Indeterminates and Incidence Matrices*

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1. Introduction

Let

$$A = [a_{ij}] \qquad (i,j = 1,\dots, n) \tag{1.1}$$

be a matrix of order n with elements in a field F and let

$$X = [x_{ij}] \qquad (i,j = 1,\dots, n) \tag{1.2}$$

be the matrix of order n whose elements are n^2 independent indeterminates over F. We call the Hadamard product

$$M = A * X = [a_{ij}x_{ij}] \tag{1.3}$$

the <u>formal</u> <u>incidence</u> <u>matrix</u> associated with A. The elements of M belong to the polynomial ring

$$F^* = F[x_{11}, x_{12}, \dots, x_{nn}]. \tag{1.4}$$

The matrix A is <u>fully</u> <u>indecomposable</u> provided that it does not contain a zero submatrix of size r by n - r. We now state at the outset one of our main conclusions.

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Theorem 1.1 *Let A be a matrix of order n with elements in a field F and let $M = A * X$ be the formal incidence matrix associated with A. Suppose that $\det(M) \neq 0$. Then A is fully indecomposable if and only if $\det(M)$ is an irreducible polynomial in F^*.*

In what follows we develop some of the basic properties of formal incidence matrices. We also summarize some recent investigations on matrices whose elements are linear forms in t independent indeterminates over F. Our results are of interest both from the algebraic and combinatorial points of view.

The motivation for much of this material appears in the much earlier investigations of Kantor [8], Frobenius [5], and Schur [16]. These authors study certain determinantal properties of matrices whose elements are linear forms in independent variables over the complex field. A more recent account of this theory is available in [10].

Despite these early origins the subject matter of this paper is still very much in its infancy. However, we anticipate that matrices in conjunction with indeterminates will play an increasingly important role in the study of various combinatorial problems.

2. The Formal Incidence Matrix

In what follows we extend in an obvious way some familiar terminology for (0,1)-matrices to matrices with elements in an arbitrary field. Let

$$A = [a_{ij}] \qquad (i = 1,\ldots, m;\ j = 1,\ldots, n) \tag{2.1}$$

be a matrix of size m by n with elements in a field F. We may regard A as the <u>incidence matrix</u> for m subsets of an n-set. Here the nonzero elements of F play the role of 1's in the standard (0,1)-matrix representation. A <u>line</u> of A denotes a row or a column of A. A <u>cover</u> of A is a set of lines of A that contain (or cover) all of the nonzero elements of A. The cover is <u>minimal</u> provided that the number of lines in the cover is minimal. The <u>term rank</u> of A is the maximal number of nonzero elements of A with no two of the elements on a line. The familiar Frobenius-König theorem asserts that <u>the term rank of</u> A <u>and the number of lines in a minimal cover are equal</u>.

Let

$$X = [x_{ij}] \qquad (i = 1,\dots, m;\ j = 1,\dots, n) \tag{2.2}$$

be the matrix of size m by n whose elements are mn independent indeterminates over F. We call the Hadamard product

$$M = A * X = [a_{ij}x_{ij}] \tag{2.3}$$

the <u>formal incidence matrix</u> associated with A. The elements of M belong to the polynomial ring

$$F^* = F[x_{11}, x_{12}, \dots, x_{mn}]. \tag{2.4}$$

The formal incidence matrix has been very useful in various combinatorial investigations [2, 4, 11, 12, 18]. The following observation of Edmonds [4] equates an important combinatorial invariant of A with an algebraic invariant of M. <u>The term rank of</u> A <u>is equal to the rank of</u> M. We note that a submatrix of M of order r has a nonzero determinant if and only if the corresponding submatrix of A has term rank r. But the rank of a matrix is the maximal order of a square submatrix with a nonzero determinant and hence the conclusion follows.

We recall that a square matrix of order n with elements in F is fully indecomposable provided that it does not contain a zero submatrix of size r by n - r. A fully indecomposable matrix does not have a cover of n lines apart from the two obvious covers of n rows and n columns.

3. Preliminary Lemmas

We first investigate some special properties concerning the factorization of homogeneous polynomials. Again let

$$X = [x_{ij}] \qquad (i,j = 1,\ldots, n) \tag{3.1}$$

be the matrix of order n whose elements are n^2 independent indeterminates over F. The associated polynomial ring is now

$$F^* = F[x_{11},x_{12},\ldots, x_{nn}]. \tag{3.2}$$

Let X_r denote a submatrix of X of order r. We form products of the indeterminates of X_r. These products each contain r indeterminates of X_r with no two of the indeterminates on a line of X_r. We designate these $u = r!$ products by

$$y_1,\ldots, y_u. \tag{3.3}$$

We are concerned with polynomials of the form

$$f = \sum_{i=1}^{u} a_i y_i, \tag{3.4}$$

where the coefficients are in F and not all of the a_i are zero. We say that the polynomial f has an <u>indeterminate pattern</u> based on X_r. All of the polynomials with an indeterminate pattern based on X_r are homogeneous and of degree r over F. Two well known polynomials with an indeterminate pattern based on X_r are $\det(X_r)$ and $\text{per}(X_r)$. The polynomials with an indeterminate pattern based on X are of the form

$$(3.5) \qquad f = \Sigma a_{\sigma(1),\ldots,\sigma(n)} x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

where σ ranges over the n! permutations of 1,..., n.

Two submatrices B and C of orders r and n - r, respectively, of a matrix A of order n are called <u>complementary</u> provided that they are formed from complementary sets of lines of A. The following lemma is of some intrinsic interest.

Lemma 3.1. *Let h be a polynomial with an indeterminate pattern based on X and suppose that in F^* we have*

$$(3.6) \qquad h = fg,$$

where f and g are polynomials of positive degrees r and n - r, respectively. Then f and g are polynomials with indeterminate patterns based on X_r and X_{n-r}, respectively, where X_r and X_{n-r} are complementary submatrices of X of orders r and n - r, respectively.

Proof. We write

$$(3.7) \qquad f = f_1 + \cdots + f_\rho, \quad g = g_1 + \cdots + g_\sigma, \quad h = h_1 + \cdots + h_\tau.$$

In (3.7) each term f_i of f is required to be a nonzero scalar multiple of a product of indeterminates and f_i and f_j are not scalar multiples for $i \neq j$. The same restrictions are placed on the terms g_i of g and h_i of h. We let F^*_{ij} denote the integral domain F^* with the indeterminate x_{ij} deleted.

We assert that f and g do not contain an indeterminate x_{ij} in common. If this were the case, then both f and g would be polynomials in x_{ij} of degree at least 1 over F^*_{ij}. But this contradicts the fact that h is a polynomial in x_{ij} of degree at most 1 over F^*_{ij}. Hence it follows that $f_i g_j$ and $f_k g_\ell$ do not contain identical indeterminates unless both $i = k$ and $j = \ell$. Thus there is no combining of terms in the product fg and we have

$$\tau = \rho\sigma. \tag{3.8}$$

It follows that f and g are homogeneous polynomials of degrees r and $n - r$, respectively.

A typical term f_i of f appears in h and has an indeterminate pattern based on a certain submatrix X' of X. It follows that a typical term g_j of g must have an indeterminate pattern based on the complementary submatrix X^* of X' with respect to X. This is the case because $f_i g_j$ is a term of h and h consists only of terms with an indeterminate pattern based on X. Finally, we assert that a second term f_k of f must also have an indeterminate pattern based on X'. This is because g_j already has its indeterminate pattern based on X^*. Hence $X' = X_r$ and $X^* = X_{n-r}$. This proves the lemma.

The following lemma is used frequently in the study of fully indecomposable matrices [1]. We include the short argument for completeness.

Lemma 3.2. *Let A be a matrix of order n with elements in a field F and let $M = A * X$ be the formal incidence matrix associated with A. Let M_{ij} denote the submatrix of M obtained by the deletion of row i and column j of M. Suppose that A is fully indecomposable. Then*

$$(3.9) \qquad \det(M_{ij}) \neq 0 \qquad (i,j = 1, \ldots, n).$$

Proof. Suppose that $\det(M_{ij}) = 0$ for some i and j. Then

$$(3.10) \qquad \operatorname{rank}(M_{ij}) = \text{term rank}(M_{ij}) \leqq n - 2.$$

It follows from the Frobenius-König theorem that M_{ij} has a cover of n - 2 lines. These n - 2 lines plus the deleted row and column give a cover of A of n lines. This cover implies that A is not fully indecomposable.

4. The Main Theorems

We now state and prove the theorem cited in Section 1.

Theorem 4.1. *Let A be a matrix of order n with elements in a field F and let $M = A * X$ be the formal incidence matrix associated with A. Suppose that $\det(M) \neq 0$. Then A is fully indecomposable if and only if $\det(M)$ is an irreducible polynomial in F^*.*

Proof. We take as our hypothesis that det(M) is an irreducible polynomial in F^*. If the matrix A were not fully indecomposable, then there would exist permutation matrices P and Q such that

$$PMQ = \begin{bmatrix} M_1 & 0 \\ * & M_2 \end{bmatrix}, \tag{4.1}$$

where M_1 and M_2 are matrices of orders r and n - r, respectively, and 0 is a zero matrix of size r by n - r. But then

$$\det(M) = \pm\det(M_1)\det(M_2), \tag{4.2}$$

and this contradicts the hypothesis that det(M) is irreducible in F^*.

Next we take as our hypothesis that the matrix A is fully indecomposable. Suppose that det(M) is reducible. Then we have

$$\det(M) = fg. \tag{4.3}$$

Now det(M) is a polynomial with an indeterminate pattern based on X. Hence by Lemma 3.1 there exist complementary submatrices X_r and X_{n-r} of X of orders r and n - r, respectively, such that f and g have indeterminate patterns based on X_r and X_{n-r}, respectively. Thus there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & * \\ * & A_2 \end{bmatrix}, \qquad PMQ = \begin{bmatrix} M_1 & * \\ * & M_2 \end{bmatrix}, \tag{4.4}$$

where

$$M_1 = A_1 * X_r, \qquad M_2 = A_2 * X_{n-r}. \tag{4.5}$$

Let us suppose that a nonzero element of PMQ does not lie within M_1 or M_2. Then by Lemma 3.2 it follows that the indeterminate in this position must appear in det(M). But this contradicts (4.3), where f and g have indeterminate patterns based on X_r and X_{n-r}, respectively. Thus the asterisks in (4.4) correspond to zero matrices and this contradicts the hypothesis that A is fully indecomposable.

Let A be a matrix of order n with elements in F and let M = A * X be the formal incidence matrix associated with A. Suppose that det(M) $\neq$ 0. Then it follows that there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ * & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & A_r \end{bmatrix}, \tag{4.6}$$

where the matrices $A_1, A_2, \ldots, A_r$ are fully indecomposable. These matrices are called the <u>fully indecomposable components</u> of A. The preceding discussion implies the following theorem of Dulmage and Mendelsohn [1, 3].

Theorem 4.2. *Let A be a matrix of order n with elements in a field F and let M = A * X be the formal incidence matrix associated with A. Suppose that det(M) $\neq$ 0. Then the fully indecomposable components of A are unique apart from order and row and column permutations within components.*

Proof. Suppose that $A_1, \ldots, A_r$ and $B_1, \ldots, B_s$ are two sets of fully indecomposable components of A. Then by Theorem 4.1 we have that

$$\det(M) = \pm f_1 \cdots f_r = \pm g_1 \cdots g_s, \tag{4.7}$$

where f_i and g_j are irreducible polynomials with indeterminate patterns based on the appropriate submatrices of X. But F^* is a unique factorization domain and this means that r = s and the f_i and the g_j are the same apart from order and scalar factors. But then the fully indecomposable components A_i and B_j are the same apart from order and row and column permutations within components.

Let A be a fully indecomposable matrix of order n with elements in F and let $M = A * X$ be the formal incidence matrix associated with A. A very natural problem emerges at this point, namely, the determination of all matrices A with the same irreducible polynomial det(M). This problem is solved for us by the following theorem of Sinkhorn and Knopp [17]. (Sinkhorn and Knopp state their result for nonnegative real matrices. But an inspection of their proof shows that the result holds for matrices with elements in an arbitrary field.) A _diagonal product_ of a matrix A of order n is a product of n elements of A with no two of the elements on a line.

Theorem 4.3. _Let A be a fully indecomposable matrix of order n with elements in a field F. Suppose that all of the nonzero diagonal products of A are equal. Then there exists a unique matrix B of order n with nonzero elements and of rank one such that_ $b_{ij} = a_{ij}$ _whenever_ $a_{ij} \neq 0$.

The preceding theorem of Sinkhorn and Knopp implies the following.

Theorem 4.4. Let A and B be fully indecomposable matrices of order n with elements in a field F and let $M = A * X$ and $N = B * X$ be the formal incidence matrices associated with A and B, respectively. Suppose that

(4.8) $$\det(M) = c \det(N) \neq 0,$$

where c is a scalar in F. Then there exist diagonal matrices D and E with elements in F such that

(4.9) $$DAE = B.$$

Proof. The hypotheses of the theorem and Lemma 3.2 imply that $a_{ij} \neq 0$ if and only if $b_{ij} \neq 0$. We now form the matrix

(4.10) $$C = [a_{ij}^{-1} b_{ij}],$$

where in (4.10) a_{ij}^{-1} is the inverse of a_{ij} for $a_{ij} \neq 0$ and all of the remaining elements of C are zero. Then (4.8) implies that all of the nonzero diagonal products of C are equal. Hence we may apply the theorem of Sinkhorn and Knopp to the fully indecomposable matrix C. This tells us that there exist nonzero elements $d_1, \ldots, d_n$ and $e_1, \ldots, e_n$ in F such that

(4.11) $$d_i e_j = a_{ij}^{-1} b_{ij}$$

whenever $a_{ij}^{-1} b_{ij} \neq 0$. Hence it follows that

(4.12) $$d_i a_{ij} e_j = b_{ij} \qquad (i,j = 1, \ldots, n).$$

But these equations are the same as (4.9).

5. Concluding Remarks

The formal incidence matrix seems ideally suited for the study of combinatorial problems related to transversal theory [11]. But matrices with indeterminate elements are also applicable to other problems of great combinatorial interest. We discuss briefly a matrix equation dealing with set intersections. Let A and B be matrices of sizes m by t and t by n, respectively, with elements in F. We now disregard our previous notation and let

(5.1) $$X = \mathrm{diag}[x_1, \ldots, x_t],$$

where $x_1, \ldots, x_t$ are t independent indeterminates over F. Then

(5.2) $$AXB = Y$$

is a matrix of size m by n such that every element of Y is a linear form in $x_1, \ldots, x_t$ over F. Ryser [15] has established the following theorem concerning the matrix equation (5.2).

Theorem 5.1. *Let Y be a matrix of size m by n such that every element of Y is a linear form in $x_1, \ldots, x_t$ over F and let $y_1, \ldots, y_u$ denote the products of $x_1, \ldots, x_t$ taken r at a time. We assume that the fixed integer r satisfies*

(5.3) $$2 \leq r \leq \mathrm{rank}(Y) - 2$$

and that every element of the rth compound matrix $C_r(Y)$ of Y is a linear form in $y_1, \ldots, y_u$ over F. Then there exist matrices A and B of sizes m by t and t by n, respectively, with elements in F such that

(5.4) $$AXB = Y.$$

We remark that Theorem 5.1 is also valid with (5.3) replaced by the requirements

(5.5) $$2 \leq r = \text{rank}(Y) - 1 \text{ and } t = \text{rank}(Y).$$

There has been considerable interest of late in intersection properties of finite sets [6, 7, 9, 13, 14]. The matrix equation (5.2) is well suited for the study of such problems. For example, suppose that A is a (0,1)-matrix of size m by t and that $B = A^T$ is the transpose of the matrix A. Then (5.2) assumes the form

(5.6) $$AXA^T = Y,$$

where Y is a symmetric matrix of order m. Moreover, if A is regarded as the incidence matrix for subsets $S_1, \ldots, S_m$ of $S = \{x_1, \ldots, x_t\}$, then the matrix Y has in its (i,j) position the sum of the indeterminates in $S_i \cap S_j$. It follows that the matrix Y gives a complete description of the intersection patterns $S_i \cap S_j$. Ryser [14] has used this observation for the study of certain combinatorial properties of set intersections. But all of these topics and especially the combinatorial applications hold great potential for further study.

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