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SYSTEMS CONTROL, INC.
260 SHERIDAN AVENUE
PALO ALTO, CALIFORNIA 94306

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DUAL CONTROL AND IDENTIFICATION METHODS FOR AVIONIC SYSTEMS — PART II, OPTIMAL INPUTS FOR LINEAR SYSTEM IDENTIFICATION

Prepared for:
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
1400 WILSON BOULEVARD
ARLINGTON, VIRGINIA 22209

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13. ABSTRACT

This report presents the results of the second part of a one-year study on Dual Control and Identification Methods for Avionic Systems. First, a general theory is developed for designing optimal inputs to identify parameters in linear dynamic systems. It is shown that the optimal energy constrained unput maximizing a weighted trace of the Fisher information matrix is an eigenfunction of a positive self-adjoint operator corresponding to its largest eigenvalue. Several numerical algorithms are described for computing the optimal inputs. A computer program based on the Riccati Equation algorithm is developed and used for computing optimal elevator deflection time histories to identify the short period stability and control derivatives of C-8 aircraft.

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I

INTRODUCTION

The importance of input selection for system identification has been recognized for a long time, though a unified mathematical treatment has emerged only recently. Some of the earlier attempts at input design were based on frequency domain methods and engineering judgment. An interesting discussion of the relative advantages and disadvantages of oscillatory versus pulse type inputs for aircraft flight testing is contained in Ref. 1. A large amount of literature exists on Pseudo Random Binary Sequence (PRBS) inputs which have been found to provide improved identification for a large number of systems.[2-4] However, PRBS inputs use very little information about the known properties of the system. Since in a number of physical systems some a priori information is available about the modes of the system (e.g., short period mode, phugoid mode, etc. of an aircraft's longitudinal motion), one can use this information to design inputs for identifying these modes more precisely. This is the basis of the approach considered in this paper.

The present work is most closely related to that of Aoki and Staley,[5] Levadi,[6] Nahi and Wallis,[7] and Levin[8] on input signal design for system identification and to that of McAulay[9] and Esposito[10] for signal synthesis. Aoki and Staley[5] consider single-input, single-output discrete-time systems. The results presented here for multi-input, multi-output continuous time linear systems are conceptually similar to theirs, but the computational methods used are entirely different. Levadi's results[6] are only applicable to the case in which the unknown parameters enter linearly in the system impulse response. Levine's results[8] are applicable when linear regression is used to estimate the unknown parameters. The optimal inputs in the present work are derived under the assumption that the input is either energy or power constraintred and an efficient estimator is used to estimate the unknown parameters. In practice, a maximum likelihood estimator may be used. The optimal input which maximizes the simple or the weighted trace of the information matrix is shown to be an eigenfunction of a positive self-adjoint operator. The resulting eigenvalue problem can be solved in a number of different ways, some of which are (i) transition matrix method, (ii) Riccati Equation method, (iii) resolvent method, and (iv) Ritz-Galerkin method. Relative advantages and disadvantages of various methods are discussed. Four analytical examples are given to illustrate the procedure for determining optimal inputs. Several numerical examples are solved to verify the theoretical results and to compare the optimal input for the short period dynamics of an aircraft with the suboptimal inputs currently in use.

The arrangement of the material is as follows: The theory of input design is contained in Section II. The problem is stated in Section 2.1 and is solved using the Maximum Principle for a single parameter in the gain matrix in Section 2.2. Stronger results are derived in Section 2.3 using Functional Analysis which also gives an operator-theoretic representation and an integral equation representation for optimal inputs. Two first order examples and a second order example are solved in closed form in Section 2.4. The extensions of these results to the identification of parameters in the system dynamics matrix are given in Section 2.5. The multi-parameter case, the process noise case, and the state variable constraints are discussed in Sections 2.6, 2.7, and 2.8.

Section III contains a description of the computer program developed to implement the results of Section II. This is followed by a description of the numerical results and suggestions for further research.

II

THEORY OF OPTIMAL INPUT DESIGN

2.1 Problem Statement

Consider a time-invariant linear dynamic system

$$\dot{x} = Fx + Gu \tag{1}$$

where x is a nxl state vector and u is a mxl control vector. F and G are n x n and n x m matrices of unknown parameters. The output of the system is denoted by a p x 1 vector $y(t)$ which is contaminated with white noise $v(t)$

$$y(t) = Hx(t) + v(t). \tag{2}$$

H is a p x n matrix and $v(t)$ is a zero mean Gaussian white noise process

$$E\{v\} = 0, \; E\{v(t)v^T(\tau)\} = R\delta(t - \tau). \tag{3}$$

Let θ denote an N x 1 vector of unknown parameters in the above system. It is required to select the input $\{u(t), 0 \le t \le T\}$ to maximize a suitable norm of the information matrix M subject to suitable constraints. The information matrix M for the unknown parameter set θ can be shown to be [7]

$$M = \int_0^T (\nabla_\theta x)^T H^T R^{-1} H(\nabla_\theta x) \, dt \tag{4}$$

where

$$\nabla_\theta = \left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \ldots, \frac{\partial}{\theta_N}\right) . \tag{5}$$

Notice that M^{-1} represents the Cramér-Rao lower bound for the covariance of an unbiased estimator of θ. Aoki and Staley [5] show that in the multi-parameter case, the maximization of tr[M] is asymptotically equivalent to the minimization of $tr[M^{-1}]$. Recently, Nahi and Napjus[18] have proposed the use of weighted trace of M as a maximization criterion. We will consider these and other optimization criteria for designing optimal inputs. Most of the important concepts, however, can be presented by considering first the case of a scalar parameter in G.

2.2 Optimal Energy-Constrained Input for One Parameter Using Maximum Principle

Let the input be energy constrained as follows:

$$\int_0^T u^T u \, dt = E \, . \tag{6}$$

There are no other constraints on u(t) and x(t). Consider first the case where θ is an unknown parameter in G. The information M is scalar and the sensitivity function $\nabla_\theta x$ is obtained as follows:

$$\nabla_\theta \dot{x} = F \, \nabla_\theta x + (\nabla_\theta G)u \tag{7}$$

$\nabla_\theta G$ is a matrix with all zero elements except a single one.

The maximization of M subject to constraint (6) is equivalent to the minimization of the performance index

$$J = \frac{1}{2}\int_0^T [-(\nabla_\theta x)^T \, H^T \, R^{-1} \, H(\nabla_\theta x) + \mu(u^T \, u - \frac{E}{T})] \, dt \tag{8}$$

where μ is a constant multiplier chosen to keep Eq. (6) satisfied. This is a linear-quadratic problem for which the Euler-Lagrange equations are easily written down.

Hamiltonian:

$$\mathscr{H} = \frac{1}{2}\,[-(\nabla_\theta x)^T H^T R^{-1} H(\nabla_\theta x) + \mu(u^T u - \frac{E}{T})] + \lambda^T[F\nabla_\theta x + \nabla_\theta G u] \quad (9)$$

where λ is a n x 1 costate vector.

$$\dot{\lambda} = -\left(\frac{\partial \mathscr{H}}{\partial \nabla_\theta x}\right)^T$$

or

$$\dot{\lambda} = - F^T \lambda + H^T R^{-1} H \nabla_\theta x \quad (10)$$

Stationarity Condition:

$$\mathscr{H}_u = 0$$

or

$$u^* = -\frac{1}{\mu}(\nabla_\theta G)^T \lambda \quad (11)$$

The boundary conditions are homogeneous.

$$\nabla_\theta x(0) = 0 \;,\; \lambda(T) = 0$$

Substituting for u^* in Eq. (7), we obtain the two point boundary value problem,

$$\frac{d}{dt}\begin{bmatrix} \nabla_\theta x \\ \lambda \end{bmatrix} = \begin{bmatrix} F & -\frac{1}{\mu}(\nabla_\theta G)(\nabla_\theta G)^T \\ H^T R^{-1} H & -F^T \end{bmatrix}\begin{bmatrix} \nabla_\theta x \\ \lambda \end{bmatrix} \quad (12)$$

Since the boundary conditions are homogeneous, the solution is trival viz. $\nabla_\theta x \equiv 0$, $\lambda \equiv 0$, $u \equiv 0$ except for certain values of μ which are the eigenvalues of the two point boundary value problem. In other words, the problem is of the Sturm-Liouville type.[12] The eigenvalues and the optimal input can be determined in a number of ways. Two possible methods are (i) the transition matrix method and (ii) the Riccati equation method.

2.2.1 Transition Matrix Method

Let $\Phi(t,0;\mu)$ denote the transition matrix of (12) for a particular μ.

$$\Phi(t,0;\mu) = \exp\begin{pmatrix} F, & -\frac{1}{\mu}(\nabla_\theta G)^T \nabla_\theta G \\ H^T R^{-1} H, & -F^T \end{pmatrix} t \qquad (13)$$

Partition $\Phi(t,0;\mu)$ into $\nabla_\theta x$ and λ parts as follows:

$$\Phi = \begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix} \qquad (14)$$

Then

$$\begin{bmatrix} \nabla_\theta x(T) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} \Phi_{xx}(T,0;\mu) & \Phi_{x\lambda}(T,0;\mu) \\ \Phi_{\lambda x}(T,0;\mu) & \Phi_{\lambda\lambda}(T,0;\mu) \end{bmatrix} \begin{bmatrix} \nabla_\theta x(0) \\ \lambda(0) \end{bmatrix} \qquad (15)$$

The second equation in (15) along with the boundary conditions gives

$$\lambda(T) = \Phi_{\lambda\lambda}(T,0;\mu)\ \lambda(0) = 0 \qquad (16)$$

For a nontrivial solution

$$|\Phi_{\lambda\lambda}(T,0;\mu)| = 0 \qquad (17)$$

Eq. (17) is the eigenvalue equation for the Hamiltonian system (12). It is a nonlinear algebraic equation in μ and can be solved by a Newton-Raphson iteration. In general, there is an infinite set of eigenvalues, but we will be only interested in the largest value of μ which will be shown to minimize J (Section 2.3).

2.2.2 Riccati Equation Method

The eigenvalues μ are functions of the interval length T. Therefore, one can fix μ and determine T for which $\Phi_{\lambda\lambda}(T,0;\mu)$ becomes singular. Another way is to use the Riccati matrix $P(t)$ defined by the relationship

$$\nabla_\theta x(t) = P(t)\lambda(t) \tag{18}$$

An equation for $P(t)$ is obtained by differentiating both sides of Eq. (18) and substituting from Eq. (12).

$$\nabla_\theta \dot{x} = \dot{P}\lambda + P\dot{\lambda}$$

or

$$[FP - \frac{1}{\mu}(\nabla_\theta G)(\nabla_\theta G)^T]\lambda = [\dot{P} + PH^T R^{-1} HP - PF^T]\lambda$$

or

$$\dot{P} = FP + PF^T - PH^T R^{-1} HP - \frac{1}{\mu}(\nabla_\theta G)(\nabla_\theta G)^T \tag{19}$$

$$P(0) = 0. \tag{20}$$

The Riccati Eq. (19) differs from the usual Riccati-equation of the Linear-Quadratic problem since the forcing term (last term) in Eq. (19) enters negatively. Eq. (18) can also be written as

$$\lambda(t) = P^{-1}(t)\nabla_\theta x(t)$$

whenever P^{-1} exists. At final time $t = T$, since $\lambda(T) = 0$,

$$P^{-1}(T) = 0. \tag{21}$$

which means that a conjugate point exists at $t = T$.

Eq. (21) provides us with a method to obtain the critical interval length T corresponding to an eigenvalue μ. The Riccati Eq. (19) is integrated forward in time for a particular μ using initial conditions (20). When the elements of $P(t)$ become very large, the critical length T corresponding to an eigenvalue is being reached. Now $P^{-1}(t)$ is integrated using the equation

$$\frac{d}{dt}(P^{-1}) = -P^{-1}\dot{P}P^{-1}$$

or

$$\frac{d}{dt}(P^{-1}) = -P^{-1}F - F^TP^{-1} + H^TR^{-1}H + \frac{1}{\mu}P^{-1}(\nabla_\theta G)(\nabla_\theta G)^TP^{-1} \tag{22}$$

At the critical interval length T, all the elements of P^{-1} go to zero. It follows from the Sturmian property[12] that the smallest T corresponds to the largest eigenvalue μ.

The Riccati Equation Method is similar to the Invariant Imbedding method of Alspaugh, Kagiwada and Kalaba(5) for determining eigenvalues in the problem of buckling of beams. By this method, one obtains a curve relating μ and T.

After the critical length T corresponding to the largest value of μ has been determined, Eq. (12) is solved forward in time using $\lambda(0)$ obtained from Eq. (16) and (17) as an eigenvector of $\Phi_{\lambda\lambda}(T,0;\mu)$ corresponding to the zero eigenvalue. Thereby, the boundary condition $\lambda(T) = 0$ is automatically satisfied. A unique value of $\lambda(0)$ is found by using the normalization of condition of Eq. (6).

2.3 Application of Functional Analysis

In the last section, the optimal input u was characterized in terms of the solution to a two point boundary value problem. In this section, we show that the optimal u is an eigenfunction of a positive self-adjoint operator corresponding to the largest eigenvalue μ.

Let A denote the operator corresponding to Eq. (7) viz.

$$A[u] = \int_0^t e^{F(t-\tau)} (\nabla_\theta G) u(\tau) d\tau \tag{24}$$

Let $A^*[\cdot]$ denote the adjoint operator to $A[\cdot]$

$$A^*[w] = (\nabla_\theta G)^T \int_t^T e^{F^T(s-t)} w(s) ds \tag{25}$$

Let $\langle u,w \rangle$ denote the inner product

$$\langle u,w \rangle = \int_0^T u^T(t) w(t) dt \tag{26}$$

The information matrix M can be written as

$$\begin{aligned} M &= \langle \nabla_\theta x, H^T R^{-1} H \nabla_\theta x \rangle \\ &= \langle Au, H^T R^{-1} H Au \rangle \\ &= \langle u, A^* H^T R^{-1} H Au \rangle \end{aligned} \tag{27}$$

The energy constraint of Eq. (6) is written as

$$\langle u,u\rangle = E$$

It is well known that M is maximized subject to the above constraint by u^* which is an eigenfunction corresponding to the largest eigenvalue of the operator $A^*H^TR^{-1}HA$. Furthermore, since $A^*H^TR^{-1}HA$ is a positive self-adjoint operator, all its eigenvalues are real and positive.[12] For finite T, the operator is also compact and has a finite maximum eigenvalue. The optimal u^* is the eigenfunction of $A^*H^TR^{-1}HA$ corresponding to this eigenvalue and normalized according to $\langle u,u\rangle = E$.

$$A^*H^TR^{-1}HA\,u = \mu u \tag{28}$$

Also,

$$\max_u M = \mu E \tag{29}$$

To show the relationship of the above eigenvalue problem with the two point boundary value of Eq. (12), define

$$z = Au \tag{30}$$

and

$$(\nabla_\theta G)^T\eta = A^*H^TR^{-1}Hz \tag{31}$$

Then, using the definition of A and A^*,

$$\dot{z} = Fz + \nabla_\theta Gu\ , \quad z(0) = 0 \tag{32}$$

$$\dot{\eta} = -F^T\eta + H^TR^{-1}Hz\ , \quad \eta(T) = 0 \tag{33}$$

From Eq. (31) and (28)

$$(\nabla_\theta G)^T \eta = A^* H^T R^{-1} HAu = \mu\, u \qquad (34)$$

or

$$u = -\frac{1}{\mu} (\nabla_\theta G)^T \eta \qquad (35)$$

Therefore

$$\left.\begin{aligned} \dot{z} &= Fz - \frac{1}{\mu} \nabla_\theta G (\nabla_\theta G)^T \eta \,, \quad z(0) = 0 \\ \dot{\eta} &= -F^T \eta + H^T R^{-1} Hz \quad , \quad \eta(T) = 0 \end{aligned}\right\} \qquad (36)$$

A comparison of Eq. (35), (36) with Eq. (11) and (12) shows that

$$z = \nabla_\theta x$$

$$\eta = \lambda.$$

We now express the operator Eq. (28) as a Fredholm Integral equation.

2.3.1 Fredholm Integral Equation for Optimal Input

From Eq. (28), a Fredholm Integral equation for optimal input u is derived as follows:

$$A^* H^T R^{-1} H\, Au = \mu\, u(t)$$

or

$$(\nabla_\theta G)^T \int_t^T ds\, e^{F^T(s-t)} H^T R^{-1} H \int_0^s d\tau e^{F(s-\tau)} \nabla_\theta G u(\tau) = \mu\, u(t)$$

or

$$\int_t^T ds \int_o^s d\tau (\nabla_\theta G)^T e^{F^T(s-t)} H^T R^{-1} H e^{F(s-\tau)} \nabla_\theta G u(\tau) = \mu\, u(t)$$

Define

$$K(t,\tau,s) = \begin{cases} (\nabla_\theta G)^T e^{F^T(s-t)} H^T R^{-1} H\, e^{F(s-\tau)} \nabla_\theta G, & s \geq t \geq \tau \\ (\nabla_\theta G)^T e^{F^T(s-\tau)} H^T R^{-1} H\, e^{F(s-t)} \nabla_\theta G, & s \geq \tau \geq t \end{cases}$$

Then

$$\int_o^T ds \int_o^T d\tau K(t,\tau,s) u(\tau) = \mu\, u(t)$$

or

$$\int_o^T d\tau \Lambda\, (t,\tau) u(\tau) = \mu\, u(t) \tag{37}$$

where

$$\Lambda(t,\tau) = \int_o^T K(t,\tau,s) ds$$

This equation is of the same type as given by Levadi(2) for the case of white noise in the observations and scalar u. It is seen that the optimal input $u(t)$ satisfies a Fredholm Integral Equation of the second kind.

The above representation suggests two other methods for obtaining μ and $u(t)$. These are the Ritz-Galerkin Method[12] and the Resolvent Method.[19]

2.3.2 Ritz-Galerkin Method

In this method, u(t) is expanded in a series of orthogonal functions.

$$u(t) = \sum_{k=1}^{L} a_k \phi_k(t)$$

where L, the number of terms in the expansion is chosen large enough to give an adequate representation for u(t) on [0, T]. One possible choice is

$$\phi_k(t) = \cos\,(k - \tfrac{1}{2})\,\frac{\pi t}{T} \quad , \qquad k = 1,\ldots,L$$

which automatically satisfies the boundary condition

$$u(T) = 0 \quad .$$

The coefficients a_k, k = 1,...,L are chosen to minimize the squared error in satisfying the integral equation. This is achieved by making the error orthogonal to ϕ_j, j = 1,...,L.

$$\int_0^T dt\left[\int_0^T \Lambda(t,\tau) \sum_{k=1}^{L} a_k\, \phi_k(\tau)d\tau - \mu \sum_{k=1}^{L} a_k\, \phi_k(t)\right]\phi_j(t) = 0$$

$$\sum_{k=1}^{L} a_k\left\{\int_0^T dt \int_0^T d\tau\Lambda(t,\tau)\, \phi_k(\tau)\, \phi_j(t) - \mu\delta_{j,k}\right\} = 0 \quad .$$

For a nontrivial solution, let

$$\left|\int_0^T dt \int_0^T d\tau\; \Lambda(t,\tau)\, \phi_k(\tau)\, \phi_j(t) - \mu\delta_{j,k}\right| = 0$$

where $|\cdot|$ denotes the determinant of L × L matrix with rows j = 1,...,L and columns k = 1,...,L. The optimal input is given by the eigenvector corresponding to the largest eigenvalue of this matrix.

2.3.3 Resolvent Method

The Resolvent Method is based on the Resolvent Identity[19]

$$R_\mu - R_{\mu'} = (\mu-\mu') R_\mu R_{\mu'}$$

where $$R_\mu = (\mu I - A^* H^T R^{-1} HA)^{-1}$$

Using this identity, initial value problems are set up in μ for the resolvent kernel and the input u. These equations are integrated forward starting from $\mu = o$ till the first pole of R_μ is encountered. The advantage of this method over the Riccati Equation method of section 2.2 is that μ is determined for a fixed T and not vice versa. (For details see Appendix A.)

2.4 Examples

We now apply the above results to two first order and a second order example.

2.4.1 First Order System with Unknown Gain

Consider the system

$$\dot{x} = -x + \theta u \tag{38}$$

where x and u are scalars and θ is the unknown gain.

$$y = x + v \quad , \tag{39}$$

where $E\{v\} = 0$, $E\{v(t)v(\tau)\} = r\,\delta(t - \tau)$.

From Eq. (11) and (12)

$$u = -\frac{1}{\mu}\lambda \tag{40}$$

$$\frac{d}{dt}\begin{bmatrix}\nabla_\theta x \\ \lambda\end{bmatrix} = \begin{bmatrix}-1 & -\frac{1}{\mu} \\ \frac{1}{r} & 1\end{bmatrix}\begin{bmatrix}\nabla_\theta x \\ \lambda\end{bmatrix} \tag{41}$$

Eigenvalues

Equation (41) gives

$$\dot{\lambda} = \frac{1}{r} \nabla_\theta x + \lambda$$

$$\ddot{\lambda} = \frac{1}{r} [-\nabla_\theta x - \frac{1}{\mu} \lambda] + [\frac{1}{r} \nabla_\theta x + \lambda]$$

$$= (1 - \frac{1}{\mu r}) \lambda \quad . \qquad (42)$$

Let $\alpha^2 = 1 - 1/\mu r$. Three cases arise:

(i) $\alpha^2 > 0$ or $1 > \frac{1}{\mu r}$

$$\lambda(t) = C_1 \sinh \alpha t + C_2 \cosh \alpha t$$

$$\nabla_\theta x(t) = r[\dot{\lambda} - \lambda]$$

$$= r[C_1\alpha \cosh \alpha t + C_2\alpha \sinh \alpha t - C_1 \sinh \alpha t - C_2 \cosh \alpha t] \quad .$$

Since

$$\nabla_\theta x(0) = 0$$

$$C_1\alpha - C_2 = 0 \quad \text{or} \quad C_2 = C_1\alpha$$

$$\lambda(T) = 0 \Rightarrow C_1[\sinh \alpha T + \alpha \cosh \alpha T] = 0 \quad .$$

For a nontrivial solution

$$\tanh \alpha T = -\alpha \quad . \qquad (43)$$

Since $\alpha > 0$, no solution to Eq. (43) exists. Therefore $C_1 = 0$ and only the trivial solution exists for this case.

(ii) $\underline{\alpha^2 = 0}$

$$\lambda(t) = C_1 + C_2 t$$

$$\nabla_\theta x(t) = r[C_2 - C_1 - C_2 t]$$

$$\nabla_\theta x(0) = r[C_2 - C_1] = 0 \Rightarrow C_2 = C_1 \quad .$$

$$\lambda(T) = C_1 + C_2 T = C_1(1 + T) = 0$$

$$\Rightarrow C_1 = 0 \quad .$$

Again the solution is trivial.

(iii) $\underline{\alpha^2 < 0:}$

Let

$$w = \left(\frac{1}{\mu r} - 1\right)^{\frac{1}{2}} \tag{44}$$

$$\lambda(t) = C_1 \sin wt + C_2 \cos wt$$

$$\nabla_\theta x(t) = r[C_1 w \cos wt - C_2 w \sin wt - C_1 \sin wt - C_2 \cos wt]$$

$$\nabla_\theta x(0) = 0 \qquad C_1 w - C_2 = 0 \quad .$$

$$\lambda(T) = 0 \qquad C_1[\sin wT + w \cos wT] = 0 \quad .$$

For nontrivial solution,

$$\boxed{\tan wT = -w} \quad . \tag{45}$$

Eq. (45) is the eigenvalue equation. The smallest root w_o of Eq. (45) corresponds to the largest value of μ. The optimal input u is obtained from Eq. (40).

$$u^* = -\frac{C_1}{\mu_o}\,[\sin w_o t + w_o \cos w_o t] \tag{46}$$

where $\mu_o = 1/r(1 + w_o^2)$. Notice that u satisfies the same second order differential equation as λ viz. Eq. (42). C_1 is determined from the condition $\int_o^T u^2 dt = E$.

This gives

$$C_1 = \sqrt{\frac{E}{T/2 - 1/4 \sin 2T}} \tag{47}$$

The Riccati equation method for this problem gives the same result. Eq. (19) gives

$$\dot{p} = -2p - \frac{p^2}{r} - \frac{1}{\mu}$$

or

$$\frac{rdp}{(p + r)^2 + r^2\left(\frac{1}{\mu r} - 1\right)} = -dt \quad .$$

Integrating both sides from 0 to t

or

$$\frac{p + r}{r\sqrt{\frac{1}{\mu r} - 1}} = \tan\left\{-\sqrt{\frac{1}{\mu r} - 1}\; t + \tan^{-1}\frac{1}{\sqrt{\frac{1}{\mu r} - 1}}\right\} \tag{48}$$

$p(T) \to \infty$ for T which satisfies

$$\sqrt{\frac{1}{\mu r} - 1}\; T = \tan^{-1}\frac{1}{\sqrt{\frac{1}{\mu r} - 1}} + (2k + 1)\frac{\pi}{2} \qquad k = 0,1,2,\ldots$$

or

$$-\sqrt{\frac{1}{\mu r} - 1} = \tan\sqrt{\frac{1}{\mu r} - 1}\; T \quad .$$

This is the same equation as Eq. (45). Notice that if $w = \sqrt{1/\mu r - 1}$ is kept fixed, the T for which w is the smallest eigenvalue can be found by integrating the above Riccati equation until $p(t) \to \infty$ for the first time. This corresponds to $k = 0$ in the above equation.

2.4.2 Levadi's Example

Levadi considers the following example in his paper.[6]

$$\dot{x} = -x + bu \tag{49}$$

$$y = x + v \tag{50}$$

where v is a correlated noise process with autocorrelation function.

$$E\{v(t)v(\tau)\} = c\, e^{-a|t-\tau|} \quad .$$

It is required to estimate b only.

Levadi's[6] results can be easily derived as follows:

v(t) can be represented as

$$\dot{v} = -av + \varepsilon \tag{51}$$

where ε is a white noise process and

$$E\{\varepsilon\} = 0 \quad , \quad E\{\varepsilon(t)\varepsilon(\tau)\} = 2ac\, \delta(t - \tau) \quad .$$

A new measurement can be generated by differentiating Eq. (50). (This procedure is similar to that of Bryson and Johansen.)[13]

$$\dot{y} = \dot{x} + \dot{v}$$

$$= (a - 1)x + bu - ay + \varepsilon \quad .$$

Now $\dot{y}$ can be regarded as a new measurement which has white noise ϵ in it. The new information matrix is:

$$M = \int_0^T \frac{1}{2ac} [(a - 1)\nabla_b x + u]^2 dt \tag{52}$$

where

$$\dot{\nabla}_b x = -\nabla_b x + u \quad . \tag{53}$$

Now the problem is in the same form as example 2.4.1 except that the performance index is slightly different.

The following equations of optimality are easily derived.

$$u = - \frac{1}{2(\mu - \frac{1}{2ac})} [\lambda - \frac{a - 1}{ac} \nabla_b x] \tag{54}$$

$$\dot{\lambda} = \frac{(a - 1)^2}{ac} \nabla_b x + \frac{a - 1}{ac} u + \lambda \quad . \tag{55}$$

An equation only in terms of u can also be obtained from Eq. (53)-(55).

$$\ddot{u} - [1 - \frac{a^2 - 1}{2\mu ac - 1}] u = 0 \quad . \tag{56}$$

The eigenvalue equation is

$$\tan(wT + \emptyset) = \frac{w}{a} \tag{57}$$

where $w = [-1 + \frac{a^2 - 1}{2 ac - 1}]^{1/2}$ (58)

$$\emptyset = \tan^{-1} w \quad .$$

The optimal input $u^* = A \sin(wt + \emptyset)$. (59)

Notice that the results for example 2.4.1 can be obtained by letting $a \to \infty$ and $2c/a \to r$, where $2c/a$ represents the area under the autocorrelation function of v.

The optimum value of w is chosen to maximize μ. From Eq. (58),

$$\mu = \frac{1}{2ac}\left[1 + \frac{a^2 - 1}{1 + w^2}\right] \quad . \tag{60}$$

It is seen from Eq. (60) that when $a^2 > 1$, the maximum of μ is attained for the smallest value of w. This corresponds to the case when the noise is wide-band. For the narrow band noise case viz. $a^2 < 1$, the second term in Eq. (60) is negative and the maximum of μ viz. $1/2ac$ is reached at $w = \infty$. The practical implication of this result is that the input should be of as high frequency as possible. Since the noise is narrow band, this increases the high frequency signal to noise ratio at the output.

2.4.3 Second Order Example

The following system represents the short-period longitudinal dynamics of an aircraft.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \delta_e \tag{61}$$

$$y = x_1 + v$$

x_1 represents pitch rate, x_2 is a linear combination of pitch rate and angle-of-attack and δ_e is the elevator deflection. The transfer function of the system from elevator to pitch rate is

$$\frac{x_1(s)}{\delta_e(s)} = \frac{b_1 s + b_2}{s^2 - a_1 s - a_2} \quad . \tag{62}$$

Typical values for the parameters are $a_1 = -2.276$, $a_2 = -2.558$, $b_1 = 1.913$, $b_2 = 1.82$. (C-8 Airplane).[15]

Optimal Input for Identifying b_1:

Sensitivity equations are

$$\begin{bmatrix} \nabla_{b_1}\dot{x}_1 \\ \nabla_{b_1}\dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} \nabla_{b_1}x_1 \\ \nabla_{b_1}x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta_e \quad . \tag{63}$$

Optimal input, $\delta_e = -\lambda_1/2\mu$

$$\dot{\lambda}_1 = \frac{2}{r} \nabla_{b_1} x_1 - \lambda_1 a_1 - \lambda_2 a_2 \tag{64}$$

$$\dot{\lambda}_2 = -\lambda_1 \quad . \tag{65}$$

Both the optimal input δ_e and λ_2 satisfy fourth order linear differential equations of the form

$$[D^4 + (\frac{1}{\mu r} - 2a_2 - a_1^2)D^2 + a_2^2]\lambda_2 = 0 \tag{66}$$

where D denotes the differential operator d/dt.

A solution for λ_2 is easily written as

$$\lambda_2(t) = C_1 \sin w_1 t + C_2 \cos w_1 t + C_3 \sin w_2 t + C_4 \cos w_2 t \tag{67}$$

where

$$w_1 = \frac{1}{\sqrt{2}} \sqrt{\sigma - \sqrt{\sigma^2 - 4a_2^2}}$$

$$w_2 = \frac{1}{\sqrt{2}} \sqrt{\sigma + \sqrt{\sigma^2 - 4a_2^2}}$$

$$\sigma = (\frac{1}{\mu r} - 2a_2 - a_1^2) = w_1^2 + w_2^2 \quad . \tag{68}$$

Also $w_1 w_2 = a_2$.

It is assumed here that $\sigma > 2a_2$ or $1/\mu r > (4a_2 + a_1^2)$ since other cases lead to hyperbolic functions which become unbounded for large T. They are rejected as possible solutions using the same reasoning as used in examples 2.4.1 and 2.4.2.

The expressions for $\lambda_1(t)$, $\nabla_{b_1}x_1$, u(t) and $\nabla_{b_2}x_2$ are easily obtained from Eq. (67) using Eq. (63)-(65). The eigenvalue equation, assuming $w_1 \neq w_2$, is obtained as

$$\begin{vmatrix} \sin w_1T & \cos w_1T & \sin w_2T & \cos w_2T \\ -w_1 \cos w_1T & w_1 \sin w_1T & -w_2 \cos w_2T & w_2 \sin w_2T \\ -w_1a_1 & a_2 + w_1^2 & -a_1w_2 & a_2 + w_2^2 \\ w_1^3 - w_1a_2 - \frac{w_1}{\mu r} & w_1^2a_1 & w_2^3 - w_2a_2 - \frac{w_2}{\mu r} & w_2^2a_1 \end{vmatrix} = 0 \qquad (69)$$

where $w_2 = - a_2/w_1$. Equation (68) is used along with (69) to select w_1 and w_2 which maximize μ. From Eq. (68),

$$\frac{1}{\mu r} = (w_1 + \frac{a_2}{w_1})^2 + a_1^2 = (w_1 - w_2)^2 + a_1^2 \quad .$$

The minimum of $1/\mu r$ is attained at $w_1^2 = -a_2$ or $w_1 = w_2$, i.e., at the undamped natural frequency of the system. However since $w_1 = w_2$ is ruled out by the solution considered here, the root of Eq. (69) closest to the undamped natural frequency should be chosen. Since $w_1w_2 = -a_2$, the two frequencies will bracket the natural frequency of the system.

The optimal input δ_e^* is

$$\delta_e^* = \frac{1}{2\mu} [C_1w_1 \cos w_1t - C_2w_1 \sin w_1t + C_3w_2 \cos w_2t - C_4w_2 \sin w_2t] \qquad (70)$$

C_1, C_2, C_3, and C_4 are determined as the eigenvector of Eq. (69) corresponding to the root w_1. They are normalized using the condition

$$\int_0^T \delta_e^2 \, dt = E \quad .$$

2.5 Extension to Unknown Parameter in F

In sections 2:2-2.4 we only considered the case in which θ is an unknown in G. Consider the case in which θ is an unknown in F. The sensitivity equation is

$$\nabla_\theta \dot{x} = F\,\nabla_\theta x + (\nabla_\theta F)x \quad . \tag{71}$$

Since Eq. (71) involves x, it is necessary to consider Eq. (1). Let x_A denote the augmented state vector

$$x_A = \begin{bmatrix} x \\ \nabla_\theta x \end{bmatrix} \tag{72}$$

then

$$\dot{x}_A = F_A x_A + G_A u \tag{72}$$

where

$$F_A = \begin{bmatrix} F & 0 \\ \nabla_\theta F & F \end{bmatrix}$$

$$G_A = \begin{bmatrix} G \\ 0 \end{bmatrix}$$

The performance index J can be written as

$$J = \frac{1}{2}\int_o^T [-x_A^T H_A^T R^{-1} H_A x_A + \mu(u^T u - \frac{E}{T})]dt$$

where

$$H_A = [0 \,\vdots\, I] \quad .$$

The two point boundary value problem becomes

$$\begin{bmatrix} \dot{x}_A \\ \dot{\lambda}_A \end{bmatrix} = \begin{bmatrix} F_A & -\frac{1}{\mu} G_A G_A^T \\ H_A^T R^{-1} H_A & -F_A^T \end{bmatrix} \begin{bmatrix} x_A \\ \lambda_A \end{bmatrix} \tag{73}$$

$$u^* = -\frac{1}{\mu} G_A^T \lambda_A$$

$$x_A(0) = \begin{bmatrix} x(0) \\ 0 \end{bmatrix}$$

$$\lambda_A(T) = 0 \quad .$$

This problem can be solved by methods similar to the ones discussed earlier in section 2.2. The dimension of the problem, however, is doubled.

2.5.1 Example

Consider a scalar system

$$\dot{x} = -\theta x + u$$

$$y = x + v \quad .$$

The optimal input for identifying the unknown time-constant θ satisfies the following differential equation.

$$[(D^2 - \theta^2)^2 - \frac{1}{\mu}]u = 0 \quad . \tag{74}$$

The optimum value of μ is determined from the boundary conditions. It is seen that the optimal input is a sum of sine and cosine functions. Notice that θ appears explicitly in Eq. (74). Therefore, the optimal input u cannot be obtained in one iteration unless a certain probability distribution is given for θ (see Aoki and Staley[5]). Otherwise, one starts with an estimate of θ to obtain a suboptimal u. Then the estimate of θ is refined from actual data and u is recomputed. This approach is similar to the Sequential Design Approach of Wald.[21]

2.6 Extension to the Multiparameter Case

In the multiparameter case, M is a matrix and the selection of a suitable norm for optimization becomes very important. This question has been discussed earlier by Aoki and Staley[5] and Nahi and Napjus[18] who have considered the following norms:

(i) $tr(M)$

(ii) $tr(M^{-1})$

(iii) $|M|$ or $|M^{-1}|$

(iv) $tr(QM)$--weighted trace.

Two sets of useful inequalities for these norms are[5,18]

$$\left(\frac{1}{N}\,tr\,M\right)^N \geq \prod_{i=1}^{N} m_{ii} \geq \det M = (\det P)^{-1} \geq \prod_{i=1}^{N} \frac{1}{p_{ii}} \geq \left(\frac{1}{N}\,tr\,P\right)^{-N} \tag{75}$$

and

$$N^2 \geq tr\,P \cdot tr\,M \geq \frac{N^2}{2}\left(1 + \frac{\alpha_{max}}{\alpha_{min}}\right) \tag{76}$$

where m_{ii} and p_{ii} are the diagonal elements of M and P; α_{max} and α_{min} are the maximum and minimum eigenvalues of M. In the second inequality, $tr\,P \cdot tr\,M$ is a figure-of-merit which provides a measure of the tightness of the inequality (75).

The approach taken in this paper applies directly to tr(M) and tr(QM) since these are linear norms of M. The other norms require iterative hill-climbing methods for solution. The presence of local minima and singular arcs make the resulting optimization problem extremely difficult to solve.

A straightforward method to maximize tr(M) is to define an augmented state vector x_A as follows:

$$\underset{(N+1)n\times 1}{x_A} = \begin{bmatrix} x \\ \nabla_{\theta_1} x \\ \vdots \\ \nabla_{\theta_N} x \end{bmatrix} \tag{77}$$

The state equation for x_A can be written as

$$\dot{x}_A = F_A x_A + G_A u \tag{78}$$

where

$$\underset{(N+1)n\times(N+1)n}{F_A} = \left[\begin{array}{c|c|c|c} F & 0 & \cdots & 0 \\ \hline \nabla_{\theta_1} F & F & & \\ \hline \vdots & & \ddots & \\ \hline \nabla_{\theta_N} F & & & F \end{array}\right] , \quad G_A = \begin{bmatrix} G \\ \nabla_{\theta_1} G \\ \vdots \\ \nabla_{\theta_N} G \end{bmatrix} \tag{79}$$

Define

$$\underset{(N+1)p\times(N+1)n}{H_A} = \begin{bmatrix} 0 & H & & 0 \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & 0 & 0 & H \end{bmatrix}$$

Then

$$tr(M) = \int_0^T tr[(\nabla_\theta x)^T H^T R^{-1} H \nabla_\theta x]\, dt = \int_0^T x_A^T H_A^T R_A^{-1} H_A x_A \, dt \tag{80}$$

where

$$R_A^{-1} = \begin{bmatrix} R^{-1} & & & \bigcirc \\ & R^{-1} & & \\ & & \ddots & \\ \bigcirc & & & R^{-1} \end{bmatrix} .$$

A comparison of Eq. (78) and (80) with Eq. (1) and (4) shows that the optimal input u for the multiparameter case can be obtained by solving the same kind of linear two point boundary value problem as for the scalar case. This method, however, leads to a high-dimensional problem. Wilkie and Perkins[14] and Denery[22] have shown that the sensitivity functions for all the parameters can be obtained by linear transformations of the sensitivity functions for few of the parameters. Using these transformations, the dimension of the problem can be kept low.

Another useful norm of the information matrix M which can be used with the above technique is the smallest eigenvalue of M. The optimal input u^* is obtained as

$$u^* = \underset{u}{\mathrm{Arg}} \{ \underset{u}{\mathrm{Max}} \ \underset{e}{\mathrm{Min}} \frac{e^T M(u) e}{e^T e} \} . \tag{81}$$

The $\underset{e}{\mathrm{Min}}$ operation gives the smallest eigenvalue of M(u). One technique to solve the above Max-Min problem is to hold one variable fixed while optimizing over the other variables. For example, one can start with an assumed vector e and maximize over u. Then change e to be the normalized eigenvector of M(u) corresponding to the smallest eigenvalue and repeat the process. Unfortunately, this simple method does not lead to uniform convergence in the present case. Other methods for the solution of this problem are currently under investigation.

2.7 Extension to Systems with Process Noise

Consider a linear dynamic system

$$\dot{x} = Fx + Gu + \Gamma\eta$$
$$y = Hx + v \qquad (82)$$

where $\eta(t)$ is a gaussian white noise forcing function

$$E\{\eta(t)\} = 0 \quad , \quad E\{\eta(t)\eta^T(\tau)\} = Q\delta(t - \tau) \quad .$$

The information matrix M in this case is given in terms of the Kalman filter for the above system.[16]

$$\dot{\hat{x}} = F\hat{x} + Gu + K(y - H\hat{x}) \qquad (83)$$

$$K = \Sigma H^T R^{-1} \qquad (84)$$

$$\dot{\Sigma} = F\Sigma + \Sigma F^T + \Gamma Q \Gamma - \Sigma H^T R^{-1} H \Sigma \qquad (85)$$

where $\hat{x}$ denotes the best filtered estimate of x and Σ denotes the covariance of $\hat{x}$. The Kalman filter provides a linear causally-invertible whitening transformation for the process y since the innovation sequence $(y - H\hat{x})$ is a gaussian white noise sequence. The likelihood function is easily written in terms of the innovation sequence.[20,17] The information matrix M is given as

$$M = \int_0^T E\{(\nabla_\theta \hat{x})^T \; H^T R^{-1} H(\nabla_\theta \hat{x})\}dt \qquad (86)$$

where $\nabla_\theta \hat{x}$ denotes the sensitivity function of the filtered estimate $\hat{x}$ with respect to the unknown parameter vector θ. Note that both K and Σ are functions of θ so that the sensitivity equations are much more complicated than for the no process noise case. Mcreover M, in general, depends on the random quantities η and v so that its expected value needs to be maximized.

A special case arises when θ contains parameters from G only and the initial state is known exactly. Since K and P do not depend upon G, the sensitivity equation has a simple form

$$\dot{\nabla}_\theta \hat{x} = (F - KH)\nabla_\theta \hat{x} + \nabla_\theta Gu \tag{87}$$

K is, in general, time-varying, but if the system is completely controllable and observable, K reaches a constant steady-state value.[16] Then Eq. (87) is essentially similar to Eq. (7) except that F is replaced by (F - KH). Thus most of the theory developed in Sections 2.2, 2.3, and 2.4 carries over to this case.

2.7.1 <u>Example</u>: Let us consider example 4.1 with additive process noise.

$$\begin{aligned} \dot{x} &= -x + \theta u + \eta \\ y &= x + v \end{aligned} \tag{88}$$

where

$$E\{\eta\} = 0 \quad , \quad E\{\eta(t)\eta(\tau)\} = q\delta(t - \tau) \quad , \quad x(0) = 0 \quad .$$

The filter sensitivity equation for θ under steady-state filter gain, $k > 0$ is

$$\nabla_\theta \hat{x} = -(1 + k)\nabla_\theta \hat{x} + u \quad , \quad \nabla_\theta \hat{x}(0) = 0 \quad . \tag{89}$$

Proceeding as in example 4.1, and defining

$$\omega = [\frac{1}{\mu r} - (1 + k)^2]^{1/2} \quad \text{or} \quad \mu = \frac{1}{r[\omega^2 + (1 + k)^2]} \tag{90}$$

it is seen that the optimal input u^* obeys Eq. (45)-(47). Notice that by increasing process noise q, the gain increases and μ decreases. Thus the information $M \approx \mu E$ for the same input energy E decreases. The frequencies ω, however, remains unchanged.

2.8 State-Variable Constraints

Linear state variable constraints can be handled either directly by adding a quadratic penalty function to the performance index or indirectly by adjusting the total input energy E. The examples 2.4.1-2.4.3 show that the amplitude of the input u is determined by E. Thus by adjusting E, the amplitude of the input u and the state x can be bounded. Of course, the inputs obtained in this fashion are not strictly optimal.

III PROGRAM DEVELOPMENT AND CONSTRUCTION

This section discusses the features and structure of the computer program written to implement the theoretical development given in Section I. The objective of the program development was to provide a completely flexible computer program which would compute the optimal input given a minimal set of input parameters. These would include the system matrices F, G, and H, a specification of those parameters for which the input is to be designed along with the partials of F, G, H with respect to these parameters, the measurement noise covariance R and either the eigenvalue μ_{max} or the time interval T. The final computer algorithm is completely general as to the system dimension and the location of the parameters for which the input is to be designed.

Section 3.1 outlines the formulation which was used in order to treat the case of a vector of unknown parameters. This is followed by Section 3.2 which gives the detailed program steps with an accompanying program flowchart. Section 3.3 describes the specialized algorithms which are built in as subroutines into the main program. Principal among these are the driving logic for a weighted trace performance criterion and the driving logic for finding the correct μ_{max} for a specified data length T.

3.1 Problem Formulation

As outlined in Section II, the performance criterion being used for determining the optimal input is the weighted trace of the information matrix, given as (with $W = \text{diag}\{w_1, w_2, \ldots, w_n\}$)

$$t_r\{WM\} = t_r\{W^{1/2}MW^{1/2}\} \tag{91}$$

$$= t_r\{\int_o^T (\nabla_\theta x \cdot W^{1/2})^T H^T R^{-1} H(\nabla_\theta x \cdot W^{1/2})\, dt\}$$

where θ is an $N \times 1$ vector of the parameters for which the optimal input is to be specified, x is an $n \times 1$ vector of the system states, $(\nabla_\theta x)$ is an $n \times N$ matrix of the partial derivatives of x with respect to θ, and W is an $N \times N$ diagonal matrix of weights. In order to specify this criterion, therefore, the quantity $(\nabla_\theta x)$ must be calculated as a function of time. A differential equation for $(\nabla_\theta x)$ can be easily found from the system equation. Since the multiplication by $W^{1/2}$ represents a column operation, this is equivalent to calculating successive equations for

$$w_i^{1/2} \frac{\partial x}{\partial \theta_i} \quad , \qquad i = 1,\ldots,N \quad .$$

This can be accomplished by the following:

$$\dot{\left(w_i^{1/2} \frac{\partial x}{\partial \theta_i}\right)} = \left(w_i^{1/2} \frac{\partial F}{\partial \theta_i}\right)x + F\left(w_i^{1/2} \frac{\partial x}{\partial \theta_i}\right) + \left(w_i^{1/2} \frac{\partial G}{\partial \theta_i}\right)u \tag{92}$$

with $\dot{x} = Fx + Gu$.

An equivalent way of formulating the problem which makes use of the simultaneous computation of $w_i^{1/2}\, \partial x/\partial \theta_i$ involves the specification of augmented F_A, G_A, and H_A matrices and x_A vector. These are defined as follows (with m the number of inputs and p the number of outputs):

$$F_A = \left[\begin{array}{c|c|c|c} F & 0 & \cdots & 0 \\ \hline w_1 \frac{\partial F}{\partial \theta_1} & F & & 0 \\ \hline \vdots & \vdots & & \vdots \\ \hline w_N \frac{\partial F}{\partial \theta_N} & 0 & & F \end{array}\right] \qquad G_A = \left[\begin{array}{c} G \\ \hline w_1 \frac{\partial G}{\partial \theta_1} \\ \hline \vdots \\ \hline w_N \frac{\partial G}{\partial \theta_N} \end{array}\right]$$

(N+1)nx(N+1)n (N+1)nxm

$$H_A = \begin{bmatrix} 0 & H & 0 & \cdots & 0 \\ 0 & 0 & H & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H \end{bmatrix} \qquad R_A = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{bmatrix}$$

Np x (N+1)n Np x Np

and

$$x_A = \begin{bmatrix} x \\ w_1 \dfrac{\partial x}{\partial \theta_1} \\ w_2 \dfrac{\partial x}{\partial \theta_2} \\ \vdots \\ w_N \dfrac{\partial x}{\partial \theta_N} \end{bmatrix}_{(N+1)n \times 1}$$

With these definitions and using Eq. (92), it is possible to write:

$$\dot{x}_A = F_A x_A + G_A u$$

$$\begin{bmatrix} w_1 \dfrac{\partial y}{\partial \theta_1} \\ w_2 \dfrac{\partial y}{\partial \theta_2} \\ \vdots \\ w_N \dfrac{\partial y}{\partial \theta_N} \end{bmatrix} = H_A x_A \quad . \tag{93}$$

If the performance criterion is now redefined as

$$t_r\{\int_o^{T_{max}} x_A^T H_A^T R_A^{-1} H_A x_A \; dt\} \tag{94}$$

it can be easily shown that this is equivalent to $t_r\{WM\}$ as defined in Eq. (91). Therefore, with the problem analytically redefined with the system matrices F_A, G_A, and H_A, the computation of the performance criterion can be performed without the use of tensor quantities. The complete theoretical formulation for the optimal input, as given in Section II, carries forward directly, with a very specific sequence of programming steps.

3.2 Steps in Optimal Input Program

As outlined in Section II, there are several possible computational techniques which can be used to solve for a consistent pair of interval length T and largest eigenvalue μ_{max}. The method that has been implemented in this program is to numerically find the first time instant at which the solution of the ricatti equation, relating $\nabla_\theta x(t)$ and $\lambda(t)$, becomes infinite. The instant at which this singularity occurs can be found with arbitrary accuracy by continually using a finer and finer step size and noting the instant at which the diagonal elements of the ricatti solution change sign.* This change in sign of the diagonal elements is one key indication that the ricatti solution has blown up through one direction (e.g., $-\infty$) and returned from the other (e.g., $+\infty$).

The complete flowchart of the computer program designed to calculate optimal inputs is given in Fig. 3.1. Many of the detailed steps have been combined into a single descriptive step since their description is beyond the scope of this report. For example, the actual technique used to integrate the ricatti equation is explained elsewhere.[23]

The only instant in the computational algorithm where the theoretical development was not followed exactly was in the calculation of the eigenvector of $\phi_{\lambda\lambda}(T,0;\mu)$ associated with the zero eigenvalue. The problem encountered was basically numerical. Very seldom, if at all, would the matrix $\phi_{\lambda\lambda}(T,0;\mu)$ exhibit an exactly zero eigenvalue. This was caused, to a large extent, by the fact that T is never the exact instant of singularity of the ricatti solution. This difficulty was resolved by choosing $\lambda(0)$ such that the product $\phi_{\lambda\lambda}(T,0;\mu)\ \lambda(0)$ had the smallest norm. If $\phi_{\lambda\lambda}(T,0;\mu)$ did, indeed have a zero eigenvalue, this would be the associated eigenvalue and

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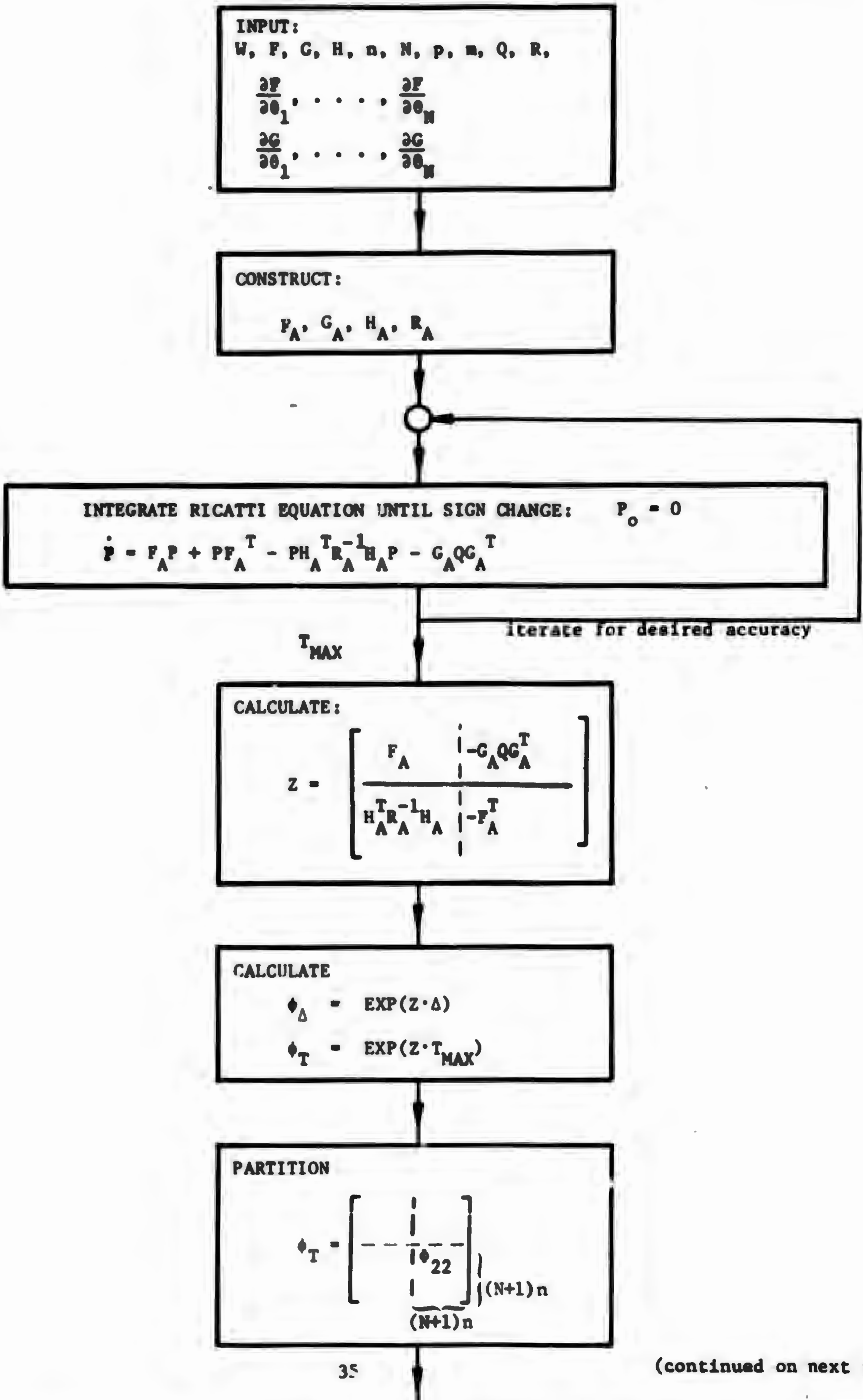
*The alternative technique of calculating the largest eigenvalue of the Hamiltonian system is much more difficult since the determinant of the transition matrix for the Hamiltonian system exhibits a very sharp zero.



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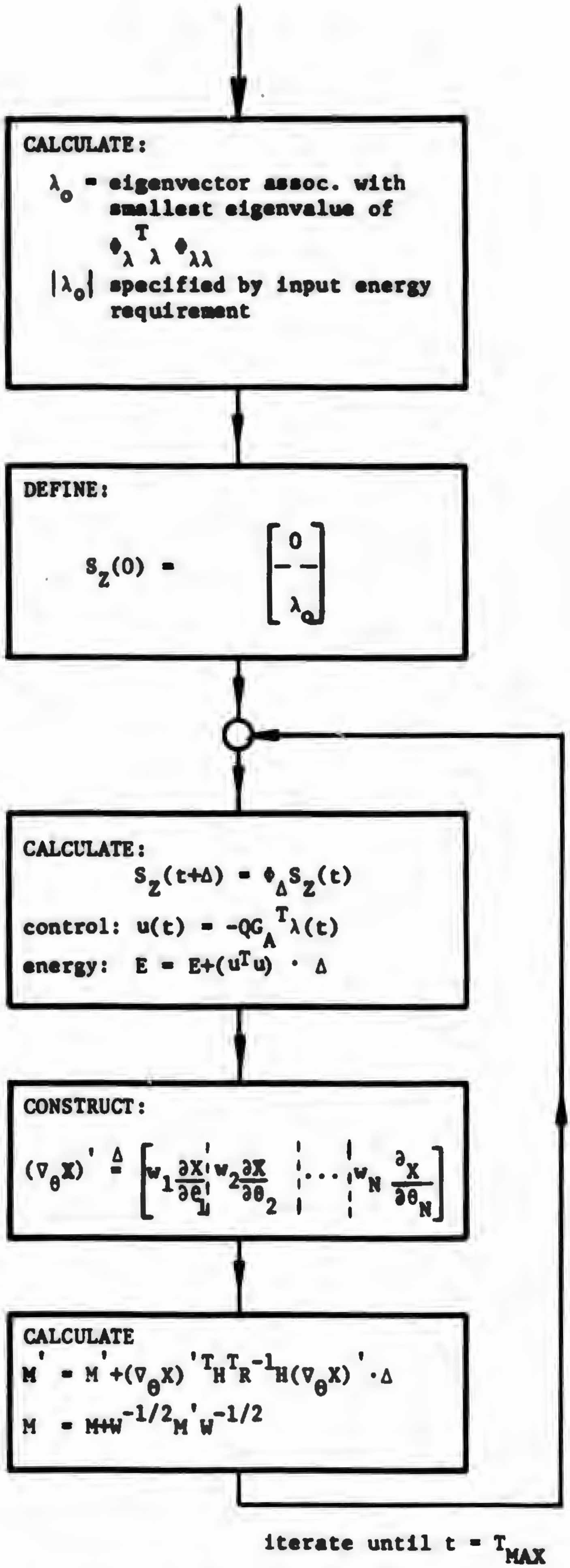


Fig. 3.1 FLOW-CHART OF OPTIMAL INPUT COMPUTER PROGRAM

$\phi_{\lambda\lambda}\ \lambda(0)$ 0. The solution of this minimum norm problem is to choose $\lambda(0)$ to be the (normalized) eigenvector associated with the smallest eigenvalue of $\phi_{\lambda\lambda}{}^T\phi_{\lambda\lambda}$. This was the technique incorporated into the optimal input program.

Two additional items should be mentioned concerning the steps in the computational algorithm. The first involves specifying some additional fact about the eigenvector $\lambda(0)$ so that it can be uniquely specified. It was shown in Section II that the control at any time point, t, is a linear functional of $\lambda(t)$. Therefore, if a particular input energy is required, the norm of $\lambda(0)$ is scaled to the proper amount.

The second item concerns the reconstruction of the M matrix as the last step in the computer algorithm. To just calculate the performance index, $t_r\{WM\}$, it would be sufficient to use

$$t_r\{\int_o^T x_A^T H_A^T R_A^{-1} H_A x_A dt\} \quad .$$

However, the information matrix M, the Cramer-Rao lower bound M^{-1}, and the det(M) give important information about the identification of the parameters which is not reflected in $t_r\{WM\}$. For this reason, the augmented state vector, x_A, is rearranged to construct the matrix $(\nabla_\theta x)W^{1/2}$, which is used to compute the information matrix M. In addition, the eigenvalues of M and the figure-of-merit $tr\{M\} \cdot tr\{M^{-1}\}$ is also computed.

3.3 Specialized Algorithms

This section describes the two specialized algorithms which have been incorporated into the main program. These are: (i) the algorithm for computing the optimal set of weights when using weighted trace as the performance criterion, and (ii) the algorithm for computing the optimal input when the data length, T, is specified.

An Algorithm for Choosing Weights:

As outlined in the second section, the purpose of the weighted trace is to obtain a closer approximation to the determinant of M as the performance criterion. This is done by bringing the ratio of the largest to the smallest diagonal element of M as close to 1 as possible. The computer algorithm takes the form of an iterative sequence of choosing weights, calculating the resulting M and then updating the weights and repeating.

The formula for the updating of the weights, once an optimal input has been computed along with the accompanying information matrix, is as follows:

$$w_{new_i} = w_{old_i} + \alpha[\frac{1}{m_{ii}} - w_{old_i}] \qquad (95)$$

where m_{ii} is the i^{th} diagonal element of M. Additional logic was subsequently added to the program to reduce the factor α by successive factors of 2 if the new set of weights failed to reduce the ratio of the largest to smallest diagonal element of M. This was made necessary since the equation (95) for updating the weights is by no means optimal.

An Algorithm for Determining μ_{max} for a Given T:

The second special algorithm built into the program enables a user to determine the μ_{max} for a specific length of data, T. The most direct, but costly, way of determining a value of μ_{max} would be to pick several values of μ_{max} and run through the program, finding the associated values of T. These pairs (μ_{max},T) could then be used to construct a μ_{max}-T curve, and from this curve, the μ_{max} for a desired T could be determined. However, this procedure would require a great many (μ_{max},T) pairs. A more direct method is to employ the optimal input program with associated zero crossing logic which would converge onto the correct μ_{max}. For the simple case of (μ_1,T_1) and (μ_2,T_2) as known associated pairs and $T_1 > T_d > T_2$ where T_d is the desired data length, the iterative equation for successive choices of μ_d is $(\mu_{d_2} \equiv \mu_2)$ [24].

$$\frac{1}{\mu_{d_{i+1}}} = \frac{(T_i - T_d)}{T_i - T_1}\frac{1}{\mu_1} - \frac{(T_2 - T_d)}{T_i - T_1}\frac{1}{\mu_{d_i}} \quad . \tag{96}$$

With a new, updated choice of μ_{d_i}, the program is run and an associated T_i is found. This new pair is used in Eq. (96) to find an updated μ_{d_i}, and so on. Once the change in values for μ_d becomes smaller than some ϵ, the procedure is stopped. For ϵ of 1%, this procedure usually requires only four or five repetitions. Of course, if the value of T_d is outside the range of the given initial pair (μ_1,T_1) and (μ_2,T_2), the logic of Eq. (96) is altered appropriately.

Examples of both these specialized algorithms are given in Section IV.

IV

NUMERICAL RESULTS

4.1 Introduction

This section deals with the numerical examples which were performed in support of the theoretical work on optimal input design. Three classes of experiments were conducted. The first was to validate the theoretical results with simple single dimensional examples for which the optimal input could be analytically calculated in closed form. The second class of experiments dealt with more difficult second order examples and the comparison of optimal inputs against other common input signals. The third class of experiments was a detailed analysis for the short period longitudinal dynamics of a C8 aircraft. For this example the weighted trace performance measure was used, the frequency spectrum of the optimal input was analyzed, and the performance was compared against doublet type of inputs, which are commonly used in flight testing.

In this section, several performance criteria are used in comparing the effectiveness of different inputs. As detailed in Section II, the principal objective of an optimal input is to enhance the ability to identify the unknown parameters and to predict the response of the system more accurately. The ideal performance measure can be shown to be the determinant of the covariance matrix of the parameter estimates.[25] However, this leads to a highly nonlinear optimization problem and requires an iterative numerical optimization technique and global search to derive the optimal input. Therefore, the basic performance criterion is chosen here as maximizing the weighted trace of the information matrix, M, where the weights are either prespecified or chosen to make all the diagonal elements of M equal. As outlined earlier, the maximization of the weighted trace is an approximation to maximizing the determinant of M. Notice that maximizing the determinant of M is equivalent to minimizing the determinant of M^{-1} which has the property of being the lower bound on the volume of the uncertainty ellipsoid (in parameter space) for the unknown parameters. It is easily shown that the smaller this uncertainty volume can be made, the more accurate will be the predicted response of the system.

Section 4.2 below gives the results for the one dimensional examples and Section 4.3 gives the results for the more complicated two dimensional cases. Section 4.4 presents the results for the C8 aircraft consisting of using the weighted trace criterion, a comparison against other inputs and a frequency domain analysis.

4.2 One Dimensional Examples

Two one dimensional examples are employed to verify the computer program by numerically computing the optimal input and comparing it with the closed form solutions (see Section 2.4).

4.2.1 Single Parameter in G Example

The system equation in the state vector form for the single parameter in G example is

$$\dot{x} = -x + \theta u \quad , \quad x(0) = 0$$

$$y = x + v$$

where v is a zero mean white noise with variance r. As given in Section II, the analytic solution for the optimal input for this example is

$$u(t) = C_1 \sin(\omega t + C_2)$$

with the boundary conditions

$$u(T_{max}) = 0$$

and

$$\int_0^T C_1^2 \sin^2(\omega t + C_2)dt = E$$

and ω is the smallest solution of the equation

$$\tan \omega T = -\omega \quad . \tag{4.4}$$

For the values $\mu_{max} = 1/2$, $r = 1$, and the frequency $\omega = 1$ this equation has the solution $T = 2.36$ sec.

The numerical result for the optimal input with the parameters μ_{max} and r specified above appears in Fig. 4.1 along with the analytic solution for the optimal input. As Fig. 4.1 indicates, the numerical solution is in close agreement with the analytic solution. The standard deviation in the parameter estimate for this example is 0.082. Notice that the parameter value itself need not be specified since it does not appear in the differential equation for $\nabla_\theta x$.

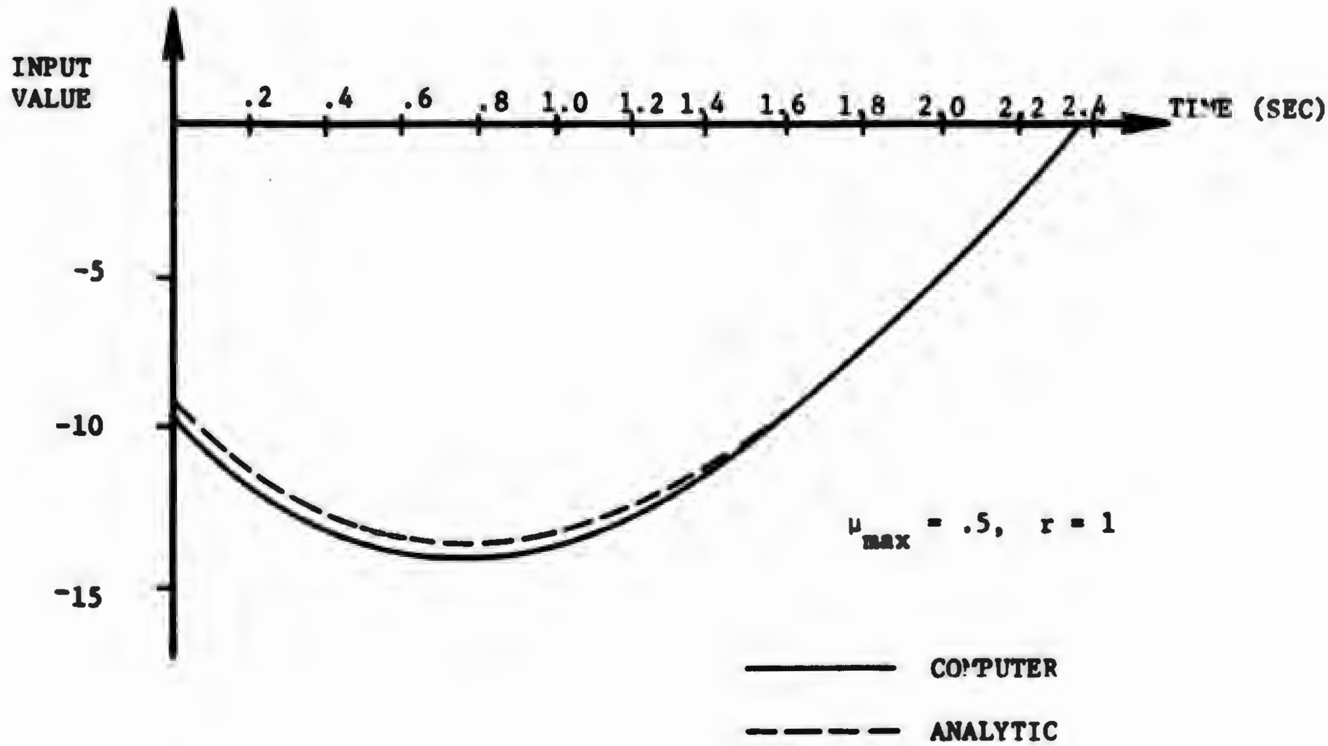
4.2.2 Single Parameter in F Example

The system equation for the single parameter in F example was the following

$$\dot{x} = -\theta x + u \qquad x(0) = 0$$

$$y = x + v$$

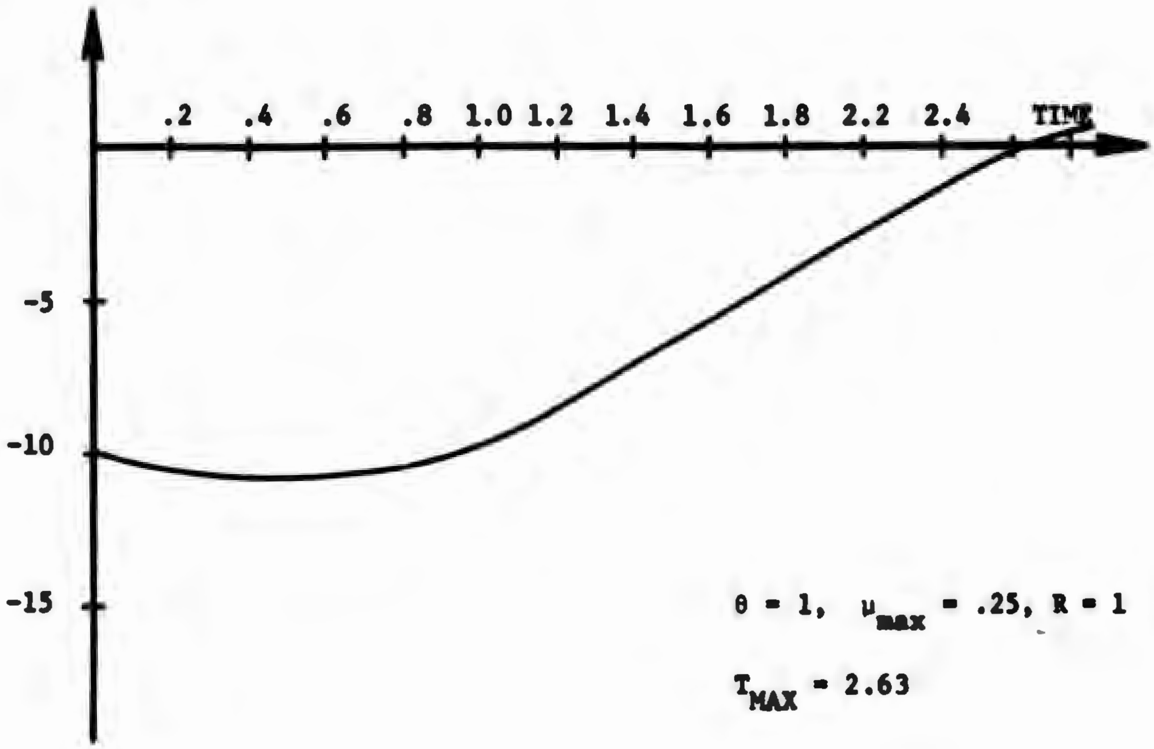
with v again a zero mean white noise of variance r. Notice that the parameter θ will appear explicitly in the equation for $\nabla_\theta x$ and therefore a numerical value must be assumed. However, for any given value of θ, the value for the optimal data length, T, can be found explicitly. With $\theta = 1$, a similar equation relating ω and T as was used in the single parameter in G case can be used. For $\theta = 1$, $r = 1$, and $\mu_{max} = 0.25$ the optimal input appears as in Fig. 4.2 for total energy of 152. Note that the T of 2.63 sec obtained from the numerical methods was slightly larger than the theoretical T computed as 2.36. This error is not as serious as might appear since the largest positive value of the input between time 2.4 and 2.63 secs is 0.018, which implies the numerically calculated input is relatively close to zero between the theoretical



OPTIMAL (ANALYTIC) INPUT = -14.1 SIN (t - 2.36)

SINGLE PARAMETER IN G

Fig. 4.1 COMPARISON OF COMPUTED AND ANALYTICAL OPTIMAL INPUTS



ENERGY IN INPUT = 152.

Fig. 4.2 OPTIMAL INPUT WITH RESPECT TO SINGLE PARAMETER IN F and numerical T values. The standard deviation of the error in estimating θ with this input is 0.026.

4.3 Two Dimensional Example--Six Parameters

This experiment involves a two dimensional example with one input and six unknown parameters. One of the important aspects of this example is the necessity of using the eigenvector associated with the minimum eigenvalue of $\phi^T_{\lambda\lambda}\ \phi_{\lambda\lambda}$ as the initial condition, $\lambda(0)$.

The system equations for this example are:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2.276 & 1 \\ -2.558 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1.913 \\ 1.82 \end{bmatrix} u$$

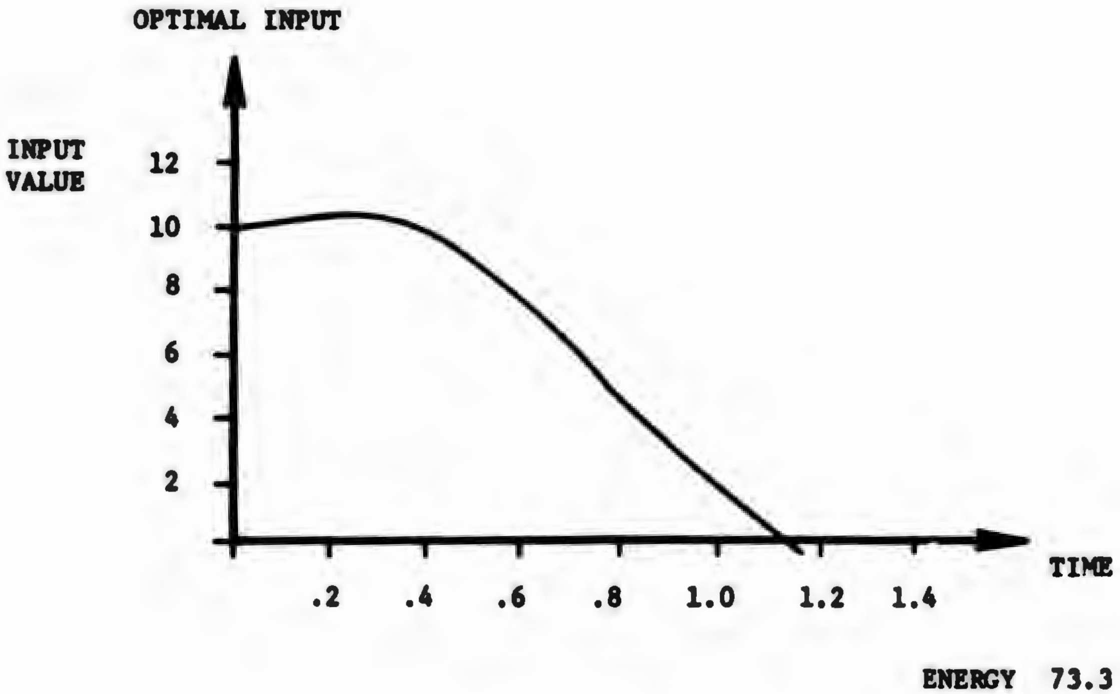
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underline{n}$$

where $\underline{n}$ is a 2x1 vector of independent, zero mean white noise with covariance $R = I \cdot 10^{-2}$.

4.3.1 Calculation of $\lambda(0)$:

The first run made uses a technique which calculates $\lambda(0)$ assuming $\phi_{\lambda\lambda}$ to have a zero eigenvalue. The resulting optimal input is shown in Fig. 4.3 for a value of $\mu_{max} = 10$. The input duration, was computed to be 1.23 seconds with an energy of 73.3. The performance criterion, tr{M}, is found to be 73.24. A check on the accuracy of the computations is to compare the values of tr{M} and μ_{max} E. As shown in Section II, they should be equal. For this particular example, they differ by less than 0.1%.

The second trial of this example involves computing $\lambda(0)$ as the eigenvalue associated with the minimum eigenvalue of $\phi^T_{\lambda\lambda}\ \phi_{\lambda\lambda}$. The norm of this $\lambda(0)$ is taken to be the same as the norm of the $\lambda(0)$ computed the previous way,



$$Q = 0.1, \; R = \begin{bmatrix} 10^{-2} & 0 \\ 0 & 10^{-2} \end{bmatrix}, \; T_{MAX} = 1.23$$

Fig. 4.3 OPTIMAL INPUT WITH RESPECT TO SIX PARAMETERS IN F, G thus assuring that the input energies would come out the same for the two cases. The resulting optimal input is shown in Fig. 4.4.

Since the integration of the Ricatti equation in no way depends on $\lambda(0)$, the point of singularity is the same in both cases, 1.23 secs. It is important to note, however, that in this latter case the zero crossing of the input is much closer to 1.23 seconds than in the previous case where $\lambda(0)$ was derived assuming $\phi_{\lambda\lambda}$ has a zero eigenvalue. If the numerical calculations were completely accurate, the zero crossing of the input would occur directly at T.

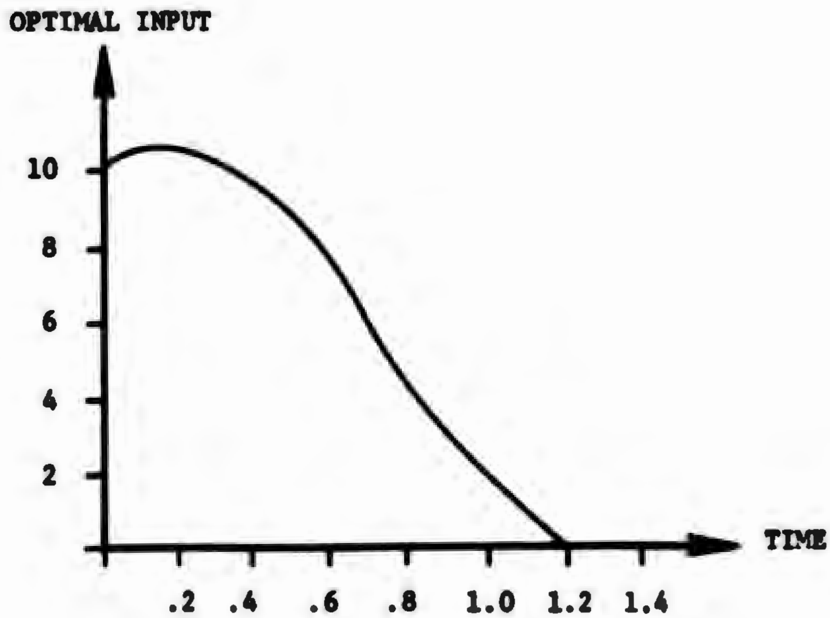
It is interesting to note that the value of tr{M} increases slightly for the latter run. The standard deviation in the parameter estimates, with the values placed in the position of the parameters in the F and G matrices, are

$$\text{Standard Deviation for } F = \begin{bmatrix} 0.158 & 0.156 \\ 0.578 & 0.54 \end{bmatrix}, \quad \text{Standard Deviation for } G = \begin{bmatrix} 0.165 \\ 0.137 \end{bmatrix}$$

The range of these 1σ values, compared to the actual parameter values, varies from 7% to 16%, except for the values in the second row of F, where the 1σ values are 23% and 54%. Increased accuracy in these parameter estimates can be achieved by increasing the input duration or increasing the measurement accuracy.

4.3.2 Comparison with a Suboptimal Input

A suboptimal input is used here for comparison purposes. The input used is shown in Fig. 4.5. The energy in the input is exactly the same as for the optimal input and the noise power was again $R = I \cdot 10^{-2}$ where I denotes the identity matrix. The computation of M, for this input, is straightforward since only the differential equations for x and $\nabla_\theta x$ must be solved. The results for the doublet input are



tr{M} = 7460 E = 73.3

Q = .01, R = 10^{-2}, T_{MAX} = 1.23

USE $\mu_{MIN_{\phi_{\lambda\lambda}}} T_{\phi_{\lambda\lambda}}$ TO CALCULATE λ(0) (AND μ(0))

Fig. 4.4 OPTIMAL INPUT WITH RESPECT TO SIX PARAMETERS IN F, G

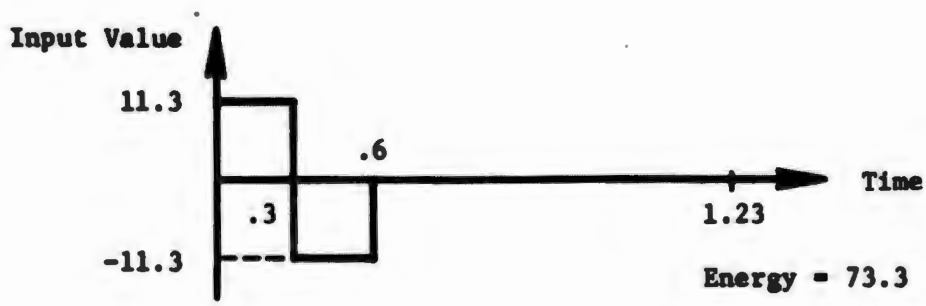


Fig. 4.5 Suboptimal Input Used For Comparison With Optimal Input of Fig. 4.4

tr{M} = 694

Standard Deviation for $F = \begin{bmatrix} 0.387 & 0.344 \\ 0.503 & 0.485 \end{bmatrix}$, Standard Deviation for $G = \begin{bmatrix} 0.112 \\ 0.102 \end{bmatrix}$

Since the standard deviations do not enter the performance measure individually a uniform improvement in the standard deviations when the optimal input is used is not expected. However, the tr{M} and tr{M^{-1}} are significantly better for the optimal input.

4.3.3 Sensitivity of Optimal Input to Assumed Parameter Values

In an actual use of the input design program, the values of the parameters fo which the input is to be designed would be unknown. Therefore, it is important to determine how sensitive the optimal input is to slight changes in the parameter values. For this purpose, the following system model is used to design the input.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1.949 & 1 \\ -2.814 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1.913 \\ 1.82 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

The parameter values here represent about a 10% change for two of the F matrix parameters. The covariance, R, of the measurement noise was fixed at the same value as was μ_{max} and the input energy. The resulting input appears in Fig. 4.6. Although, the data length came out to be 1.2 sec, which is 0.3 seconds off, the overall shape of optimal input with the altered parameter values was extremely close to the original input shown in Fig. 4.4. This implies that at least for this limited class of examples, small changes in the parameter values do not materially affect the optimal input.

4.4 Short Period Longitudinal Dynamics of C-8 Aircraft

4.4.1 This experiment deals with finding the optimal input with respect to the parameters in the short period equations of a C8 aircraft. The state variables are the angle-of-attack, α, and the pitch rate, q, and the input is the elevator command, δ_e. The equations for the short period dynamics of the C8 aircraft are

$$\begin{bmatrix} \dot{q} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -1.588 & -.562 \\ 1 & 0.737 \end{bmatrix} \begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} -1.66 \\ 0.005 \end{bmatrix} \delta_e$$

and the measurement equations are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} n_q \\ n_\alpha \end{bmatrix}$$

In determining the power spectral densities of n_q and n_α, the values given in Reference [15] (1 deg./sec. error in q, and 2^o in α) are multiplied by two times the correlation time of the noise sources, which is assumed to be 0.01 secs. The measurement noise spectral density matrix is therefore given as

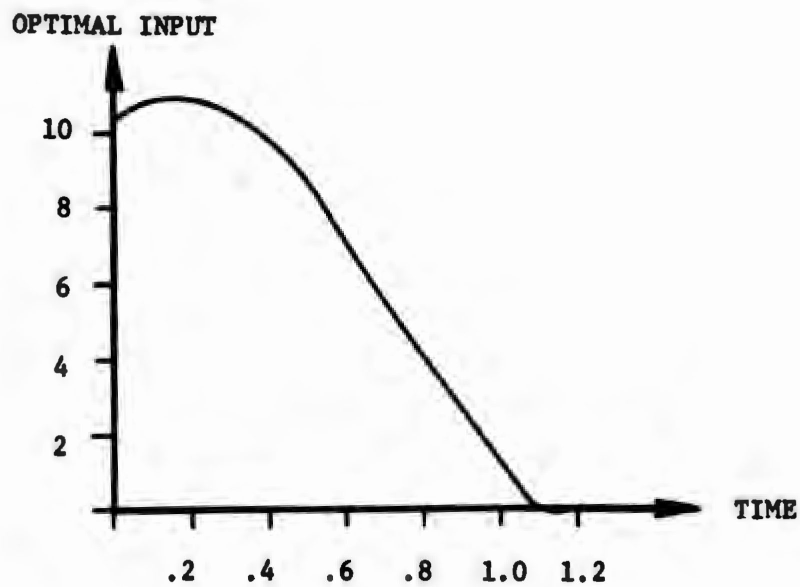


Fig. 4.6 EFFECT OF VARIATION IN TWO F PARAMETER VALUES ON THE OPTIMAL INPUT

$$R = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.04 \end{bmatrix}$$

In this experiment the data length T is fixed at 4 secs. The appropriate μ_{max} is found from a μ_{max}-T curve shown in Fig. 4.7. The value of μ_{max} associated with a T of 4 secs is 0.015. The shape of the μ_{max}-T curve in Fig. 4.7 is characteristic of the general relationship between these two variables.

For the μ_{max} and R values indicated above the optimal input with respect to the three parameters in the F matrix and the two parameters in G is given in Fig. 4.8. The energy of the input is 311 and tr{M} = 20,460. The check value of μ_{max} E is approximately 20,200, indicating a numerical error of 0.1%. The determinant of M is computed to be 1×10^{15}, with the ratio of the largest to smallest eigenvalue of M being almost three orders of magnitude. The eigenvalues themselves indicated that much of the relative uncertainty in the parameter estimates is concentrated in two of the five dimensions. The standard deviations of the parameter estimates, are

$$\begin{matrix}\text{Standard} \\ \text{Deviation}\end{matrix} \text{ for } F = \begin{bmatrix} 0.167 & 0.0639 \\ & 0.035 \end{bmatrix}, \quad \begin{matrix}\text{Standard} \\ \text{Deviation}\end{matrix} \text{ for } G = \begin{bmatrix} 0.095 \\ 0.025 \end{bmatrix}$$

The time histories of the states α and q, resulting from the optimal input are shown in Fig. 4.9. The energy of the input is so comprised that the α does not exceed 10°. This method of energy limitation is the most direct way of applying state constraints; although, as mentioned in Section II, the penalty function approach can also be used.

4.4.2 Fourier Transform of the Optimal Input

Since it was specified that the input be designed with respect to the parameters both in F and G, it would be reasonable to infer that the input would have a low frequency component for identifying the parameters in G and

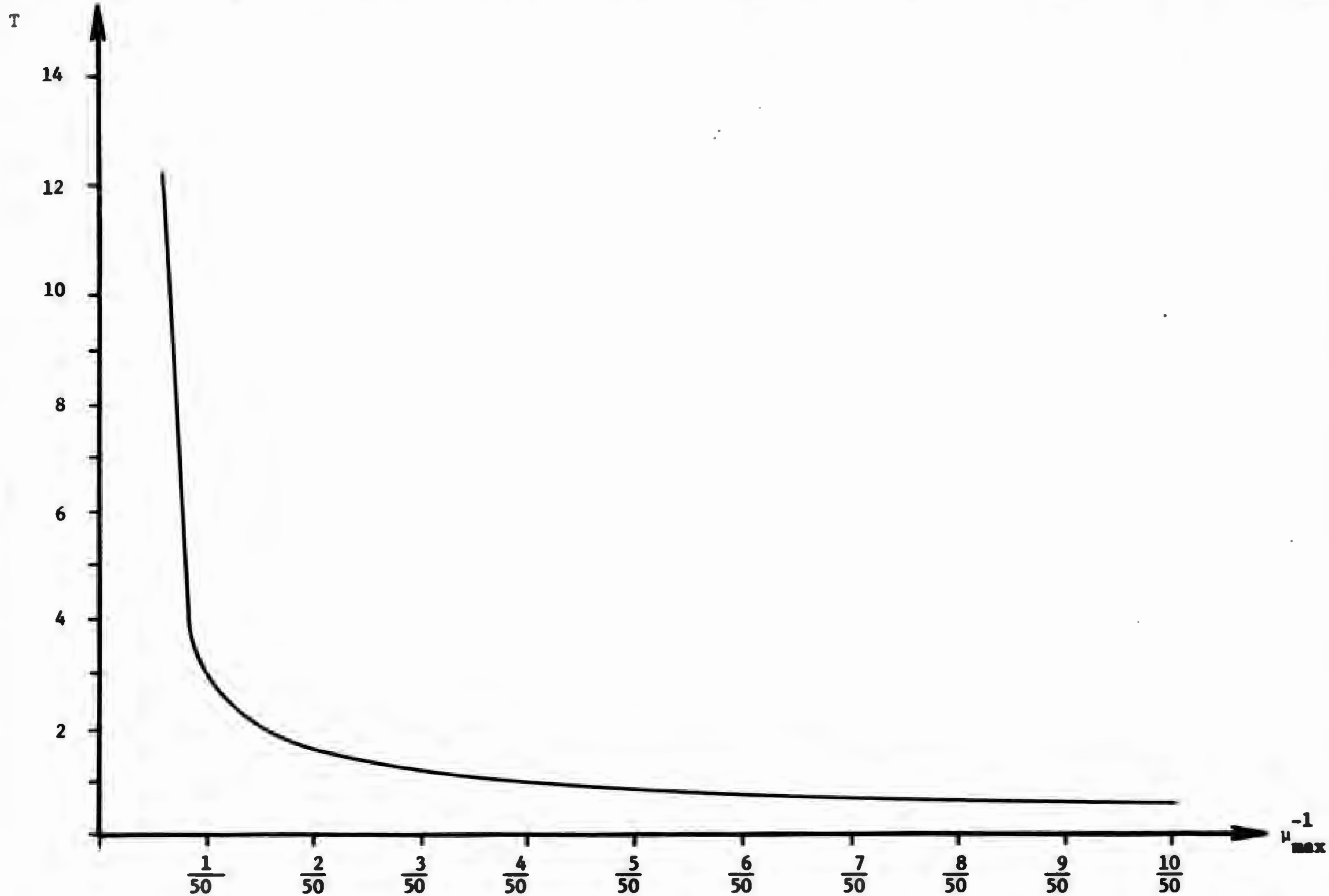
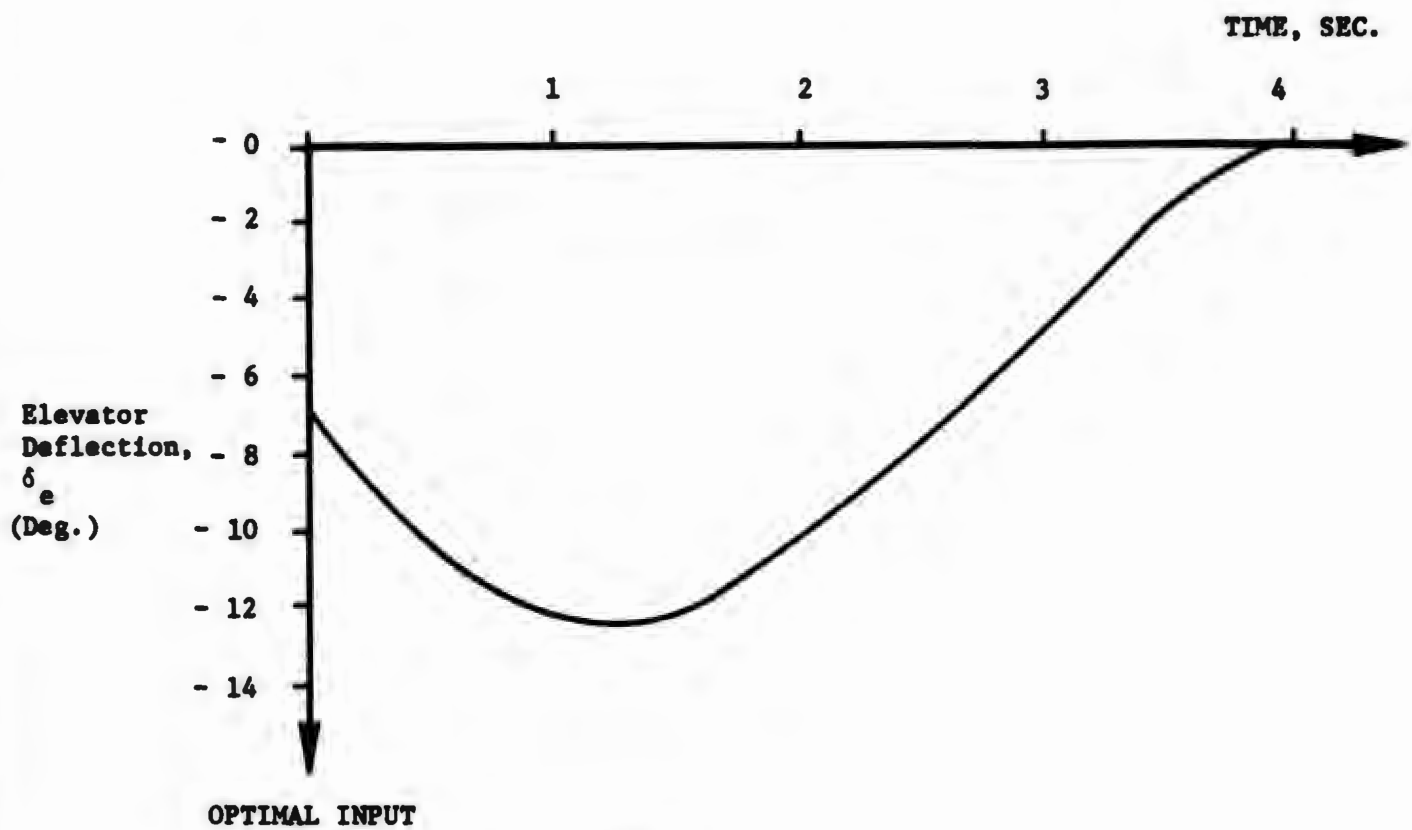


Fig. 4.7 μ_{max}^{-1} VS. T CURVE FOR A 2 STATE/5 PARAMETER MODEL



Tr{M} = 20460 E = 311

Fig. 4.8 OPTIMAL INPUT FOR SHORT PERIOD LONGITUDINAL DYNAMICS

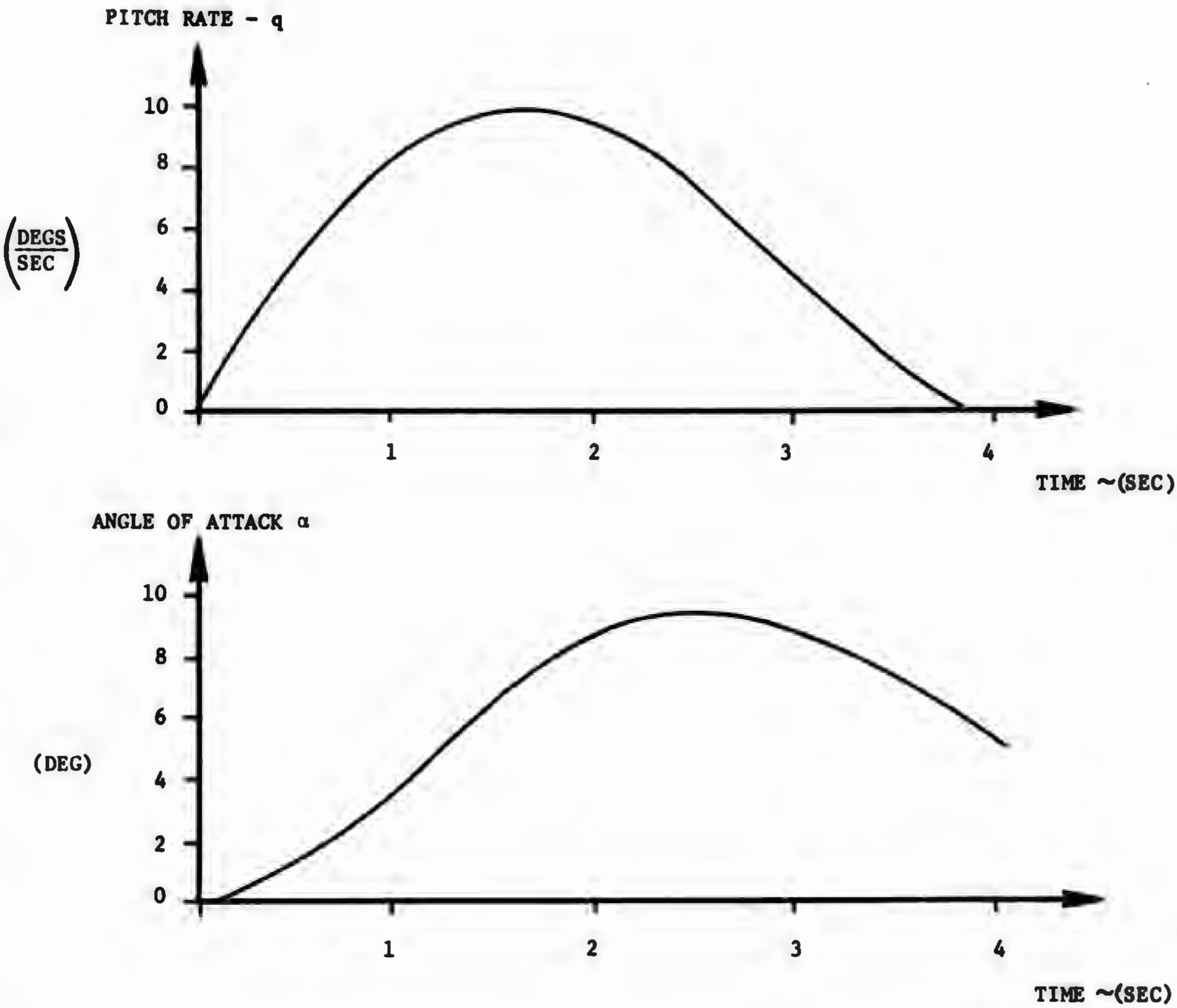


Fig. 4.9 PITCH RATE AND ANGLE-OF-ATTACK TIME HISTORIES WITH OPTIMAL INPUT

a high frequency component for identifying the parameters in F. The actual fourier transform of the input signal is given in Fig. 4.10. The vertical scale has been reduced to 1/10 its actual height in order to illustrate the smaller variations. The DC component is 0.98. The small peak in the transform occurs at a frequency of 0.375 cyc/sec, which is close to the short period frequency of the aircraft.

The most important point demonstrated by the frequency domain analysis is that the sinusoidal like component of the input signal occurred at a very specific frequency. It is well known that the maximum signal power can be obtained from a second order system if it is excited at its actual frequency. Therefore, in order to maximize the sensitivity of the output signal to the parameters in the F matrix (which is given by the M matrix), or in other words, to maximize the component of the output signal due to the parameters in F, the input signal had a specific component set at the actual system frequency.

4.4.3 Weighted Trace Criterion:

The third part of the experiment involved using the weighted trace criterion to derive the optimal input and choosing the weights to make the diagonal elements of the information matrix equal. As detailed in Section III, the performance criterion is tr{WM} where W is a diagonal matrix of weights chosen to set $\omega_1 m_{11} = \omega_2 m_{22} = \ldots = \omega_p m_{pp}$. When all the diagonal elements of WM are equal, maximizing the trace of WM is equivalent to maximizing the product of the diagonal elements which is a better approximation to det(M).

Since the input which maximizes the performance criterion depends on the values of the weights, which in turn affect the input, an iterative scheme is used to update the weights until convergence is achieved.

The optimal input and the state time histories for a T of 0.77 sec and an energy of 62.61, and unity weights are given in Figs. 4.11 and 4.12 The trace of M for this input is 611 and the determinant of M is 8.11×10^3.

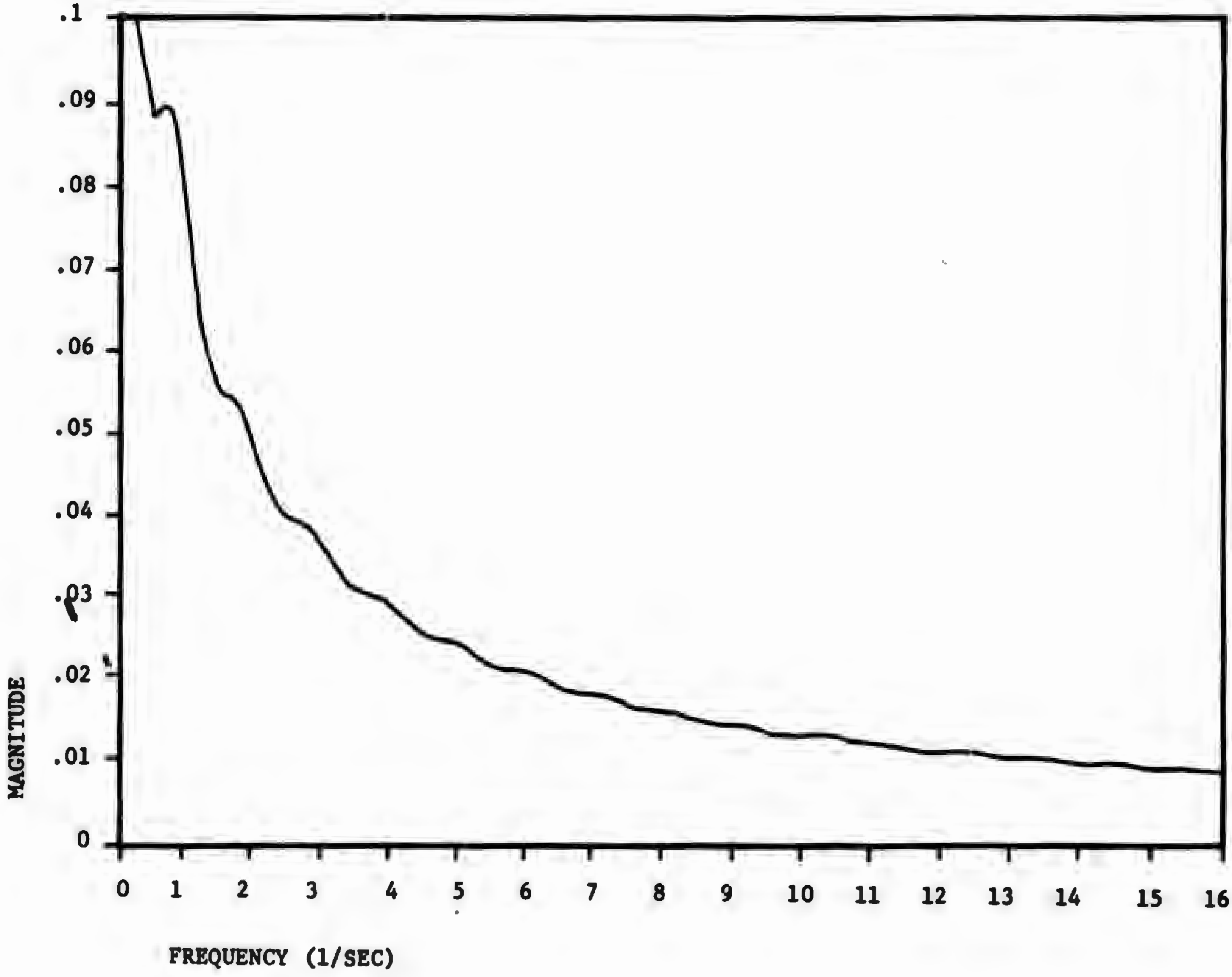


Fig. 4.10 FOURIER TRANSFORM OF OPTIMAL INPUT

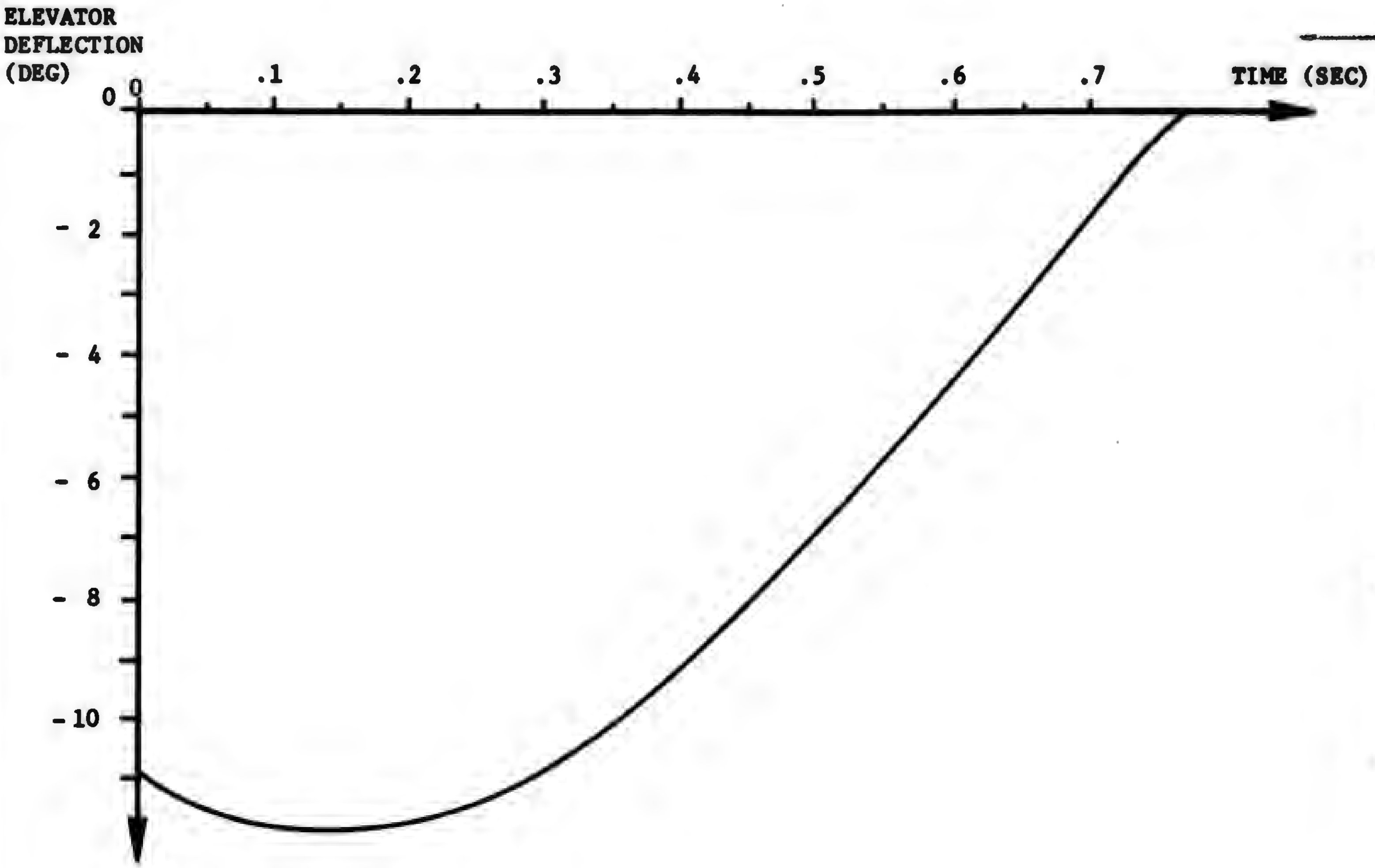


Fig. 4.11 OPTIMAL ELEVATOR DEFLECTION WITH UNITY WEIGHTS

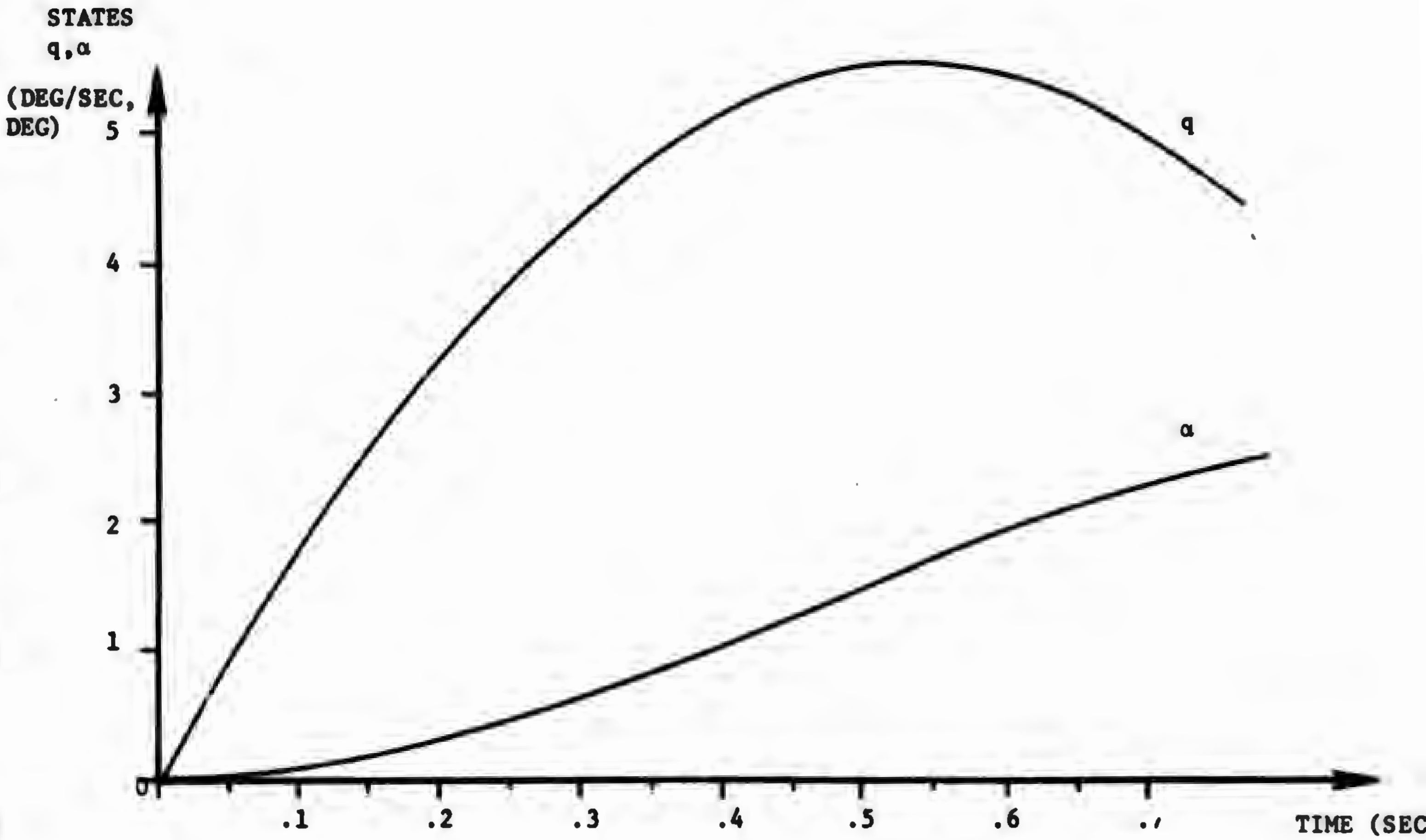


Fig. 4.12 OPTIMAL STATE TIME HISTORIES FOR UNITY WEIGHTS

The ratio of the largest to smallest eigenvalue of M is 1270. The eigenvalues themselves indicate that the parameter uncertainty is quite disproportionate along two of the eigenvector directions. The standard deviation in the parameter estimates is given below.

$$\begin{matrix}\text{Standard}\\\text{Deviation}\end{matrix} \text{ for } F = \begin{bmatrix} 2.03 & 5.57 \\ & 1.43 \end{bmatrix} , \quad \begin{matrix}\text{Standard}\\\text{Deviation}\end{matrix} \text{ for } G = \begin{bmatrix} 0.381 \\ 0.129 \end{bmatrix}$$

Using the weighted trace criterion, and 12 iterations to bring the ratio of the largest to smallest element of WM down to 1.14, the optimal input and state time histories given in Fig. 4.13 are obtained. The determinant of M is calculated to be 2.27×10^4 which is five times greater than the unity weights determined. The volume of the uncertainty ellipsoid decreases by the same factor. Another indication of this is the fact that the ratio of the largest to smallest eigenvalue of M has been reduced by a factor of 2 to 640, with the largest eigenvalue itself being reduced by a factor of 2. The standard deviations for the parameter estimates are as follows:

$$\begin{matrix}\text{Standard}\\\text{Deviation}\end{matrix} \text{ for } F = \begin{bmatrix} 1.56 & 4.04 \\ & 1.23 \end{bmatrix} , \quad \begin{matrix}\text{Standard}\\\text{Deviation}\end{matrix} \text{ for } G = \begin{bmatrix} 0.304 \\ 0.122 \end{bmatrix}$$

Notice that the parameters which are poorly estimated with unity weights are assigned higher weights. As a result, the lengths of the error ellipsoid along each of the axes have become more uniform and the total volume of the uncertainty ellipsoid has decreased.

It is clear from this example that using a weighted trace criterion does result in an input which can, in an overall sense, identify the unknown parameters with improved accuracy. This improvement is measured by the increase in the value of the determinant of M.

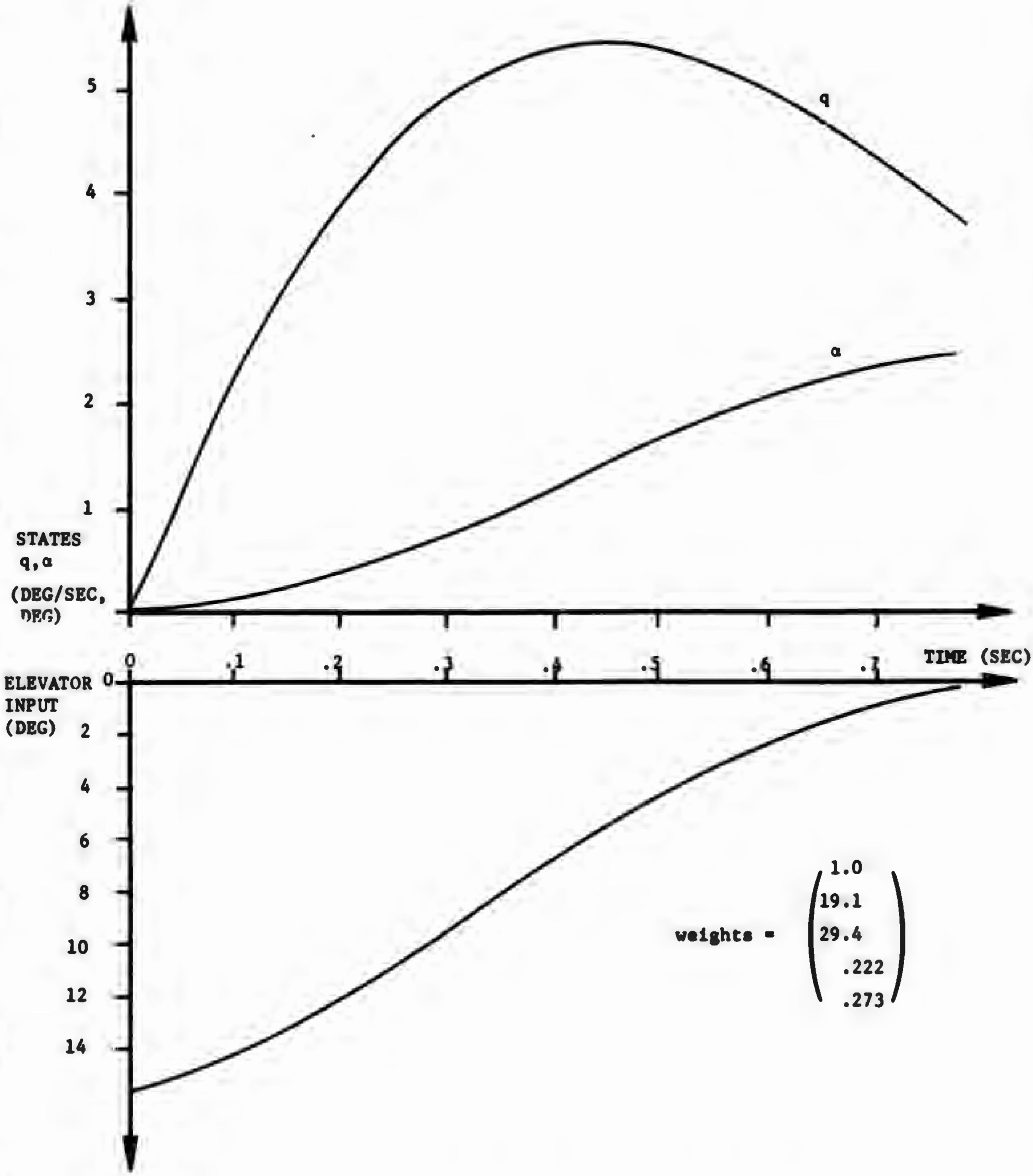


Fig. 4.13 OPTIMAL INPUT AND STATE TIME HISTORIES
- WITH WEIGHTED TRACE

4.4.4 Comparison with Doublet Input

The last step of the numerical experiment was to compare the optimal inputs, for both the unity and weighted trace criteria, against a common input. The data length is kept at 0.77 secs and the input energy is 62.61. The input used is shown in Fig. 4.14.

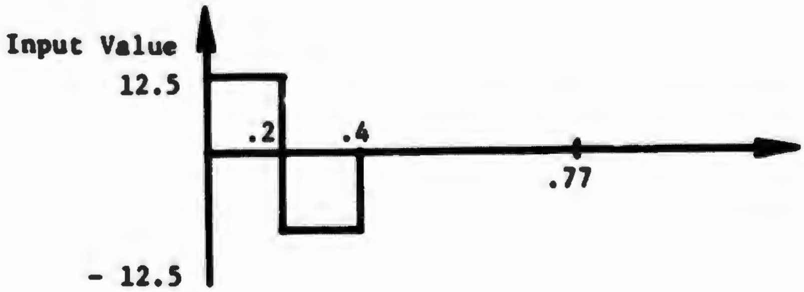


Fig. 4.14 DOUBLET INPUT USED FOR COMPARISON WITH OPTIMAL INPUT

The determinant of M for the doublet input is 1.19×10^2, which is two orders of magnitude less than that obtained by the optimal weighted trace input. The standard deviations of the parameter estimate are

$$\text{Standard Deviation for } F = \begin{bmatrix} 0.955 & 2.62 \\ & 1.78 \end{bmatrix}, \quad \text{Standard Deviation for } G = \begin{bmatrix} 0.232 \\ 0.228 \end{bmatrix}$$

4.5 C-8 Monte-Carlo Simulation

A complete C-8 Monte Carlo simulation involving generation of flight test data using optimal and doublet inputs, indentification of parameters and comparisons based on sample statistics is currently under development. The results of this investigation will be reported elsewhere.[26]

V

CONCLUSION

5.1 Summary of Results

This report has considered the design of optimal inputs for linear system identification. The criterion used for optimization is a weighted trace of the Fisher information matrix for estimating the unknown parameters in a linear model. The inputs are assumed to be energy or power constrained. General results characterizing the optimal input are developed and several numerical algorithms are given for their computation. It is shown that the optimal input is an eigenfunction of a positive self-adjoint operator corresponding to the largest eigenvalue. Four different numerical algorithms are described for the computation of optimal inputs. These are

1. A Transition Matrix method,
2. Riccati Equation method,
3. A Resolvent method,
4. Ritz-Galerkin method.

Numerical results are obtained by using the Riccati Equation Method and the Resolvent Method.

Three analytical examples involving first and second order system dynamics are solved in closed-form. The nature of the solutions gives important insights into the nature of the optimal inputs. Extensions to other norms of the information matrix and to systems with process noise are also indicated.

Section III of the report describes the details of a computer program for designing optimal inputs based on the theoretical results of Section II. The results of using this program for designing elevator flight test inputs to identify the stability and control derivatives of a C-8 aircraft are given in Section IV. Also included are several other examples to verify the theoretical results and show their sensitivity to a priori assumptions on the parameter values.

5.2 Recommendations for Further Research

Following areas are recommended for future research.

Process Noise Case: When process noise is present in the state equations, the information matrix M, in general, becomes data-dependent. This requires using the expected value of M as the criterion for optimization. Moreover, the expression for the information matrix M is given in terms of the Kalman filter for the system. The resulting optimization problem is much more difficult to solve numerically. A special case was considered in Section 2.6.

Dimensionality Problem: The dimensionality problem arises due to the augmentation of the state vector with all the sensitivity functions. It has been shown by Wilkie and Perkins[14] and Denery[15] that this is unnecessary since most of the sensitivity functions can be obtained by linear transformations of a few of the sensitivity functions. Thus the dimension of the augmented state vector can be kept fixed even though the number of parameters may increase. It would be interesting and useful from an applications viewpoint to use the techniques of Ref. [14] and [15] and, if necessary, to develop new techniques in order to solve the dimensionality problem.

Max-Min Criterion: As mentioned earlier, (Section 2.5) the maximization of the minimum eigenvalue of the information matrix M is a better criterion than trace (M). However, since the saddlepoint property is not satisfied (i.e., max-min is not equal to min-max), the optimization problem is not easy to solve. It is recommended that a gradient procedure be developed to search over the function space of inputs. Notice, however, that the minimization problem is finite dimensional and is easily solved by eigenvector methods.

Random Inputs: The random inputs currently in use for process dynamic identification are Pseudo Random Binary Sequence (PRBS) and white noise processes. These inputs, however, are not optimal under all situations and better identification can be achieved by varying the frequency spectrum or the correlation functions of the random inputs. Preliminary work in this direction has been done by Box and Jenkins[27] and Watts[28]. It would be interesting to relate this work with the deterministic input design work reported herein in order to develop efficient methods for designing random inputs.

VI

PUBLICATIONS UNDER THIS CONTRACT

In the following pages we describe the technical publications which have resulted from this contract.

(1) "Optimal Inputs for Linear System Identification, Part I - Theory," by R. K. Mehra (to be presented at the 1972 Joint Automatic Control Conference, Stanford, California).

<u>Abstract</u>

The theory of input design for linear system identification is developed using the Maximum Principle, Sturm-Liouville Theory and Functional Analysis. The optimal energy-constrained inputs maximizing the trace of the information matrix are characterized equivalently (1) as the solution to a homogeneous two point boundary value problem (Sturm-Liouville Problem) (2) as an eigenfunction of a positive self-adjoint operator and (3) as the solution to a Fredholm Integral equation of the second bind. Several numerical algorithms for computing optimal inputs are outlined. The choice of a suitable norm of the information matrix is discussed. The extensions to process noise and state variable constraint problems are indicated. Four analytical examples are included to describe the nature of optimal inputs.

(2) "Optimal Inputs for Linear System Identification, Part II - Numerical Results and Applications," by R. K. Mehra, D. E. Stepner, and J. S. Tyler (in preparation).

<u>Abstract</u>

This paper presents the numerical results of applying the input design theory developed in Part I to various analytical and practical examples. The application considered is the design of elevator deflection

time histories for identifying longitudinal stability and control derivatives of a C-8 aircraft. The sensitivity of the optimal input to assumed parameter values is studied. A number of performance criteria are developed for comparing the optimal input with other suboptimal inputs (step, doublet, etc.). A complete Monte Carlo simulation of the C-8 longitudinal equation is performed with data generated by using both optimal and doublet inputs and identifying the parameters from the noisy data.

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APPENDIX A

A Resolvent Method for Computing Optimal Inputs
by John Casti

In this appendix an approach centering upon the well known resolvent equation for bounded linear operators on a Banach space is outlined for obtaining optimal inputs. An initial value procedure is given which may be easily implemented on a digital computer to produce the desired characteristic elements. The results of a numerical experiment are also given to indicate the utility of the approach.

1. THE RESOLVENT EQUATION

To develop an initial value system suitable for computing the characteristic values and functions associated with the integral equation of section 2.3.1, we shall employ a small amount of elementary functional analysis.

Let A be the bounded, self-adjoint integral operator generated by the kernel ∇, i.e.,

$$[Af]\ (t) = \int_0^T \Lambda(t,\tau)\ f(\tau)\ d\tau \qquad \text{(A.1)}$$

Equation (37), section 2.3.1, may be written in the operator form

$$(\mu I - A)u = 0, \qquad \text{(A.2)}$$

The resolvent operator associated with A is defined as

$$R(\alpha) = \left(\frac{1}{\alpha}\ I - A\right)^{-1}, \quad \text{where} \quad \alpha = \frac{1}{\mu} \qquad \text{(A.3)}$$

It is a well known theorem of functional analysis that the resolvent operator is a meromorphic function of α and that Eq. (A.2) has non-trivial solutions only for those values of α which are poles of R_α (Fredholm Alternative theorem). Consequently, if a system of differential equations can be obtained for R_α, viewing α as the independent variable, the characteristic values of the operator A may be obtained by integrating such a system until the dependent variable becomes unbounded. The value of α for which this occurs will then be a characteristic value of A. In the next section we will display such a system which will be seen to be a special case of the more general resolvent equation

$$R(\alpha) - R(\alpha') = \left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right) R(\alpha)\ R(\alpha') \tag{A.4}$$

valid for any bounded, self-adjoint operator on a Banach space.

2. INITIAL VALUE SYSTEM FOR R_α

Consider the inhomogeneous form of integral equation (A.1)

$$u(t) = g(t) + \alpha \int_0^T A(t,\tau)\ u(\tau)\ dt,$$

where g is a bounded function on [0,T].

It is a simple exercise to see that the solution of this equation is given by

$$u(t,\alpha) = g(t) + \alpha \int_0^T R(t,\ \tau,\ \alpha)\ \ g(\tau)\ \ d\tau, \qquad 0 \leq t \leq T, \tag{A.5}$$

where, by a slight abuse of notations, we write $R(t,\tau,\alpha)$ for the kernel of the operator R_α discussed in the previous section. Since we are intereste interested in the behavior of u and R as functions of α, we explicitly indicate this independence in Eq. (A.5). It is also an easy matter to establish that $R(t,\tau,\alpha)$ satisfies the integral equation.

$$R(t,\tau,\alpha) = \Lambda(t,\tau) + \int_0^T \Lambda(t,s)R(s,\tau\ \alpha)\, ds, \qquad 0\leq t,\tau\leq T . \tag{A.6}$$

It is shown in Ref. (19) that an initial value system in α can be derived from Eq. (A.5) and (A.6). The resulting equations are

$$\frac{dR}{d\alpha}(t,\tau,\alpha) = \int_0^T R(t,s,\alpha)\, R(s,\tau,\alpha)\, ds , \tag{A.7}$$

$$\frac{du}{d\alpha}(t,\alpha) = \int_0^T R(t,s,\alpha)\, u(s,\alpha)\, ds , \qquad \alpha > 0 . \tag{A.8}$$

The initial conditions at $\alpha = 0$ are

$$R(t,\tau,0) = \Lambda(t,\tau), \tag{A.9}$$

$$u(t,0) = g(t), \qquad 0\leq t, \quad \tau\leq T. \tag{A.10}$$

Note that in these equations t and τ are just parameters, α being the independent variable. Thus, Eqs. (A.7) - (A.10) comprise an initial value system for ordinary differential equations.

The solution procedure is to integrate Eqs. (A.7) and (A.8), subject to the conditions (A.9) and (A.10) from $\alpha = 0$ to $\alpha = \alpha_1$, the first value of α for which R becomes unbounded. At this point the function u, being the solution of the inhomogeneous equation, will also be unbounded. The number α_1 will be the smallest characteristic value of the integral operator generated by Λ, provided Λ is symmetric in (t,τ) and is positive. To obtain the associated characteristic function, we save the function $u(t,\alpha_1-\epsilon)$, $\epsilon<<1$. It is easy to establish that the solution to the inhomogeneous problem for this value of α will arbitrarily closely approximate the characteristic function associated with α_1 by making $\alpha_1-\varepsilon$ sufficiently small. Numerical experiments with this procedure will be described in the following section.

3. NUMERICAL RESULTS

The above equations have been used to calculate the smallest characteristic value and associated function for an integral operator.

To make computational use of Eqs. (A.7) - (A.10), the integrals that appear must be replaced by finite sum approximations. In the experiments that follow, Gaussian quadrature of order N = 7 was used. Thus, Eqs. (A.7) and (A.8) become

$$R_{ij}(\alpha) = \sum_{k=1}^{7} R_{ik}(\alpha)\, R_{kj}(\alpha)\, a_k \tag{A.11}$$

$$u_i(\alpha) = \sum_{k=1}^{7} R_{ik}(\alpha)\, u_k(\alpha)\, a_k \tag{A.12}$$

$$i,j = 1,2,\ldots,7,$$

where the notation

$$R_{ij}(\alpha) = R(t_i,\ \tau_j,\alpha)$$

$$u_i(\alpha) = u(t_i,\alpha),\ i,j = 1,2,\ldots,7,$$

has been used. Here the numbers $\{t_i,\tau_j\}$, $\{a_i\}$ are respectively the nodes and weights of the Gaussian quadrature scheme of order seven. The appropriate initial conditions become

$$K_{ij}(0) = \Lambda(t_i,\ \tau_j), \tag{A.13}$$

$$u_i(0) = g(t_i),\ i,j = 1,2,\ldots,7. \tag{A.14}$$

The integration scheme employed is an Adams-Moulton predictor-corrector with step size $\Delta = 0.01$.

The experiment performed corresponds to Example 2. The integral operator is generated by the kernel $\Lambda(t,\tau) = \frac{1}{2} e^{t+\tau}[e^{-2\tau} - e^{-3\pi/2}]$, with interval length $T = 3\pi/4$. The closed form solution of this problem is $\lambda_{min} = 2.0$ with associated characteristic function $u(t) = \sin t + \cos t$. Utilizing the above formulation, the functions $R_{ij}(\alpha)$ exceeded the maximum machine number ($\sim 10^{30}$) between $\alpha = 1.96$ and $\alpha = 1.97$. Since it can be readily seen in this case that α_{min} is a simple pole of the resolvent, the functions R_{ij} should change sign as α passes through the pole. This behavior was observed in going from $\alpha = 1.96$ to $\alpha = 1.97$. Undoubtedly a finer integration step size could have refined the value 1.96+ to something much closer to the actual value. To check the accuracy of the characteristic function, we note that since the characteristic function is only determined up to a constant multiplier, division of the approximate unnormalized characteristic function $u(\alpha_1 - \epsilon)$ by the true solution should be a constant. The results of this experiment are shown in Table A.1.

Table A.1 Check on Characteristic Function

t_i	$u_i(1.96)(\times 10^{-14})$	$u_i(1.96)/(\sin t_i + \cos t_i)$ $(\times 10^{-14})$
0.0599558	4.95270	4.680646
0.3045014	5.96290	4.755808
0.6999722	6.84300	4.856441
1.1780972	6.41560	4.9102876
1.6562223	4.44060	4.874256
2.0516931	2.02860	4.7843655
2.2962386	0.399210	4.7110157

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