of course subject to a frequency interpretation. The ratio of successes to total trials over the very long term can be nearly equated to the probability of the event.

(2) "Subjectivist" which will refer to the group which while dealing with the frequency application of probability to games of chance and other applicable situations, also believes that probability theory can be applied to questions of degree of belief in propositions (e.g., the probability that there is life on Mars). This faction believes it is sensible to talk of probability distributions of parameters based upon degree of belief in the location of the parameter; this permits both frequency and non frequency information to be combined using the central framework of Bayes Theorem. (Ref 213:345).

It is not the intent of this work to delve into the controversy but merely point out its existence. A rather comprehensive discussion of the pros and cons of each of the approaches is presented by Hahn, in the proceedings of the 1965 General Electric Seminar (Ref 214:Sec VIII).

In spite of the controversy, Bayes Theorem and the more inclusive field of what is sometimes referred to as "Bayesian Statistics" have been described as an important step forward in removing some of the constraints of classical theory (Ref 178). Some authors point out that "Bayesian techniques complement classical, statistical methods rather than replacing them." (Ref 74:5). This is especially true when the techniques are applied to reliability estimation and assessment problems.

Application to Reliability Problems

Schulhof and Lindstrom have stated three primary factors which have generated the need to replace time honored classical methods with more modern schemes:

1. Products are becoming more complex and correspondingly expensive.
2. Time is at a great premium.
3. High reliability is being both demanded and achieved. (Ref 187:684).

The paradox is one of greater requirements but fewer resources to verify achievement. The "Classicist" by ignoring prior subjective estimates on the range of the failure rate of an item in essence makes the implicit assumption that the item may possess any failure rate (e.g., $0 < \lambda < \infty$) (Ref 153:3). This hypothesis is extremely costly from the standpoint of the inordinate amount of testing required to narrow in on the true value for a reasonable degree of assurance.

The Bayesian approach recognizes the value of past experience and admits any data - theoretical limits, quantified beliefs, intuitions, whatever - into the analysis. This prior information reduces the scope of sampling to a finite range with resultant savings in time and money. Iterative analyses may be performed to provide periodic updates concerning a system's reliability.

Bayesian methods can use knowledge gained from development testing to indicate the reliability of equipment at each stage of its development. Tests may also be conducted during design development for purposes other than reliability estimation with full expectation that the data can be integrated into a reliability estimation procedure with reasonable statistical validity. It is possible to ascertain by these methods whether a specified system reliability requirement has a reasonably good chance of being achieved. Conversely, these methods will show whether special effort, such as redesign or modification, is warranted, by indicating the existence of a low probability of design reliability achievement.

By having this objective information available concurrently with each phase in design development, the designer is enabled to make necessary improvements at an earlier state. This makes revisions more compatible with costs, schedules, tool design, and other important factors, and makes achievement of the contractual reliability requirement more certain. As a consequence, the number of tests required to achieve and demonstrate a reliable design can be reduced, resulting in time and cost savings. (Ref 74:5).

In summary, the following are considered practical advantages of using Bayes techniques for estimating reliability:

1. The estimates take both predictions and test data into account.
2. The estimates give reasonable results for little or no test data.
3. The estimates agree with the classical estimates for large samples. (Ref 187:634).

Of course, there are drawbacks to using Bayesian techniques. In addition to the issues raised by the "Classicists," there is the obvious problem of erroneous initial information. What happens if incorrect assumptions are made or faulty logic was used to formulate a considered opinion? How accurate can an overall final estimate or assessment be if based on imprecise inputs? Fortunately, Bayesian methods have an inherent corrective feature since as the quantity of subsequent test data increases, the initial estimates become overshadowed. If desired, the effect of incorrect prior information can be reduced to any stated level by sufficient additional testing. As previously stated, Bayesian estimates agree with "Classicist" estimates after an extensive amount of test and operational life data has been obtained.

The question which must be answered now is "How does one quantify his subjective appraisal of an item's reliability in a manner that is usable?" This is accomplished by using available reliability predictions and related statements of uncertainty surrounding them. Basically, the parameters of the applicable system failure density function are considered to be random variables. A probability distribution function for a parameter is established based on the point estimate and expected range of the predicted value. This parameter distribution is known as the prior distribution and is subject to modification as test data becomes available. Typical prior distributions encountered in reliability and instructions on how to generate them are the subject of the next section.

IV. Prior Distributions

The heart of a Bayesian statistical inference is that a probability distribution is assumed to exist for the unknown true state of nature θ . This prior distribution essentially reflects the amount of knowledge, or degree of belief, before the results of experimentation are available. If absolutely no information is known, all values of θ are equally likely, and logic dictates a Uniform prior distribution. However, in virtually all reliability problems, there exists a significant amount of information from generic or similar parts documented in various handbooks. Also, according to Gottfried and Weiss, "experience indicates that the failure rate of any device is bounded on the left (non-negative) and skewed to the right - larger failure rates than expected are less surprising than smaller values." (Ref 100:603). Although there is no widely agreed rules on how to select a suitable prior distribution, Babillis and Smith have established what is considered to be minimal criteria:

1. The prior distribution must adequately reflect what is actually known before test data becomes available. That is, the distribution must be consistent with the available prior knowledge of a component's reliability.
2. The prior distribution should not imply any assumptions about unknown information concerning the reliability of a component. In other words it should remain as maximally noncommittal and as unprejudiced as possible concerning things which are considered unknown.

3. The prior distribution should not lead to absurd conclusions concerning the component's reliability when it is modified by the data. The selected apriori distribution should not lead to results which are inconsistent with what is known or intuitively felt... However, it is important to realize that these inconsistencies may only be evaluated in terms of what was originally thought to be known. If pertinent information is withheld in establishing the prior distribution, this same information may not be used to discredit the resulting posterior distribution.

4. The resulting equations should be tractable by available mathematical methods. This is purely a pragmatic consideration based solely on the desire for an answer. Through this same crack in the otherwise logical framework also creeps a certain element of empiricism which requires drawing upon experience to get the technique started. (Ref 6:357).

Although the above criteria are general, they do serve the purpose of narrowing the selection process. Perhaps someday simple algorithms for establishing priors will exist. As of this writing there is considerable effort underway to find ways and means for formulating probability distributions for reliability indices. For example, Rome Air Development Center has a current study contract with Hughes Aircraft Company to develop methods for fitting prior distributions to empirical data and combining priors from similar but not identical equipment. The reported results of initial efforts are quite promising. It has been concluded that it is entirely feasible to fit prior distributions to equipment level Mean Time Between Failure, MTBFs, although the amount of data currently in existence to do so is somewhat limited. It was also determined that the probability of MTBF is usually well described by an Inverted Gamma Distribution when the assumed equipment failure time is exponentially distributed. (Ref 182).

Since MTBF or its reciprocal, the failure rate λ , are the most

commonly used measures for time dependent reliability, one merely has to determine which member of the family of Gamma type distributions is applicable to the situation. How does one translate his knowledge concerning the most likely single value and range of uncertainty for MTBF or λ into a Gamma distribution?

Assigning Gamma Parameters

The Gamma family of probability distributions meets all the previously stated criteria. In the most common form, it is described by two parameters and is very flexible. Also, the selected parameters combine readily with Poisson sampling statistics (the Poisson process is an experiment which observes f failures in t time) which is a prerequisite to meet the fourth criterion for priors. If a Gamma prior is updated with Poisson data the resultant posterior is also a Gamma distribution. This property is very convenient for calculation purposes.

Now that the suitability of a Gamma form has been somewhat justified, an explanation on how it fits into the overall decision making procedure is in order. If an equipment failure rate λ is assumed to be a random variable, then

$$P(\theta) = g(\lambda; a, b) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} \quad (4-1)$$

for $a, b, \lambda > 0$

The probability of a particular outcome of the Poisson experiment ($x = f$ observed failures in t time), given the true state of nature ($\theta = \lambda$) can be expressed as

$$P(x/\lambda) = p(f/t, \lambda) = \frac{(\lambda t)^f e^{-\lambda t}}{f!} \quad (4-2)$$

The remaining term, $P(x)$, of the Bayes formula can be determined in the following manner

$$\begin{aligned} P(x) &= \int P(x/\theta)P(\theta)d\theta \\ &= \int_0^{\infty} p(f/t, \lambda)g(\lambda; a, b)d\lambda \end{aligned} \quad (4-3)$$

Thus, from the Bayes relationship (3-1)

$$P(\theta/x) = P(\lambda/f, t) = \frac{\frac{(\lambda t)^f e^{-\lambda t} b^a \lambda^{a-1} e^{-b\lambda}}{f! \Gamma(a)}}{\int_0^{\infty} \frac{(\lambda t)^f e^{-\lambda t} b^a \lambda^{a-1} e^{-b\lambda}}{f! \Gamma(a)} d\lambda} \quad (4-4)$$

Now that this specific case has been defined, the general expressions involving θ and x will no longer be used, and the expression for the posterior distribution can be reduced to

$$P(\lambda/f, t) = \frac{\lambda^{a+f-1} e^{-\lambda(b+t)}}{\int_0^{\infty} \lambda^{a+f-1} e^{-\lambda(b+t)} d\lambda} \quad (4-5)$$

by letting $y = \lambda(b+t)$ the denominator becomes

$$\int_0^{\infty} \left(\frac{y}{b+t} \right)^{a+f-1} e^{-y} \frac{dy}{b+t}$$

which can be solved by rearranging terms,

$$\left(\frac{1}{b+t} \right)^{a+f} \int_0^{\infty} y^{a+f-1} e^{-y} dy = \left(\frac{1}{b+t} \right)^{a+f} \Gamma(a+f) \quad (4-6)$$

Substituting this result in equation (4-5)

$$\begin{aligned} P(\lambda/f, t) &= \frac{(b+t)^{a+f} \lambda^{a+f-1} e^{-\lambda(b+t)}}{\Gamma(a+f)} \\ &= g(\lambda; a+f, b+t) \end{aligned} \quad (4-7)$$

With this proof, a better appreciation for the interpretation of the meaning of the parameters a and b is possible. Since a and f are equivalent quantities, the former can be considered what has been termed as a "pseudo-failure"; likewise, b can be thought of as "pseudo-time" (Ref 214:5-6). So by assigning values for a and b , one may assert his extent of subjective confidence in predicted equipment failure rates. This may be done by examining the mean and variance of the Gamma form defined in equation (4-1),

$$\begin{aligned}\mu &= \frac{a}{b} \\ \sigma^2 &= \frac{a}{b^2} = \frac{\mu}{b}\end{aligned}\tag{4-8}$$

If the predicted value λ_p is equated to the mean, then

$$\begin{aligned}\lambda_p &= \frac{a}{b} \\ \sigma^2 &= \frac{\lambda_p}{b}\end{aligned}\tag{4-9}$$

Thus, by assigning a and b , one is really stating his belief concerning the number of "pseudo-failures" and the amount of "pseudo-time" reflected in the predicted value. For example, if $\lambda_p = 50 \times 10^{-6}$, then the following combinations are possible and are listed in order of increasing certainty from left to right:

$$\frac{a}{b} : \frac{0.1}{2000}, \frac{0.5}{10,000}, \frac{1}{20,000}, \frac{5}{100,000}, \frac{10}{200,000}$$

The importance of carefully selecting values for a and b that realistically correspond to prior knowledge cannot be overemphasized. Understatements are costly because of extensive sampling required to narrow the resultant wide range (i.e., large σ^2). Conversely,

overstatements are ill advised because of the significant contribution that the prior will have on the posterior and thus the decision to be made. With these cautions, three methods will now be discussed which address how to select values for a and b .

Upper Bound Method. Two known bits of information must be provided if two unknown quantities (in this case a and b) are to be determined. The predicted failure rate λ_p represents only half of the required input. This value must be combined with another statement relating to additional prior knowledge about the failure rate. One way to encode a conviction of belief concerning a reliability prediction is to assign a probability, ρ , that the true failure rate is no greater than a particular upper limit λ_u . Once this is done, then λ_p can be considered to be either the mean or median value of the prior distribution and published tables or graphs can be used to select values for a and b which satisfy the probability statement.

(Ref 214:A2-37). The value a is the same for any λ_p and λ_u combination with an identical discrimination ratio λ_u/λ_p and extent of certainty, ρ . It was found convenient for later use to tabulate an assortment of what is considered typical λ_u/λ_p and ρ . The values for a for discrimination ratios from 1.50 to 10 and ρ from .67 to .99 are given for λ_p equal to the mean in Table I and λ_p equal the median in Table II.

Coefficient of Variation Method. Another way to describe uncertainty associated with a prediction is to state the extent of possible inaccuracy in terms of an estimated standard deviation. If an engineer states "I think λ_p is the true failure rate but I could

Table I

Values for a if λ_p Assigned Median

$\psi = \frac{\lambda_u}{\lambda_p}$	$p = P(\lambda \leq \lambda_u)$							
	.67		.75		.90		.99	
	a	V_u	a	V_u	a	V_u	a	V_u
1.50	1.00	1.10	2.50	3.30	9.0	12.8	29.0	43.0
1.75	.80	.90	1.50	2.10	5.7	8.7	14.0	24.0
2.00	.70	.80	1.00	1.40	3.0	5.3	9.0	21.0
2.50	.60	.60	.70	.92	1.7	3.4	4.9	11.5
3.00	.45	.45	.50	.65	1.3	2.6	3.2	8.8
4.00	.37	.30	.40	.50	1.2	2.5	1.5	5.7
5.00	.33	.19	.35	.40	.7	2.3	1.3	5.5
7.50	.30	.18	.32	.35	.5	2.2	.9	4.6
10.00	.29	.17	.30	.33	.4	2.1	.7	4.0

Table II

Values for a if λ_p Assigned Mean

$\psi = \frac{\lambda_u}{\lambda_p}$	$p = P(\lambda \leq \lambda_u)$			
	.67	.75	.90	.99
1.50	-	.90	7.0	28.0
1.75	-	.65	3.0	13.0
2.00	-	-	2.0	8.0
2.50	-	-	0.9	4.2
3.00	-	-	0.6	2.5
4.00	-	-	0.3	1.2
5.00	-	-	-	0.9
7.50	-	-	-	0.5
10.00	-	-	-	0.2

be off by a factor of m then λ_p and m can be used to compute a and b as follows:

$$\lambda_p = \frac{a}{b}$$

$$\sigma = m \lambda_p = \sqrt{\frac{a}{b^2}} \quad (4-10)$$

from which

$$a = \frac{1}{m^2}$$

$$b = \frac{1}{m^2} \lambda_p \quad (4-11)$$

Also from equation (4-10) it is noted that m is the ratio of the standard deviation to the mean which is defined to be the coefficient of variation. To assess the impact that the choice of m will have on the final decision, the mean of the posterior must be examined. The revised estimate of the failure rate λ_p may be written

$$\lambda_p = \frac{a + f}{b + f} = \frac{f + \frac{1}{m^2}}{t + \frac{1}{m^2} \lambda_p} \quad (4-12)$$

Thus, if there is no uncertainty associated with λ_p then $m = 0$ reflects total confidence. In this case $\lambda_p = \lambda_p$ irrespective of test data. Conversely, $m = \infty$ corresponds to total ignorance and $\lambda_p = f/t$, which is the best estimate obtained from test data only.

Babillis and Smith have stated:

For the time being the Gamma prior is being applied with either $m = 0.75$ or $m = 1.75$ depending on whether the test data reflects no failures or some failures respectively. General experience with this system has been favorable, and development efforts have been planned to provide substantiation and or refinement of the approach. (Ref 6:361).

Also, Feduccia reports that based on data dealing with observed vs predicted MTBFs of over one hundred ground electronic equipment and systems, the value of m was found equal to 1.38. If similar data exists for other types of equipment, then a more meaningful value for m can be determined from the sample means and standard deviation.

Equivalent Test Time Method. The last technique to be discussed in this paper addresses the predetermined contribution that a predicted value will have in the final decision. The basic approach reflected in this method is to relate the uncertainty associated with the prediction with the amount of usage experience likely to be encountered before the final decision. If one has high confidence in initial predictions, he has less need for additional experimentation. Conversely, if predictions are considered somewhat inaccurate, then greater reliance on test results is in order. Usually, the time associated with a test program is either specified or can easily be estimated. Then any amount of "pseudo-time" can be assigned to be compatible with a desired extent of contribution to the total time on which the final decision will be based. For example, if a 10,000 hour test program is expected and the decision to be made is to be based on 90% test results and 10% predictions then $b = 1000$ would be an appropriate assignment.

If the preselected portion of expected test time, b , is combined with λ_p , a unique Gamma distribution can be defined. The obvious need is a relationship between prediction uncertainty and the extent that the prediction should contribute to the posterior estimate. There are no established guidelines on this issue, however, it is

suggested that after careful and considered thought, a relationship could be devised for a particular program. For illustrative purposes in the next section, a hypothetical relationship is given in Table III below.

TABLE III

Equivalent Test Time for Priors

Degree of Belief in Predicted Value	Percent Contribution of Prior Parameter b	Divisor for Test Time T
0.99	0.75	0.33
0.90	0.50	1.00
0.75	0.25	3.00
0.67	0.17	5.00

In addition to time dependent reliability problems for which Gamma distributions apply, there are cases when time is not a factor. In these situations, it has been shown that distributions from the Beta family meet the previously stated criteria for priors.

Assigning Beta Parameters

As with the Gamma family, a Beta distribution is also defined by two parameters and thus a wide range of priors is possible. The assigned parameters combine readily with Binomial sampling statistics (the Binomial process is an experiment which observes s successes in n trials). If a Beta prior is updated with Binomial data, the resultant posterior is also a Beta distribution. In this case, if the success ratio p is assumed to be a random variable, then

$$\beta(p; \phi, \eta) = \frac{(\eta-1)! p^{\phi-1} (1-p)^{\eta-\phi-1}}{(\phi-1)! (\eta-\phi-1)!} \quad (4-13)$$

$$\text{for } 0 \leq p \leq 1$$

$$\phi, \eta > 0$$

The probability of a particular outcome of the Binomial experiment given p , can be expressed as follows

$$b(s/n, p) = \frac{n! p^s (1-p)^{n-s}}{s! (n-s)!}$$

It can be shown that the posterior distribution, $P(p/s, n)$, is also Beta, $\beta(p; \phi + s, \eta + n)$. The mean and variance of the two distributions are:

Prior

$$p = \frac{\phi}{\eta}$$

$$\sigma^2 = \frac{\phi(\eta - \phi)}{\eta(\eta - 1)}$$

Posterior

$$p = \frac{\phi + s}{\eta + n}$$

$$\sigma^2 = \frac{(\phi + s)(\eta - \phi + n - s)}{(\eta + n)(\eta + n - 1)}$$

The same rationale used to assign a and b for a prior Gamma distribution can be used to select ϕ and η for a prior Beta distribution. This discussion on the Beta distribution is presented for comparison with the time dependent situation for completeness only. The example in the next section does not illustrate the practical application of this distribution.

V. Bayesian Reliability Assessment Example

Now that the necessary building blocks have been explained, they will be used to construct a framework for the periodic assessment of system reliability based on analytical predictions and results of only limited testing. The proposed scheme will be presented by way of a hypothetical example.

The Original Low-level Detector, OLD, system has been in service for the past twenty years. Because of break throughs in technology pertaining to electronic jamming devices, this system is now only marginally effective in tracking targets. Also, the OLD design is of vacuum tube vintage and has required increasing amounts of downtime for maintenance in recent years. Logistics costs to support this system are inordinate because of limited sources of supply, since most tube manufacturers now concentrate on the production of solid state devices. A modern more effective system is currently being developed to replace the obsolete OLD system.

The Network for Early Warning, NEW, system prototype has been assembled and factory testing is now in progress. Because of the urgent need for the NEW system, the development and acquisition contract was structured for concurrency. The terms and conditions of the contract stipulate that the release of funds for production will be predicated on satisfactory demonstration of system performance capability. The procuring agency and contractor have agreed on the portion of the total test program which must be accomplished prior to

production release. It was decided that the formal reliability test required to verify achievement of the contractual quantitative requirement could be deferred until after production commitment. However, an assessment of the inherent reliability designed into the NEW system must be made to determine if a minimal acceptable level has been achieved. If a threshold value cannot be reached, it will be necessary to redefine the system requirements and initiate a redevelopment effort for an alternative replacement for the OLD system. This contingency system will be designated XYZ. Now that the foundation has been laid for this case study, the remaining presentation will adhere to the established blueprint for solving decision problems.

Definition of the Problem

The primary objective of the designated procuring activity is the timely introduction of a cost-effective means of satisfying the operational need for sufficient warning of an advancing aggressor. Although the NEW system is based on existing technology to a great extent, it does contain some innovative features which constitute technical risks. There are many component types in the NEW system which are state-of-the-art devices and thus have questionable reliability characteristics. Although there are other uncertainties associated with the NEW development effort, it will be assumed that they will be addressed separately. Thus the problem will be defined: "From a reliability viewpoint, determine the advisability of committing funds for production of the NEW system." Thus A is the set of actions that a decision maker might take regarding the release of production funds. For illustrative purposes, the following choices are assumed:

a_1 = Release the funds; proceed into production.

a_2 = Wait three months for results of performance testing in progress.

a_3 = Wait six months for results of formal/reliability test.

a_4 = Cancel program; initiate development of the contingency system.

To evaluate these options, the technical risks associated with NEW system reliability must be evaluated.

Initial Assessment of Uncertainties

The unknown quantity in this decision situation is the true system reliability. So in this case specific values for the system failure rate, λ , will constitute the possible states of nature. The contractual requirement for the NEW system is 10,000 failures per million hours, fpmh. To assess the probability that the inherent reliability is less than, equal to, or greater than this specified value, it is necessary to calculate a point estimate prediction and address the variability associated with it. At the critical design review for the NEW system, the information presented in Tables IV and V was presented by the contractor.

Table IV

NEW System Reliability Predictions

Item	Unit/Subsystem	Failure Rates ($\times 10^{-6}$)		Pr ($\lambda \leq \lambda_u$) ρ
		λ_{pi}	λ_{ui}	
A	Array Antenna	2500	10000	.75
B	Beam Selector/Steerer	750	3000	.90
C	Control/Display Console	700	2100	.90
D	Data Processor	300	600	.99
E	Emitter/Detector	600	3000	.75
F	Frequency Randomizer	500	5000	.90

Table V

Parameters for a System Prior Distribution

Item	Qty Per Sys n_i	a_i	$b_i = \frac{\lambda_{ui}}{\lambda_{pi}}$	$\mu_i = \frac{a_i}{b_i}$ ($\times 10^{-6}$)	$n_i \mu_i$ ($\times 10^{-6}$)	$\sigma_i^2 = \frac{n_i \mu_i}{b_i}$ ($\times 10^{-8}$)
A	1	.40	50	8000	8000	16000
B	2	1.20	835	1435	2870	344
C	5	1.30	1240	1050	5240	423
D	1	9.00	29100	310	310	1
E	2	.35	133	2630	5260	3957
F	1	.40	420	950	950	227
					22,630	20,952

These tabulated data can be used to determine estimates for the system failure rate mean value, $\hat{\mu}_s$, and an upper bound, $\hat{\lambda}_{su}$, from the following relationships,

$$\hat{\mu}_s = \sum_{i=A}^F n_i \mu_i \quad (5-1)$$

$$\hat{\sigma}_s^2 = \sum_{i=A}^F \sigma_i^2 \quad (5-2)$$

$$\hat{\lambda}_{su} = \frac{2 \hat{\mu}_s + \frac{Z^2 \hat{\sigma}_s^2}{\hat{\mu}_s} + \sqrt{4Z^2 \hat{\sigma}_s^2 + \frac{Z^4 \hat{\sigma}_s^4}{\hat{\mu}_s^2}}}{2} \quad (5-3)$$

where Z = Normal deviate for the one sided confidence level of interest (e.g., for $u = .90$, $Z = 1.282$)

However, in decision problems there are usually several ranges of interest instead of just two (i.e., the intervals of $0 < \lambda < \hat{\lambda}_{su}$ and $\hat{\lambda}_{su} < \lambda < \infty$). In this example there will be six $P(\theta_j) = P(\lambda_j)$ divisions,

$$0 < \lambda_1 \leq 0.003$$

$$0.003 < \lambda_2 \leq 0.005$$

$$0.005 < \lambda_3 \leq 0.015$$

$$0.015 < \lambda_4 \leq 0.045$$

$$0.045 < \lambda_5 \leq 0.055$$

$$0.055 < \lambda_6 < \infty$$

To determine the $P(\lambda_j)$ it is convenient for later comparison to assume that the system failure rate distribution is also Gamma. A Gamma distribution can be transformed to a Chi-squared, χ^2 , distribution with d degrees of freedom by letting $d = 2(a + 1)$ and $\chi^2(d) = 2b$. (Ref 153:31). In this case the values a_s and b_s for the system failure rate distribution can be determined from $\hat{\mu}_s$ and $\hat{\sigma}_s^2$, since

$a_s = \hat{\mu}_s^2 / \hat{\sigma}_s^2$ and $b_s = \hat{\mu}_s / \hat{\sigma}_s$. Then χ^2 tables can be used to find $P(\chi_j^2, d) = P(\lambda \leq \lambda_0)$. The $P(\lambda_j)$ can be determined from the relationships $P(\lambda_j) = P(\chi_{j-1}^2, d)$ and $P(\chi_0^2, d) = 0$. For example, using equations (5-1) and (5-2) and the data listed in Table V, $P(\lambda_3)$ can be computed as follows,

$$a_s = \frac{\hat{\mu}_s^2}{\hat{\sigma}_s^2}$$

$$= \frac{(.022630)^2}{.00020952}$$

$$= 2.46$$

$$b_s = \frac{\hat{\mu}_s}{\hat{\sigma}_s}$$

$$= \frac{(.022630)}{.00020952}$$

$$= 108$$

$$\chi_2^2 = 2b_s \lambda_{u2}$$

$$= 2(108)(.005)$$

$$= 1.08$$

$$P(\chi_2^2, d) = P(1.08, 2(2.46) + 2)$$

$$= P(1.08, 6.92)$$

$$= .0070$$

$$\chi_3^2 = 2(108)(.015)$$

$$= 3.24$$

$$P(\chi_3^2, d) = P(3.24, 6.92)$$

$$= .1331$$

$$\begin{aligned}
 P(\lambda_3) &= P(x_{3,d}^2) - P(x_{2,d}^2) \\
 &= .1381 - .0070 \\
 &= .1311
 \end{aligned}$$

The above sequence was repeated for the other $P(\lambda_j)$ and the results are listed in Table VI along with assigned point values, λ_{aj} , for each interval which will be used in subsequent calculations.

Table VI

Values for $P(\lambda_j)$ Based on Predictions

j	$x_j^2 = 2b_s \lambda_{uj}$	$P(x_j^2, 2a_s + 2)$	$P(\lambda_j)$	λ_{aj}
1	.65	.0014	.0014	.002
2	1.08	.0070	.0056	.004
3	3.24	.1381	.1311	.010
4	9.72	.7936	.6555	.030
5	11.88	.8959	.1023	.050
6	∞	1.0000	.1041	.060

Comparison of Consequences

Now that all prior knowledge concerning NEW system reliability has been quantified, the impact of the various λ_j on each of the a_i can be determined. In this case, the primary concern is to minimize anticipated life cycle costs since it has been assumed that the performance effectiveness of the NEW system is acceptable. There are many ways to estimate life cycle costs, most of which are rather elaborate and time consuming if any reasonable degree of precision is desired. The essential elements consist of development and acquisition costs and operation and maintenance, O&M, expenses. The

factors which relate to these basic costs for each of the systems in question are provided in Table VII. The most significant portion of total life cost is the maintenance expense. Table VIII contains the costs of monthly maintenance (assuming \$5000 per repair) for each assigned failure rate, λa_j , and various system quantities.

Table VII
Life Cycle Cost Factors

Factor \ System	OLD	NEW	XYZ
Prototype Costs, P	-	\$ 50M	\$ 75M
Price per System, A	-	\$ 2M	\$2.5M
Delay Costs/Quarter, D	-	\$ 1M	-
Cancellation Costs, C _j	-	\$0-75M *	
O&M (Cost per Repair), R	\$ 5K	\$ 5K	\$ 5K
Observed/Specified λ	.05	.01	
Number of Systems, N	60	60	60
Delivery Rate (Systems Per Month)	-	5	5
Development Time (Months)	-	0	24
Production Lead Time (Months)	-	15	12

* $C_1 = 75M$; $C_2 = 50M$; $C_3 = 25M$; $C_4 = C_5 = C_6 = 0$

Table VIII

Monthly Maintenance Expense, $R(N, \lambda)$ (In the millions of dollars)

Num of Sys N	System Failure Rate λ					
	.002	.004	.01	.03	.05	.06
5	.036	.072	.18	.54	.9	1.08
10	.072	.144	.36	1.08	1.8	2.16
15	.108	.216	.54	1.62	2.7	3.24
20	.144	.288	.72	2.16	3.6	4.32
25	.180	.360	.90	2.70	4.5	5.4
30	.216	.432	1.08	3.24	5.4	6.48
35	.252	.504	1.26	3.78	6.3	7.58
40	.288	.576	1.44	4.32	7.2	8.64
45	.324	.648	1.62	4.86	8.1	9.72
50	.360	.720	1.80	5.40	9.0	10.80
55	.396	.792	1.98	5.94	9.9	11.88
60	.432	.864	2.16	6.48	10.8	12.96

Another important consideration in this particular case is the impact that additional time has on effecting reliability improvements.

Potential enhancement of reliability is dependent on many factors, including current technology, available resources, physical space, etc. There is no universal reliability growth model that applies to all types of equipment in any stage of development. For illustrative purposes, a simple linear relationship of estimated reliability improvement versus time is considered sufficient. To account for the additional time advantage in subsequent calculations, an estimated reliability improvement factor, $I(M, \lambda)$ will be used. Values for $I(M, \lambda)$ for $M = 0-30$ months and $\lambda = .004, .01, .03, .05$ and $.06$ can be obtained from Figure 1. For simplification, $I(M, .002) = 1$.

Now that all the pertinent information and relationships regarding individual cost elements for both existing and planned systems have been discussed, the expressions for computing the C_{ij} can be presented. Specifically, the C_{ij} represents total life cycle costs for a ten-year period commencing with scheduled production release. The delivery rates are such that a mix of existing and replacement occurs only during one year starting with the first increment. During this year, there is an average of 30 each present and replacement systems. For the alternative decisions, the corresponding consequences are:

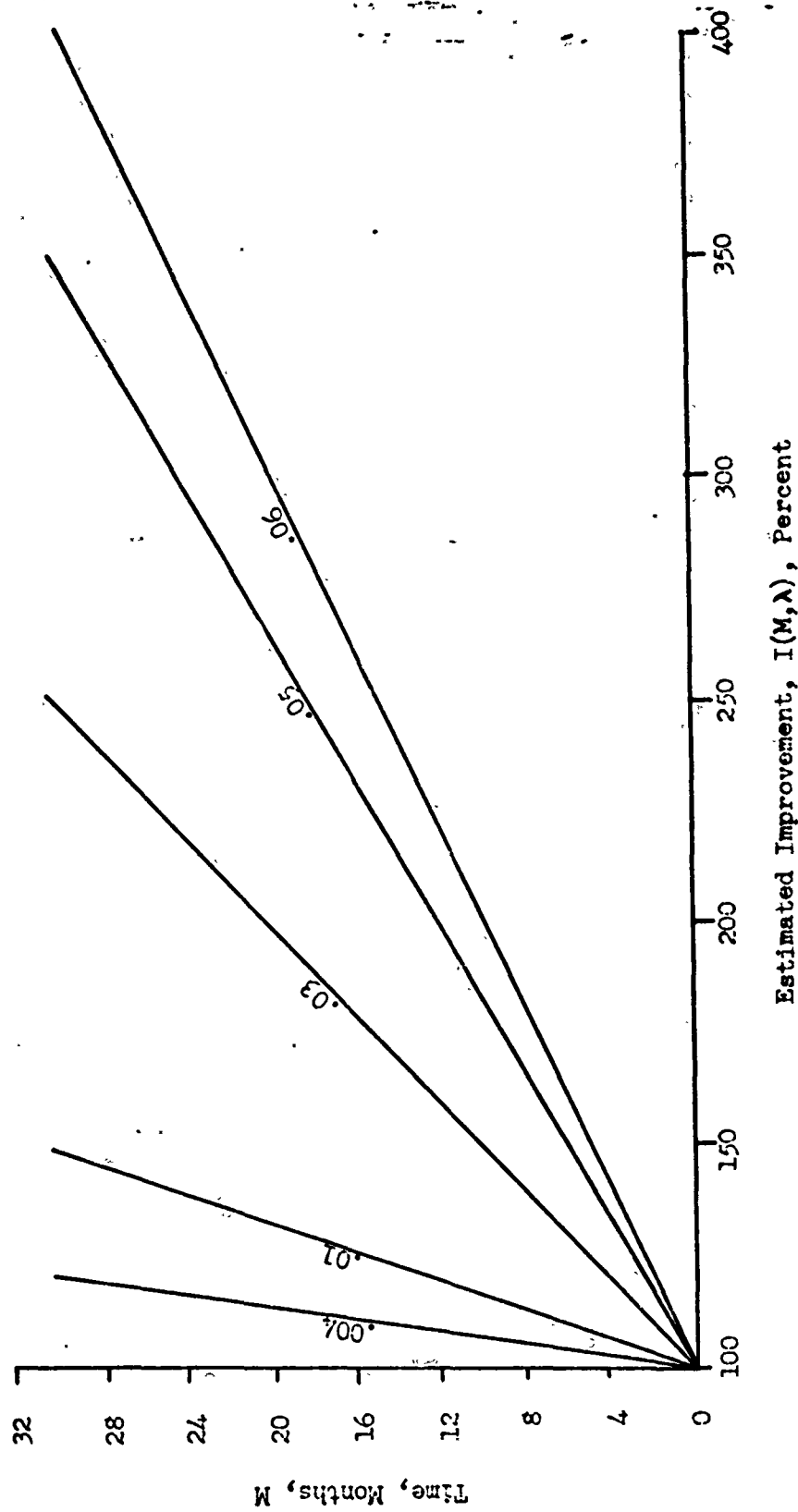


Figure 2. Reliability Growth Curves for Initial $\lambda = .00\%$, .01, .03, .05 and .06

$$C_{1j} = Pn + 60An + 15R(60, .05) + 12R(30, .05) + 12R(30, \lambda_{aj}) + 105R(60, \lambda_{aj}) \quad (5-4)$$

$$C_{2j} = Pn + 60An + D + 18R(60, .05) + 12R(30, .05) + \left[\frac{12R(30, \lambda_{aj}) + 102R(60, \lambda_{aj})}{I(3, \lambda_{aj})} \right] \quad (5-5)$$

$$C_{3j} = Pn + 60An + 2D + 21R(60, .05) + 12R(30, .05) + \left[\frac{12R(30, \lambda_{aj}) + 99R(60, \lambda_{aj})}{I(6, \lambda_{aj})} \right] \quad (5-6)$$

$$C_{4j} = Pn + Px + 60Ax + 36R(60, .05) + 12R(30, .05) + \left[\frac{12R(30, \lambda_{aj}) + 84R(60, \lambda_{aj})}{I(24, \lambda_{aj})} \right] \quad (5-7)$$

Thus, from equation (5-6), C_{3j} (in millions of dollars) can be determined as follows,

$$\begin{aligned} C_{3j} &= 50 + 120 + 2 + 21(10.8) + 12(5.4) + \\ &\quad \left[\frac{12(1.08) + 84(2.16)}{1.1} \right] \\ &= 670 \end{aligned}$$

The remaining C_{ij} were calculated in a similar manner and the results are listed in Table IX. Since the C_{ij} are expressed in quantitative terms, assignment of utility measures is unnecessary.

Table IX
Production Release Decision Matrix
(C_{ij} in millions of dollars)

System Failure Decision Rate	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
a_1 Proceed	445	493	624	1116	1596	1836
a_2 Wait 3 mos.	477	521	652	1039	1363	1498
a_3 Wait 6 mos.	509	551	670	987	1220	1315
a_4 Cancel	843	846	893	994	1053	1072

Calculation of Expected Value

The solution to this production release decision problem is to take the action, a_1 , which is most likely to result in the lowest life cycle cost. This selection can be made by comparing the expected value of each alternative, $E(a_i)$. The $E(a_i)$ can be computed from the data in Table IX by using equation (2-1),

For example,

$$\begin{aligned}
 E(a_1) &= \sum_{j=A}^F C_{1j} P(\lambda_j) \\
 &= (445)(.0014) + (493)(.0056) + \\
 &\quad (624)(.1311) + (1116)(.6555) + \\
 &\quad (1596)(.1023) + (1836)(.1041) \\
 &= \$1171M \quad (5-9)
 \end{aligned}$$

Similarly, the other $E(a_i)$ were computed and the following values were obtained for $E(a_2)$, $E(a_3)$, $E(a_4)$ respectively: \$1069M, \$1000M, \$994M. Thus, the logical decision based on predicted reliability data only is a_4 or cancel the NEW program and initiate procurement of the XYZ system.

Second Assessment of Uncertainties

However, in addition to these analytical predictions, logs of individual unit/subsystem operating times and failures have been maintained as portions of the NEW system have progressed through various phases of testing. The specific information reported is listed in Table X.

Table X

Cumulative Unit/Subsystem Test Data

Unit/Subsystem	A	B	C	D	E	F
Total Failures f_i	5	1	4	0	1	1
Total Time t_i	1500	2000	4750	900	2400	1500

These data also reflect a measure of system reliability and may be used independent of the prior information to determine estimates for $P(\lambda_j)$. Again the χ^2 distribution may be used by letting $d = 2(f + 1)$ and $\chi_j^2(d) = 2t\lambda_{0j}$. If it is assumed that the data in Table X represent $f = 12$ failures in approximately $t = 1000$ equivalent hours of system operation, then the $P(\lambda_j)$ can be determined using the same procedure as before. These $P(\lambda_j)$ - based solely on the observed test data - are listed in Table XI. When these probabilities are used in the decision matrix (Table IX), the values for $E(a_j)$ are: \$755M, \$755M, \$755M, and \$920M. Therefore the test results are inconclusive as far as a definite decision is concerned.

Table XI

Values for $P(\lambda_j)$ Based on Test Data

j	$\chi_j^2 = 2t\lambda_{0j}$	$P(\chi_j^2, 2f + 2)$	$P(\lambda_j)$	λa_j
1	6	.0000	.0000	.002
2	10	.0020	.0020	.004
3	30	.7324	.7304	.010
4	90	1.0000	.2676	.030
5	110	1.0000	.0000	.050
6	∞	1.0000	.0000	.060

Final Assessment of Uncertainties

The test data may also be combined with the prior information recorded in Tables IV and V to produce a more accurate revised system failure rate distribution. The parameters and statistics of the

posterior distributions will be distinguished from prior values by using script letters and bold type. The posterior parameters for the unit/subsystem failure rate distributions are $A_1 = a_1 + f_1$ and $B_1 = b_1 + t_1$. The posterior system failure rate distribution mean and variance are μ_s and σ_s^2 , respectively. In this particular case, when the individual unit/subsystem "pseudo" and actual failures and operating times are combined, the resultant values for A_1 and B_1 are those listed in Table XII. The values for μ_s and σ_s^2 are determined by following the same procedure used to obtain μ_3 and σ_3^2 . Likewise, the system Gamma parameters, A_s and B_s , can be computed the same way as before. These quantities may then be used to determine the individual $P(\lambda_j)$ from expressions associated with Bayesian one sided upper confidence limits given by Nagy (Ref 153:29).

$$\begin{aligned} x_j^2(d) &= 2B_s \lambda_{uj} \\ d &= 2A_s \end{aligned} \quad (5-10)$$

Again, since the λ_{uj} are stipulated, the values for x_j^2 can be calculated and the x^2 tables can be used to find $P(x_j^2, 2A_s)$. The values for $P(\lambda_j)$ which result from these computations are listed in Table XIII. When these revised estimates are used in equation (5-8) to compute the $E(a_1)$ the results (in millions of dollars) are: 652, 674, 688, and 898. Thus, the preferred course of action is clearly a, and production release may be granted without delay. Note that this decision is the reasonable choice although μ_s is greater than the failure rate specified in the contract. In this case, it has been illustrated that the planned deployment of the NEW system with an estimated reliability of approximately only 87% the required level.

is more cost effective than the continued use of the OLD system or the three year delay required for the XYZ system.

Table XII

Parameters for a System Posterior Distribution

Item	n_i	A_i	B_i	$\theta_i = \frac{A_i}{B_i}$ ($\times 10^{-6}$)	$n_i \theta_i$ ($\times 10^{-6}$)	$\theta_i^2 = \frac{n_i \theta_i}{B_i}$ ($\times 10^{-8}$)
A	1	5.40	1550	3484	3484	224.8
B	2	2.20	2835	776	1552	54.7
C	5	5.30	5990	885	4425	74.0
D	1	9.00	30000	300	300	1.0
E	2	1.35	2533	533	1066	42.1
F	1	1.40	1920	729	729	38.0
					11,556	434.6

Table XIII

Values for $P(\lambda_j)$ Based on Combined Inputs

j	$x_j^2 = 2B_s \lambda_{0j}$	$P(x_j^2, 2A_s)$	$P(\lambda_j)$	λa_j
1	15.95	.0000	.0000	.002
2	26.59	.0001	.0001	.004
3	79.77	.9425	.9424	.010
4	239.31	1.0000	.0575	.030
5	292.49	1.0000	.0000	.050
6	∞	1.0000	.0000	.060

VI. Assessment Technique Analysis and Refinement

The sample information used in the example presented was obviously selected to illustrate a point. There is absolutely no guarantee that the pooling of reliability predictions and test data will always result in such a drastic change in the preferred course of action. However, the scheme outlined does produce better informed decisions which are normally less costly because less experimentation is usually required. The reason for this is better appreciated if the contributions of the two inputs are analyzed.

Examination of Input Contributions

As mentioned in Section III, the quantification of prior information (in this case reliability predictions) narrows the range of exploration for the true state of nature (system failure rate). The outcomes of experimentation (failure and time data) provide a further reduction to the area of consideration by decreasing the variability of the initial estimates. These contributions are apparent when the unit/subsystem and system failure rate densities are examined. In the example, the greatest amount of uncertainty was contributed by items A and E. The prior and posterior failure rate density functions for these two items are plotted in Figures 3 and 4 respectively. In each instance, the posterior variance is considerably smaller than the prior variance. The magnitudes of difference can be determined by comparing the values for σ_j^2 and σ_j^2 (listed in Tables V and XII) for these two and the other items. (Also shown in Figures 3 and 4 are $g(\lambda; A, B)$ for other selected test data for comparative purposes.)

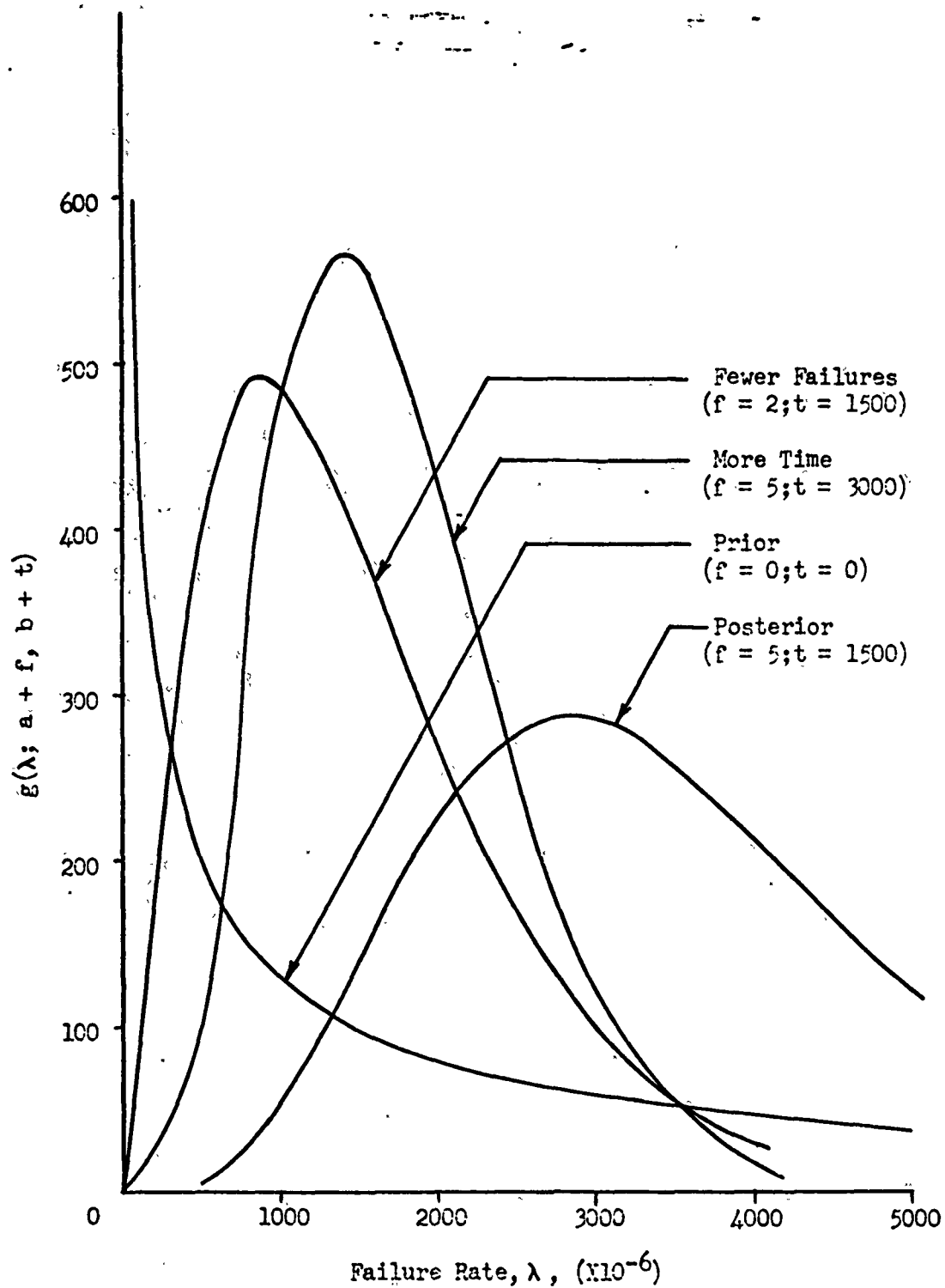


Figure 3. Effect of Test Data on Antenna Array Gamma Distribution

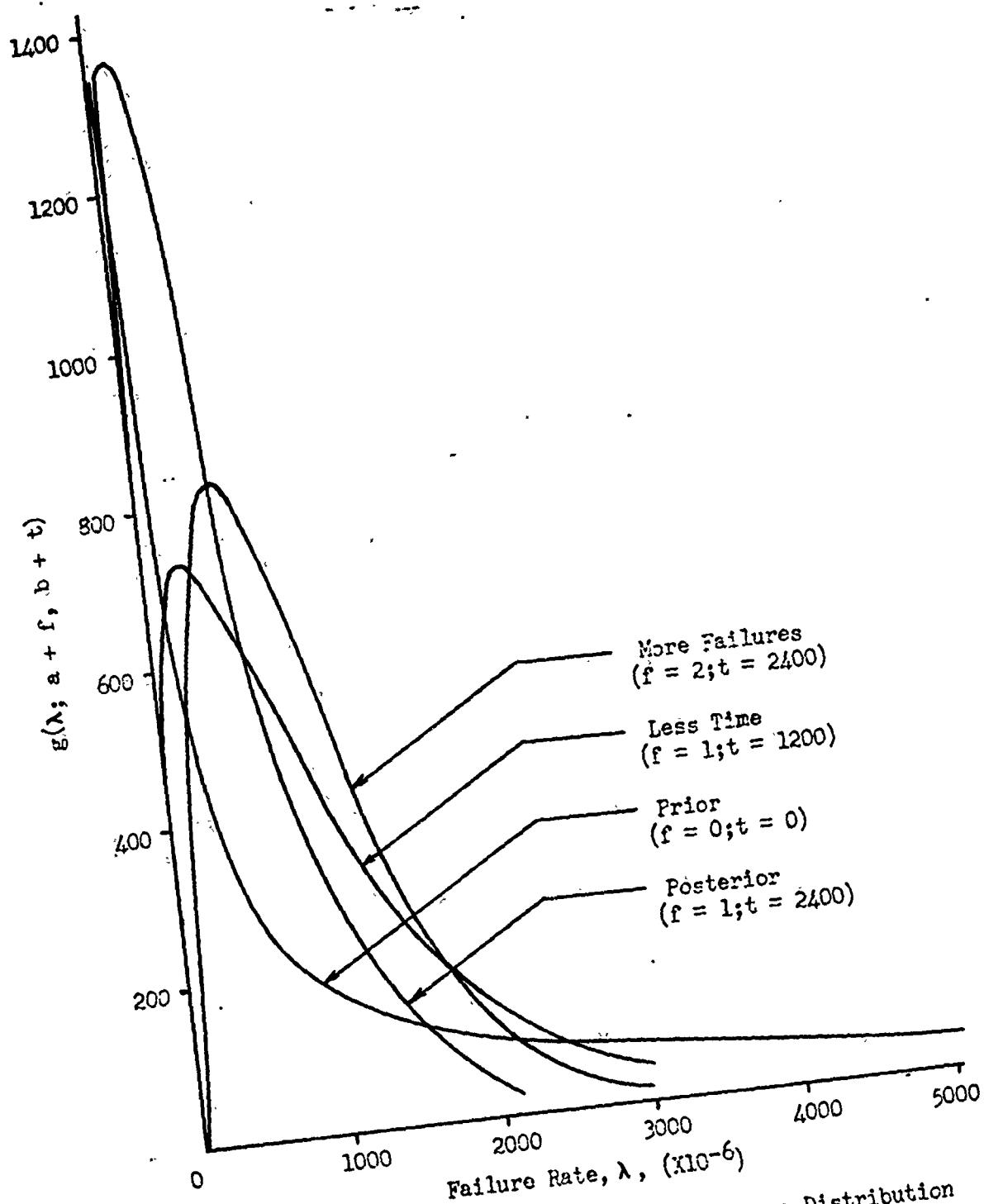


Figure 4. Effect of Test Data on Emitter/Detector Gamma Distribution

The values of σ_s^2 and σ_j^2 may also be found in Tables V and XII. These quantities dictate the shapes of the prior and posterior system failure rate densities. Again the posterior variance is less than the prior, and therefore, the range of the true system failure rate is smaller which results in a more accurate estimate of reliability. Another factor which influences the accuracy of the $P(\lambda_j)$ quantities is the assumed system distribution.

Gamma Vs Normal System Distributions

As previously stated, a Gamma system failure rate distribution was assumed as a matter of convenience. Another statistical distribution which is widely known and easily applied is the Normal distribution. However, in order to employ the Normal distribution two parameters must be specified or estimated. To analyze the sensitivity of the assessment technique used in the example, four additional sets of sample information were assumed and resultant choices compared with original decisions. Rather than assuming a particular value for the standard deviation of the hypothetical test data, it was decided to create sufficient information from which a sample standard deviation could be computed. The additional sets of information include two unfavorable sets and two favorable sets as compared with the original example set. The unfavorable data include both the case in which the same number of failures are observed in half the time and the case in which twice the failures are observed in the same amount of time. The favorable data include both the case in which half the number of failures are observed in the same amount of time and the case in which the same number of failures are observed in twice the time. The

sample failure information for these four new cases plus the original case is presented in Table XIV.

For each set of data, a set of $P(\lambda_j)$ was computed twice - once assuming a Gamma system distribution and again assuming a Normal distribution. Then the Normal distribution was assumed, the sample μ_s and σ_s^2 were calculated from the following relationships,

$$\mu_s = \frac{1}{m} \sum_{j=1}^m \frac{1}{t_j}$$

$$\sigma_s^2 = \frac{\sum_{j=1}^m (1/t_j - \mu_s)^2}{m - 1} \quad (6-1)$$

For example, using the data in Table XIV in the first and third columns,

$$\begin{aligned} \mu_s(f, .5t) &= \frac{1}{12} \left[\frac{2}{35} + \frac{5}{40} + \frac{4}{45} + \frac{1}{50} \right] \\ &= .024253 \\ \sigma_s^2(f, .5t) &= \frac{1}{11} \left[2(.028571 - .024253)^2 + 5(.025000 - .024253)^2 \right. \\ &\quad \left. + 4(.022222 - .024253)^2 + (.020000 - .024253)^2 \right] \\ &= .002605 \end{aligned}$$

To obtain the $P(\lambda_j)$, it is necessary to compute $Z_j = (\lambda_{uj} - \mu_s) / \sigma_s$ and then use standard statistical tables to obtain $P(Z_j) = P(\lambda \leq \lambda_{uj})$. The $P(\lambda_j)$ can then be determined from the relationships $P(\lambda_j) = P(Z_j) - P(Z_{j-1})$ and $P(Z_0) = 0$.

The results of recomputing the $E(a_i)$ for each set of data are listed in Tables XV and XVI when Gamma and Normal distributions, respectively, are assumed to apply. Analysis of these results indicates that the assessment technique yields the same decision for

Table XIV
Sample Failure Information

Time Between Failure (hrs)	Number of Failures				
	Example Case (f,t)	Unfavorable Data (f,.5t)	Unfavorable Data (2f,t)	Favorable Data (.5f,t)	Favorable Data (f,2t)
25			1		
30			2		
35		2	3		
40		5	8		
45		4	5		
50		1	3		
55			1		
60	1		1		
70	2				
75	1				
80	1				
85	1				
90	3				
95	2				
100	1				
135				1	1
150				1	1
155				1	2
165					3
170					1
175				1	1
180				1	2
205				1	1

Table XV

 $E(a_1)$ Values Assuming Gamma System Distribution

$P(\lambda_j)$ Data Source	$E(a_1)$	$E(a_2)$	$E(a_3)$	$E(a_4)$
Prior Information (Predictions)	1171	1069	1000	994*
Unfavorable Test Data (12/500)	1101	1027	976*	990
Unfavorable Test Data (24/1000)	1111	1039	934*	993
Example Case Test Data (12/1000)	755*	755*	755*	920
Favorable Test Data (6/1000)	595*	622	643	950
Favorable Test Data (12/2000)	632*	662	685	950
Unfavorable Posterior (12/500)	1081	1011	964*	988
Unfavorable Posterior (24/1000)	1116	1039	937*	994
Example Case Posterior (12/1000)	652*	674	688	899
Favorable Posterior (6/1000)	617*	645	664	976
Favorable Posterior (12/2000)	650*	680	702	962

*Minimum: $E(a_1)$

Table XVI

 $E(a_i)$ Values Assuming Normal System Distribution

$P(\lambda_j)$ Data Source	$E(a_1)$	$E(a_2)$	$E(a_3)$	$E(a_4)$
Prior Information (Predictions)	983	926	891*	952
Unfavorable Test Data (12/500)	1116	1039	987*	994
Unfavorable Test Data (24/1000)	1098	1024	975*	990
Example Case Test Data (12/1000)	704*	711	716	907
Favorable Case Test Data (6/1000)	610*	638	658	828
Favorable Case Test Data (12/2000)	661*	680	692	900
Unfavorable Posterior (12/500)	1079	1010	963*	926
Unfavorable Posterior (24/1000)	1092	1020	972*	989
Example Case Posterior (12/1000)	648*	671	685	892
Favorable Posterior (6/1000)	615*	643	662	890
Favorable Posterior (12/2000)	619*	647	665	977

*Minimum $E(a_i)$

either distribution except when only the prior information is used to compute the $P(\lambda_j)$. Also the extent of discrimination between the $E(a_1)$ computed from the same data base appears to be remarkably similar for the two assumed distributions. Comparison of the $E(a_1)$ based on unfavorable vs favorable test results reveals that the intuitive decrease in calculated value occurs as fewer failures are observed in greater time. Overall, the assessment technique seems realistic and practical.

Other Practical Considerations

In addition to the features of the technique presented with the example and discussed above, there are two other refinements which are considered worthy of discussion. One pertains to the consideration of different opinions when assigning parameters of the unit/subsystem prior failure rate distributions. The other relates to the treatment of different types of test data when determining operating times.

Weighting Prior Assignments. There are many instances in which different individuals provide inputs for the various pieces of equipment that make up a system. The analyst might also wish to combine the inputs from several sources in order to formulate a single overall estimate for an item failure rate. Fox advances the idea of assigning weighting factors to each contributor (Ref 84:3). These weights can be based on either the contributors accuracy and consistency of previous predictions or his extent of participation if a joint assessment is required (i.e., a prior based on combined government, consultant, contractor inputs). In either case, if a weighting factor ϕ is assigned to the k^{th} of r individuals, then the parameters for a

Gamma prior can be computed from the following expressions,

$$a = \sum_{k=1}^r \phi_k a_k$$

$$b = \sum_{k=1}^r \phi_k b_k$$

Weighting Heterogeneous Test Data. The other weighting factor considered important is a multiplier which accounts for the difference in severity of the various types of tests to which an item is usually subjected. The concept of attaching more significance to data obtained under more difficult environments has been proposed by Pozner. He states this may be done "by weighting the time experience in particular environments by the k factor corresponding to these environments. These k factors are environmental failure rate acceleration factors such as those in MIL-STD-756." (Ref 162:139). Test severity weights need not be of the magnitude generally associated with acceleration factors. The important consideration is that the value computed or assigned reflect the extent of additional exposure experienced by the equipment beyond that normally expected. As with the reliability predictions, test severity weights based on considered opinion or technical judgment of competent engineers may be used. Again, if several subjective inputs are to be combined, the contributors' weights can be included in deriving an overall set of test severity weights for the various types of tests to be performed. In general, a value k for a particular test can be determined from the following expression,

$$K = \sum_{j=1}^r \gamma_j k_j$$

where γ_j = contributors weight

k_j = individual test severity weight

r = number of contributors

Note that the term used for the contributor's weight is different than the one used in assigning priors. This was done purposefully to distinguish between judgements pertaining to reliability predictions and those concerning difficulty of test environments.

With the addition of these two weighting factors a recap of the revised assessment technique for the general case is in order.

Revised General Approach

The Bayesian reliability assessment technique may be implemented in a variety of situations. It is not necessary for the situations to be based on the need to make a particular decision as was illustrated. The procedures outlined and methods presented may also be used for periodic determination of the status of reliability achievement. In summary, the following step-by-step activities are required to obtain and update reliability estimates using subjective evaluation and Bayes techniques:

- (1) Quantification of all prior knowledge concerning failure rate predictions. This effort may be a singular or combined input. If combined, contributor weights may be used to obtain single values for prior parameters for calculation purposes. The assertion of uncertainty must be carefully considered in light of potential

impact when combined with measured data. Parameters for prior distributions may be determined by a variety of methods and Tables I and II should be beneficial.

- (2) Determination of a system prior distribution. This may be accomplished by summing the means and variances of constituent element failure rate distributions. If only a one-sided upper confidence limit for the system failure rate is of interest, equation (5-3) may be used. If discrete intervals are desired, then either a Gamma or Normal system distribution may be assumed and particular probabilities may be computed as illustrated.
- (3) Collection of time and failure data. This is the most critical activity associated with any estimation task. Posterior estimates are only meaningful if they are based on accurate and complete information. In order to obtain high quality data, special emphasis must be placed on the proper training and motivation of personnel responsible for maintaining records. There is also the question of which types of anomalies to consider and which ones to censor. The definition of relevance is many times a subject for negotiation if the products of the assessment effort are used for acceptance purposes. Also, test times may be adjusted to reflect severity of equipment exposure. This should be done before tests are started to avoid undue bias based on outcomes.

- (4) Determination of a system posterior distribution. This involves adding the constituent element "pseudo" failures and times with corresponding observed values to modify the prior parameters. System reliability indices can be computed from the posterior distribution in the same manner as from the prior distribution.
- (5) Analysis of sensitivity. This optional task may be performed if there is doubt concerning the impact that a particular quantity has on the overall results. If more precision is desired than that achievable from assuming either a Gamma or Normal distribution, then Monte Carlo techniques can be employed to determine exact confidence bounds.

The tasks outlined above are quite general and are considered to be useful for widespread applications. To facilitate the recording of the inputs necessary for a Bayesian assessment effort, sample worksheets are included as Appendix B.

VII. Conclusions and Recommendations

Conclusions

It has been shown that Bayesian statistics used in conjunction with decision theory offer a suitable framework for solving cost effectiveness type problems involving uncertainty. When the uncertainty is reliability, there is considerable advantage to be realized in combining predictions with test data to obtain greater precision in the reliability estimate. Therefore, it is concluded that the implementation of Bayesian techniques in the solution of reliability decision problems can produce more conclusive results with the added advantages of being economically practical and intuitively appealing.

Recommendations

Based on the findings of this study, it is recommended that the reader desiring to further pursue the field of Bayesian statistics consider the following topics:

- (1) Development of simple algorithms and other aids to establish prior distributions based on predictions.
- (2) Investigation of other closed distribution forms to fit the system failure rate.
- (3) Development of a simple generalized Monte Carlo model which can be used to determine exact system failure rate bounds.

Also, it is suggested that this study be used as:

- (1) an aid in the study of Bayesian statistics and decision theory;

- (2) as a source of references and possible thesis topics
for future Air Force Institute of Technology students.

Bibliography

1. Aggarwal, O. P. "Bayes and Minimax Procedures in Sampling From Finite and Infinite Populations-I." Annals Mathematical Statistics, 30:206-218 (March 1959).
2. Aitchison, J. "Bayesian Tolerance Regions." Journal Royal Statistical Society (B), 26:161-175 (Number 2, 1964).
3. AMCP-702-5. Planning Guide for Demonstration and Assessment of Reliability and Durability. Washington, D.C.: Army Material Command, February 1969. AD710215.
4. Amstadter, B. L. Reliability Mathematics: Fundamentals; Practices; Procedures. New York: McGraw Hill Book Co., Inc., 1971.
5. Antelman, G. R. "Insensitivity to Non-Optimal Design in Bayesian Decision Theory." Journal of the American Statistical Association, 60:584-601 (June 1965).
6. Babllis, R. A., and A. M. Smith. "Application of Bayesian Statistics in Reliability Measurements." Annals of Reliability and Maintainability, Volume 4, Practical Techniques and Application. Washington, D. C.: Spartan Books, Inc., 1965, pp. 357-365.
7. Balaban, H. S. "A Demonstration Bayesian Approach to Reliability Demonstration." Annals of Assurance Sciences; Proceedings of the Eighth Reliability and Maintainability Conference. New York: Gordon and Breach Science Publishers, Inc., 1969, pp. 497-506.
8. ----- "A Bayesian Approach for Designing Component Life Tests." Proceedings 1967 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1967, pp. 59-74.
9. Barish, N. N. Economic Analysis for Engineering and Managerial Decision-Making. New York: McGraw Hill Book Co., Inc., 1962.
10. Barnard, G. A. "Thomas Bayes - a Biographical Note." Biometrika, 45:293-315 (1958).
11. Bartholomew, C. S. "Reliability and Program Decision Making." Proceedings 1967 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1967, pp. 148-161.

12. Bartholomew, D. C. "A Comparison of Some Bayesian and Frequentist Inferences." Biometrika, 52:19-35 (1965).
13. Bell, C. F. Reliability of Aircraft as Determined by Operational Field Tests: The Need for Proper Test Design and Data Requirements. Technical Paper N. P-4054. Santa Monica, California: RAND Corporation, April 1969. AD686414.
14. Bellinger, D. Q., et al. Reliability Prediction and Demonstration for Ground Electronic Equipment. Technical Report No. RADC-TR-68-280. Redondo Beach, California: TRW Systems, November 1968.
15. ----- Reliability Prediction and Demonstration for Missile and Satellite Electronics. Technical Report No. RADC-TR-68-281. Redondo Beach, California: TRW Systems, November 1968. AD844973.
16. Benton, A. W. An Investigation of the Characteristics of Bayesian Confidence Intervals for Attribute Data. Technical Report No. ARDC-TM-14. Aberdeen Proving Ground, Maryland: Aberdeen Research and Development Center, November 1969. AD698466.
17. Bernberg, J. G. "Bayesian Statistics: A Review." Journal of Accounting Research, 2:108-118 (Spring 1964).
18. Berndt, G. D. Some Models and Techniques for Evaluating System Reliability. Technical Memorandum No. SAC-OA-TM-69-2. Offutt Air Force Base, Nebraska: Strategic Air Command, May 1969. AD852217L.
19. Bhattacharya, S. K. "Bayesian Approach to Life Testing and Reliability Estimation." Journal American Statistical Association, 62:48-62 (March 1967).
20. Blake, R. E. "Predicting Structural Reliability for Design Decisions." AIAA Journal of Spacecraft and Rockets, 4:392-398 (March 1967).
21. Bogdanoff, D. A. Bayesian Inference for the Weibull Distribution. PhD Thesis. Oregon: Oregon State University, June 1971. Order No. 71-12,658.
22. Bohrer, R. "On Bayes Sequential Design with Two Random Variables." Biometrika, 53:469-475 (1963).
23. Bonis, A. J. "Bayesian Reliability Demonstration Plans," in Annals of Reliability and Maintainability, Volume 5, Achieving Systems Effectiveness. New York: American Institute of Aeronautics and Astronautics, 1966, pp. 861-873.
24. Boot, J. C. G. Statistical Analysis for Managerial Decisions. New York: McGraw-Hill Book Co., Inc., 1970.

25. Box, G. E. P., and G. C. Tiao. Bayesian Inference. Reading, Massachusetts: Addison-Wesley, (Scheduled for publication 1973).
26. Bracken, J. "Percentage Points of the Beta Distribution for Use in Bayesian Analysis of Bernoulli Process." Technometrics, 8: 687-694 (November 1966).
27. Bram, J. Confidence Limits for System Reliability. Technical Report No. OEG-79. Washington, D.C.: Operations Evaluation Group, Center for Naval Analyses, February 1968. AD666560.
28. Bratcher, T. L. A Bayesian Treatment of a Multiple Comparison Problem for Binomial Probabilities. Technical Report No. THEMIS-SMU-TR-49. Dallas, Texas: Department of Statistics, Southern Methodist University, 24 November 1969. AD700229.
29. Breipohl, A.M., et al. "A Consideration of the Bayesian Approach in Reliability Evaluation." IEEE Transactions on Reliability, R-14:107-113 (October 1965).
30. Breipohl, A. M., and W. C. McCormick, Jr. "Bayesian Estimation of Time Varying Reliability," in Annals of Assurance Sciences: Proceedings of 7th Reliability and Maintainability Conference. New York: American Society of Mechanical Engineers, 1968, pp. 347-351.
31. Breipohl, A. M. Probabilistic Systems Analysis. New York: John Wiley and Sons, Inc., 1970.
32. Brender, D. M. "Reliability Testing in a Bayesian Context." 1966 IEEE International Convention Record Part 9, 14: 125-136 (March 1966).
33. ----- "The Prediction and Measurement of System Availability - A Bayesian Treatment." IEEE Transactions on Reliability, R-17: 127-138 (September 1968).
34. ----- "The Bayesian Assessment of System Availability - Advanced Applications and Techniques." IEEE Transactions on Reliability, R-17:138-147 (September 1968).
35. Briggs, W. G. "Reliability Statements Under Uncertainty." The Logistics Review, 3:11-19 (January/February 1967).
36. ----- Statistical Decision Theory for Logistics Planning. Technical Report No. E-1350. Cambridge, Massachusetts: Instrumentation Laboratory, Massachusetts Institute of Technology, May 1963.
37. Britt, P. S., and E. L. Ibbotson. A Bayesian Approach to Determining the Sample Size for Maintainability Demonstration. Thesis No. GSA/SM/69-2. Wright-Patterson Air Force Base, Ohio: Air Force Institute of Technology, June 1969. AD857547.

38. Cambi, E. The Reliability of a Space Vehicle Launcher System - A Theoretical Approach. Technical Memorandum No. ELDO-TM-107. Paris, France: European Space Vehicle Launcher Development Organization, March 1968.
39. Canavos, G. C., and C. P. Tsokos. "A Study of an Ordinary and Empirical Bayes Approach to Reliability Estimation in the Gamma Life Testing Model," in Annals of Assurance Sciences: Proceedings 1971 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1971, pp. 343-349.
40. Canfield, R. V. "A Bayesian Approach to Reliability Estimation Using a Loss Function." IEEE Transactions on Reliability, R-19: 13-16 (February 1970).
41. Cartland, J. C., Jr. An Ad Hoc Bayesian Method for Determining Lower Confidence Limits. Masters Thesis. Monterey, California: U.S. Naval Postgraduate School, September 1970. AD713075.
42. Chen, C. H. "A Theory of Bayesian Learning Systems." IEEE Transactions on System Science and Cybernetics, SSC-5:30-37 (January 1969).
43. Chernoff, H., and S. N. Ray. "A Bayes Sequential Sampling Inspection Plan." Annals Mathematical Statistics, 36:1387-1407 (October 1965).
44. Chernowitz, G., et al. Guide for Development of Statistical Techniques for Defining Significant Variables Affecting Product Quality and Reliability. Technical Report No. APJ-3761. Ridgefield, New Jersey: American Power Jet Company, June 1964. AD632011.
45. Clarke, R. W. A General Computational Algorithm for Bayesian Confidence Bounds, with Application to the Weibull, Lognormal, and Gamma Densities. Technical Report No. WVT-6911. Watervliet Arsenal, New York: Benet R & E Laboratories, May 1969. AD688862.
46. Clevenston, M. L. Subpopulation Identification Using the Bayesian Approach. Technical Memorandum No. SC-TM-67-748. Albuquerque, New Mexico: Sandia Corp., September 1967. AD830663L.
47. Cole, P. V. Z. The Use of Bayes Theories in Reliability Estimates. Technical Report No. IDNP-3/7.70.00.00-Y6-02. Oahu, Hawaii: Quality Evaluation Laboratory, Naval Ammunition Depot, December 1968. AD850575L.
48. ----- A Distributed Estimate for a Probability Via Any Complex Model. Technical Report No. IDNP-3/7.70.00.00-Y6-03. Oahu, Hawaii: Quality Evaluation Laboratory, Naval Ammunition Depot, October 1968. AD850576L.

49. Cortisiero, J. V. Evaluating Weapon System Accuracy from a Classical-Bayesian Approach. Thesis No. GSA/SM/70-5. Wright-Patterson Air Force Base, Ohio: Air Force Institute of Technology, June 1970. AD874194.
50. Couture, D. J. Some Practical Empirical Bayes Procedures for Use in Weibull Reliability. PhD Thesis. Texas: Texas Tech University, December 1970. Order No. 71-17,894.
51. Davis, G. M. "Decision-Making Under Conditions of Uncertainty - An Application to Intermodel Carrier Selection." The Logistics Review, 7:9-26 (Winter 1971).
52. Deeley, J. J., and W. J. Zimmer. "Partial Prior Information and Shorter Confidence Intervals," in Annals of Assurance Sciences: Proceedings of the Eighth Reliability and Maintainability Conference. New York: Gordon and Breach, Scientific Publishers, Inc., 1969, pp. 488-496.
53. Deeley, J. J., et al. "On the Usefulness of the Maximum Entropy Principle in the Bayesian Estimation of Reliability." IEEE Transactions on Reliability, R-19:110-115 (August 1970).
54. Deeley, J. J., and W. J. Zimmer. "Some Comparisons of Bayesian and Classical Confidence Intervals in the Exponential Case," in Annals of Assurance Sciences: Proceedings of 5th Reliability and Maintainability Conference. New York: American Society of Mechanical Engineers, 1968, pp. 366-371.
55. DeFinetti, B. "Foresight: Its Logical Laws, Its Subjective Sources," in Studies in Subjective Probability, edited by Kyburg, H. E., Jr., and H. E. Smokler. New York: John Wiley, 1964.
56. DeGroot, M. H. Optimal Statistical Decisions. New York: McGraw-Hill Book Co., Inc., 1970.
57. DeHart, J. H., and H. D. McLoughlin. "Using Bayesian Methods to Select a Design With Known Reliability Without a Confidence Interval," in Annals of Reliability and Maintainability. Volume 5 - Achieving Systems Effectiveness. New York: American Institute of Aeronautics and Astronautics, Inc., 1966, pp. 611-617.
58. Dempster, A. P. A Generalization of Bayesian Inference. Technical Report No. TR-20. Cambridge, Massachusetts: Department of Statistics, Harvard University, November 1967. AD664659.
59. DeVitt, C. M., et al. Reliability Prediction and Demonstration for Airborne Electronics. Technical Report No. RADG-TR-68-223. Culver City, California: Aerospace Group, Hughes Aircraft Co., August 1968. AD841095.

60. Dixon, D. O., and R. P. Bland. A Bayes Solution for the Problem of Ranking Poisson Parameters. Technical Report No. THEMIS-SMU-TR-57. Dallas, Texas: Department of Statistics, Southern Methodist University, March 1970. AD704797.
61. Drake, A. W. Research in the Control of Complex Systems. Technical Report No. MIT-DSR-79153. Cambridge, Massachusetts: Operations Research Center, Massachusetts Institute of Technology, May 1967. AD700763.
62. ----- "Bayesian Statistics for the Reliability Engineer." Proceedings 1966 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, 1966, pp. 315-320.
63. Draper, N. R., and I. Guttman. Bayesian Estimation of the Binomial Parameter N. Technical Report No. UWIS-DS-71-260. Madison, Wisconsin: Department of Statistics, University of Wisconsin, January 1971. AD717606.
64. ----- The Value of Prior Information. Technical Summary Report No. MRC-TSR-841. Madison, Wisconsin: Mathematics Research Center, University of Wisconsin, January 1968. AD679928.
65. Druas, T. M. Reliability Demonstration Test Plans for Aerospace Systems Based on Bayesian Concepts. PhD Thesis. Los Angeles: University of Southern California, 1970. Order No. 70-23.154.
66. Dunsmore, I. R. "A Bayesian Approach to Calibration." Journal of Royal Statistical Society (B), 30: 396-405 (Number 2, 1968).
67. ----- "A Bayesian Approach to Classification." Journal Royal Statistical Society (B), 28:568-577 (Number 3, 1966).
68. Duroux, J. W. The Prediction of Operational Success. Technical Report No. SAMSO-TR-69-185. El Segundo, California: Engineering Science Operations, Aerospace Corp., June 1969. AD689748.
69. Dyckman, T. R., et al. Management Decision Making Under Uncertainty; An Introduction to Probability and Statistical Decision Theory. New York: MacMillan, 1969.
70. Earnest, C. M. Estimating Reliability After Corrective Action: A Bayesian Viewpoint. Masters Thesis. Monterey, California: U. S. Naval Postgraduate School, May 1966. AD487411.
71. Easterling, R. G. "On the Use of Prior Distribution in Acceptance Testing," in Annals of Reliability and Maintainability, Volume 9 Assurance Technology Spinoffs. New York: Society of Automotive Engineers, Inc., 1970, pp. 31-35.

72. Ebeling, D. G. "A Risk Analysis Procedure for Calculating Failure Rates vs. Time," in Annals of Reliability and Maintainability, Volume 9 - Assurance Technology Spinoffs. New York: Society of Automotive Engineers, Inc., 1970, pp. 36-45.
73. Elfving, G. Robustness of Bayes Decisions Against the Choice of Prior. Technical Report No. TR-122. Palo Alto, California: Department of Statistics, Stanford University, November 1966. AD642613.
74. Ellis, J. W. Application of Bayesian Statistics to Reliability. Technical Report No. D2-22574 Rev A. Seattle, Washington: Aerospace Division, The Boeing Co., July 1963. AD466817L.
75. ElMawaziny, A. H., and A. H. Buehler. "Confidence Limits for the Reliability of Series Systems." Journal of the American Statistical Association, 62:1452-1459 (December 1967).
76. Engleman, J. H. "A Bayesian Time-to-Failure Distribution," in Annals of Assurance Sciences; Proceedings 1971 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1971, pp. 350-355.
77. Ericson, W. A. "Subjective Bayesian Models in Sampling Finite Populations (with discussion)." Journal of Royal Statistical Society (B), 31:195-233 (Number 2, 1969).
78. Evans, R. A. "Classical Confidence Intervals Have a Bayesian Interpretation," in Annals of Assurance Sciences; Proceedings 1970 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1970, pp. 343-347.
79. Everson, R. C. A Method for Computing the Shortest Confidence Interval for Any Beta Distribution. Technical Report No. IDEP 347.40.00.00-Y6-06. Oahu, Hawaii: Quality Evaluation Laboratory, Naval Ammunition Depot, December 1968. AD850851L.
80. Farrell, R. H. "Towards a Theory of Generalized Bayes Tests." Annals Mathematical Statistics, 39:1-22 (February 1968).
81. Feduccia, A. J. A Bayesian/Classical Approach to Reliability Demonstration. Technical Report No. RAEC-TR-70-72. Griffiss Air Force Base, New York: Rome Air Development Center, June 1970. AD871969.
82. Ferguson, I. S. Mathematical Statistics: A Decision Theoretic Approach. New York: Academic Press, 1967.
83. Fishburn, P. C. Decision and Value Theory. New York: John Wiley and Sons, Inc., 1964.

84. Fox, B. L. A Bayesian Approach to Reliability Assessment. Research Memorandum No. RM-5084-NASA. Santa Monica, California: RAND Corporation, August 1966. N66-36262.
85. Fragola, J. An Illustration of Bayesian Analysis of a Weibull Process. Technical Report No. IDEP 347.40.00.00-K4-24. Bethpage, New York: Grumman Aircraft Engineering Corporation, June 1970. AD880190L.
86. Freund, J. E. Mathematical Statistics (Second Edition). Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1971.
87. Gabbert, J. T., and R. G. Krutchkoff. "Supplementary Sample Nonparametric Empirical Bayes in a Quality Control Situation." Biometrika, 57:67-69 (April 1970).
88. Gaver, D. P., Jr. Bayesian Analysis of System Readiness. Technical Report No. 103. Pittsburgh, Pennsylvania: Management Sciences Research Group, Carnegie Institute of Technology, July 1967. AD656031.
89. Gaver, D. P., Jr., and M. Mazumdar. Some Bayes Estimates of Long-Run Availability in a Two-State System. Research Report No. RR-161. Pittsburgh, Pennsylvania: Management Sciences Research Group, Carnegie-Mellon University, May 1969. AD691404.
90. George, S. L. Partial Prior Information: Some Empirical Bayes and G-Minimax Decision Functions. Technical Report No. TR-48. Dallas, Texas: Department of Statistics, Southern Methodist University, 30 October 1969. AD698501.
91. Germeyer, Y. B., et al. Assured Evaluations of System Reliability with Incomplete Information Concerning Reliability of Elements. Technical Report No. FTD-HT-67-263. Wright-Patterson Air Force Base, Ohio: Foreign Technology Division, September 1967. AD673758.
92. Girshick, M. A., and H. Rubin. "A Bayes Approach to a Quality Control Model." Annals Mathematical Statistics, 23:114-125 (1952).
93. Gnedenko, B. V. Problems in Theory of Testing Products for Quality and Reliability. Technical Report No. FSTC-HT-23-227-68. Washington, D. C.: Army Foreign Science and Technology Center, 1968. AD844347L.
94. Good, I. J. The Estimation of Probabilities; An Essay on Modern Bayesian Methods. Cambridge, Massachusetts: Massachusetts Institute of Technology Press, 1965.
95. Goode, H. H. "Deferred Decision Theory," in Recent Developments in Information and Decision Processes, edited by R. E. Machol and P. Gray. New York: MacMillan Co., 1962, pp. 71-91.

96. Good, I. J. "How to Estimate Probabilities." Journal of the Institute of Mathematics and Its Applications, 2:364-383 (December 1966).
97. Gottfried, P., et al. Evaluation of Reliability Prediction Techniques for Entire Flight Control Systems. Technical Report No. AFFDL TR-67-183. Bethesda, Maryland: Booz-Allen Applied Research, Inc., March 1968. AD829292.
98. ----- Reliability Prediction Techniques for Flight Control Systems. Technical Report No. AFFDL-TR-67-20. Bethesda, Maryland: Booz-Allen Applied Research, Inc., April 1967. AD815590.
99. Gottfried, P. "A Reliability Demonstration Plan with Incentives," in Proceedings 1966 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1966, pp. 584-593. IEEE Catalog No. 7C26.
100. Gottfried, P., and D. W. Weiss. "Reliability Prediction with Inadequate Data," in Annals of Reliability and Maintainability 1967, Volume 6 - All Systems Co. New York: Society of Automotive Engineers, Inc., 1967, pp. 600-607.
101. Greenbolt, B. J., and J. C. Hung. "A Structure for Management Decision Making," in IEEE Transactions on Engineering Management, EM-17:145-158 (November 1970).
102. Guild, R. D. Reliability Testing and Equipment Design Using Bayesian Models. PhD Dissertation. Evanston, Illinois: Northwestern University, 1968. Order No. 69-6932.
103. Guthrie, D., Jr., and M. V. Johns, Jr. "Bayes Acceptance Sampling Procedures for Large Lots." Annals Mathematical Statistics, 30: 896-925 (December 1959).
104. Guttman, I. Tolerance Regions: A Survey of Its Literature Part 6. The Bayesian Approach. Technical Report No. TR-127-PT-6. Madison, Wisconsin: Department of Statistics, University of Wisconsin, August 1968. AD686294.
105. Hamilton, C. W. "Bayesian Procedures and Reliability Information," in Aerospace Reliability and Maintainability Conference, A Volume of Technical Papers. New York: American Institute of Aeronautics and Astronautics, Inc., 1963, pp. 278-283.
106. Hamilton, C. W., and J. E. Drennan. "Research Toward a Bayesian Procedure for Calculating System Reliability," in Proceedings of Third Annual Aerospace Reliability and Maintainability Conference. New York: Society of Automotive Engineers, Inc., 1964, pp. 614-620.
107. Harris, C. M., and N. D. Singpurwalla. "Life-Time Distributions Derived from Stochastics Hazard Functions." IEEE Transactions on Reliability, R17:70-79 (June 1968).

108. Hartley, H. D. "In Dr. Bayes' Consulting Room." The American Statistician, 17:22-24 (February 1963).
109. Herd, G. R., et al. "The Uncertainty of Reliability Assessments," in Proceedings of the Tenth National Symposium on Reliability & Quality Control. New York: The Institute of Electrical and Electronic Engineers, Inc., 1964, pp. 33-40.
110. Hershman, R. L., and M. Freitag. A Bayesian Model for Troubleshooting Electronic Equipment Research Report January-July 1966. Technical Report No. NEL-1412. San Diego, California: Navy Electronics Laboratory, November 1966. AD645577.
111. Hill, B. M. "Information for Estimating the Proportions in Mixtures of Exponential and Normal Distributions." Journal of the American Statistical Association, 58:918-932 (December 1963).
112. Hiltz, P. A. Considerations of the System/Equipment Reliability Demonstration Problem. Technical Report No. SD-69-1; IDEP 347.40. 00.00-FI-57. Downey, California: North American Rockwell Corporation, February 1969. AD856202L.
113. Hinojoss, A., and G. K. Mikasa. A Bayesian Distributed Estimate for Reliability Using Variable Test Data. Technical Report No. IDEP 347.40.00-00-Y6-05. Oahu, Hawaii: Quality Evaluation Laboratory Naval Ammunitions Depot, October 1968. AD850578L.
114. Hoel, D. G., and M. Mazumdar. "A Class of Sequential Tests for an Exponential Parameter." Journal of the American Statistical Association, 64:1549-1559 (December 1969).
115. Holla, M. S. "Bayesian Estimates of the Reliability Function." Australian Journal of Statistics, 8:32-35 (April 1966).
116. Horowitz, I. Decision Making and the Theory of the Firm. New York: Holt, Rinehart and Winston, Inc., 1970.
117. Howard, R. A. "Bayesian Statistical Models for Systems Engineering." IEEE Transactions on Systems Science and Cybernetics, SSC-1:36-40 (November 1965).
118. Ijiri, Y., and R. Kaplan. Sequential Models in Probabilistic Depreciation. Research Report No. RR-160. Pittsburgh, Pennsylvania: Management Sciences Research Group, Carnegie-Mellon University, April 1969. AD689371.
119. Isken, J., and J. Saboe. "Reliability Improvement Through Effective Nondestructive Screening," in Annals of Assurance Sciences; Proceedings 1970 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1970, pp. 326-330.

120. Jacks, H. G. "Total Confidence Limits on Observed Reliability," in Annals of Assurance Sciences; Proceedings 1971 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1971, pp.338-342.
121. Johnson, C. R. A Duality Property for Bayes Rules with Applications. Technical Report No. THEMIS-SMU-TR-75. Dallas, Texas: Department of Statistics, Southern Methodist University, June 1970. AD711800.
122. Johnson, J. D., and L. T. Stewart. "Failure Prediction From Interval Data," in Annals of Assurance Sciences; Proceedings 1971 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, 1971, pp. 356-361.
123. Kassouf, S. T. Normative Decision Making. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1970.
124. Keefe, A. A., and H. T. Ohara. "Confidence Intervals for System Reliability from Component Testing Using Converse Hypergeometric Probability Distributions." Journal of the Electronics Division, ASQC, 5:21-32 (April 1967).
125. Kerr, R. B. "Bayesian Identification of Systems Parameters." Journal of Engineering Mathematics, 4:273-281 (July 1970).
126. Kerridge, D. F. "Bounds for the Frequency of Misleading Bayes Inferences." Annals Mathematical Statistics, 34:1109-1110 (September 1963).
127. King, W. R. Probability for Management Decisions. New York: John Wiley and Sons, Inc., 1968.
128. Kraft, C. H., and C. VenEeden. "Bayesian Bio-assay." Annals Mathematical Statistics, 35:886-890 (June 1964).
129. Kyburg, H. E., Jr. Probabilistic Theory. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1969.
130. Larson, H. J. Bayesian Methods and Reliability Growth. Technical Report No. TR/RP-75. Monterey, California: U.S. Naval Postgraduate School, March 1967. AD653460.
131. ----- Conditional Distribution of True Reliability After Corrective Action. Technical Report/Research Paper No. 61. Monterey, California: U.S. Naval Postgraduate School, January 1966. AD629084.
132. Lee, T. C., et al. "Maximum Likelihood and Bayesian Estimation of Transition Probabilities." Journal American Statistical Association, 63:1162-1179 (December 1968)

133. Lemon, G. H., and R. G. Krutchkoff. "An Empirical Bayes Smoothing Technique." Biometrika, 56:361-365 (August 1969).
134. Lentner, M. M., and R. J. Buehler. "Some Inferences About Gamma Parameters with an Application to a Reliability Problem." Journal of the American Statistical Association, 58:670-677 (September 1963).
135. Lieberman, G. J. "Some Problems in Reliability Estimation," in Proceedings Third Annual Aerospace Reliability and Maintainability Conference. New York: Society of Automotive Engineers, Inc., 1964, pp. 136-140.
136. Lindley, D. V. "The Use of Prior Probability Distributions in Statistical Inference and Decision," in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1. Berkeley, California: University of California Press, 1961, pp. 453-469.
137. -----. Making Decisions. London: John Wiley and Sons, Ltd., 1971.
138. -----. Introduction to Probability and Statistics from a Bayesian Viewpoint; Part I, Probability; Part II, Inference. London: Cambridge University Press, 1965.
139. Locke, L. G. "Bayesian Statistics." Industrial Quality Control, 20:18-22 (April 1964).
140. Locks, M. O., and B. Sherman. Design, Testing and Estimation in Complex Experimentation. II Reliability Growth Processes - Estimation Theory and Decision Theoretic Formulations. Technical Report Nos. R-6078-2, ARL 65-116. Canoga Park, California: Rocketdyne, June 1965. AD618516.
141. Lukgashchenko, V. I., and A. N. Terpilovskiy. "How Preliminary Information is Taken Into Account When Evaluating the Reliability of Complex Systems," in The Reliability of Complex Technical Systems. Washington, D.C.: Joint Publications Research Service, June 1967, pp. 267-279. N67-32209.
142. MacFarland, W. J. "Use of Bayes Theorem in Its Discrete Formulation for Reliability Estimation Purposes," in Annals of Assurance Sciences: Proceedings of 7th Reliability and Maintainability Conference. New York: American Society of Mechanical Engineers, 1968, pp. 352-365.
143. Mann, N. R. "Computer-Aided Selection of Prior Distribution for Generating Monte Carlo Confidence Bounds on System Reliability." Naval Research Logistics Quarterly, 17:41-54 (March 1970).

144. Maritz, J. S. "Empirical Bayes Estimation for the Poisson Distribution." Biometrika, 56:349-359 (August 1969).
145. Martin, J. J. Bayesian Decision Problems and Markov Chains. New York: John Wiley and Sons, Inc., 1967.
146. Mastran, D. V. "A Bayesian Approach for Assessing the Reliability of Air Force Re-Entry Systems," in Annals of Assurance Sciences; Proceedings of 7th Reliability and Maintainability Conference. New York: American Society of Mechanical Engineers, 1968, pp. 380-383.
147. Mazzilli, F., et al. RADC Reliability Notebook Volume I. Technical Report No. RADC-TR-67-108-Vol I. New York: Computer Applications, Inc., November 1968. AD845304.
148. Miller, R. N. "Decision Theory in Reliability and Project Management," in Annals of Assurance Sciences; Proceedings 1971 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1971, pp. 376-382. IEEE Catalog No. 71C2-R.
149. Morgan, B. W. An Introduction to Bayesian Statistical Processes. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1968.
150. Morris, L. A. Estimation of Hazard Rate from Incomplete Data. Technical Report No. X9-1923/201. Anaheim, California: Autonetics, October 1969. AD870403L.
151. Morris, W. T. Management Science: A Bayesian Introduction. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1968.
152. Mount, R. L. "Emphasis on 'Change'...at 16th Annual Symposium on Reliability." Quality Assurance, Volume 9, No. 3:50-52 (March 1970).
153. Nagy, G. The Potential of Bayesian Statistics in Reliability Engineering. Technical Report No. GER-14133. Akron, Ohio: Goodyear Aerospace Corporation, June 1968. AD870431L.
154. Odiorne, G. S. Management Decisions by Objectives. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1969.
155. Olsson, J. E. "Implementation of Bayesian Reliability Measurement Program," in Annals of Assurance Sciences; Proceedings of 7th Reliability and Maintainability Conference. New York: American Society of Mechanical Engineers, 1968, pp. 372-379.
156. Pantz, T. L. An Application of Bayesian Technique for Early Reliability Measurements. Washington, D.C.: Naval Ship Systems Command, 1970. AD709386.

157. Peers, H. W. "Confidence Properties of Bayesian Interval Estimates." Journal of Royal Statistical Society (B), 30:535-544 (Number 3, 1968).
158. Petrasovits, A., and R. G. Cornell. Bayesian Analysis for An Exponential Surveillance Model. Technical Report No. NASA-CR-113891 TR-20. Tallahassee, Florida: Department of Statistics, Florida State University, August 1970. N70-41139.
159. Pierce, D. A., and J. L. Folks. "Sensitivity of Bayes Procedures to the Prior Distribution." Operations Research, 17:344-350 (March-April 1969).
160. Pollack, S. M. "A Bayesian Reliability Growth Model." IEEE Transactions on Reliability, R-17:187-198 (December 1968).
161. Porter, S. R. Reliability Programs for Aerospace Systems and the Bayes Theorem to Assure Reliability. Technical Report No. SEF-IN-67-5. Wright-Patterson Air Force Base, Ohio: Systems Engineering Group, August 1967. AD664586.
162. Pozner, A. W. "A New Reliability Assessment Technique." Transactions Twentieth Annual Technical Conference. New York: American Society for Quality Control, 1966, pp. 188-201.
163. Press, S. J. Statistical Estimation by the Empirical Bayes Methods: Some Extensions and Logistical Applications. Research Memorandum No. RM-4442-PR. Santa Monica, California: RAND Corp., June 1965. AD617606.
164. Prior, R. J., and G. J. Schick. "Reliability and Confidence of Serially Connected Systems," in Canaveral Council of Technical Societies Third Space Congress. Los Angeles: University of Southern California, 1966, pp. 352-360. N66-36532.
165. Pugh, E. L. "The Bayesian Approach to Reliability - Confidence Relation for Exponential Failure." Operations Research, 8:721-724 (September-October 1960).
166. Raiffa, H. Decision Analysis, Introductory Lectures on Choices Under Uncertainty. Reading, Massachusetts: Addison-Wesley, 1968.
167. ----- "Bayesian Decision Theory," in Recent Developments in Information and Decision Processes, edited by R. E. Machol, and P. Gray. New York: MacMillan Co., 1962, pp. 92-101.
168. Raiffa, H., and R. Schlaifer. Applied Statistical Decision Theory. Boston: Graduate School of Business Administration, Harvard University, 1961.

169. Reed, A. C., et al. Proposed Military Standard for Reliability Demonstration Testing. Technical Report No. TOR-669(6303)-1. El Segundo, California: Aerospace Corporation, September 1965. AD474617.
170. Richmond, S. B. Operations Research for Management Decisions. New York: The Roland Press Co., 1968.
171. Robbins, H. "An Empirical Bayes Approach to Statistics." Proceedings of Third Berkeley Symposium on Mathematical Statistics and Probability, 1. Berkeley, California: University of California Press, 1956, pp. 157-163.
172. ----- "The Empirical Bayes Approach to Statistical Decision Problems." Annals of Mathematical Statistics, 35:1-20 (March 1964).
173. Robbins, H., and E. Samuel. "Testing Statistical Hypothesis - The 'Compound' Approach," in Recent Developments in Information and Decision Processes, edited by R. E. Machol, and P. Gray. New York: MacMillan Co., 1962, pp. 63-70.
174. Rutherford, J. R., and R. G. Krutchkoff. "The Empirical Bayes Approach: Estimating the Prior Distribution." Biometrika, 54: 326-328 (1967).
175. Salt, T. L., and M. H. Monahan. "Modification of Gas Turbine Component Life Predictions Utilizing Experience and the Discrete Formulation of Bayes Theorem," in Annals of Assurance Sciences: Proceedings of the Eighth Reliability and Maintainability Conference. New York: Gordon and Breach, Science Publishers, Inc., 1969, pp. 63-68.
176. Saunders, S. C. On Confidence Limits for the Performance of a System When Few Failures Are Encountered. Technical Report No. IDEP 347.40.00.00-C6-10. Seattle, Washington: Mathematics Research Laboratory, Boeing Company, November 1967. AD668983.
177. Savage, L. J. "The Foundations of Statistics Reconsidered," in Studies in Subjective Probability, edited by Kyburg, H. E. Jr., and H. E. Smokler. New York: John Wiley and Sons, 1964, pp.
178. ----- "Bayesian Statistics," in Recent Developments in Information and Decision Processes, edited by R. E. Machol, and P. Gray. New York: MacMillan Co., 1962, pp. 161-194.
179. Schafer, R. E. "Bayes Single Sampling Plans by Attributes Based on Posterior Risks." Naval Research Logistics Quarterly, 14: 81-88 (March 1967).

180. Schäfer, R. E. Development of Bayes Single Sampling Plans by Attributes. PhD Thesis. Cleveland, Ohio: Case Western Reserve University, 1968. Order No. 69-9371.
181. ----- "Note on the Uniform Prior Distribution for Reliability." IEEE Transactions on Reliability, R-19:76-77 (May 1970).
182. Schafer, R. E., et al. Bayesian Reliability Demonstration - Phase I: Data for the A Priori Distribution. Technical Report No. RADC-TR-69-359. Griffiss Air Force Base, New York: Rome Air Development Center, February 1970. AD866166.
183. Schafer, R. E., and N. D. Singpurwalla. "A Sequential Bayes Procedure for Reliability Demonstration," in Annals of Assurance Sciences; Proceedings of the Eighth Reliability and Maintainability Conference, Denver, Colorado, July 7-9 1969. New York: Gordon and Breach, Science Publishers, Inc., 1969, pp. 507-514.
184. Schafer, R. E. "Bayesian Operating Characteristic Curves for Reliability and Quality Sampling Plans," in Proceedings Tenth National Symposium on Reliability and Quality Control. New York: Institute of Electrical and Electronic Engineers, Inc., 1964, pp. 555-559.
185. Schick, G. J. "Bayesian Concepts for Reliability and Confidence," in Annals of Assurance Sciences: Proceedings, 1968 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1968, pp. 397-405. IEEE Catalog No. 68C 33-R.
186. Schulhof, R. J., and D. L. Lindstrom. "Application of Bayesian Statistics in Reliability." Proceedings, 1966 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, Inc., 1966, pp. 684-695. IEEE Catalog No. 7C26.
187. Selman, V. "Game Theory Applied to Reliability Problems," in Proceedings Eleventh National Symposium on Reliability & Quality Control. New York: The Institute of Electrical and Electronic Engineers, Inc., 1965, pp. 248-253.
188. Shooman, M. L. Probabilistic Reliability - An Engineering Approach. New York: McGraw-Hill Book Co., Inc., 1968.
189. Simkins, D. J. "Figure of Merit of Bayesian Analysis and Modeling," in Annals of Assurance Sciences; Proceedings, 1971 Annual Symposium on Reliability. New York: The Institute of Electrical and Electronic Engineers, 1971, pp. 394-397. IEEE Catalog No. 71 C2-R.

190. Simmons, E. D., and C. E. Wisler. A Decision Theory Approach to Acceptance Testing. Technical Memorandum No. NMC-TM-67-77. Point Mugu, California: Naval Missile Center, January 1968. AD665029.
191. Sineath, R. M., Jr., and R. A. Venditti. "Saturn V System Reliability Analysis," in Annals of Assurance Sciences; Proceedings 1969 Annual Symposium on Reliability. New York: Institute of Electrical and Electronic Engineers, Inc., 1969, pp. 567-571. IEEE Catalog No. 69C 8-R.
192. Smith, A. M. "Risk Assessment in Complex Unattended Aerospace Systems," in Annals of Reliability and Maintainability - 1967, Volume 6 - All Systems Go? New York: Society of Automotive Engineers, Inc., 1967, pp. 478-486.
193. Soland, R. M. "Bayesian Analysis of the Weibull Process with Unknown Scale and Shape Parameters." IEEE Transactions on Reliability, R-18:181-184 (November 1969).
194. ----- "Bayesian Analysis of the Weibull Process with Unknown Scale Parameter and Its Application to Acceptance Testing." IEEE Transactions on Reliability, R-17:84-90 (June 1968).
195. ----- Use of the Weibull Distribution in Bayesian Decision Theory. Technical Paper No. RAC-TP-225. McLean, Virginia: Advanced Research Department, Research Analysis Corporation, August 1966. AD668677.
196. ----- Bayesian Analysis of the Weibull Process with Unknown Scale Parameter. Technical Paper No. RAC-TP-215. McLean, Virginia: Advanced Research Department, Research Analysis Corporation, August 1966. AD643816.
197. Solomon, D. L. Partially Bayes Estimates. Technical Report No. FSU-M152. Tallahassee, Florida: Department of Statistics, Florida State University, January 1969. AD685595.
198. Spragins, J. "A Note on the Iterative Application of Bayes Rule." IEEE Transactions on Information Theory, IT-11:544-549 (October 1965).
199. Springer, M. D. "Bayesian Confidence Limits for Reliability of Redundant Systems When Tests Are Terminated at First Failure." Technometrics, 10:29-36 (February 1968).
200. Springer, M. D., and W. E. Thompson. "Bayesian Confidence Limits for System Reliability," in Annals of Assurance Sciences: Proceedings of the Eighth Reliability and Maintainability Conference. New York: Gordon and Breach, Science Publishers, Inc., 1969, pp. 515-523.

201. Springer, M. D., and W. E. Thompson. "Bayesian Confidence Limits for the Product of N Binomial Parameters." Biometrika, 53:611-613 (1966).
202. ----- "Bayesian Confidence Limits for the Reliability of Cascade Exponential Systems." IEEE Transactions on Reliability, 16:86-89 (September 1967).
203. Taylor, A. C. "A Bayesian Approach to Equipment Replacement." Industrial Management Review, 10:33-43 (Spring 1969).
204. Thatcher, A. R. "Relationships Between Bayesian and Confidence Limits for Predictions (with discussion)." Journal Royal Statistical Society (B), 26:176-210 (Number 2, 1964).
205. Tribus, M. Rational Descriptions, Decisions and Designs. New York: Pergamon Press, 1969.
206. ----- "The Use of the Maximum Entropy Estimate in the Estimation of Reliability," in Recent Developments in Information and Decision Process, edited by Machol, R. E., and P. Gray. New York: MacMillan Co., 1962, pp. 102-140.
207. Tribus, M., et al. "The Use of Entropy in Hypothesis Testing," in Proceedings of the Tenth National Symposium on Reliability & Quality Control. New York: The Institute of Electrical and Electronic Engineers, Inc., 1964, pp. 579-590.
208. Typaldos, A. A., and D. E. Brunley. Point Estimation of Reliability From Results of a Small Number of Trials. Research Memorandum No. RM-3044PR. Santa Monica, California: The RAND Corporation, May 1962. AD276150.
209. Varde, S. D. "Life Testing and Reliability Estimation for the Two Parameter Exponential Distribution." Journal of American Statistical Association, 64:621-631 (June 1969).
210. VonMises, R. "On the Correct Use of Bayes Formula." Annals Mathematical Statistics, 13:156-165 (1942).
211. Wallenius, K. T. "Sequential Reliability Assurance in Finite Lots." Technometrics, 11:61-74 (February 1969).
212. Walter, J. P. Bayesian Statistical Model Theory for Mechanical Systems. Technical Report No. LMEC-69-8. Canoga Park, California: Liquid Metals Engineering Center, Atomics International, 31 August 1969. N70-20430.
213. Weir, W. T. "Bayesian Reliability Evaluation?" in Annals of Assurance Sciences; Proceedings of the 7th Reliability and Maintainability Conference. New York: American Society of Mechanical Engineers, 1968, pp. 344-346.

214. Weir, W. T. Proceedings of the Second Missile and Space Division Seminar on Bayes' Theorem and Its Application to Reliability Measurement. Technical Report No. 65SD995. Philadelphia: Re-Entry Systems Department, General Electric Co., January 1966. AD481645.

215. ----- On Bayes Theorem and Its Application to Reliability Evaluation. Technical Report No. 67SD213. Philadelphia: Re-Entry Systems Department, General Electric Co., April 1967. N68-86487.

216. Wetherill, G. B. "Bayesian Sequential Analysis." Biometrika, 48:281-292 (1961).

217. ----- "Some Remarks on the Bayesian Solution of the Single sample Inspection Scheme." Technometrics, 2:341-352 (August 1960)

218. Williams, N. Compilation of Mathematical Techniques as Applied to Reliability Analysis. Technical Report No. IDEP-347.40.00.00-FL-60. Downey, California: North American Rockwell Corporation, February 1970. AD870602L.

219. Zacks, S. "Bayes Sequential Design of Stock Levels." Naval Research Logistics Quarterly, 16:143-155 (June 1969).

APPENDIX A

General Reference Bibliography

1. Agrawala, A. K. Learning With a Probabilistic Teacher. Technical Report No. TR-611. Cambridge, Massachusetts: Division of Engineering and Applied Physics, Harvard University, May 1970. AD708062.
2. Aigner, D. J. Principles of Statistical Decision Making. New York: The MacMillan Co., 1968.
3. Alens, M. Compound Bayes Learning Without a Teacher. PhD Thesis. Palo Alto, California: Stanford University, 1967. Order No. 68-6383.
4. Altman, P. M. E. "Exact Bayesian Analysis of a 2X2 Contingency Table and Fisher's 'Exact' Significance Test." Journal of Royal Statistical Society (B), 31:261-269 (Number 2, 1969).
5. Amster, S. J. "A Modified Bayes Stopping Rule." Annals Mathematical Statistics, 34:1404-1413 (December 1963).
6. Anderson, T. W. "On Bayes Procedures for a Problem with Choice of Observations." Annals Mathematical Statistics, 35:1128-1135 (September 1964).
7. Ando, A., and G. M. Kaufman. "Bayesian Analysis of the Independent Multi-Normal Process - Neither Mean nor Precision Known." Journal American Statistical Association, 60:347-358 (March 1965).
8. Anscombe, F. J., and R. J. Aumann. "Definition of Subjective Probability." Annals Mathematical Statistics, 34:199-205 (March 1963).
9. Anscombe, F. J. "Bayesian Inference Concerning Many Parameters With Reference to Supersaturated Designs." Bulletin of the International Statistical Institute, 40:721-733 (1963).
10. ----. "Tests of Goodness of Fit." Journal of the Royal Statistical Institute, B, 25:81-94 (Number 1, 1963).
11. ----. "Bayesian Statistics." The American Statistician, 15:21-24 (February 1961).
12. Antelman, G. R. Bayes Decision Theory: Insensitivity to Non-Optimal Design. Technical Report No. TR-16. Minneapolis, Minnesota: Minnesota University, October 1963. AD296171.

13. Arrow, K. J., et al. "Bayes and Minimax Solutions of Sequential Decision Problems." Econometrica, 17:213-244 (July-October 1949).
14. Balaban, H. "Bayesian Methods in Reliability and Life Testing." ASQC Electronics Division Newsletter, 2:3-8 (October 1968).
15. Barlow, R. E. Some Recent Developments in Reliability Theory. Technical Report No. ORC-68-19. Berkeley, California: Operations Research Center, University of California, July 1968. AD675034.
16. Barnard, G. A., et al. "Statistical Inference," in The Future of Statistics. New York: Academic Press, Inc., 1968, pp. 139-160.
17. Bartholomew, D. J., and E. E. Bassett. "A Comparison of Some Bayesian and Frequentist Inferences II." Biometrika, 53:262-264 (1966).
18. Basu, A. P. "Estimates of Reliability for Some Distributions Useful in Life Testing." Technometrics, 6:215-219 (May 1964).
19. Bather, J. A. "Bayes Procedures for Deciding the Sign of a Normal Mean." Proceedings of the Cambridge Philosophical Society, 58: 599-620 (1962).
20. Bennett, G. K. Smooth Empirical Bayes Estimation with Application to Weibull Distribution. Technical Report No. NASA-TM-X-58048. Houston, Texas: Manned Spacecraft Center, National Aeronautics and Space Administration, June 1970.
21. Bhattacharya, S. K. "Bayes Approach to Compound Distribution Arising From Truncated Mixing Densities." Annals Institute Statistical Mathematics, 20:375-381 (1968).
22. Bhattacharya, S. K., and M. S. Holla. "On a Discrete Distribution with Special Reference to the Theory of Accident Proneness." Journal of American Statistical Association, 60:1060-1066, (December 1965).
23. Bickel, P. J., and D. Blackwell. "A Note on Bayes Estimates." Annals Mathematical Statistics, 38:1907-1911 (December 1967).
24. Bickel, P. J., and J. A. Yahav. "Asymptotically Optimal Bayes and Minimax Procedures in Sequential Estimation." Annals Mathematical Statistics, 39:442-456 (April 1968).
25. Birnham, A. "Another View on the Foundations of Statistics." The American Statistician, 16:17-21 (February 1962).
26. Bland, R. P., and T. L. Bratcher. "A Bayesian Approach to the Problem of Ranking Binomial Probabilities." SIAM Journal on Applied Mathematics, 16:843-850 (July 1968).

27. Blöchl, D. A., and G. S. Watson. "A Bayesian Study of the Multi-Normal Distribution." Annals Mathematical Statistics, 38:1423-1435 (October 1967).
28. Blum, J. R., and J. Rosenblatt. "On Partial A Priori Information in Statistical Inference." Annals of Mathematical Statistics, 38:1671-1678 (1967).
29. Blumenthal, S. "Interval Estimation of the Normal Mean Subject to Restrictions, When the Variance is Known." Naval Research Logistics Quarterly, 17:485-505 (December 1970).
30. Bohrer, R. E. On Bayes Sequential Design of Experiments. PhD Thesis. Chapel Hill, North Carolina: North Carolina University, 1966. Order No. 67-965.
31. Bond, N. A., Jr., and J. W. Rigney. "Bayesian Aspects of Troubleshooting Behavior." Human Factors, 8:377-383 (October 1966).
32. Borch, K. H. The Economics of Uncertainty. Princeton, New Jersey: Princeton University Press, 1968.
33. Bowen, R. R. "Bayesian Decision Procedure for Interfering Digital Signals." IEEE Transactions on Information Theory, IT-15:506-507 (July 1969). AD698400.
34. Box, G. E. P. Bayesian Approaches to Some Bothering Problems in Data Analysis. Technical Report No. TR-56. Madison, Wisconsin: Department of Statistics, University of Wisconsin, October 1965. AD626853.
35. Box, G. E. P., and N. R. Draper. "The Bayesian Estimation of Common Parameters from Several Processes." Biometrika, 52:355-365 (1965).
36. Box, G. E. P., and T. L. Henson. Model Fitting and Discrimination. Technical Report No. 211. Madison, Wisconsin: Department of Statistics, University of Wisconsin, July 1969.
37. Box, G. E. P., and G. C. Tiao. Statistical Analysis and Design of Experiments. Scientific Report No. AFOSR-69-2735TR. Boston: Graduate School of Business Administration, Harvard University, October 1969. AD696237.
38. ----- "Bayesian Estimation of Means for the Random-Effect Model." Journal American Statistical Association, 63:173-181 (March 1968).
39. ----- "Bayesian Analysis of a Three-Component Hierarchical Design Model." Biometrika, 54:109-125 (1967).

40. -----, "A Bayesian Approach to the Importance of Assumptions Applied to the Comparison of Variance." Biometrika, 51:153-167 (1964).
41. -----, "A Further Look at Robustness Via Bayes Theorem." Biometrika, 49:419-433 (1962).
42. Braga-Illa, A. "A Simple Approach to the Bayes Choice Criterion: The Method of Extreme Probabilities." Journal American Statistical Association, 59:1227-1230 (December 1964).
43. Brick, D. B., W. N. Furey, and E. G. Henrichson, Jr. Research in Adaptive Pattern Recognition. Technical Report No. Rept 100-30. Burlington, Massachusetts: Infotcn, Inc., 28 August 1970. AD711675.
44. Brown, M. B. "The Two-Means Problem - A Secondary Bayes Approach." Biometrika, 54:85-91 (1967).
45. Brown, R. V. Research & The Credibility of Estimates: An Appraisal Tool for Executives & Researchers. Boston: Harvard Business School, 1969.
46. Canner, P. L. "Selecting One of Two Treatments When the Responses are Dichotomus." Journal of American Statistical Association, 65:293-306 (March 1970).
47. Caplen, R. A Practical Approach to Quality Control. London: Business Books, 1969.
48. Chance, N. A. Statistical Methods for Decision Making. Homewood, Illinois: R. D. Irwin, 1969.
49. Chao, M. T. "Asymptotic Behavior of Bayes' Estimators." Annals of Mathematical Statistics, 41:601 (April 1970).
50. Christensen, C. S. "An Algorithm for Telemetry Decommuation Using Bayesian Decisions." Proceedings 3rd International Conference on System Sciences Part 2. Edited by B. S. M. Granborg. North Hollywood, California: Wester Periodicals Co., 1970, pp. 822-824.
51. Christopher, P. F. "Bayes Approach to Frequency Optimization for Satellite Communication." Proceedings of the IEEE, 56:2186-2187 (December 1968).
52. Clemmer, B. A., and R. G. Krutchkoff. "The Use of Empirical Bayes Estimators in a Linear Regression Model." Biometrika, 55:525-534 (1968).

53. Cornell, R. G. Bayesian Estimation of Population Size with Removal Sampling. Technical Report No. ONR TR-43. Tallahassee, Florida: Department of Statistics, Florida State University, January 1970. AD712029.
54. Cornfield, J. "The Bayesian Outlook and Its Application (French Summary)." Biometrika, 25:617-642 (Number 4, 1969).
55. ----- "A Bayesian Test of Some Classical Hypotheses with Applications to Sequential Clinical Trials." Journal American Statistical Association, 61:577-594 (September 1966).
56. Dear, R. E. Bayes Estimation for Some Stimulus Sampling Models. Technical Memorandum No. TM-1734/002/00. Santa Monica, California: Systems Development Corporation, February 1965. AD615119.
57. DeFinetti, B. "The Bayesian Approach to the Rejection of Outliers," in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability I. Berkeley, California: University of California Press, 1961, pp. 199-210.
58. DeGroot, M. H., and M. M. Rao. "Bayes Estimation with Convex Loss." Annals Mathematical Statistics, 34:839-846 (September 1963).
59. Dickey, J. M. "A Bayesian Hypothesis - Decision Procedure." Annals Institute of Statistical Mathematics, 19:367-369 (1967).
60. Dickey, J. M., and B. P. Lientz. Exact Bayesian Tests of Sharp Hypotheses. Technical Report No. SDC-SP-3236. Santa Monica, California: Systems Development Corporation, October 1968. AD680109.
61. Donaghue, P. J. System Identification by Bayesian Learning. PhD Thesis. East Lansing, Michigan: Michigan State University, 1968. Order No. 68-17074.
62. Draper, N. R., and I. Guttman. "Some Bayesian Stratified Two-Phase Sampling Results," in Biometrika, 55:131-139 (1968).
63. ----- "Bayesian Stratified Two-Phase Sampling Results: K Characteristics." Biometrika, 55:587-589 (1968).
64. ----- Transformation of Life Test Data. Technical Report No. TR-60. Madison, Wisconsin: Department of Statistics, University of Wisconsin, December 1965. AD482000.
65. Dreze, J. H. "Decision Theory and Bayes Statistics (Entscheidungs Theorie und "Bayessche Statistik")." Jahrbuch fur National okonomie und Statistik, 182:216-223 (December 1968).
66. Duncan, D. B. "A Bayesian Approach to Multiple Comparisons." Technometrics, 7:171-222 (May 1965).

67. Duncan, D. B. "Bayes Rules for a Common Multiple Comparisons Problem and Related Student-t Problems." Annals Mathematical Statistics, 32:1013-1033 (December 1961).
68. Dunsmore, I. R. "Regulation and Optimization." Journal of Royal Statistical Society (B), 31:160-170 (Number 1, 1969).
69. Easterling, R. G. Sample Size Determination for Bayesian Tolerance Intervals. PhD Thesis. Stillwater, Oklahoma: Oklahoma State University, 1967. Order No. 68-6395.
70. Edwards, W., et al. "Bayesian Statistical Inference for Psychological Research." Psychology Review, 70:193-242 (1963).
71. Ehrenfeld, S. "Some Experimental Design Problems in Attribute Life Testing." Journal of the American Statistical Association, 57:668-679 (September 1962).
72. Emory, W., and P. N. Land. Making Management Decisions. Boston: Houghton Mifflin, 1968.
73. Epstein, B., and A. Schiff. Improving Availability and Readiness of Field Equipment Through Periodic Inspection. Technical Report No. UCRL 50451. Livermore, California: Lawrence Radiation Laboratory, University of California, July 1968.
74. Evans, I. G. "Bayesian Estimation of Parameters of a Multivariate Normal Distribution." Journal Royal Statistical Society (B), 27: 279-283 (Number 2, 1965).
75. ----- "Bayesian Estimation of the Variance of a Normal Distribution." Journal Royal Statistical Society (B), 26:63-68 (Number 1, 1964).
76. Fabius, J. "Asymptotic Behavior of Bayes Estimates." Annals Mathematical Statistics, 35:846-856 (June 1964).
77. Farrell, E. J. "Improving the Reliability of Digital Devices with Redundancy: An Application of Decision Theory." IRE Transactions on Reliability and Quality Control, 11:44-50 (May 1962).
78. Farrell, R. H. "On Bayes Characters of a Standard Model II Analysis of Variance Test." Annals of Mathematical Statistics, 40:1094-1097 (June 1969).
79. ----- "Weak Limits of Sequences of Bayes Procedures in Estimation Theory," in Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, 1. Berkeley, California: University of California Press, 1967, pp. 83-112.

80. Fertig, K. W. A Result Concerning Bayesian Prior Distributions and Confidence Bounds on the Reliability of Serial Systems with Exponential Failure Times. Research Report No. RR69-6. Canoga Park, California: Rocketdyne, 1969.
81. Fishburn, P. C. Utility Theory for Decision Making. New York: John Wiley and Sons, Inc., 1970.
82. Fox, R. J. Contributions to Compound Decision Theory and Empirical Bayes Squared Error Loss Estimation. Technical Report No. SLP-13 RM-214; RJF-1. East Lansing, Michigan: Statistical Laboratory, Michigan State University, September 1968.
83. Freedman, D. A. "On the Asymptotic Behavior of Bayes Estimates in the Discrete Case, II," in Annals Mathematical Statistics, 38: 1281-1283 (August 1967).
84. ----- "On the Asymptotic Behavior of Bayes Estimates in the Discrete Case," in Annals Mathematical Statistics, 34:1356-1403 (December 1963).
85. Fukunaga, K., and T. F. Krile. "Calculation of Bayes' Recognition Error for Two Multivariate Gaussian Distributions." IEEE Transactions on Computers, C-18:220-229 (March 1969).
86. Garner, J. B. Robust Bayesian Inference for Hypergeometric Parameter. Technical Report No. RR-61/JEG-8. Sheffield, England: Department of Probability and Statistics, Sheffield University, April 1969.
87. ----- A Bayesian Test for Equality of Medians. Technical Report No. RR-59/JEG-6. Sheffield, England: Department of Probability and Statistics, Sheffield University, April 1969.
88. ----- The Product of Independent Binomial Parameters. Technical Report No. RR-53/JEG-4. Sheffield, England: Department of Probability and Statistics, Sheffield University, December 1968.
89. Geisser, S. "A Bayes Approach for Combining Correlated Estimates." Journal American Statistical Association, 60:602-607 (June 1965).
90. ----- "Bayesian Estimation in Multivariate Analysis." Annals Mathematical Statistics, 36:150-159 (February 1965).
91. Gershick, M. A., and L. J. Savage. "Bayes and Minimax Estimates for Quadratic Loss Functions." Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. Berkeley, California: University of California Press, 1961, pp. 53-74.
92. Ghosh, J. K. "Bayes Solutions in Sequential Problems for Two or More Terminal Decisions and Related Results." Calcutta Statistical Association Bulletin, 13:101-122 (1964).

93. Gillis, F. E. Managerial-Economics; Decision Making Under Certainty for Business and Engineering. Reading, Massachusetts: Addison-Wesley Publishing Co., 1969.
94. Glier, R. Two Papers on Adaptive Inventory Control. Palo Alto, California: Decision Studies Group, January 1968. AD666574.
95. Gnedenko, B. V., et al. Mathematical Methods of Reliability Theory. New York: Academic Press, 1969.
96. Goldman, T. A. Bayesian Supply Policies for Service-Life Parts. Interim Research Memo IRM-36. Washington D.C.: Operations Evaluation Group, Center for Naval Analysis, March 1963. AD405106.
97. Good, I. J. "A Bayesian Significance Test for Multinomial Distributions (with Discussion)." Journal Royal Statistical Society (B), 29:399-431 (Number 3, 1967).
98. Gottfried, P. "Product Risks: Prediction Techniques," in Proceedings of the 1970 Product Liability Prevention Conference. Milwaukee, Wisconsin: American Society for Quality Control, Inc., 1970, pp. 3-14.
99. Greenberg, S. A., and I. Bosinoff. Optimum Decision Criteria for Reliability Tests. Bedford, Massachusetts: The MITRE Corporation, May 1967.
100. Greenwood, W. T. Decision Theory and Information Systems; An Introduction to Management Decision Making. Cincinnati, Ohio: Southwestern Publishing Co., 1969.
101. Guttman, I., and G. C. Tiao. "A Bayesian Approach to Some Best Population Problems." Annals Mathematical Statistics, 35:825-835 (June 1964).
102. Hadley, G. Introduction to Probability and Statistical Decision Theory. San Francisco: Holden-Day, 1967.
103. Hajek, J. Limiting Properties of Likelihoods and Inference. Technical Report No. ONR IR-42. Tallahassee, Florida: Department of Statistics, Florida State University, January 1970. AD712028.
104. Hald, A. "Bayesian Single Sampling Attribute Plans for Continuous Prior Distributions." Technometrics, 10:667-683 (November 1968).
105. ----- "The Mixed Binomial Distribution and the Posterior of P for a Continuous Prior Distribution." Journal of the Royal Statistical Society, B, 30:359-367 (Number 2, 1968).
106. ----- "Asymptotic Properties of Bayesian Single Sampling Plans." Journal Royal Statistical Society (B), 29:162-173 and 586 (Numbers 1 & 3, 1967).

107. HaId, A. Attribute Sampling Plans Based on Prior Distributions and Costs. Copenhagen, Denmark: Copenhagen University, May 1966. AD635451.
108. ----- Single Sampling Inspection Plans with Specified Acceptance Probability and Minimum Average Costs. Technical Report No. 6. Copenhagen, Denmark: Copenhagen University, December 1964. AD609993.
109. ----- "Efficiency of Sampling Inspection Plans for Attributes." Bulletin of the International Statistical Institute, 40:681-697 (1964).
110. ----- Bayesian Single Sampling Attribute Plans for Discrete Prior Distribution. Technical Report No. 5. Copenhagen, Denmark: Copenhagen University, June 1964. AD602396.
111. ----- "The Compound Hypergeometric Distribution and A System of Single Sampling Inspection Plans Based on Prior Distributions and Costs." Technometrics, 2:275-339 (August 1960).
112. Hamburg, M. "Bayesian Decision Theory and Statistical Quality Control," in Transactions 15th Annual Convention American Society for Quality Control, 1961, pp. 181-190.
113. Harris, C. M., and N. D. Singpurwalla. "On Estimation in Weibull Distributions with Random Scale Parameters." Naval Research Logistics Quarterly, 16:405-410 (September 1969).
114. ----- Life Distributions Derived from Stochastic Hazard Functions. Technical Paper No. RAC-TP-280. McLean, Virginia: Research Analysis Corporation, October 1967. AD664142.
115. Hartigan, J. "Invariant Prior Distributions." Annals Mathematical Statistics, 35:836-845. (June 1964).
116. Hartley, H. O., and J. N. Rao. A New Estimation Theory for Sample Surveys. Technical Report No. TR-1. College Station, Texas: Texas A and M University, 1968. AD698468.
117. ----- A New Estimation Theory for Sample Surveys, II. Technical Report No. TR-2. College Station, Texas: Texas A and M University, 1969. AD698484.
118. Hildreth, C. G., and J. Y. Lu. A Monte Carlo Study of the Regression Model with Autocorrelated Disturbances. Research Memorandum RM-5728-PR. Santa Monica, California: The RAND Corporation, April 1969. AD686729.
119. Hirshleifer, J. "The Bayesian Approach to Statistical Decision, An Exposition." Journal of Business, 34:471-489 (October 1961).

120. Ho, Y. C., and R. C. K. Lee. "A Bayesian to Problems in Stochastic Estimation and Control," in Preprints of Papers, 1964 Joint Automatic Control Conference. New York: The Institute of Electrical and Electronic Engineers, Inc., 1964, pp. 382-387.
121. Hoadley, B. "A Bayesian Look at Inverse Linear Regression." Journal of American Statistical Association, 65:356-369 (March 1970).
122. Hoadley, A. B. "The Compound Multinomial Distribution and Bayesian Analysis of Categorical Data from Finite Populations." Journal of the American Statistical Association, 64:216-229 (March 1969).
123. Hodges, J. L., Jr., and E. L. Lehman. "The Use of Previous Experience in Reaching Statistical Decisions." Annals Mathematical Statistics, 23:396-407 (1952).
124. Holla, M. S. "Discrete Distributions with Prior Information." Annals of the Institute of Statistical Mathematics, 20:151-157 (1968).
125. ----- "Life Estimation by Bayesian Method." Calcutta Statistical Association Bulletin, 15:158-164 (December 1966).
126. Holland, J. D. "The Reverend Thomas Bayes, F. R. S. (1702-61)." Journal Royal Statistical Society (A), 125:451-461 (1962).
127. Huntsberger, D. V. Elements of Statistical Inference. Boston: Allyn and Bacon, Inc., 1969.
128. Husa, G. W., and A. P. Sage. Adaptive Bayes Filtering with Unknown Prior Statistics. Technical Report No. AFOSR-70-0608TR. Dallas, Texas: Information and Control Sciences Center, Southern Methodist University, April 1969. AD705247.
129. Hymans, S. H. Probability Theory with Applications to Econometrics and Decision Making. Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1967.
130. Jaarsma, D. The Theory of Signal Detectability: Bayesian Philosophy, Classical Statistics, and the Composite Hypothesis. Technical Report No. Rept-D3674-22-T. Ann Arbor, Michigan: Cooley Electronics Laboratory, Michigan University, February 1970. AD703327.
131. Jaynes, E. T. "New Engineering Applications of Information Theory," in Proceedings, First Symposium on Engineering Applications of Random Function and Probability, edited by Bogdanoff, J. and F. Kozin. Lafayette, Indiana: Purdue University Press, 1962.
132. Jazwinski, A. H. Stochastic Processes and Filtering Theory. New York: Academic Press, 1970.

133. Johns, M. V., Jr. "Non-Parametric Empirical Bayes Procedures." Annals of Mathematical Statistics, 28:649-669 (September 1957).
134. Kepner, C. H., and B. B. Tregoe. The Rational Manager; A Systematic Approach to Problems Solving and Decision Making. New York: McGraw-Hill Book Co., Inc., 1965.
135. Kiefer, J., and R. Schwartz. "Admissible Bayes Character of T^2 , R^2 and Other Fully Invariant Tests for Classical Multivariate Normal Problems." Annals Mathematical Statistics, 36:747-770 (June 1965).
136. Klotz, J. M., et al. "Mean Square Efficiency Estimators of Variance Components." Journal of the American Statistical Association, 64:1383-1402 (December 1969).
137. Krutchkoff, R. G. "A Supplementary Sample Nonparametric Empirical Bayes Approach to Some Statistical Decision Problems." Biometrika, 54:451-458 (1967).
138. Lapinski, A. Application of Reliability Acceptance Test Criteria Nomograph. Technical Report No. IDEP-347.40.00.00-K4-21. Bethpage, New York: Grumman Aircraft Engineering Corporation, October 1969. AD870588L.
139. Lasky, P. H. New Concepts of Flight Testing for Reliability. Technical Note No. SED-TN-410. Menlo Park, California: Systems Evaluation Department, Stanford Research Institute, September 1970. AD876077L.
140. Lee, T. The Specification and Sampling Properties of Classical and Bayesian Transition Probability Estimators. PhD Thesis. Urbana, Illinois: Illinois University, 1967. Order No. 68-8145.
141. Lieberman, G. J. "The Status and Impact of Reliability Methodology." Naval Research Logistics Quarterly, 16:17-35 (March 1969).
142. Lin, T., and S. S. Yau. "Bayesian Approach to the Optimization of Adaptive Systems." IEEE Transactions on Systems, Science and Cybernetics, SSC-3:77-85 (November 1967).
143. Lindley, D. V., and G. M. El-Sayyad. "The Bayesian Estimation of a Linear Functional Relationship." Journal of the Royal Statistical Society, B, 30:190-202 (Number 1, 1968).
144. ----- "The Bayesian Analysis of Contingency Tables." Annals Mathematical Statistics, 35:1622-1643 (December 1964).
145. ----- "Fiducial Distributions and Bayes' Theorem." Journal of the Royal Statistical Society, B, 20:102-107 (Number 1, 1958).

146. Loechner, R. H., and A. P. Basu. Bayesian Analysis of Time Truncated Samples. Technical Report No. TR 201. Madison, Wisconsin: Statistics Department, University of Wisconsin, May 1969.
147. Locks, M. O. The Decision Theory Approach to Complex Experimentation. Canoga Park, California: Rocketdyne, April 1964. AD809152.
148. Lorden, G. "Integrated Risk of Asymptotically Bayes Sequential Tests." Annals Mathematical Statistics, 38:1399-1422 (October 1967).
149. Luckie, P. T., et al. Investigation of a Bayesian Approach Applied to a Specific Intelligence Problem. Technical Report No. 401511-R-3. State College, Pennsylvania: HRB-Singer, Inc., June 1968. AD839284L.
150. Marciano, J. P., and J. Voranger. "On the Optimization of the Size of a Sample (Sur l'optimisation de la taille d'un echantillonnage)." Revue Francaise d'Informatique et de Recherche Operationelle, 3:107-112 (1969).
151. Maritz, J. S. Empirical Bayes Methods. New York: Barnes & Noble, Inc., 1970.
152. ----- "On the Smooth Empirical Bayes Approach to Testing of Hypothesis and the Compound Decision Problem." Biometrika, 55: 83-100 (1968).
153. ----- "Smooth Empirical Bayes Estimation for Continuous Distributions." Biometrika, 54:435-450 (1967).
154. ----- "Smooth Empirical Bayes Estimation for One-Parameter Discrete Distributions." Biometrika, 53:417-429 (1966).
155. Martel, R. J. Optimum Bayes Strategies in Truncated Life Testing. Paper Presented at 27th National Meeting of the Operations Research Society of America, Boston, 6-7 May 1965.
156. Martin, D. W., and C. F. Gettys. "Feedback and Response Mode in Performing a Bayesian Decision Task." Journal of Applied Psychology, 53:413-418 (October 1969).
157. Mastran, D. V., et al. Monte Carlo Approach to Reliability Statements Based on Attribute Data. Kelly Air Force Base, Texas: Directorate of Special Weapons, San Antonio Air Material Area, 1967.
158. Mazumdar, M. Optimal Sequential Plans Based on Prior Distributions and Costs. Technical Report No. TR-3 (PhD Thesis). Ithaca, New York: Department of Industrial Engineering and Operations Research, Cornell University, April 1966. AD634341.

159. McBride, A. L., and A. P. Sage. "Optimum Estimation of Bit Synchronization." IEEE Transactions on Aerospace and Electronic Systems, AES-5:525-536 (May 1969).
160. McGlothlin, W. H. Development of Bayesian Parameters for Spare Parts Demand Prediction. Research Memorandum RM-3699-PR. Santa Monica, California: The RAND Corporation, July 1963. AD411356.
161. McGlothlin, W. H., and Radner. The Use of Bayesian Techniques for Predicting Spare Parts Demand. Research Memorandum RM-2536. Santa Monica, California: The RAND Corporation, March 1960.
162. McLendon, J. R., and A. P. Sage. "Computational Algorithms for a Pseudo-Bayes Digital Radar Receiver." IEEE Transactions on Aerospace and Electronic Systems, AES-6:815-820 (November 1970).
163. ----- A Pseudo Bayes Approach to Digital Detection and Likelihood Ratio Computation. Technical Report No. AFOSR-70-06C4TR. Dallas, Texas: Information and Control Sciences Center, Southern Methodist University, December 1969. AD703715.
164. McNolty, F. "Reliability Density Functions When the Failure Rate is Randomly Distributed." Sankhya, A, 26:287-292 (1964).
165. Middleton, D. Bayes Ambiguity Functions: Some Simple Applications to Resolution and Radar Countermeasures. Technical Note No. TM-1969-16. Lexington, Massachusetts: Lincoln Laboratory, Massachusetts Institute of Technology, February 1969. AD686420.
166. Mikkalevich, V. S. "Sequential Bayes Solutions and Optimum Methods of Statistical Acceptance Control." Theory Probability Applications, 1: 395-421 (Number 4, 1956).
167. Miller, J. R., III. Professional Decision-Making: A Procedure for Evaluating Complex Alternatives. New York: Praeger Publishers, 1970.
168. Miyasawa, K. "An Empirical Bayes Estimator of the Mean of a Normal Population." Bulletin of the International Statistical Institute, 38:181-188 (1961).
169. Mood, A. M., and F. A. Graybill. Introduction to the Theory of Statistics (2nd edition). New York: McGraw Hill Book Co., Inc., 1963.
170. Moriguti, S., and H. Robbins. "A Bayes Test of $p = \frac{1}{2}$ versus $p = \frac{1}{2}$." Report of Statistical Applications Research Union of Japanese Scientist and Engineers, 9:39-60 (1962).
171. Morris, C. Admissible Bayes Procedures and Classes of Epsilon Bayes Procedures for Testing Hypotheses in a Multinomial Distribution. Technical Report No. IR-55. Palo Alto, California: Department of Statistics, Stanford University, August 1955.

172. Morrison, N. Introduction to Sequential Smoothing and Prediction. New York: McGraw-Hill, Inc., 1959.
173. Nahi, N. E., and R. M. Gagliardi. "Threshold Determination in Sequential Detection with Fixed Error Rates." Proceedings International Conference on Communications, Volume I. New York: The Institute of Electrical and Electronic Engineers, Inc., 1970, pp. 18-26 to 18-27.
174. Naresky, J. J. "Reliability and Maintainability Research in the United States Air Force," in Annals of Reliability and Maintainability, Volume 5, Achieving Systems Effectiveness. New York: American Institute of Aeronautics and Astronautics, Inc., 1966, pp. 769-787.
175. Neyman, J. "Two Breakthroughs in the Theory of Statistical Decision Making." Review of the International Statistical Institute, 30:11-27 (1962).
176. Novick, M. R. "Multiparameter Bayesian Indifference Procedures." Journal of the Royal Statistical Society (B), 31:29-64 (Number 2, 1969).
177. Novick, M. R., and J. E. Grizzle. "A Bayesian Approach to the Analysis of Data from Clinical Trials." Journal American Statistical Association, 60:81-96 (March 1965).
178. Orth, P. J. Reliability Demonstration and Evaluation Techniques. San Bernardino, California: Technology Division, Aerospace Corporation, November 1965.
179. Overall, J. E. "Classical Statistical Hypothesis Testing Within the Context of Bayesian Theory." Psychological Bulletin, 71:285-292 (April 1969).
180. Patrick, E. A., et al. Computer Analysis and Classification of Waveforms and Pictures. Part I: Waveforms. Technical Report No. RADC-TR-69-279. Lafayette, Indiana: School of Electrical Engineering, Purdue University, September 1969. AD695835.
181. Petrasovits, A., and R. G. Cornell. Approximations to the Bayes Estimate for a Quantal Assay with Simple Exponential Tolerance Distribution. Technical Report No. NASA-CR113-62; TR-21. Tallahassee, Florida: Department of Statistics, Florida State University, September 1970. N70-41241.
182. Petterson, C. R., and R. G. Swensson. "Intuitive Statistical Inferences About Diffuse Hypotheses." Organizational Behavior and Human Performance, 3:1-11 (February 1965).
183. Pettit, R. H. "A Bayes Estimator in a Decision-Directed Adaptive Detection Problem." Frequency Technology, 7:3-21 (January 1969).

184. Phi Delta Kappa Symposium on Educational Research - Ninth. Bayesian Statistics. Ithaca, Illinois: Peacock, F. T., Publishers, Inc., 1970.
185. Phillips, H. E. Sensitivity Analysis on the Multiple Action Problem. PhD Thesis. Seattle, Washington: Washington University, 1968. Order No. 69-7075.
186. Pitt, J. M., and B. F. Womack. "Additional Features of an Adaptive, Multicategory, Pattern Classification System." IEEE Transactions on Systems Science and Cybernetics, SSC-5:183-191 (July 1969).
187. Pollock, S. M. A Bayesian Reliability Growth Model. Technical Report No. NPS-TR/RP-50. Monterey, California: U.S. Naval Postgraduate School, June 1967. AD663279.
188. Polovko, A. M. Fundamentals of Reliability Theory. New York: Academic Press, 1968.
189. Portnoy, S. L. Formal Bayes Estimation with Application to a Random Effects Analysis of Variance Model. Technical Report No. TR-142A Revised. Stanford, California: Department of Statistics, Stanford University, February 1969. AD689228.
190. Pratt, J. W. "Bayesian Interpretation of Standard Inference Statements (with Discussion)." Journal Royal Statistical Society (B), 27:169-203 (Number 2, 1965).
191. Pugh, E. L. "The Best Estimate of Reliability in the Exponential Case." Operations Research, 11:57-61 (January-February 1963).
192. Purves, R. A., and D. A. Freedman. "Bayes' Method for Bookies." Annals of Mathematical Statistics, 40:1177-1186 (August 1969).
193. Raj, D. Sampling Theory. New York: McGraw-Hill, Inc., 1968.
194. Rappaport, A. Information for Decision Making; Quantitative and Behavioral Dimensions. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1970.
195. ----- "Sequential Decision Making in a Computer Controlled Task." Journal of Mathematical Psychology, 1:351-374 (July 1964).
196. Ray, S. N. "Bounds on the Maximum Sample Size of a Bayes Sequential Procedure." Annals Mathematical Statistics, 36:859-878 (June 1965).
197. ----- Some Sequential Bayes Procedures for Comparing Two Binomial Parameters When Observations Are Taken in Pairs. Chapel Hill, North Carolina: Department of Statistics, University of North Carolina, 1963.

198. Rencher, A. C. The Empirical Bayes Approach to Analysis of Variance and Linear Regression. PhD Thesis. Blacksburg, Virginia: Virginia Polytechnic Institute, 1968. Order No. 69-4600.
199. Roberts, H. V. "Probabilistic Prediction." Journal of American Statistical Association, 60:50-62 (March 1965).
200. Rolph, J. E. "Bayesian Estimation of Mixing Distributions." Annals of Mathematical Statistics, 39:1289-1302 (August 1968).
201. Rosenblatt, J. R. "Confidence Limits for the Reliability of Complex Systems," in Statistical Theory of Reliability, edited Zelen, M. Madison, Wisconsin: University of Wisconsin Press, 1963, pp. 115-137.
202. Rubin, H. Decision-Theoretic Evaluation of Some Non-Parametric Methods. Technical Report No. SER-193. Lafayette, Indiana: Department of Statistics, Purdue University, July 1969. AD691801.
203. ----- Decision-Theoretic Approach to Some Multivariate Problems. Technical Report No. SER-167. Lafayette, Indiana: Department of Statistics, Purdue University, August 1968. AD674524.
204. Rutherford, J. R. "An Empirical Bayes Approach to the Non-Central t Distribution," in Proceedings of the Symposium on Empirical Bayes Estimation and Computing in Statistics. Lubbock, Texas: Texas Tech. Press, 1970, pp. 61-70.
205. Rutherford, J. R., and R. G. Krutchkoff. "Some Empirical Bayes Techniques in Point Estimation." Biometrika, 56:133-137 (1969).
206. Sacks, J. "Generalized Bayes Solutions in Estimation Problems." Annals Mathematical Statistics, 34:751-768 (September 1963).
207. Samuel, E. "An Empirical Bayes Approach to the Testing of Certain Parametric Hypotheses." Annals Mathematical Statistics, 34:1370-1385 (December 1963).
208. Sasaki, K. Statistics for Modern Business Decision Making. Belmont, California: Wadsworth Publishing Co., 1968.
209. Savage, L. J. The Foundations of Statistics. New York: John Wiley and Sons, Inc., 1954.
210. Scarf, H. "Bayes Solutions of the Statistical Inventory Problem." Annals Mathematical Statistics, 30:490-508 (June 1959).
211. Schlaifer, R. Analysis of Decisions Under Uncertainty. New York: McGraw-Hill Book Co., Inc., 1969.
212. Schum, D. A. Concerning the Simulation of Diagnostic Systems Which Process Complex Probabilistic Evidence Sets. Technical Report No. AMRL TR-69-10. Columbus, Ohio: Human Performance Center, Ohio State University, April 1969. AD691238.

213. Schum, D. A., et al. Aided Human Processing of Inconclusive Evidence in Diagnostic Systems; A Summary of Experimental Evaluation. Technical Report No. AMRL TR-69-11. Columbus, Ohio: Human Performance Center, Ohio State University, May 1969. AD691239.
214. ----- "Subjective Probability Revisions Under Several Cost-Payoff Arrangements." Organizational Behavior and Human Performance, 2: 84-104 (February 1967).
215. ----- "Research on a Simulated Bayesian Information Processing System." IEEE Transactions on Human Factors in Electronics, HFE-7: 37-48 (March 1966).
216. Schmitt, S. A. Measuring Uncertainty: An Elementary Introduction to Bayesian Statistics. Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1969.
217. Schwartz, L. "On Bayes Procedures." Zeitschrift fur Wahrscheinlichkeitstheorie und Verwandte Gebiete, 4:10-26 (1965).
218. Schwartz, G. "Asymptotic Shapes of Bayes Sequential Testing Regions." Annals Mathematical Statistics, 33:224-236 (March 1962).
219. Schwartz, R. E. "Invariant Proper Bayes Tests for Exponential Families." Annals of Mathematical Statistics, 40:270-283 (February 1969).
220. Sclove, S. L. Remarks on the Problem of Homogenization of Bernoulli Trials. Technical Report No. TR-137. Stanford, California: Department of Statistics, Stanford University, June 1968. AD673676.
221. Scott, A. "A Multi-Stage Test for a Normal Mean." Journal of Royal Statistical Society (B), 30:461-468 (Number 3, 1968).
222. Sexsmith, R. G. Structural Decisions for Consistent Reliability Allocation. Paper Presented at Annual General Meeting of the Engineering Institute of Canada. Vancouver, British Columbia, 9-13 September 1969.
223. Shah, H. C. "Reliability Estimates from Run-Out or Non-Failure Data Using Information Theory," in Annual Technical Conference Transactions. Milwaukee, Wisconsin: American Society for Quality Control, Inc., 1968, pp. 425-434.
224. Sheridan, T. B. "On How Often the Supervisor Should Sample." IEEE Transactions on Systems Science and Cybernetics, SSC-6:140-145 (April 1970).
225. Skibinsky, M. "Some Properties of a Class of Bayes Two-Stage Tests." Annals Mathematical Statistics, 31:332-351 (June 1960).

226. Smith, C. A. B. "Personal Probability and Statistical Analysis." Journal of the Royal Statistical Society, A, 128:469-499 (Part 4, 1965).
227. ----- "Consistency in Statistical Inference and Decision." Journal of the Royal Statistical Society, B, 23:1-25 (Number 1, 1961).
228. Smith, E. B. A Bayesian Procedure for Detection of Slippage. Technical Report No. TR-8. Pittsburgh, Pennsylvania: Carnegie Institute of Technology, February 1963. AD405026.
229. Smith, G. L. On the Theory and Methods of Statistical Inference. Technical Report No. NASA-TR-251. Moffett Field, California: Ames Research Center, April 1967. N67-22039.
230. Smith, M. W. An Optimum Discrete Space Sequential Search Procedure Which Considers False Alarms and False Dismissal Instrument Errors. Technical Report No. THEMIS-SMU-TR-35. Dallas, Texas: Department of Statistics, Southern Methodist University, May 1969. AD688410.
231. Smith, M. W., and J. E. Walsh. Optimum Sequential Search with Discrete Locations and Random Acceptance Errors. Technical Report No. TR-44. Dallas, Texas: Department of Statistics, Southern Methodist University, August 1969. AD694441.
232. Soland, R. M. Bayesian Analysis of the Weibull Process with Unknown Scale and Shape Parameters. Technical Paper No. RAC-TP-359. McLean, Virginia: Advanced Research Department, Research Analysis Corporation, April 1969. AD687289.
233. ----- Optimal Bayesian Stratified Sampling by Nonlinear Programming. Technical Paper No. RAC-TP-221. McLean, Virginia: Advanced Research Department, Research Analysis Corporation, June 1966. AD637593.
234. Solomon, H., et al. Optimal Design of Sampling From Finite Populations; A Critical Review and Development of New Research Areas. Technical Report No. TR-140. Palo Alto, California: Department of Statistics, Stanford, California, November 1968. AD681030.
235. Spooner, R. L. The Theory of Signal Detectability: Extension to the Double Composite Hypothesis Situation. Technical Report No. TR-192, 3674-16-1. Ann Arbor, Michigan: Cooley Electronics Laboratory, University of Michigan, April 1968. AD672920.
236. Springer, M. D., and W. E. Thompson. "Bayesian Confidence Limits for the Product of N Binomial Parameters." Biometrika, 53:611-613 (1966).

237. Stein, C. "Approximation of Improper Prior Measures by Prior Probability Measures," in Bernoulli, 1713; Bayes, 1763; Laplace, 1813. Anniversary Volume, edited by J. Neyman and L. M. LeCam. Berlin: Springer-Verlag, 1965, pp. 217-240.
238. Strawderman, W. E. On the Existence of Proper Bayes Minimax Estimators of the Mean of a Multivariate Normal Distribution. Technical Report No. TR-170. Palo Alto, California: Department of Statistics, Stanford University, January 1971. AD716958.
239. Suppes, P. "The Role of Subjective Probability and Utility in Decision Making," in Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, V. Berkeley, California: University of California Press, 1956, pp. 61-73.
240. Tan, W. Y. Bayesian Analysis of a Generalized Regression Model of Potthof and Roy. Technical Report No. UWIS-DS-70-235. Madison, Wisconsin: Department of Statistics, University of Wisconsin, May 1970. AD708404.
241. Tiao, G. C., and M. M. Ali. Effect of Non-Normality on Inferences About Variance Components. Technical Report No. TR-185. Madison, Wisconsin: Department of Statistics, Wisconsin University, November 1968. AD686295.
242. Tiao, G. C., and S. Fienberg. Bayesian Estimation of Latent Roots and Vectors, with Special Reference to Bivariate Normal Distribution. Technical Report No. TR-139. Madison, Wisconsin: Department of Statistics, University of Wisconsin, May 1968. AD679203.
243. Tiao, G. C. Bayesian Comparison of Means of a Mixed Model with Application to Regression Analysis. Technical Report No. TR-49. Madison, Wisconsin: Department of Statistics, Wisconsin University, April 1965. AD625271.
244. Tiao, G. C., and W. Y. Tan. "Bayesian Analysis of Random Effect Models in the Analysis of Variance, II. Effect of Autocorrelated Errors." Biometrika, 53:477-495 (1966).
245. ----- "Bayesian Analysis of Random Effect Models in the Analysis of Variance, I. Posterior Distribution of Variance Components." Biometrika, 52:37-53 (1965).
246. Tiao, G. C., and A. Zellner. "Bayes Theorem and the Use of Prior Knowledge in Regression Analysis." Biometrika, 51:219-230 (1964).
247. ----- "On the Bayesian Estimation of Multivariate Regression." Journal Royal Statistical Society (B), 26:277-285 (Number 2, 1964).
248. Watson, G. S. "Some Bayesian Methods Related to X^2 ." Bulletin of the International Statistical Institute, 41:64-76 (1966).

249. Wald, A. Statistical Decision Functions. New York: John Wiley and Sons, Inc., 1950.
250. ----- Sequential Analysis. New York: John Wiley and Sons, Inc., 1947.
251. Wald, A., and J. Wolfowitz. "Bayes Solutions of Sequential Decision Problems." Annals Mathematical Statistics, 21:82-99 (1950).
252. Wasan, M. T. Parametric Estimation. New York: McGraw-Hill Book Co., Inc., 1970.
253. Weiler, H. "The Use of Incomplete Beta Functions for Prior Distributions in Binomial Sampling." Technometrics, 7:335-347 (August 1965).
254. Wetherill, G. B., and G. E. G. Campling. "The Decision Theory Approach to Sampling Inspection." Journal of the Royal Statistical Society, B, 28:381-416 (Number 3, 1966).
255. White, D. J. Decision Theory. Chicago: Aldine Publishing Co., 1969.
256. Wijsman, R. A. "Continuity of the Bayes Risk." The Annals of Mathematical Statistics, 41:1083-1085 (June 1970).
257. Winkler, R. L. "The Assessment of Prior Distributions in Bayesian Analysis." Journal American Statistical Association, 62:776-800 (September 1967).
258. Yen, E. H. Some Terminology for Reliability Demonstration Testing. Technical Report No. IDEP 347.40.00.00-K4-22. Bethpage, New York: Grumman Aircraft Engineering Corporation, November 1969. AD870587L.
259. Yereance, R. A. "Reliability Facts and Factors - Bayesian Statistics." Systems Design, 9:6-7 (April 1965).
260. Young, S. Management: A Decision-Making Approach. Belmont, California: Dickenson Publishing Co., 1968.
261. Zacks, S. "Bayes and Fiducial Equivariant Estimators of Common Mean." Annals of Mathematical Statistics, 41:59-69 (February 1970).
262. ----- "Bayesian Design of Single and Double Stratified Sampling for Estimating Proportion in Finite Population." Technometrics, 12:119-130 (February 1970).
263. ----- "Bayes Sequential Designs of Fixed Size Samples From Finite Populations." Journal of the American Statistical Association, 64:1342-1349 (December 1969).

264. Zacks, S. Bayes Sequential Designs of Fixed Size Samples from Finite Populations. Technical Report No. PB-183938. Albuquerque, New Mexico: Mathematics and Statistics Department, New Mexico University, October 1968. N70-10197.
265. ----- "Bayes Sequential Design of Fractional Factorial Experiments for the Estimation of a Subgroup of Pre-Assigned Parameters." Annals Mathematical Statistics, 39:973-982 (June 1968).
266. Zellner, A., and V. K. Chetty. "Prediction and Decision Problems in Regression Models From the Bayesian Point of View." Journal American Statistical Association, 60:608-616 (June 1965).
267. Zellner, A., and G. C. Tiao. "Bayesian Analysis of the Regression Model with Autocorrelated Errors." Journal American Statistical Association, 59:763-778 (September 1964).

APPENDIX B

BAYESIAN RELIABILITY ASSESSMENT
SAMPLE WORKSHEETS

Worksheet A

QUANTITATIVE PREDICTION/UNCERTAINTY DATA

[illegible]

Worksheet B

PRIOR KNOWLEDGE SUMMARY

System _____ Unit/Subsystem _____					
Analyst	Weight ϕ	a	b	ϕa	ϕb
$a = \sum \phi a$	1.00	$b = \sum \phi b$	Σ		

Worksheet C

PRIOR DISTRIBUTION PARAMETER DATA

System.

[illegible]

Worksheet D

TEST SEVERITY WEIGHTING FACTORS

System

Worksheet D									
TEST SEVERITY WEIGHTING FACTORS									
System _____									
Analyst	Contributor Weight γ	Test 1		Test 2		Test 3		Test 4	
		k	γk	k	γk	k	γk	k	γk

Worksheet E

UNIT/SUBSYSTEM FAILURE RECORD

Worksheet E													
UNIT/SUBSYSTEM FAILURE RECORD													
System _____		Unit/Subsystem _____						Month _____					
Type Test/ Data Source	Proc No	Weight K	Week 1		Week 2		Week 3		Week 4		Totals		
			t	f	t	f	t	f	t	f	t	Kt.	f
$\lambda_P =$			$t_m = \sum K t_i$								$r_m = \sum f$		

Worksheet F

[illegible]

Worksheet G

POSTERIOR DISTRIBUTION PARAMETER DATA

System

[illegible]

Vita

Lewis Ray White [REDACTED]
[REDACTED]
[REDACTED]

[REDACTED] Virginia, [REDACTED] which he enrolled at Virginia Polytechnic Institute, Blacksburg, Virginia. There he participated in the school's Cooperative Engineering Program under which he alternated his academic quarters with periods of work as a student trainee at Norfolk Naval Shipyard, Portsmouth, Virginia. In June 1961, he was graduated with the Degree of Bachelor of Science in Mechanical Engineering and a month later was commissioned a Second Lieutenant in the USAF Reserve. He was ordered to active duty in November 1961 and was assigned as a Project Engineer to the Air Force Plant Representative Office at the Convair Division of General Dynamics Corporation, San Diego, California before attending the Air Force Institute of Technology.

[REDACTED]
[REDACTED]

This thesis was typed by Miss Louise J. Houle