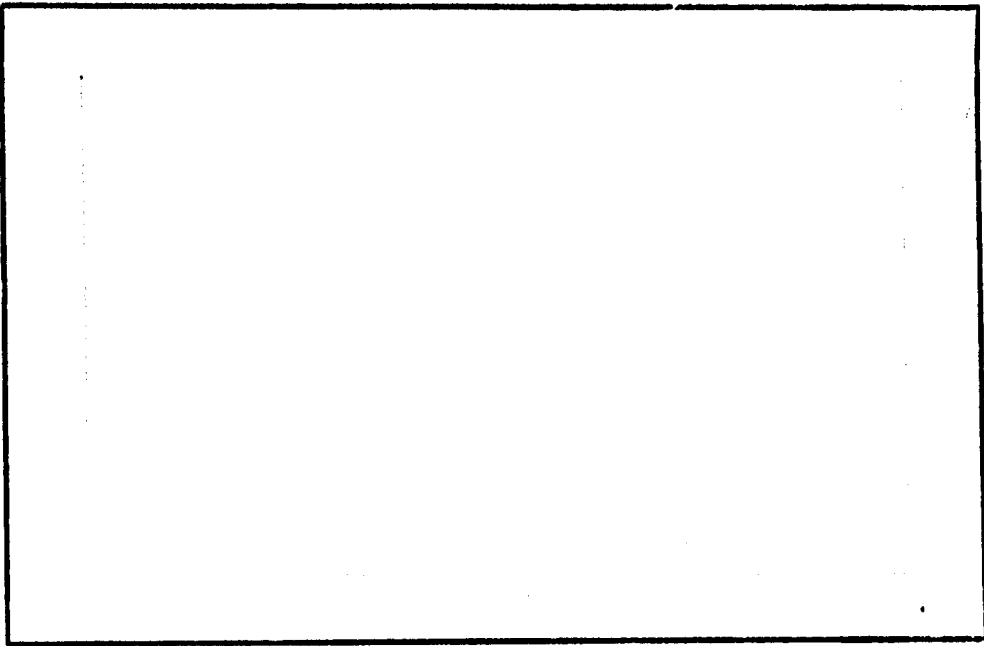


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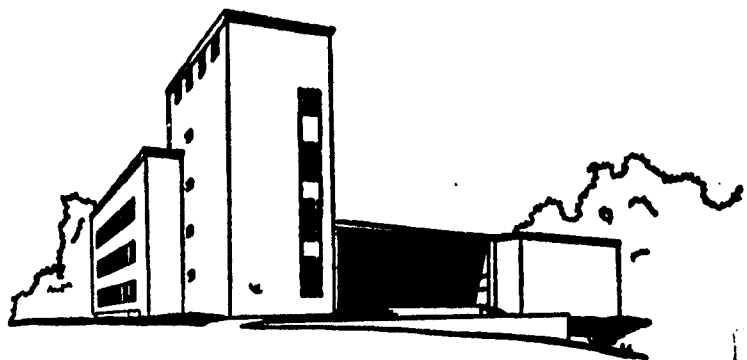
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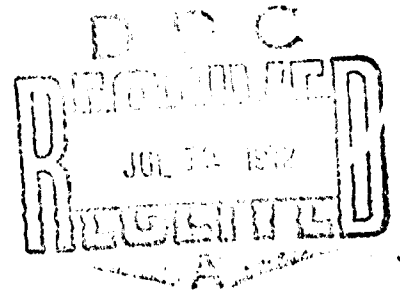
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by

Claude-Alain Burdet

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## Simple Polaroids for Non-Convex Quadratic Programming

### Abstract

This paper presents an application of polaroid sets; in the first section we show how bilinear polaroid bifunctions can be defined for an arbitrary quadratic function. The second section establishes two properties (convexity and validity) of the corresponding polaroid sets; this allows one to define valid cutting planes for the quadratic program: "optimize an arbitrary quadratic function over an arbitrary (closed) set of feasible solutions." A third section describes the structure of polaroids in relation to a given polyhedral feasible set.

IV

## Simple Polaroids for Non-Convex Quadratic Programming

by Claude-Alain Burdet

### 1) Simple polaroid bifunctions

Definition 1: A bifunction  $f$  is called simple if the function  $f(x; \cdot)$  is linear, i.e.

$$f(x; \alpha y^1 + \beta y^2) = \alpha f(x; y^1) + \beta f(x; y^2)$$

Definition 2: A function  $g$  defined on  $\mathbb{R}^n$  is said polarized by  $f$  if the following relation holds true  $\forall x \in \mathbb{R}^n$  :

$$g(x) = f(x; x)$$

Consider a quadratic function  $g$  of the form

$$g(x) = x^T C x + 2d^T x$$

with  $d$  an  $n$ -vector and  $C$  a symmetric  $n$  by  $n$  matrix.

A classical manipulation reduces  $g$  to the form

$$g(x) = \tilde{x}^T \tilde{C} \tilde{x}$$

where  $\tilde{x} = (1, x)$  and  $\tilde{C} = \begin{pmatrix} 0 & d^T \\ d & C \end{pmatrix}$ , symmetric. We may therefore, without loss of generality, restrict our analysis to quadratic functions of the form:

$$g(x) = x^T C x ;$$

2.

in fact,  $C$  need not even be assumed symmetric in the following developments.

Note that no assumptions concerning definiteness of  $g$  is required either.

We now define a polaroid bifunction  $f$  which polarizes  $g$  :

$$f(x;y) = x^T C y$$

Property 1:  $f$  is simple.

Proof: Immediate from the definition 1 since  $f$  is bilinear.

Q.E.D.

Polaroid sets are defined (see [1]) in terms of a polaroid bifunction  $f$ , for an arbitrary (closed) set  $P$  and a parameter  $k \in \mathbb{R}$ . For our present purposes, we only need a definition related to the above bilinear bifunction  $f$  :

Definition 3: The simple polaroid set  $P^*(k)$  is defined by

$$P^*(k) = \{y \mid f(x;y) = x^T C y \leq k, \forall x \in P\}$$

2) Convexity and validity of the simple polaroids.

Property 2: The simple polaroid  $P^*(k)$  is a convex set.

Proof: Let  $y^3 = \lambda y^1 + (1-\lambda) y^2$ , with  $y^1$  and  $y^2 \in P^*(k)$ ,

and  $\lambda \in [0,1]$ ; we need show  $y^3 \in P^*(k)$ , i.e.  $\forall x \in P$ :

$$x^T C y^3 = \lambda x^T C y^1 + (1-\lambda) x^T C y^2 \leq \lambda k + (1-\lambda) k = k$$

Q.E.D.

Furthermore, the polaroid  $P^*(k)$  can be considered as a valid outer-domain, in the sense that it contains no point  $x \in P$  with a value  $g(x)$  greater than  $k$ ;

Property 3:  $P \cap P^*(k) \subset \text{lev}_k g$ .

Proof: Take  $y \in P \cap P^*(k)$ : then  $x^T C y \leq k$ ,  $\forall x \in P$ ; in particular for  $y \in P$  one has

$$y^T C y = g(y) \leq k \quad \text{Q.E.D.}$$

Note that no particular assumptions concerning  $P$  need be made; thus the cutting planes described below are valid for any subset of  $P$  as well; this is clearly a useful property in the context of integer quadratic programming, for instance.

2.1 Cutting planes.

Suppose that  $P$  is polyhedral and characterized by the following explicit format:

$$x = \bar{b} - \bar{A}t \geq 0, \quad t \geq 0$$

the  $n$ -vector  $\bar{b}$  and the  $n$  by  $n$  matrix  $\bar{A}$  are derived from the full linear programming tableau corresponding to an arbitrary feasible basic representation, by choosing the rows and columns relevant to the initial variables  $x$ ;  $t$  denotes the  $n$ -vector of non-basic variables.

Consider now the mathematical programming problem

$$\text{maximize } g(x), \text{ subject to } x \in P;$$

since we assumed a feasible basis, one has, for  $t = 0$ ,  $x = \bar{b} \in P$  and thus the value  $g(\bar{b})$  will be a lower bound for the above problem. Hence we may assume the given parameter  $k$  to satisfy  $k \geq g(\bar{b})$ .

As customary in integer and/or concave programming we now construct the intersection cut generated by the outer-domain  $P^*(k)$ .

For each non-basic variable  $t_j$ ,  $j \in \bar{N}$ , consider the ray

$$u_j = \bar{b} - \lambda_j \bar{a}_j, \quad \lambda_j > 0$$

where  $\bar{a}_j$  is the column of  $\bar{A}$  belonging to  $t_j$ .

Proposition 1: Let  $u_j \in P^*(k)$ ,  $\forall j \in \bar{N}$ ; then the cut

$$\sum_{j \in \bar{N}} \lambda_j^{-1} t_j \geq 1, \quad t_j \geq 0 \text{ is valid, i.e. it does not cut}$$

off any solution  $x \in P$  with  $g(x) > k$ .

Proof: Since  $g(\bar{b}) \leq k$ , one has  $\bar{b} \in P^*(k)$ ; on the other hand,  $u_j \in P^*(k)$  by hypothesis; hence the entire simplex  $S$ ,

$$S = \{x \mid \sum_{j \in \bar{N}} \lambda_j^{-1} t_j \leq 1, \quad t_j \geq 0\},$$

is contained in  $P^*(k)$  (Property 2).

The cut off portion  $S \cap P$  then satisfies

$$P \cap S \subset P \cap P^*(k) \subset \text{lev}_k g \text{ (Property 3)}$$

Q.E.D.

Since it is obviously desirable to generate as deep a cut as possible, one will determine the value  $\lambda_j^* > 0$ ,  $j \in \bar{N}$  which corresponds to the intersection of the ray  $u_j$  with the boundary of  $P^*(k)$  (i.e.  $f(x; u_j^*) = k$ ); thus one requires

$$x^T C u_j^* = x^T C (\bar{b} - \lambda_j^* \bar{a}_j) = k;$$

Assuming that there exists a point  $u_j(\lambda)$ ,  $\lambda \in [0, 1]$  which lies in  $\text{Int } P^*(k)$ , we can infer that the intersection point  $u_j^* = u_j(\lambda_j^*) \in \text{bd } P^*(k)$  with  $\lambda_j^* > 0$  is unique. (This assumption is easily satisfied by choosing  $k > \bar{b}^T C \bar{b}$ , for instance.)

The intersection problem merely reduces to finding one point  $x \in P$  such that  $x^T C (\bar{b} - \lambda_j^* \bar{a}_j) = k$ ; this can be solved by the following parametric linear programming scheme:

Step 0: Set  $\lambda_j = \lambda_j^0 = \min_{\bar{a}_{ij} > 0} \frac{\bar{b}_i}{\bar{a}_{ij}}$  (simplex criterion)

(Any other  $\lambda_j^0 > 0$  can also be used as initial value.)

Step 1: Solve the L.P.:

$$\max z(x) = x^T C (\bar{b} - \lambda_j \bar{a}_j), \quad \lambda_j \text{ fixed}$$

subject to  $x \in P$

Let  $\bar{x}$ ,  $\bar{z} = z(\bar{x})$  denote the optimal solution.

**Step 2:** Compute the correction  $\Delta\lambda_j$

$$\Delta\lambda_j = \frac{\tilde{z}-k}{\tilde{x}^T \bar{c}_j}$$

and set  $\lambda_j^* = \lambda_j + \Delta\lambda_j$

If  $\lambda_j^* = \lambda_j$  (i.e.  $\Delta\lambda_j = 0$ , because  $\tilde{z}=k$ ) then stop!

Otherwise update the value  $\lambda_j$  by setting

$\lambda_j^{\text{new}} := \lambda_j^*$ , and go to Step 1.

It can be shown that the polyhedral nature of the set  $P$  implies finiteness of the above iterative procedure.

**2.2 Applications:** In conclusion, we note that a deep polaroid cut can be generated for the optimization of a quadratic function subject to linear constraints. This cut can be implemented in a variety of circumstances, for various types of problems and in various kinds of algorithms.

- **Problems:** The present polaroid cuts apply to a more general class of problems than the strictly concave quadratic programs of [4] since definiteness is not required; furthermore, in the concave case, they can be shown (see [2]) to dominate uniformly Hoang Tuy's cuts. Moreover, our polaroid cuts apply to any kind of problem where  $F$  represents a superset of the feasible set; typical applications of this type occur when the original problem contains some extraneous conditions (integrality, zero-one, extreme point conditions, etc. . . . see [5,6]).

- **Algorithms:** A first straightforward implementation can be thought of in terms of a method of cuts accompanied by a dual operation to restore primal feasibility (this procedure is well-known, particularly in integer programming). As usual, this approach may present here some convergence difficulties.

Because they do not require definiteness, the present cuts were also found useful in curtailing a face decomposition tree search (see [7,8]). Clearly they can also be used in any other branch and bound approach.

The next section establishes further properties of the simple polaroids  $P^*$ , and relates them to further types of algorithms (see also [3,9]).

3) The facial structure of the simple polaroids  $P^*$  .

Proposition 2: Suppose  $C$  non-singular; then to every point

$\tilde{x} \in \text{bd}(\text{conv } P)$  there corresponds a hyperplane (halfspace)

$$\tilde{x}^T C y \leq k$$

which supports  $P^*(k)$  at a point  $y^* \in \text{bd } P^*(k)$  .

Proof:

By definition,  $P^*(k)$  lies in the halfspace  $x^T C y \leq k$  ,

for any given  $x \in P$  . Consider now a hyperplane

$x^T u \leq k$  supporting the set  $\text{conv } P$  at the point  $x = \tilde{x}$  ;

let  $y^* = C^{-1} u$  .

Then  $\tilde{x}^T C y^* = k$  and  $x^T C y^* \leq k \quad \forall x \in P$  ; thus  $y^*$  lies

on the boundary of  $P^*(k)$  and  $\tilde{x}^T C y \leq k$  supports  $P^*(k)$

at  $y = y^*$  .

Q.E.D.

Corollary 2.1: Suppose  $C$  singular; let  $y^*$  define an objective function  $z(x) = x^T C y^*$  which satisfies:

$$\max_{x \in P} z = z(\tilde{x}) = k \quad , \quad (\tilde{x} \in P) .$$

Then the hyperplane  $\tilde{x}^T C y \leq k$  supports  $P^*(k)$  at  $y = y^*$  .

Proof:

Similar to that of Proposition 2.

Q.E.D.

Corollary 2.2: Suppose  $P$  closed, convex, polyhedral; i.e.,

$P = \{x \mid Ax \leq b\}$  , with suitable  $A$  and  $b$  . Let a face  $F$

of  $P$  be represented by the system  $\alpha x = \beta$  where  $\alpha$  and

$\beta$  are obtained from  $A$  and  $b$  respectively, by deleting

the appropriate rows. Then to  $F$  there corresponds a face

$F^*$  of  $P^*(k)$  which is characterized by

$$F^* = \{y \mid Cy = \alpha^T \mu, \beta^T \mu = k, \mu \geq 0\}$$

Proof:

By hypothesis one has for  $\mu \geq 0$ :  $\mu^T \alpha x = \mu^T \beta$ ,  $\forall x \in F$  and by transposition  $x^T \alpha^T \mu = \beta^T \mu$ ; thus if  $\mu$  is chosen in such a way that  $\beta^T \mu = k$ , one obtains for  $P^*(k)$  the supporting hyperplane(s)  $x^T u \leq k$ , with  $u = \alpha^T \mu$ ,  $\mu \geq 0$ . And for all  $y$  such that  $Cy = u$  one has

$$x^T Cy \leq k, \forall x \in F \text{ indicating that } y \in F^* \subset \text{bd } P^*(k)$$

Q.E.D.

Note that the relation  $Cy = \alpha^T \mu$ , defines the respective dimensions of corresponding faces  $F$  and  $F^*$ ;

Corollary 2.3: Let  $C$  be non-singular. Then to a  $p$ -dimensional face  $F$  of  $P$ , there corresponds a  $(n-p-1)$  dimensional face  $F^*$  of  $P^*(k)$ .

Proof: One has  $F^* = \{y \mid y = C^{-1} \alpha^T \mu, \mu \geq 0, \beta^T \mu = k\}$ ; by assumption the matrix  $C^{-1} \alpha^T$  has rank  $(n-p)$ , but the  $(n-p)$  vector  $\mu$  has only  $(n-p-1)$  degrees of freedom on account of  $\beta^T \mu = k$ .

Q.E.D.

(Note: as in corollary 2.1, a similar statement can be made here when  $C$  is singular, indicating that the dimension of  $F^*$  is  $\geq (n-p-1)$ ).

An interesting example of corollary 2.3 is obtained for  $p = n - 1$ , i.e. the facets of  $P$ , which correspond to the rows of the matrix  $A$  (defining inequalities); indeed, one obtains the  $(n-n+1-1) = 0$  dimensional faces  $F^*$ , i.e. the vertices of  $P^*(k)$ . Hence the vertices of  $P^*(k)$  can be immediately given by the relation

$$v = (C^{-1} \alpha_q^T) \frac{k}{b_q} \quad \text{where } \alpha_q \text{ is any (single) row of } A .$$

This property holds in particular for the classical polar sets where the matrix  $C$  is assumed definite positive (it is the identity matrix for instance in [10]). It is used in [3] to construct a new algorithm for strictly concave quadratic programming, which is based on the vertices of the polar set.

In this context, the results of section 3 can be seen to introduce a new tool for the following problems:

- semi-, and indefinite quadratic programs: a linear term may produce a singular matrix  $C$  ; in an algorithm based on extreme points  $v$  (like [3]), one may replace here the vertex  $v$  by any point  $w$  satisfying

$$Cw = \alpha_q^T \frac{k}{b_q}$$

This holds for an arbitrary indefinite quadratic problem.

- non-linear extreme point and/or integer programming: when the feasible set  $S$  is only a subset of  $P$  (of integers, for instance), we know that  $P^*(k)$  is still valid because the inclusion theorem (see [1]):

$$S \subset P \Rightarrow S^* \supset P^*$$

may be invoked; simple polaroids may therefore be expected to contribute to the solution of such less tractable mathematical programming problems.

In conclusion, we establish a property which helps visualizing the correspondence  $P \rightarrow P^*$  which turns out to be involutory under the proper assumptions:

Property 4: Assume  $C$  symmetric and non-singular; then

$$(P^*(k))^*(k) = \text{conv} (\{0\} \cup P)$$

( $P$  is assumed closed)

Proof:

First let us note that  $0 \in P^*(k)$ ; furthermore, we know that a polaroid is a closed convex set (property 2); hence,  $P^{**}$  must be convex and contain the origin.

Let  $Q = \text{conv}(\{0\} \cup P)$ ; we show that  $Q^*(k) = P^*(k)$ :

the inclusion theorem (see [1]) implies from  $P \subset Q$  that

$Q^* \subset P^*$ ; furthermore, for  $y \in P^*(k)$  one has

$x^T C y \leq k$ ,  $\forall x \in P$ . An arbitrary point  $q \in Q$  can be described by

$$q = \lambda x + (1-\lambda)0 = \lambda x, \quad \lambda \in [0,1]$$

hence  $q^T C y = \lambda x^T C y \leq x^T C y \leq k$ ,  $\forall q \in Q$

this implies that, for any  $y \in P^*(k)$ , one has  $y \in Q^*(k)$ ; thus  $P^* = Q^*$ .

It now remains to show that  $(Q^*(k))^*(k) = Q$ .

By definition the convex set  $(Q^*(k))^*(k)$  is generated

by the following family of hyperplanes (halfspaces)

$$u^T x \leq k$$

where  $u^T = y^T C$  (i.e.  $u = C^T y$ ), with  $y \in Q^*(k) = P^*(k)$ .

Consider now a family of supporting hyperplanes (halfspaces)  $v^T q \leq k$  which defines  $Q$ ; the existence of such a family presents no difficulty since every convex set can be considered to be the intersection of a collection of halfspaces. On the other hand, one has

$$q^T C y \leq k, \quad \forall q \in Q, \quad y \in Q^*(k);$$

since  $q^T C y = y^T C^T q = v^T q \leq k$ , we note that the vector

$v = C y$ ,  $y \in Q^*(k)$ , generates a hyperplane in the

latter family; furthermore, the definition of  $Q^*(k)$  indicates that every  $v$  of this family generates an element  $y = C^{-1}v \in Q^*(k)$  which is uniquely defined when  $C$  is non-singular; hence, assuming symmetry (i.e.  $C = C^T$ ), one sees that both sets  $Q$  and  $Q^{**}$  are characterized by the same family of hyperplanes (half-spaces)  $u^T x \leq k$ , with  $u = Cy = v$ ,  $y \in Q^*(k)$  implying  $Q = Q^{**}$ .

Q.E.D.

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