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MODIFIED DIPOLES : II
NUMERICAL SOLUTIONS

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II. NUMERICAL SOLUTIONS

Peter S. Kao

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MODIFIED DIPOLES
II. NUMERICAL SOLUTIONS

By

Peter S. Kao

Division of Engineering and Applied Physics
Harvard University · Cambridge, Massachusetts

ABSTRACT

The 'modified dipole' has its origin in the consideration of the general properties of a satellite antenna which bears great resemblance to a dipole modified to incorporate at the center a conducting volume which is used to radiate electromagnetic waves and to house a power supply and radio frequency generators, etc. The object of this research is to pursue a theoretical and experimental exploration of the effects induced by the presence of the conducting volume on the antenna performance, i.e., input characteristics, current distribution along the surfaces of the entire radiating structure and radiation properties.

In Volume I a mathematical model consisting of a perfectly conducting sphere from which project the ends of a thin biconical antenna is chosen to simulate the actual sphere-centered thin dipole. The conical antenna is driven at its junction with the sphere by a rotationally symmetric electric field maintained across the gap by a biconical transmission line excited by the TEM mode. The attractive features of this model include the fact that it has surfaces that permit a simple specification of boundary conditions and, hence, a rigorous formulation for the electromagnetic fields and a shape such that its properties should come reasonably close to those of a modified cylindrical antenna as the cone angle becomes quite small.

The measurements of both input admittances and current distributions on modified dipoles (with either conical or cylindrical

antenna projecting from the sphere) are also presented in Volume I. Comparisons were also made between modified conical and cylindrical antennas with the same sphere radii and antenna heights. The radius of the cylindrical antenna is the same as the smaller end of the cone. The fact that the admittance curves for modified cylindrical and conical antennas involve only slight shifts suggests that by introducing an equivalent antenna length that is a little longer than the actual physical length of the conical antenna a good approximation is obtained for the cylindrical antenna.

An infinite set of algebraic equations was solved numerically in Volume II for small cone angles. Comparisons were made between the modified conical antenna and its limiting biconical antenna which provides both an extrapolatory numerical check for the modified conical antenna with shrinking central sphere and an understanding of the underlying physical phenomena. Theoretical and experimental results are in very good agreement.

Accepted for the Air Force
Joseph R. Waterman, Lt. Col. USAF
Chief, Lincoln Laboratory Project Office

1. INTRODUCTION

The formal solution for the b_k's and with them the determination of the driving point admittance, the current distribution and the far-field pattern of the modified conical antenna with half cone angle θ_0 and central sphere of radius b is contained in the infinite set of linear equations, (1-57) and (1-64) which appear in Volume I of this report. No general, exact solution of this set of equations is available for cones with arbitrary angles θ_0. Hence, a general formula for the input admittance of the modified dipole cannot be provided. Thus, although the analysis of the problem is not restricted in its formulation, mathematical limitations make an exact solution unavailable. However, if the half cone angle θ_0 is sufficiently small (i.e., $\theta_0 < 4^o$), which is of special interest in the present problem, a numerical solution of (1-57) and (1-64) can be carried out. In this problem an exact result can be obtained only by an infinite sequence of steps. Since the process necessarily has to be truncated after a certain finite number of steps, truncation errors are unavoidable. There are, however, three sources of truncation errors in the approach taken to this problem. First, there is the error introduced when the infinite matrix is truncated to one of finite dimensions. Secondly, error is introduced by truncating the infinite sum (over v) in the calculation of each matrix element. Thirdly, the double series representing the input admittance, current distribution and near fields is twice truncated; once over v and once over k. In addition, because of the formidable amount of computation required to solve the truncated infinite matrix, small round-off errors may cause errors in the computed solution out of all proportion to their size. Above all, the truncated system of linear equations has to be solvable (i.e., the coefficient matrix must be non-singular) before problems of accuracy and convergence of the computed solution can be considered.

An exact error analysis relating different sources of errors to the final results is very difficult, if not impossible, when many equations are involved. It cannot be made here. Hence, after all possible precautions have been taken to prevent error from growing out of proportion and to

preserve the accuracy of the final solution, the computed results are to be justified by comparison with experimental measurements.

A brief review of the major steps taken in the course of the numerical calculation and their associated theories follows:

(A) Existence of the Solution

Once the infinite set of linear equations is truncated, a set of n simultaneous linear equations in n unknown is at hand. It is

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (2\text{-}1a)$$

Equation (2-1a) is conveniently written in matrix form as follows:

$$A X = b \qquad (2\text{-}1b)$$

where $A = [a_{ij}]$ is the n x n matrix of coefficients, $X^T = (x_1 \ldots x_n)$ and $b^T = (b_1 \ldots b_n)$ with T denoting the transpose. Let A_b denote the n x (n+1) matrix which has the column vector b appended as an (n+1)st column to A. The rank of any matrix A is r(A). Then the basic theory on the existence of a solution of (2-1) states that the system of linear equations has a solution if and only if

$$r(A) = r(A_b) \qquad (2\text{-}2)$$

This familiar theorem may be proved by using Gaussian elimination. This will not be pursued here [1]-[2].

The theoretical proof of the existence of a unique solution for the simultaneous linear equations does not in any way guarantee the correctness of a computed solution of (2-1a, b). There is no hard-and-fast rule in carrying out a numerical solution for simultaneous linear equations. Different approaches need be employed for different matrices with different characteristics; e. g., direct methods are usually used for matrices which are filled but not large, while iteration methods are useful for matrices that are sparse and filled etc.

(B) Ill-Condition

In numerical calculations, a class of matrices of coefficients is encountered which is called ill-conditioned such that no matter how accurately it is calculated, the solution may still be grossly in error. If the matrix A is normalized so that the largest term in magnitude has the order of magnitude of unity and is such that A^{-1} contains some very large elements, the matrix and therefore, the system of equations is ill-conditioned. Conversely, if the largest element in magnitude of A^{-1} has the order of magnitude of unity, the matrix may be said to be well-conditioned. This can easily be seen by writing (2-1b) in the following form:

$$X = A^{-1}b \tag{2-3}$$

Suppose that A^{-1} has very large elements, one of which is $a^{-1}_{ji} = A_{ij}/|A|$. $|A|$ denotes the determinant of A and A_{ij} is the cofactor of a_{ij}. The assumption that a^{-1}_{ji} is large means that A_{ij} must be large relative to $|A|$. Since one of the terms in the expansion of $|A|$ about the ith row and jth column is $a_{ij}A_{ij}$, a small error in a_{ij} may cause a large relative error in $|A|$ and therefore a large relative error in a^{-1}_{ji}. This in turn can cause a large relative error in X. Similarly, a small change in an element of b could cause a large change in X.

(C) Truncation Error

The errors caused by truncating the infinite sum by finding an approximation (or an upper bound) to the remainder in each series after truncation will be examined. Having found these, an estimate could be found of the relative error in each matrix element by comparing the remainder with the corresponding truncated sum. Finally, the remainder estimate is added to the corresponding matrix element, the system is solved and its solution compared with that of the unaltered matrix. This comparison yields an estimate of the error one might expect in the field pattern, current distribution and input admittance etc.

(D) Intermediate Round-offs

One way of checking the round-off error in the course of computation is by substituting the calculated values into the left-hand members of the original equations. The presence of the deviations between the resultant members and original right-hand members serves to indicate the presence of errors due to intermediate round-offs. Again, the relationship between the magnitudes of these deviations and the magnitude of the errors in the solution column is not a simple one. Use will be made of this simply as a check if the round-off errors are too far out of proportion to afford a reasonable solution.

(E) Convergence of the Solution

Before considering the convergence of the solution of the system of linear equations, error checks were made for several cases with various combinations of the extreme sizes of antennas and inner spheres. The maximum size of matrices has been studied so that the different kinds of errors mentioned above can be kept within one per cent. It has been found that for most cases a 40 x 40 matrix is still well within the acceptable range and the solutions show rapid convergence when the matrix size exceeds 15 x 15.

The convergence of the solution of the system of linear equations will be dealt with empirically by solving the same problem with matrices of increasing size and examining the successive solutions for convergence.

2. ROOTS OF THE CHARACTERISTIC EQUATION $L_v(\theta_0) = 0$

The application of the boundary condition given by equation (1-35) in Volume I together with the property of anti-symmetry of E_R with respect to the equatorial plane (i. e., $\theta = \pi/2$ - plane) leads to the following characteristic equation [see equation (1-30) in Volume I]:

$$L_v(\theta_0) = L_v(\pi - \theta_0) = 0$$

which is independent of the presence of the conducting sphere at the center. Once the half cone angle θ_0 is fixed, an infinite sequence of discrete values of v is obtained from the above equation. The present analysis will be restricted to small half cone angles (i. e., $\theta_0 < 4$).

In the neighborhood of the point $\theta = \pi$, Legendre's functions can be expressed in the following way [3] :

$$P_v(-\cos\theta_0) = \frac{\sin v\pi}{\pi} F(-v, v+1; 1; \sin^2\frac{\theta_0}{2}) \cdot H(v, \theta_0)$$

$$+ \sum_{a=1}^{\infty} \frac{(-v)\cdots(-v+a-1)(v+1)\cdots(v+a)}{(a!)^2} \phi(v, a)(\sin\frac{\theta_0}{2})^{2a} \quad (2\text{-}4)$$

where $\theta_0 = \pi - \theta$ is in the neighborhood of zero

$$H(v, \theta) = 2\sin\frac{\theta}{2} + \psi(v+1) + \psi(-v) + 2\gamma \quad (2\text{-}5a)$$

and

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \quad (2\text{-}5b)$$

which is the Gauss hypergeometric series whose circle of convergence is the unit circle $|z| = 1$.

$$\phi(v, r) = \frac{1}{v+1} + \cdots + \frac{1}{v+r} + \frac{1}{-v} \cdots + \frac{1}{-v+r-1}$$

$$- 2(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{r}) \quad (2\text{-}5c)$$

and $\psi(z)$ is the logarithmic derivative of the Gamma function; i. e.,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (2\text{-}5d)$$

or

$$= -\gamma + (z-1)\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+z)} \tag{2-5e}$$

where γ denotes Euler's or Mascheroni's constant equal to 0.57722.

The expression (2-4) will be single-valued, on account of the cross-cut of the plane of $\mu = \cos\theta$ along the real axis from -1 to $-\infty$.

Using the reflection formula for the ψ-function, namely,

$$\psi(z+1) = \psi(-z) - \pi \cot \pi z \tag{2-6}$$

(2-4) is rearranged in the following form;

$$L_v(\cos\theta_0) = \sin v\pi \; G(v, \theta_0) \tag{2-7a}$$

$$= \sin v\pi \; [G_1(v, \theta_0) + G_2(v, \theta_0)] \tag{2-7b}$$

where

$$G_1(v, \theta_0) = F(-v, v+1; 1; \sin^2 \frac{\theta_0}{2}) \cdot F(v, \theta_0) \tag{2-8a}$$

$$F(v, \theta_0) = C(\theta_0) + \frac{2}{\pi} \psi(v+1) - \tan \frac{v\pi}{2} \tag{2-8b}$$

In (2-8b)

$$C(\theta_0) = \frac{2}{\pi} [\ln(\sin \frac{\theta_0}{2}) + \gamma] \tag{2-8c}$$

$$G_2(v, \theta_0) = \sum_{a=1}^{\infty} (-1)^a \frac{(v-a+1)\cdots v(v+1)\cdots(v+a)}{a! \cdot a!} \frac{\phi(v, \theta_0)}{\pi} \tag{2-8d}$$

Thus, finding the roots of the characteristic equation is now equivalent to setting $G(v, \theta_0) = 0$. Before actually carrying out the numerical calculation of the roots of (2-7a), it is instructive first to survey the equation qualitatively. Some general properties of these roots regardless of the cone angle θ_0 (as long as it is small) can be derived by looking at (2-7a, b) more carefully. The vanishing of (2-7a) can also be accomplished by finding the intersections of two functions, i.e., $-G_1(v, \theta_0)$ and $G_2(v, \theta_0)$ with v as a variable.

It is convenient to look into the restrictions upon v and θ_0 such that the numerical solution of $G(v, \theta_0) = 0$ is feasible and a sufficient number of roots is obtained to provide accuracy in the calculation of the matrix elements, the input admittance and the current distribution. It has been found that both the functions G_1 and G_2 are well behaved (i.e., the series converges quite rapidly) and the above-mentioned requirements were met over the range of interest so long as v is not too large in magnitude and the cone angle θ_0 is sufficiently small so that the product of these two quantities does not exceed the order of magnitude of unity [i.e., $v \sin \frac{\theta_0}{2} \leq 0(1)$].

The function $-G_1(v, \theta_0)$ behaves essentially like $\tan \frac{v\pi}{2}$ with its values continuously adjusted by first adding to it a slowly-decreasing quantity [i.e., $-C(\theta_0) - \frac{2}{\pi}\psi(v+1)$] and then multiplying it by a slowly-varying function [i.e., $F(-v, v+1; 1; \sin^2 \frac{\theta_0}{2}) \sim (1 - v(v+1) \sin^2 \frac{\theta_0}{2} + \cdots)$]. Therefore, $-G_1(v, \theta_0)$ is a continuous, monotonically increasing function within each equally-spaced segment which has a total length of two and the odd integers as its end points. $G_1(v, \theta_0)$ becomes singular in the neighborhood of the end points and changes sign whenever it crosses the end points. The derivative of $G_2(v, \theta_0)$ with respect to v is proportional to $(4 - 3v^2 \sin^2 \frac{\theta_0}{2}) \cdot (v \sin \frac{\theta_0}{2})$ which is positive as long as $v \sin \frac{\theta_0}{2} \leq 1$. So $G_2(v, \theta_0)$ is also a monotonically increasing continuous function over the range of interest. Accordingly, the functions $-G_1(v, \theta_0)$ and $G_2(v, \theta_0)$ intersect only once in each segment. Therefore, the form of the roots is an odd integer plus a small number which increases with v and θ_0.

The simplification usually employed [4] to obtain an approximation of the roots of (2-7) is actually to take the leading term of $G_1(v, \theta_0)$ [i.e., $F(v, \theta_0)$] and retain only the two terms in F[i.e., $\tan(v\pi/2)$ and $\ln(\theta_0/2)$] under the assumption that $\ln(\sin \frac{\theta_0}{2})$ approaches infinity for sufficiently small angles θ_0. Therefore, the approximate roots are

$$v_{app.} = k + \frac{1}{\ln(\sin \frac{\theta_0}{2})} \qquad (2\text{-}9)$$

This approximation is in error due to the fact that even for half cone angle θ_0 as small as 0.1°, $|\ln(\sin\frac{\theta_0}{2})|$ (= 7.044 in this case) is still of the same order of magnitude as the two neglected terms, i.e., (v+1) and γ. Furthermore, $\ln(\sin\frac{\theta_0}{2})$ becomes even smaller than $\psi(v+1)$ as one proceeds to find larger v's. In this range, the higher-order terms of G become comparable in magnitude and can no longer be neglected.

In Fig. 2-1, the results from the approximate formula (2-9) are compared with those calculated from the exact formula (2-7a) for various cone angles. Generally speaking, (2-9) provides quite a poor approximation of the accurate formula except for very thin cones for which the approximate formula gives the correct trend but the results are shifted from the accurate ones by a constant.

The numerically calculated v's were checked by substituting them in a subroutine which generated Legendre functions of fractional orders. At $\theta = \theta_0$, the generated values of $L_v(\theta_0)$ (which are equal to zero theoretically) are of the order of 10^{-5} which indicates the accuracy of the solution.

3. COMPUTATION OF FRACTIONAL ORDER LEGENDRE FUNCTION

The computational difficulties with the fractional order Legendre functions lie in the following facts:

(a) The Legendre function, defined in the interval (-1, 1), can be expressed by the hypergeometric series with various elements, i.e.,

$$P_v(\cos\theta) = F(-v, v+1; 1; \sin^2\frac{\theta_0}{2}) \qquad (2\text{-}10)$$

The above expansion can be employed for the calculation of values of the function $P_v(\cos\theta)$ only in a limited region of variation of the argument $\sin\frac{\theta}{2}$ and order v. With an increase in v and θ, the convergence of the series grows worse, which makes it of little use for calculation with large values of v and θ.

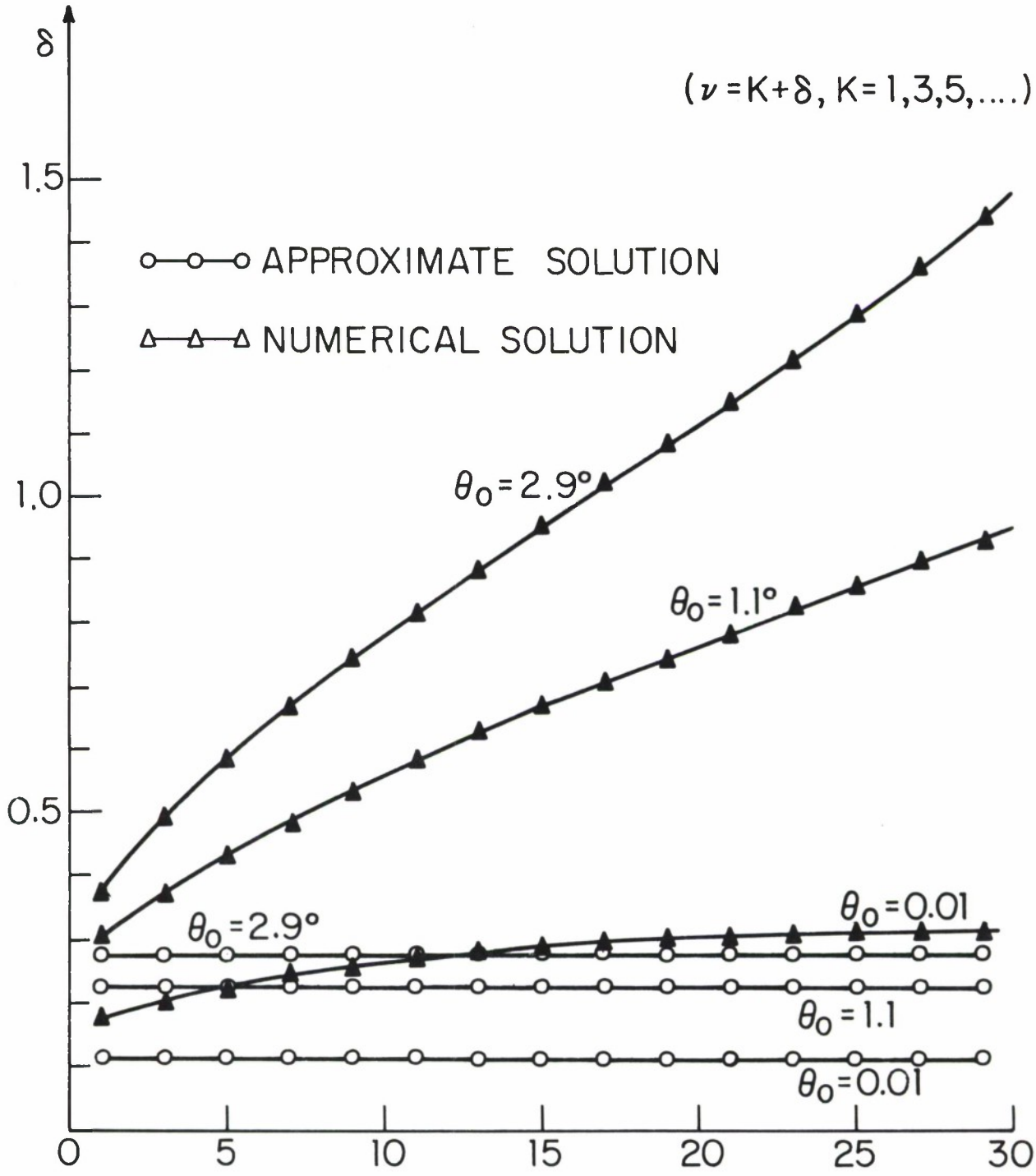


FIG. 2-1 ROOTS OF CHARACTERISTIC EQUATION $L_\nu(\theta_0)=0$

(b) The process in generating higher-order functions by a recurrence relation is unstable, since the generation is carried out with rounded values which propagate in the recurrence process of increasing order and hence cause loss of significant figures.

The Legendre function $P_v(\cos\theta)$ assumes various representations suitable for numerical calculation in different ranges of v and θ with the aid of definite and contour integrals. In this work use is made of representation of $P_v(\theta)$ in the form of Mehler-Fock, namely,

$$P_v(\cos\theta) = \frac{2}{\pi} \int_0^{\theta} \frac{\cos(v+1/2)t}{2(\cos t - \cos\theta)}\, dt \qquad (2\text{-}11)$$

which provides (under different conditions for v and θ) rapidly converging series for numerical calculation.

Equation (2-11) can be expanded in positive powers of (v+1/2) by substituting in it in place of cos(v+1/2)t its series expansion. After changes of variables and rearrangement of terms, one obtains

$$P_v(\cos\theta) = \sum_{m=0}^{\infty} B_m (v+1/2)^{2m} \qquad (2\text{-}12)$$

where

$$B_m = (-1)^m \frac{2^{(2m+1)}}{\pi (2m)!} \int_0^{1/2\pi} \frac{[\sin^{-1}(p\sin x)]^{2m}}{(1-p^2\sin^2 x)^{1/2}}\, dx \qquad (2\text{-}13)$$

and

$$p = \sin(\theta/2) \qquad (2\text{-}14)$$

In the last integral, the function under the integral sign becomes infinite only when $p = 1$, that is for $\theta = \pi$. Consequently, outside the neighborhood of the point $\theta = \pi$, it is possible to calculate this integral by the usual numerical methods.

One limitation in using (2-12) for computing Legendre functions is its slowness in convergence as v becomes larger.

An expansion of Legendre's function of (2-11) can be expressed in a series of Bessel's function [5], namely,

$$P_\nu(\cos\theta) = \sum_{m=0}^{\infty} a_m(\theta)\, J_m[(\nu+1/2)\theta]\,(\nu+1/2)^{-m} \tag{2-15}$$

where the a_m's are elementary functions, regular in $0 \leq \theta \leq \pi$, and

$$a_0(\theta) = \frac{\theta}{\sin\theta} \tag{2-16a}$$

$$a_m(\theta) = (2m-1)!!\; a_0(\theta)\, \theta^m \psi_m(\theta) \tag{2-16b}$$

$\psi_m(\theta)$ is a combination of functions of ϕ_m which is obtained from $2(\cos t - \cos\theta)$ in terms of a series of $(\theta^2 - t^2)$, i. e.,

$$2(\cos t - \cos\theta) = (\theta^2 - t^2)\,\frac{\sin\theta}{\theta}\left[1 + \sum_{m=2}^{\infty} (\theta^2 - t^2)^{\nu-1}\phi_m(\theta)\right] \tag{2-17}$$

where

$$\phi_m(\theta) = \frac{1}{m!}\,\frac{1}{2^{(m-1)}}\,\frac{J_{m-\frac{1}{2}}(\theta)}{\theta^{(m-1)}\, J_{\frac{1}{2}}(\theta)} \tag{2-18}$$

Taking the inverse of the square root of (2-17), one obtains

$$\frac{1}{\sqrt{2(\cos t - \cos\theta}} = \frac{1}{\sqrt{\theta^2 - t^2}}\,\frac{\theta}{\sin\theta}\sum_{m=0}^{\infty} (\theta^2 - t^2)^m \psi_m(\theta) \tag{2-19}$$

Therefore, the combination of (2-17) and (2-19) gives

$$\psi_0 = 1$$

$$\psi_1 = \frac{\phi_2}{2}$$

$$\psi_2 = -\frac{\phi_3}{2} + \frac{3\phi_2^2}{8}$$

$$\psi_3 = -\frac{\phi_4}{2} + \frac{3}{4}\,\phi_2\,\phi_3 - \frac{1}{2}\,\phi_2^3$$

$$\psi_4 = -\frac{1}{2}\phi_5 + \frac{3}{8}(\phi_3^2 + 2\phi_2\phi_3) - \frac{15}{16}\phi_3\phi_2^2 + \frac{35}{128}\phi_2^4$$

$$\psi_5 = -\frac{1}{2}\phi_6 + \frac{3}{4}(\phi_2\phi_5 + \phi_3\phi_4) - \frac{5}{16}(3\phi_2\phi_3^2 + 3\phi_4\phi_2^2) + \frac{35}{32}\phi_2^3\phi_3 + \frac{63}{250}\phi_2^5 \qquad (2\text{-}20)$$

By using the recurrence relation for Bessel's functions of the first kind, namely,

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z) \qquad (2\text{-}21)$$

and substituting (2-21) into (2-15), we finally obtain an expression for $P_\nu(\cos\theta)$;

$$P_\nu(\cos\theta) = J_0[(\nu+1/2)\theta]\left[a_0 - \frac{a_2}{(\nu+1/2)^2} - \frac{4a_3}{(\nu+1/2)^4\theta} + \cdots\right] + J_1[(\nu+1/2)\theta]\left[\frac{a_1}{(\nu+1/2)} - \frac{(a_3\theta - 2a_2)}{(\nu+1/2)^3\theta} + \frac{8a_3}{(\nu+1/2)^5\theta^2} + \cdots\right] \qquad (2\text{-}22)$$

At first glance, (2-22) does not assume any advantage over either of the more general expressions (2-10) and (2-11). But the asymptotic expansion for large values of ν is a convenient representation of the function $P_\nu(\cos\theta)$ from the computational point of view. It gives a good approximation to the functions themselves and can also be used for the computation of their derivatives and integrals with respect to the variable ν. Furthermore, the coefficients of the representation are associated with functions of spherical Bessel's functions and integral order Bessel's functions of the first kind J_0 and J_1, which are available as built-in functions on the IBM 360/65 machine in use at the Harvard Computing Center. Hence, (2-22) is both a convenient and accurate formula for computation.

In summary, it is necessary to use a suitable representation for Legendre's functions of fractional order in different regions in the v-θ plane for purposes of computation with desired accuracy. Therefore, as shown in Fig. 2-2, the v-θ plane is divided into four regions. In region I, the hypergeometric expansion (2-10) is generally suitable as long as the angle is small and the order is not large; also the product of v and $\sin(\theta/2)$ must be less than one. In region II, (2-4) is a suitable expansion in generating Legendre's function with the same restrictions upon v and θ as in region I. In region III, use is made of the expansion (2-12) as long as the order is not large ($v < 10$). Finally, in region IV, (2-22) provides a good tool which is also valid in the neighborhood of $\theta = 0$ and π for large values of v. Therefore, in computing $L_v(\theta) = P_v(\theta) - P_v(\pi - \theta)$, formulae for different regions must be used simultaneously, e. g., for small θ, formulae for region I and II must be used.

The quantity Q_{rk} in the second set of linear equations is generally a small quantity for small half cone angle θ_0. For $k \neq r$, Q_{rk} can easily be expressed in a close form, i. e.,

$$Q_{rk} = -2(1-\mu_0^2)\frac{[k(k+1)P_r'(\mu_0)P_k(\mu_0) - r(r+1)P_r(\mu_0)P_k'(\mu_0)]}{[k(k+1) - r(r+1)]} \tag{2-23}$$

and for $k=r$

$$Q_{rk} = 2[-(1-\mu_0^2)P_k'(\mu_0)P_k(\mu_0) + 2k(k+1)(1-\mu_0^2)H(k,\theta_0)] \tag{2-24}$$

where $H(k,\theta_0)$ is a polynomial of $\sin^2\frac{\theta_0}{2}$ of order k, namely,

$$H(k,\theta_0) = \sum_{m=0}^{\infty} \frac{C_m^k}{m+1}(\sin^2\frac{\theta_0}{2})^m \tag{2-25}$$

where

$$C_m^k = \sum_{n=0}^{m} A_n^k A_{m-n}^k \tag{2-26a}$$

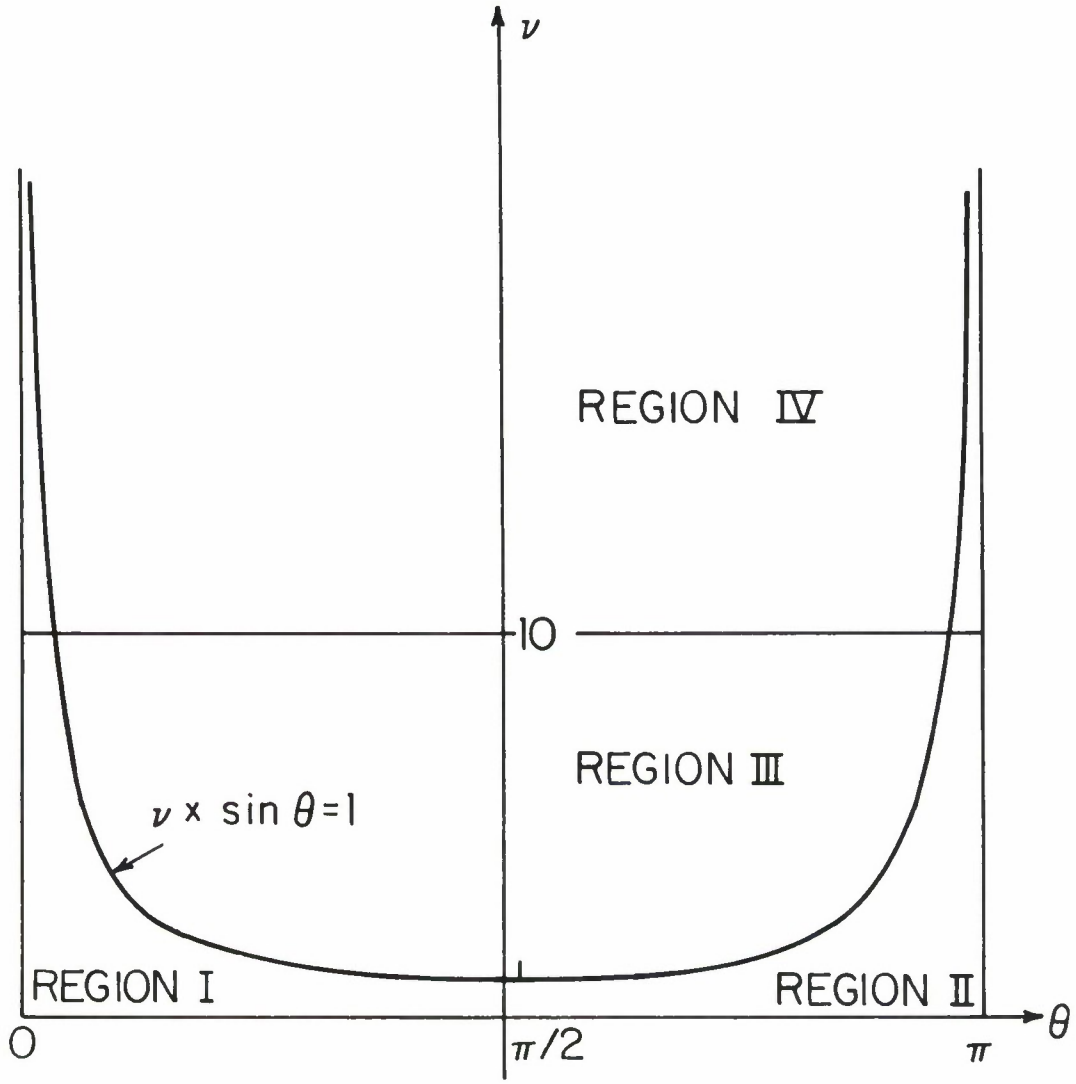


FIG. 2-2 COMPUTATION OF FRACTIONAL ORDER LEGENDRE FUNCTION

and

$$A_m^k = (-1)^m \frac{\Gamma(k+m+1)}{\Gamma(m+1)\,\Gamma(k-m+1)} \qquad (2\text{-}26b)$$

Hence, $A_m^k = 0$ for $m > k$.

The generation of Bessel's functions of fractional order is not significantly different from that for functions of integral order. It will not be discussed here.

4. THE BICONICAL ANTENNA

As a preliminary to the later discussions of the modified dipole, the problem of the thin, center-fed biconical dipole will be treated first. Based upon equations (1-90), (1-97) and (1-99) given in Volume I, a numerical solution for the input impedance and current distribution with various cone angles and antenna heights can be obtained.

Most of the considerations here will provide an opportunity to develop the technique which will be employed later for the more complicated case of the modified dipole. Furthermore, they will also serve as a numerical extrapolatory check for the modified dipole with a shrinking central sphere.

Three sets of linear algebraic equations describing the same biconical structure will be solved numerically for various antenna lengths and cone angles with proper error estimate for the elements of the coefficient matrices and the terms with infinite summations.

The general terms of the coefficient matrix in (1-90) include the sum

$$g(r,k,v) = \sum_v \frac{(2v+1)}{v(v+1)} \frac{(1-\mu_0^2)}{[k(k+1)-v(v+1)][r(r+1)-v(v+1)]} \frac{\partial v}{\partial \mu_0} \qquad (2\text{-}27)$$

To discuss this case for large v, some approximate formulae are introduced for the Legendre's functions which have been given by Schelkunoff [6] provided θ_0 is not near 0 or π,

$$P_v(\cos\theta) \approx \frac{\cos(q - \pi/4)}{(\frac{\pi}{2} p \sin\theta)^{1/2}} \tag{2-28a}$$

and

$$P_v(-\cos\theta) \approx \frac{\cos(q\pi - q\theta_0 - \pi/4)}{(\frac{\pi}{2} p \sin\theta)^{1/2}} \tag{2-28b}$$

where

$$p = [v(v+1)]^{1/2} \qquad \text{and} \qquad q = [(v+1/2)^2 + 1/4]^{1/2}$$

Thus,

$$L_v(\theta) \approx \frac{\sin(q\pi/2 - \pi/4)\, \sin(\pi/2 - \theta)}{(\frac{\pi}{2} p \sin\theta)^{1/2}} \tag{2-29}$$

Using the fact that $L_v(\theta_0) = 0$, it follows that for v very large

$$(1-\mu_0^2) \frac{(2v+1)}{v(v+1)} \frac{\partial v}{\partial \mu_0} \approx \frac{4 \sin^2\theta_0}{(\pi - 2\theta_0)} \tag{2-30}$$

The factor $1/[k(k+1) - v(v+1)]\,[r(r+1) - v(v+1)]$ tends to be quite large and sensitive to the value of v when v is near k or r. Let N be the matrix size, then the largest value that k or r may have is (2N+1). Let v_m be the value of v at which the series (2-27) is terminated. An examination of (2-27) shows that the most significant values occur when v-k and/or v-r are small as v passes through the entire range of values obtained from $L_v(\theta_0) = 0$. Therefore, it is necessary to require that v_m be greater than (2k+1) and (2r+1) in all computations.

Let R(r, k, v) denote the remainder such that

$$R(r, k, v_m) = \sum_{v=v_m}^{\infty} g(r, k, v) \tag{2-31}$$

From (2-27) and (2-30), one obtains for

(i) off-diagonal elements, i.e., $k \neq r$

$$g(r, k, v) \approx \frac{4 \sin^2 \theta_0}{(\pi - 2\theta_0)} \left[A_1 \left(\frac{1}{v+k} - \frac{1}{v-k-1} \right) + A_2 \left(\frac{1}{v+r} - \frac{1}{v-r-1} \right) \right] \tag{2-32}$$

where

$$A_1 = \frac{1}{(2k+1)(k+r+1)(r-k)} \tag{2-33a}$$

and

$$A_2 = \frac{1}{(2r+1)(k+r+1)(k-r)} \tag{2-33b}$$

From (2-30), one finds that the separation of zeros of $L_v(\theta_0)$ is approximately

$$\Delta v = \frac{2\pi}{(\pi - 2\theta_0)} \tag{2-34}$$

then

$$R(r, k, v_m) \approx \frac{\pi - 2\theta_0}{\pi} \int_{v_m}^{\infty} g(r, k, v)\, dv$$

$$= \frac{2 \sin^2 \theta_0}{\pi} \left[A_1 \ln \left(\frac{v_m - k - 1}{v_m + k} \right) + A_2 \ln \left(\frac{v_m - r - 1}{v_m + r} \right) \right] \tag{2-35}$$

(ii) For diagonal terms, i.e., $k = r$

$$g(r, k, v) \approx \frac{4 \sin^2 \theta_0}{-2\theta_0} \frac{1}{(2k+1)^2} \left(\frac{1}{v-k-1} - \frac{1}{v+k} \right)^2 \tag{2-36}$$

thus for large v,

$$R(k, k, v_m) \approx \frac{2\sin^2\theta_0}{\pi} \frac{1}{(2k+1)^2} \left[\frac{1}{v_m+k} + \frac{1}{v_m-k-1} - \frac{2}{k+1} \ln\left(\frac{v_m+1}{v_m-k-1}\right) \right] \tag{2-37}$$

For those terms with infinite summation over v in (1-97) and (1-99), the approximate formulae are introduced for large v and $v \gg x$, i.e., Meissel's formula [7]

$$J_v(x) \approx \left(\frac{\pi x}{2}\right)^{1/2} \frac{x^{(v+1/2)} \exp\left[(v+1/2)^2 - x^2 - (v+1/2)\right]}{\left[1-\left(\frac{x}{v+3/2}\right)^2\right]^{1/4} \left[1+(1-x^2/v^2)^{1/2}\right]^v}$$

$$(v-x \gg v^{-1/3}) \tag{2-38}$$

Actually, since v is much larger than x, a further approximation can be made, namely,

$$[(v+1/2)^2 - x^2]^{1/2} - v \approx -\frac{x^2}{2v}$$

$$\left[1-\left(\frac{x}{v+1/2}\right)^2\right]^{1/4} \approx 1 - \frac{x^2}{4v^2}$$

$$\left[1+\sqrt{1-\left(\frac{x}{v}\right)^2}\right] \approx 2^v \exp\left(-\frac{x^2}{4v}\right)$$

thus,

$$J_v(x) \approx \pi^{1/2} \frac{x^{(v+1)} \exp\left[-\frac{x^2}{4(v+1/2)}\right]}{2^{(v+1)}\,\Gamma(v+3/2)} \tag{2-39}$$

Similarly,

$$J'_\nu(x) \sim \frac{\nu+1}{x} J_\nu(x) \tag{2-40}$$

Again, let $g_1(r,k,\nu)$ and $g_2(r,k,\nu)$ denote the remainders for the first and second set of linear equations, respectively; one finds for:

(i) the off-diagonal elements, i.e., $k \neq r$

$$g_1(r,k,\nu) = \frac{2\nu+1}{\nu(\nu+1)} \frac{J'_\nu(\beta_0 h)}{J_\nu(\beta_0 h)} \frac{\frac{\partial \nu}{\partial \mu_0}}{[k(k+1) - \nu(\nu+1)][r(r+1) - \nu(\nu+1)]} \tag{2-41}$$

and

$$g_2(r,k,\nu) = g_1(r,k,\nu) \left[\frac{J_\nu(\beta_0 h)}{J'_\nu(\beta_0 h)}\right]^2 \tag{2-42}$$

For ν large and $\nu >> \beta_0 h$, one gets

$$R_1(r,k,\nu_m) \approx \frac{\sin^2\theta_0}{\pi \beta_0 h (k^2 - r^2)} \left[\ln\left(\frac{\nu_m^2 - r^2}{\nu_m^2 - k^2}\right) + \frac{1}{k}\ln\left(\frac{\nu_m + k}{\nu_m - k}\right) - \frac{1}{r}\ln\left(\frac{\nu_m + r}{\nu_m - r}\right)\right] \tag{2-43}$$

Similarly,

$$R_2(r,k,\nu_m) \approx \frac{2\beta_0 h \sin^2\theta_0}{\pi} \left[A_1 \left\{\frac{1}{k-1}\ln\left(\frac{\nu_m + k}{\nu_m + 1}\right) + \frac{1}{k+2}\ln\left(\frac{\nu_m - k - 1}{\nu_m + 1}\right)\right\} + A_2 \left\{\frac{1}{r-1}\ln\left(\frac{\nu_m + r}{\nu_m + 1}\right) + \frac{1}{r+2}\ln\left(\frac{\nu_m - k - 1}{\nu_m + 1}\right)\right\}\right] \tag{2-44}$$

(ii) the diagonal terms, i.e., $k = r$

$$R_1(k,k,v_m) \approx \frac{\sin^2\theta_0}{\pi\beta_0 h}\left[\frac{1}{2r^2}\ln\left(\frac{v_m-k}{v_m+k}\right)+\frac{1-v_m/r^2}{v_m^2-r^2}\right] \tag{2-45}$$

and

$$R_2(k,k,v_m) \approx \frac{2\beta_0 h\sin^2\theta_0}{\pi}\left[\frac{-1}{(2k+1)^2(k-1)}\frac{1}{v_m+k}\right.$$

$$\left.+\frac{1}{(2k+1)^2(k+2)}\frac{1}{(v_m-k-1)}\right] \tag{2-46}$$

Having found the error estimate due to the truncation of the infinite sum in the calculation of each matrix element, it has been possible to get the relative error introduced in each element by comparing the error estimates with the corresponding truncated matrix elements and hence keep the relative error in each element of the matrix less than a specified value (one part in 10^3 in our calculation). Furthermore, by adding the error estimates to each element of the matrix and solving the resulting equations, it is possible to determine how sensitive the solution is to the degree of uncertainty in the matrix elements.

Before solving (1-90), it is of interest to consider an approximate solution derived originally by Schelkunoff [6] by assuming a sinusoidal current distribution along the thin biconical antenna and later from a different point of view by Tai [4] by neglecting all the off-diagonal elements of the coefficient matrix of (1-90) under the assumption that the half cone angle θ_0 is small so that the diagonal elements are dominant. The fact that the same formula was obtained by both for d_k in (1-90) signifies nothing more than that the approximations involved in the two approaches are essentially the same. In other words, a sinusoidal current distribution in itself implies sufficiently severe restrictions on the magnitude of the cone angle. As in the thin cylindrical antenna [4], a sinusoidal current distribution can only be accomplished with a line of ideal generators distributed along the length of the antenna which, in practice, does not exist and thus the assumed distribution is acceptable only for determining certain properties of very thin antennas over a very

limited range of length.

The approximate formula for the input impedance of a thin biconical antenna is [4],

$$Z_0 = Z_c \left(\frac{Z^e_{mc} \sin\beta_0 h - j Z_c \cos\beta_0 h}{Z_c \sin\beta_0 h - j Z^e_{mc} \cos\beta_0 h} \right) \tag{2-47}$$

where

$$Z^e_{mc} = R^e_{mc} + j X^e_{mc} \tag{2-48}$$

$$R^e_{mc} = 30 \left[2\mathrm{Cin}\, 2\beta_0 h + (\mathrm{Si}\, 4\beta_0 h - 2\mathrm{Si}\, 2\beta_0 h) \sin 2\beta_0 h + (2\mathrm{Cin}\, 2\beta_0 h - \mathrm{Cin}\, 4\beta_0 h) \cos 2\beta_0 h \right] \tag{2-49a}$$

$$X^e_{mc} = 30 \left[2\mathrm{Si}\, 2\beta_0 h + (\ln 4 - \mathrm{Cin}\, 4\beta_0 h) \sin 2\beta_0 h - \mathrm{Si}\, 4\beta_0 h \cdot \cos 2\beta_0 h \right] \tag{2-49b}$$

where $\mathrm{Si}\, x = \int_0^x \frac{\sin u}{u}\, du$ is the integral sine and $\mathrm{Cin}\, x = \int_0^x \frac{1-\cos u}{u}\, du$.

Generally speaking, the coefficient matrix of (1-90) has symmetric elements with its dominant part lying on and near the main diagonal. For each row in the upper triangular matrix of coefficients, the diagonal element is always the dominant quantity with the other matrix elements rapidly decreasing in value as the distance from the diagonal is increased. The size of the diagonal elements, however, also decreases in the direction of larger row numbers, but with a much smaller rate than that of the off-diagonal ones.

It is instructive to look into the elements of the coefficient matrix of (1-90) for the case of a quarter wave conical dipole. Note that, while all the off-diagonal elements are decreasing with shrinking cone angles, the diagonal elements continue to increase substantially. Therefore, in

the limit, this justifies the approximation made by Tai in deriving (2-47) for extremely small cone angles ($\theta_0 < 0.01$). Whereas, on the other hand, when the angle becomes larger, the opposite trend occurs and the dominance in value seems to spread from the diagonal elements to the ones near them. Hence, the neglect of off-diagonal terms will cause great errors in the final solution. A typical example of a 5 x 5 matrix, normalized by its maximum element for the cone angles 1^o and 0.1^o, is shown below

$$[C]_{\theta_0 = 1^o} = \begin{vmatrix} 1.000 & 0.040 & 0.020 & 0.010 & 0.007 \\ & 0.230 & 0.070 & 0.005 & 0.003 \\ & & 0.096 & 0.003 & 0.002 \\ & & & 0.056 & 0.0013 \\ & & & & 0.034 \end{vmatrix}$$

$$[C]_{\theta_0 = 0.1^o} = \begin{vmatrix} 1.000 & 0.010 & 0.005 & 0.003 & 0.002 \\ & 0.160 & 0.018 & 0.001 & 0.0008 \\ & & 0.083 & 0.0006 & 0.0004 \\ & & & 0.050 & 0.0003 \\ & & & & 0.0320 \end{vmatrix}$$

A concentration of the dominant values is readily observed from the above examples. But even for angles as small as 0.1^o, element C_{31} which is small compared to C_{11} is still of comparable magnitude with respect to element C_{33}. Hence, the neglect of C_{31} will cause at least 10% error in the final solution for b_3. For large cone angles, elements in the first row, and hence first column, become quite comparable to the corresponding diagonal elements, which implies that a greater error is introduced by neglecting the off-diagonal terms.

The results presented in Fig. 2-3 are the calculated input impedance from (1-90) for the biconical antenna with cone angles $\theta_0 = 1.5^o$, 1.1^o and 0.1^o. The impedance variation is given as a function of the electric length of the antenna (i.e., $\beta_0 h$) with particular points where the electric

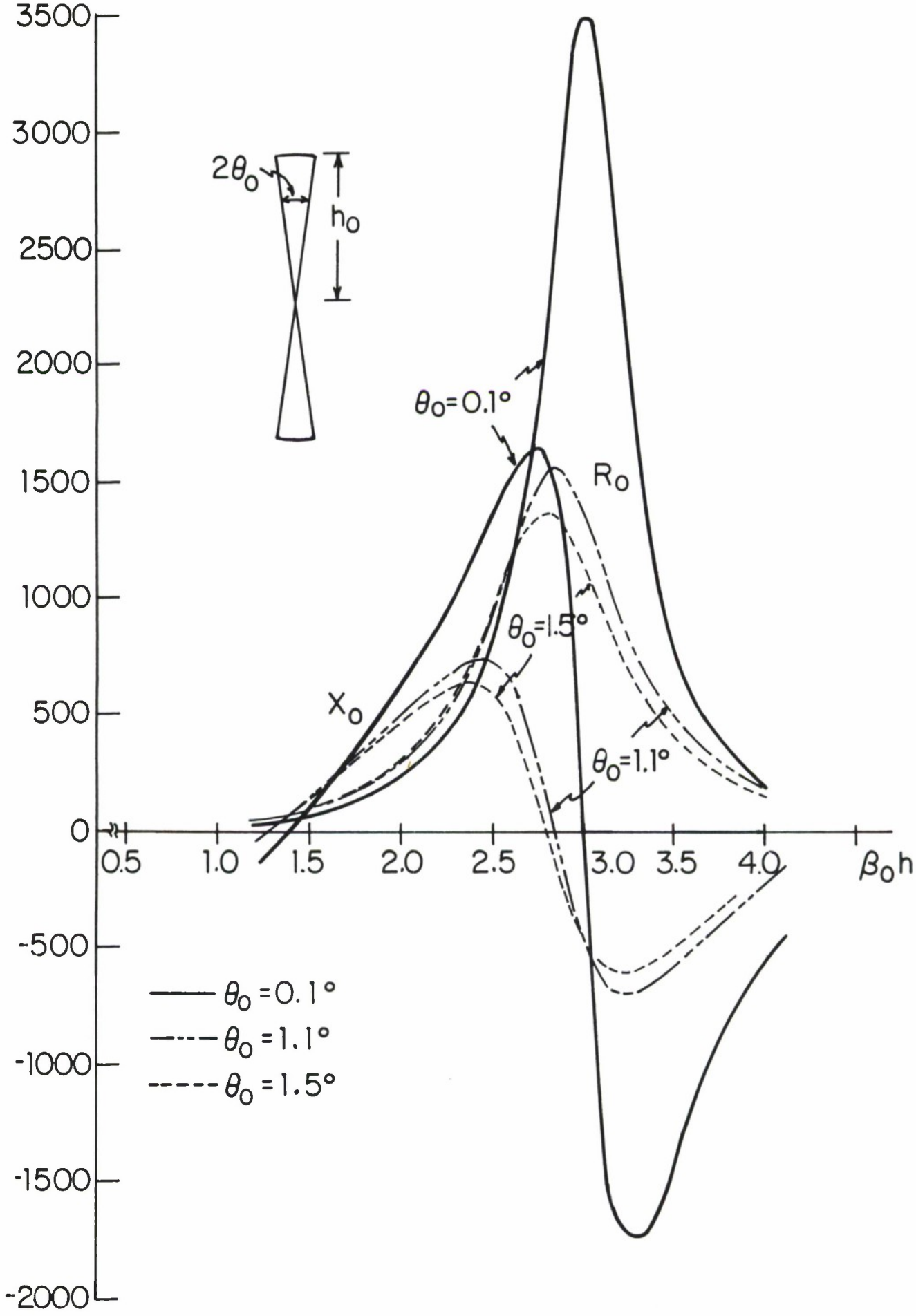


FIG. 2-3 INPUT IMPEDANCES OF THIN BICONICAL ANTENNA

length attains the values $\lambda/4$ and $\lambda/2$ specially noted. Larger cone angles tend to flatten the impedance curve and shorten both the resonant and antiresonant lengths of the antenna. A discussion of the characteristics of the curves for the impedance $Z_0 = R_0 + jX_0$ may be centered about certain critical points. Of these, the most important are the values of $\beta_0 h$ for which the reactance vanishes and which involve some interesting points which have counterparts in the cylindrical antenna.

Input resonance is characterized by

$$X_0 = 0, \quad R_0 = (R_0)_{res.} \quad \text{and} \quad X_0 = (X_0)_{res.} \tag{2-50}$$

The condition of input resonance occurs first when $\beta_0 h$ is somewhat smaller than $\pi/2$. It is seen that the resonant length of a biconical antenna moves toward $\pi/2$ as $1/\theta_0$ (and hence Z_c) increases. It is significant to note that when $\beta_0 h = \pi/2$ the input impedance given by the approximate formula (2-47) is

$$Z_0 = 73.13 + j\,153.7 \text{ ohms} \tag{2-51}$$

Thus the input impedance of the thin biconical antenna as given by (2-47) is a constant independent of the cone angle θ_0 when h is a quarter wave in length and θ_0 is sufficiently small.

For the cylindrical antenna, such constancy obtains in the modified zeroth order approximation when for $\beta_0 h = \pi/2$, i.e.,

$$(Z_0)_{cylindrical} = 73.13 + j\,42.5 \text{ ohms} \tag{2.52}$$

It was observed by King [1] that the curve of Z_0 at $\beta_0 h = \pi/2$ approaches the value of 73.13 + j 42.5 ohms as $\Omega = 2\ln(2h/a)$ increases. The same phenomenon is observed for the biconical antenna as the corresponding characteristic impedance Z_c increases. This is shown in Fig. 2-4 in which the impedance curves tend to converge to the value of 73.13 + j 153.7 ohms as the cone angle becomes vanishingly small. This suggests that the formula (2-47) may be no better than a modified zeroth-order approximation.

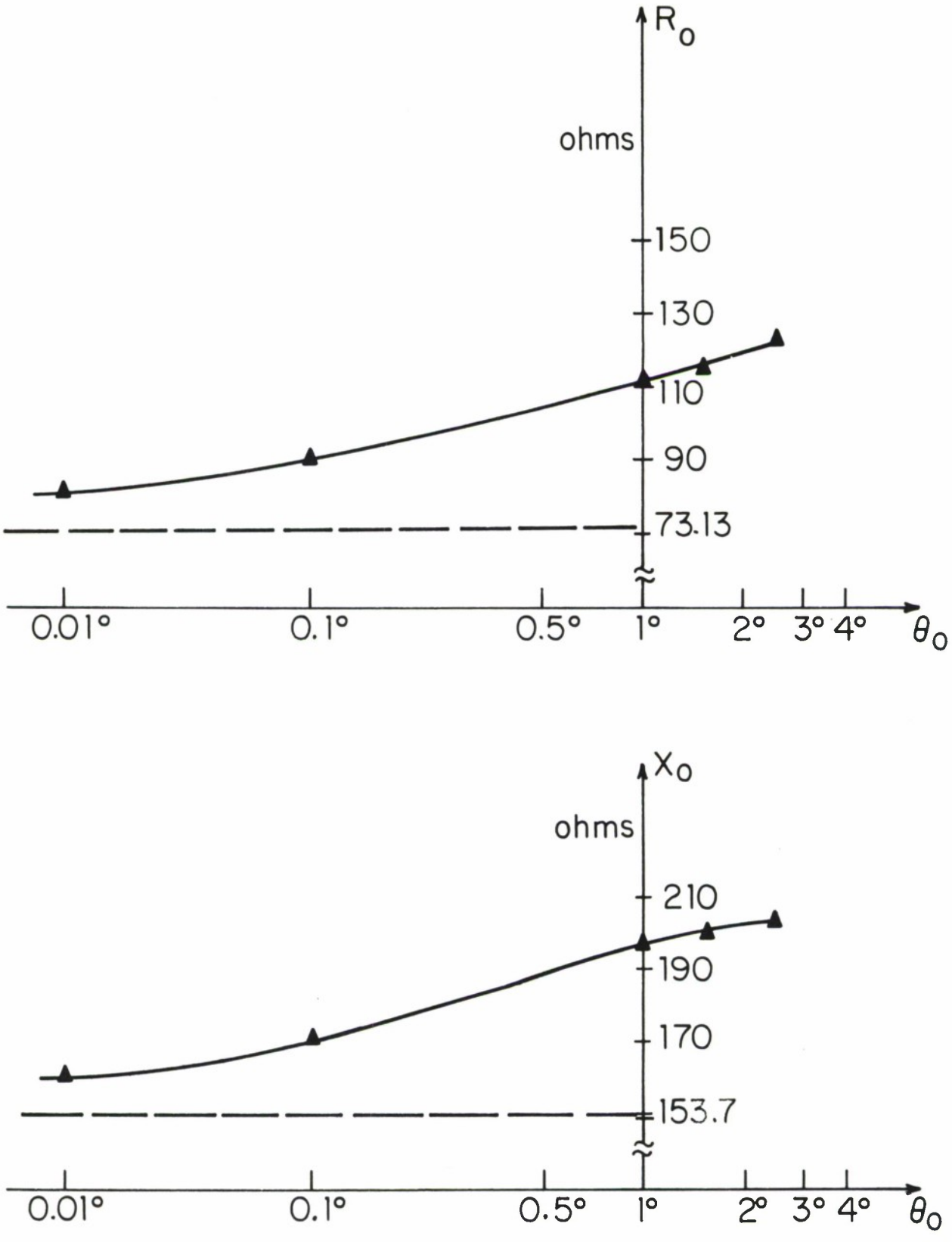


FIG. 2-4 INPUT IMPEDANCE OF $\lambda/4$ THIN CONICAL ANTENNA VS. CONE ANGLE θ_o

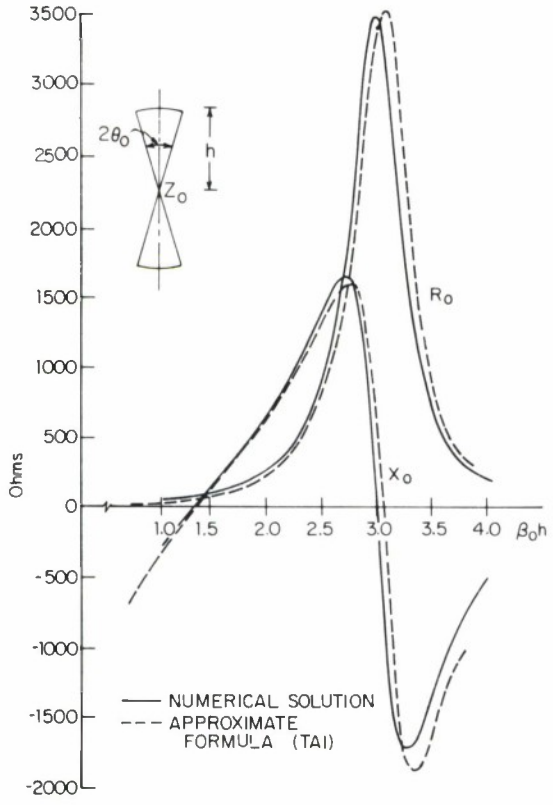
Input antiresonance is characterized by

$$X_0 = 0, \quad R_0 = (R_0)_{\text{antires.}} \quad \text{and} \quad \beta_0 h = (\beta_0 h)_{\text{antires.}} \tag{2-53}$$

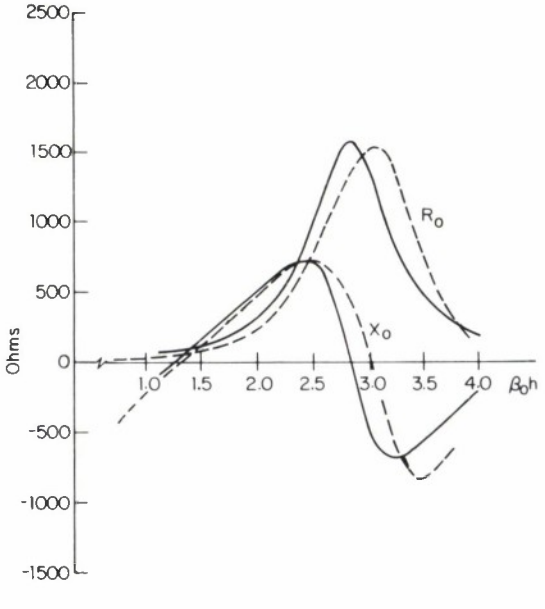
It occurs at values of $\beta_0 h$ that are less than π by an amount that increases with decreasing Z_c in a manner resembling the occurrence of resonance near $\pi/2$. The reactance X_0 has a maximum for $\beta_0 h$ slightly less than $(\beta_0 h)_{\text{antires.}}$. The resistance has its maximum very near to anti-resonance. The fact that the ratio $|X_{min}|/|X_{max}|$ characterizing the behavior of the reactance near antiresonance is always greater than one for the cylindrical antenna is not necessarily true of the biconical antenna. The ratio $|X_{min}|/|X_{max}|$ for the biconical antenna is quite different; in fact, $|X_{min}|/|X_{max}|$ is greater than one only for small cone angles ($\theta_0 < 1^o$) and becomes less than one with increasing θ_0. This implies that the capacitive lobe of the reactance curve increases with θ_0 and becomes larger than the associated inductive lobe. In other words, the capacitive loading effect of the upper and lower conical surfaces which act more or less as a capacitor with electric lines E_θ between them to the point generator are much larger in the biconical case than in the cylindrical one.

The difference in the behavior of the reactance curves might be ascribed to the quite different driving condition in these two types of antenna; for the biconical antenna, there exists a point generator at the center while for the cylindrical dipole, the excitation is actually over a circular surface with vanishing separation between them. This suggestion is later justified in the experimental comparison between modified conical and cylindrical dipoles in which both are driven by a circular radial electric field at some distance away from the origin. Both real and imaginary parts of the impedance curves are more comparable than observed here.

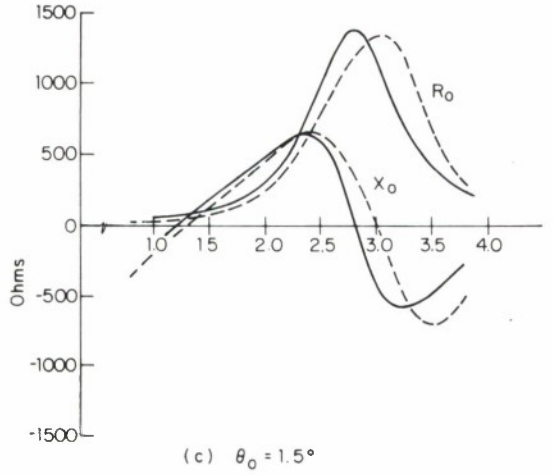
A comparison of the input impedance of the thin biconical antenna as computed from (1-90) with the corresponding approximate results using (2-47) is shown in Fig. 2-5. It is seen that the approximate results lead to lower resistance near anti-resonance and resonance and larger



(a) $\theta_0 = 0.1°$



(b) $\theta = 1.1°$



(c) $\theta_0 = 1.5°$

FIG. 2-5 INPUT IMPEDANCE FOR THIN CONICAL ANTENNA

resonant and anti-resonant length than the computed results. The reactance behaves in an analogous manner. In general, the approximate formula leads to results comparable to those with (1-90) only at very small cone angles. This suggests that for cone angles greater than 0.1^{o}, (1-90) must be used for studying the behavior at resonance and anti-resonance.

In Fig. 2-6, is shown the variation of the cylindrical antenna impedance as a function of antenna length with various values of the parameter Ω and a comparison with a biconical antenna with half angle $\theta_0 = 1.1^{o}$. The comparison is interesting since by judicious adjustment of the radius of the cylindrical antenna, a measure of correspondence can be achieved. It is obvious from Fig. 2-6 that for the cylindrical antenna with radius equal to or greater than the maximum radius of the biconical antenna a_ℓ, the impedance behavior is quite different from that of the biconical antenna, whereas the impedance curves tend to be comparable as the radius of cylindrical antenna remains within the range of 1/4 or 1/5 of a_ℓ. Again the resistance curves seem to match each other better than the reactance curves. One way of measuring the comparability is by defining the average of the antenna as follows [6], i.e.,

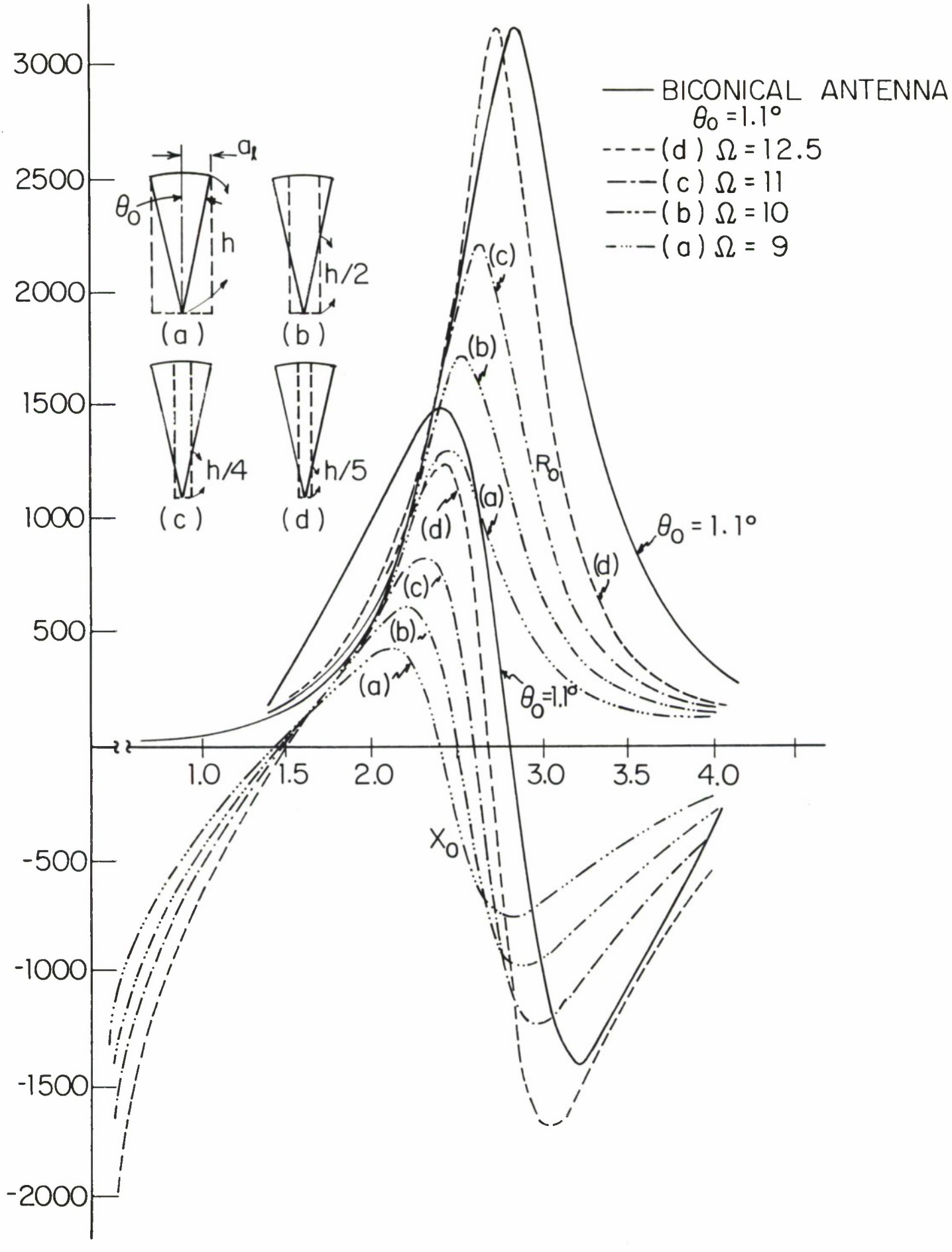
$$< \Omega >_{ave.} = \frac{2}{h} \int_0^h \ln(\frac{2R}{r})\, dR \qquad (2\text{-}54)$$

where r is the radius of the section at radial distance R from the driving point at the origin. Accordingly, the average Ω for the cylindrical antenna with radius a is

$$< \Omega >_{ave.} = 2\,[\ln(\frac{2R}{a}) - 1] \qquad (2\text{-}55)$$

and that for the conical antenna with angle θ_0 is

$$< \Omega >_{ave.} = 2 \ln(2 \cot\theta_0) \qquad (2\text{-}56)$$



G. 2-6 INPUT IMPEDANCE OF CONICAL AND CYLINDRICAL DIPOLE

It is seen in Fig. 2-6 that the impedances of the biconical and cylindrical antennas are comparable when both have approximately the same $< \Omega >$ (for curve (d) $< \Omega > = 12.5$ and $< \Omega >_{conical} = 14.1$). This means that for purpose of comparison, one must choose a cylindrical antenna with a radius at least smaller than half the maximum radius of the cone antenna. No attempt has been made here to pursue further a theoretical study of the relation between these two radiating structures. The general properties provide a qualitative guide of the extent to which these two radiating antennas behave reasonably alike, so that information obtained from one can be generally applied to predict the behavior of the other. This is essential later in interpretating the experimental comparison between modified cylindrical and conical antennas in which both the cone angle and antenna radius are fixed by the measuring set-up.

Numerical values for (1-97) and (1-99) for a corresponding biconical structure are also obtained along much the same lines as for (1-90). All three sets of equations give the same results with differences less than 1 per cent among them. For each equation, matrices of order N-2, N-1, N etc. are solved separately and the solution inspected for convergence. It was found if $v_m >$ max. (2k+1, 2r+1) that the solutions converge both rapidly and monotonically. However, if $v_m <$ max. (2k+1, 2r+1), the solutions tend to oscillate widely and show little tendency to converge.

The choice of the size of the matrices depends upon the accuracy required for the solution on the one hand and the computation time on the other. As a compromise, a less elegant method was adopted. That is, the extreme cases are examined with great care (i. e., the longest antenna). In addition to checking among themselves for convergence in the course of a numerical solution, solutions from a truncated matrix were continously checked with those of the modified matrix in which each element was corrected by adding the error estimates. To proceed one step further, solutions from a matrix of size N (which is the number when solutions show convergence of 1 per cent) are compared with solutions from a matrix of size (N+10). It is required that their difference lie within 1 per cent. Although it is not known exactly how this truncated solution converges to the true solution, at least a self-consistent one is obtained

in this manner. The number N is then used for antennas of shorter lengths without repeating the same test for convergence.

As a double check for convergence, observations were made particularly at $\beta_0 h = 1.6$ and $\theta_0 = 1.1^{\circ}$. Numerical results from (1-90) and (1-99) for a series of matrix sizes are shown in Fig. 2-7. It is observed that values of (1-90) tend to converge toward the final value in a monotonically increasing way with an increasing size of the matrices while the solutions from (1-99) also approach the same value in much the same but monotonically decreasing way.

The mutual checking among these three sets of linear equations suggests that the procedures used in the derivation of (1-97) and (1-99) for the biconical antenna and hence, in quite a similar manner for (1-57) and (1-64) for the modified dipole are justified.

5. THE MODIFIED CONICAL DIPOLE

Based upon (1-57), (1-64), (1-73) and (1-76), a numerical solution for the current distributions along both antenna and sphere with various sizes can be obtained. Also the input conductance and susceptance of the antenna are readily known as the real and imaginary parts of the total current at the driving point. In the process of the numerical computation, much of the analysis and many of the results from the preceding section can readily be utilized with some necessary modifications.

The general terms of the coefficient matrix of (1-57) and (1-64) are the following forms:

$$g_1(r,k,\nu) = \frac{(2\nu+1)}{\nu(\nu+1)} \frac{M_\nu'(\beta_0 a)}{M_\nu(\beta_0 a)} \frac{1}{[k(k+1)-\nu(\nu+1)]} \cdot \frac{1}{[r(r+1)-\nu(\nu+1)]} \frac{\partial \nu}{\partial \mu_0} \qquad (2\text{-}57)$$

and

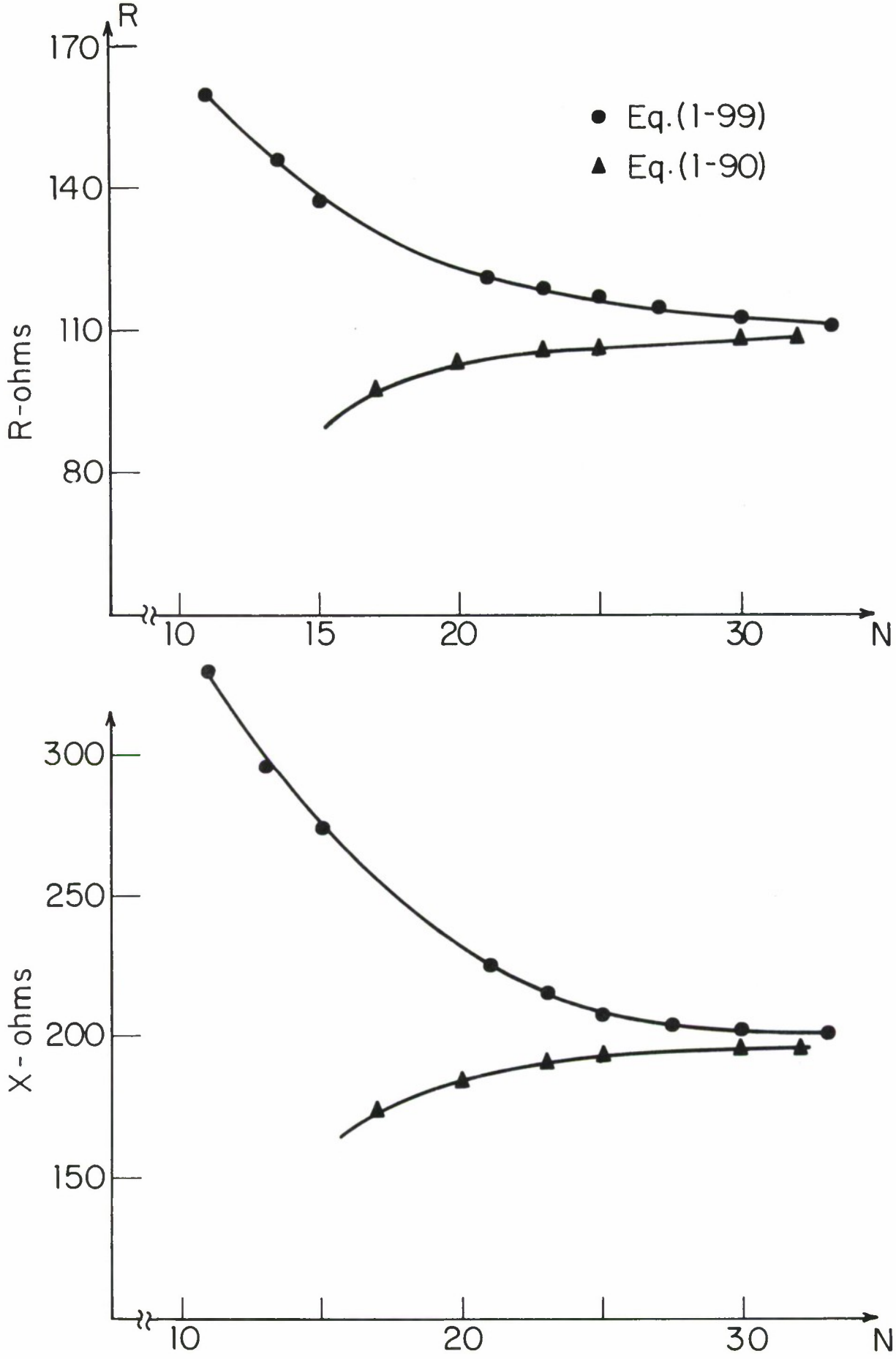


FIG. 2-7 CONVERGENCE OF NUMERICAL SOLUTION FOR $\lambda/4$ BICONICAL ANTENNA ($\theta_0 = 1.1°$)

$$g_2(r, k, v) = g_1(r, k, v) \left[\frac{M_v(\beta_0 a)}{M'_v(\beta_0 a)} \right]^2 \tag{2-58}$$

where subscripts 1 and 2 denote the first and second sets of linear equations respectively.

The above equations differ from those for the corresponding biconical antenna only in the ratio $M'_v(\beta_0 a)/M_v(\beta_0 a)$.

By applying the approximate formulae for the Bessel's function of the second kind for large v, i. e.,

$$N_v(x) \approx (\pi)^{1/2} 2^v \Gamma(v + 1/2) x^{-v} \exp \frac{x^2}{4(v+1/2)} \tag{2-59a}$$

and

$$N'_v(x) \approx -\frac{v}{x} N_v(x) \tag{2-59b}$$

and by using the fact that $v >> x$, one obtains

$$\frac{M'_v(x)}{M_v(x)} \approx \frac{(v+1)}{x} = \frac{J'_v(x)}{J_v(x)} \tag{2-60}$$

The fact that $M'_v(x)/M_v(x)$ and $J'_v(x)/J_v(x)$ have the same asymptotic behavior for large v implies that all of the formulae for error estimates in the preceding section [(2-35) - (2-46)] can be used directly for the modified conical antenna with $\beta_0 h$ replaced by $\beta_0 a$. Furthermore, the same error bound obtained for the two cases suggests that the coefficient matrix associated with the modified dipole should have essentially the same convergence properties as that for the biconical one; especially for a small central sphere in which D_{r2} and D'_{r2} are small relative to D_{r1} and D_{r2} [(1-57) and (1-64)].

Additional terms (i. e., D_{r2} and D'_{r2}) which characterize the presence of the central sphere for the modified dipole are also present in the form of an infinite summation in the driving terms of (1-57) and (1-64).

For a small central sphere, these terms are of the order $(\beta_0 b)^{\nu_1}$ [where ν_1 is the first root of the characteristic equation $L_\nu(\theta_0) = 0$] and vanish in the limiting case of the biconical antenna. On the other hand, D_{r2} and D'_{r2} become more important as the central sphere becomes larger.

The error estimates for D_{r2} and D'_{r2} will be examined first. Thus,

$$D_{r2} = \frac{-j V_0^e}{60} P_r(\theta_0)(1-\mu_0^2) \sum_\nu C_\nu \frac{(2\nu+1)}{\nu(\nu+1)} \frac{1}{[r(r+1) - \nu(\nu+1)]} \cdot$$

$$\frac{1}{M_\nu(\beta_0 a)\, J'_\nu(\beta_0 b)} \frac{\partial \nu}{\partial \mu_0} \qquad (2\text{-}61)$$

again, for large ν, one gets

$$C_\nu \approx \frac{-\sin(\nu\theta_0)}{\nu\, fn(\theta_0, \theta_1)\, \sin\theta_0 \cdot \sin(\theta_0+\theta_1)^{1/2}} \qquad (2\text{-}62a)$$

$$M_\nu(\beta_0 a)\, J'_\nu(\beta_0 b) \approx \frac{\nu}{2\nu+1} \left(\frac{a}{b}\right)^\nu \qquad (2\text{-}62b)$$

Therefore, the upper bound of the remainder of D_{r2} is R_{1D},

$$R_{1D}(r, \nu_m) \sim \frac{4}{\pi} \frac{1}{fn(\theta_0, \theta_1)} \left(\frac{b}{a}\right)^{\nu_m} \frac{1}{[\sin\theta_0 \cdot \sin(\theta_0+\theta_1)]^{1/2}} \cdot$$

$$\frac{1}{r(2r+1)} \ln\left(\frac{\nu_m}{\nu_m+r}\right) + \frac{1}{(2r+1)(r+1)} \ln\left(\frac{\nu_m}{\nu_m-2r-1}\right) \qquad (2\text{-}63)$$

Similarly for D'_{r2},

$$R_{2D}(r,\nu_m) \approx \frac{4a}{\pi}\left(\frac{b}{a}\right)^{\nu_m} \frac{1}{fn(\theta_0,\theta_1)[\sin\theta_0 \cdot \sin(\theta_0+\theta_1)]^{1/2}} \cdot$$

$$\left[\frac{1}{(r+1)\nu_m} + \frac{1}{(r+1)^2}\ln\left(\frac{\nu_m - r - 1}{\nu_m}\right) - \frac{1}{r^2}\ln\left(\frac{\nu_m + r}{\nu_m}\right)\right] \tag{2-64}$$

The errors caused by the double truncation which occurs in the computation of the fields [i.e., $(B_\Phi)_{1C}$ and $(E_\theta)_{1C}$] and the input admittance (i.e., Y_{1C}) can be estimated in the following way. A comparison of the real part of the complex power radiated as seen from the driving point with the far-field radiated power computed from (1-89) gives:

$$G_0 = 240 \sum_k{}' \frac{|b_k|^2}{(2k+1)(k+1)k} \frac{1}{|R_k(\beta_0 a)|^2} \tag{2-65}$$

The right-hand side converges extremely fast. It was found that less than ten terms were needed to achieve accuracy of one out of 10^5. It is reassuring to note that in the calculation, the left-hand side of (2-65) agrees within the range of relative error one part out of 10^3 with the right-hand side.

It is interesting to note that the computation on the right-hand side of (2-65) involves only the radiation field coefficients (i.e., b_k's) which is the solution of an infinite set of linear equations while the left-hand side is a result involving all the fields inside antenna region. The excellent agreement between these two demonstrates the internal consistency and accuracy of the computation.

(A) Input Admittance

The input admittances of modified conical antennas with half cone angle $\theta_0 = 1.1^\circ$ and driving gap $\theta_1 = 1.5^\circ$ and antenna heights ranging from

$\beta_0 h$ = 0.6 to 4.0 and spherical radius ranging from $\beta_0 b$ = 0.15 to 1.5 have been obtained (numerical data are listed in Appendix A).

In Fig. 2-8, are shown graphs of the input admittance of a modified conical antenna as a function of $\beta_0 h$ with the sphere radius as a parameter. This is convenient theoretically and also for a later experimental check in conjunction with a partially fixed installation for which the frequency is varied (i.e., in the experimental set-up, the physical size of the central sphere, center conductor and hence the cone angle are fixed while the antenna length varies in discrete steps).

Because the input admittance of the modified conical antenna is defined as that seen at one driving terminal, the admittance curves in Fig. 2-8 have to be rescaled by a factor 1/2 in order to compare them with that for a corresponding biconical antenna.

In general, the admittance curves are broadened and flattened as the center sphere is enlarged. More specifically, the magnitude of the maximum values of $|G_{max}|$ and $|B_{max}|$ and the minimum $|B_{min}|$ decrease as the radius of the central sphere increases until approximately at $\beta_0 b = 1$ where the circumference of the sphere equals one wavelength, and the magnitudes of $|G_{max}|$, $|B_{max}|$, and $|B_{min}|$ start growing again with increases in the size of the sphere.

The behavior of the susceptance near resonance and anti-resonance is also of interest with a change in the radius b. In Fig. 2-9, are shown the resonant and anti-resonant lengths as a function of $\beta_0 b$. It is seen that $(\beta_0 h)_{antires.}$ first decreases with increases in $\beta_0 b$ until $\beta_0 b$ approximately equals one, when $(\beta_0 h)_{antires.}$ begins to increase again. On the other hand, $(\beta_0 h)_{res.}$ remains at much the same value for $\beta_0 b < 0.5$, after which $(\beta_0 h)_{res.}$ increases rapidly. The increasing trend of $(\beta_0 h)_{res.}$ is stopped and reserved also at approximately $\beta_0 b = 1$. The relative constancy of $(\beta_0 h)_{res.}$ for $\beta_0 b < 0.5$ might be the result of simultaneous capacity loading due to a central modification which tends to shift $(\beta_0 h)_{res.}$ toward the right and a thickening on the average of the protruding portion which tends to pull $(\beta_0 h)_{res.}$ back toward smaller values.

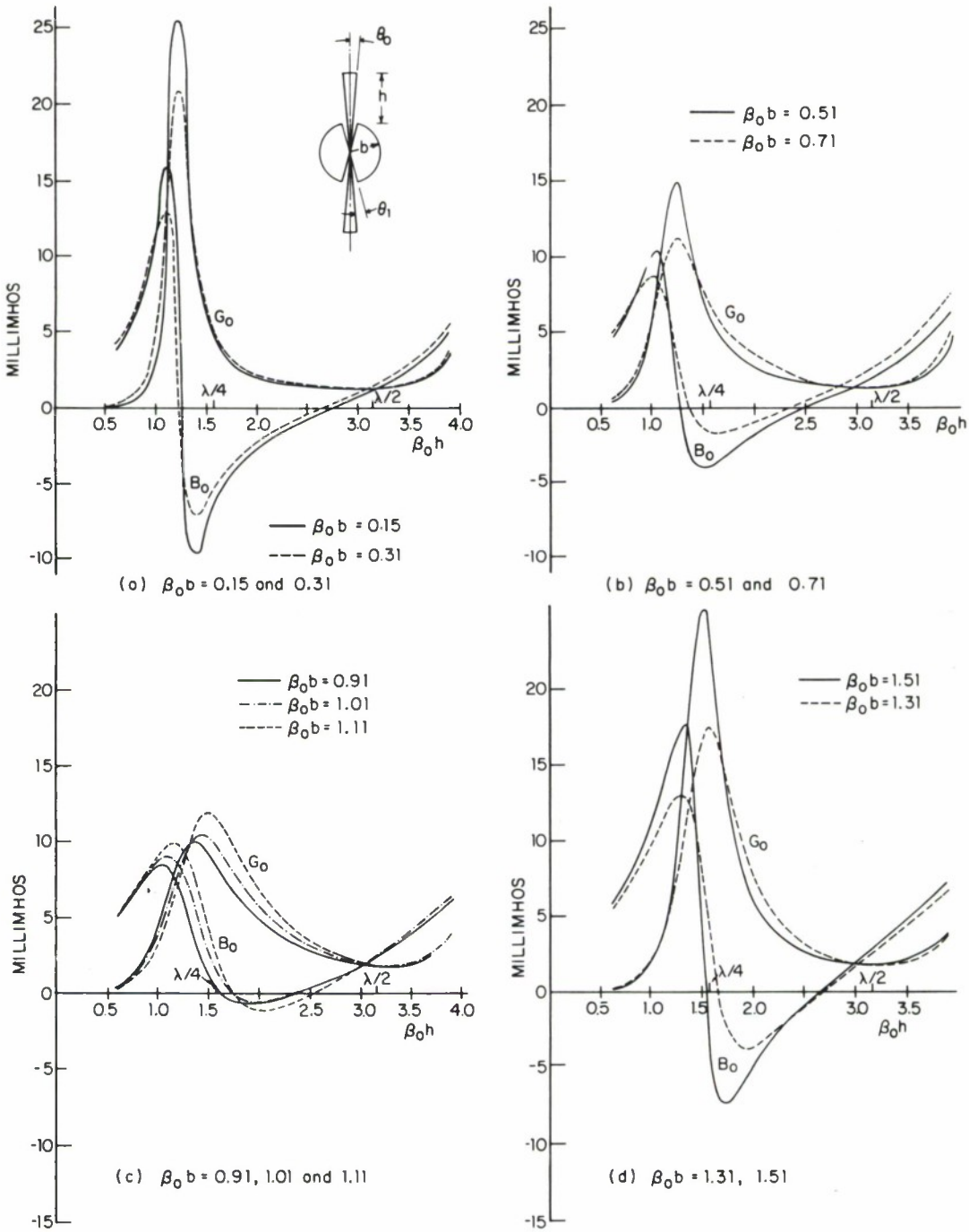


FIG. 2-8 INPUT ADMITTANCE OF MODIFIED CONICAL DIPOLE

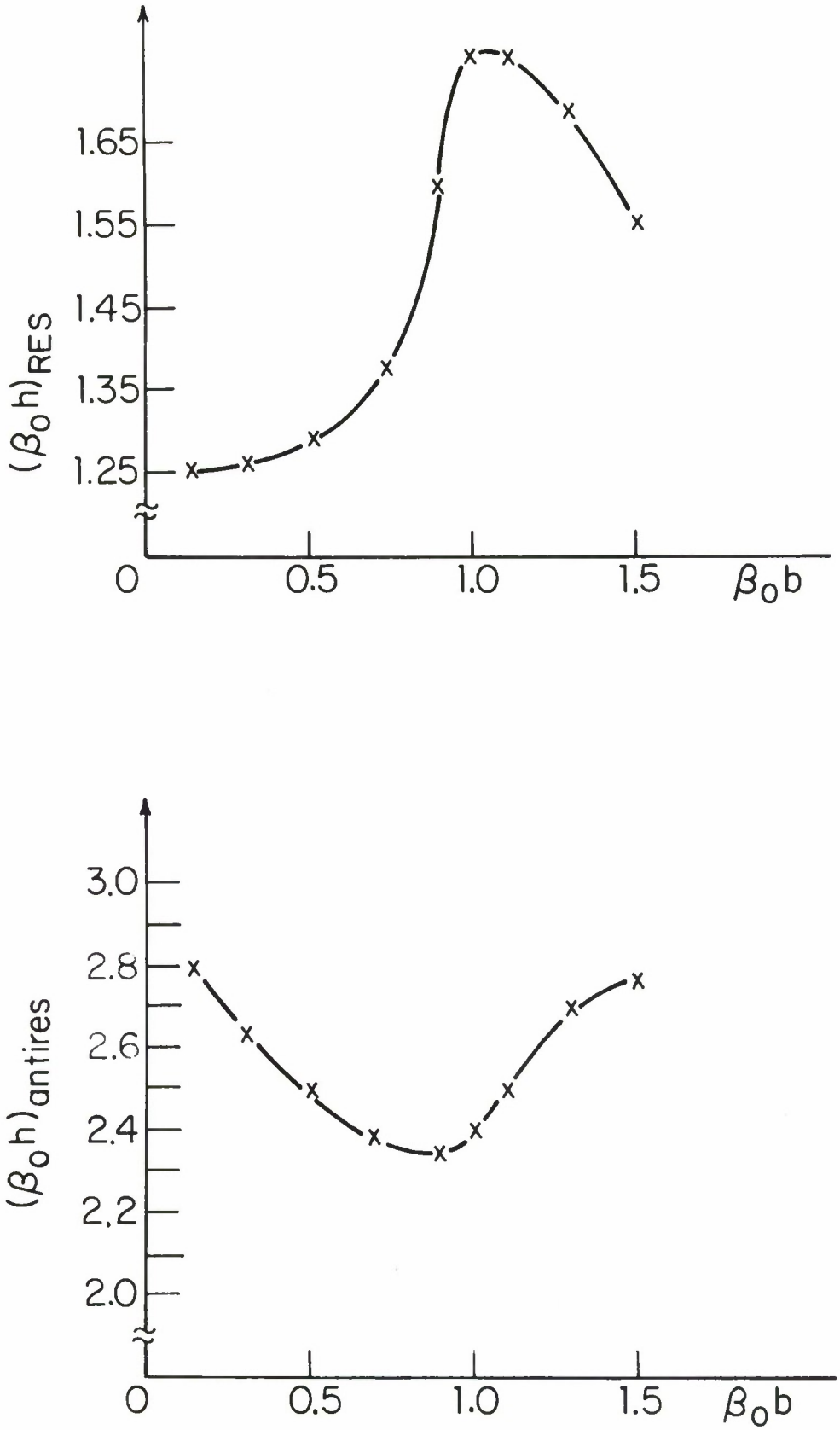


FIG. 2-9 VARIATION OF $(\beta_0 h)_{res.}$ AND $(\beta_0 h)_{antires}$ VS. $\beta_0 b$

In the neighborhood of $(\beta_0 h)_{res.}$ the slope of the susceptance curves decreases with increases in $\beta_0 b$. The flattening of the curve is accompanied by a shift of the point of zero susceptance ($\beta_0 b > 0.5$) from smaller values toward $\pi/2$. The fact that $(\beta_0 h)_{res.}$ becomes even larger than $\pi/2$ as $(\beta_0 b)$ becomes greater than 0.8 suggests that the central modification itself becomes such a substantial part of the whole radiating structure that the performance of the antenna protruding out from the sphere is changed beyond any resemblance with the corresponding dipole.

In the neighborhood of $\beta_0 b = 1$, the susceptance stays fairly constant over the range of $\beta_0 h$ from $\pi/2$ to π even though the associated conductances differ significantly (e.g., at $\beta_0 h = 0.91$, the susceptance varies within 0.7 millimhos of a mean value near 0.2 millimhos, whereas the conductance ranges between 2 and 8 millimhos). This observation confirms the experimental results for the modified cylindrical antenna reported by Iizuka [8].

Consider specifically the effect on the input characteristics of a sudden change from a point generator in the biconical case to a radial ring driving a modified conical antenna. In Fig. 2-10, the input impedance is shown near $(\beta_0 h)_{antires.}$ for both the biconical and the modified conical antenna with a small central modification ($\beta_0 b = 0.15$). Although the real parts are still comparable with each other, one remarkable change in the imaginary parts is the ratio $|X_{min}|/|X_{max}|$ which is larger than one in the biconical case but is now less than one. This means that with a small central modification which provides the modified conical antenna with a driving condition quite similar to that of the cylindrical antenna, the input characteristics of the two become more comparable.

In Fig. 2-11, is shown the variation of antenna admittance with a fixed length of protrusion as a function of the radius of the central sphere. Again, the driving point admittance of the antenna is seen to change significantly with the size of the sphere. Generally speaking, for $\beta_0 b < 1$, the change in susceptance with respect to $\beta_0 b$ is larger than that of the conductance. Specifically, for $\beta_0 b < 0.2$, there are practically no

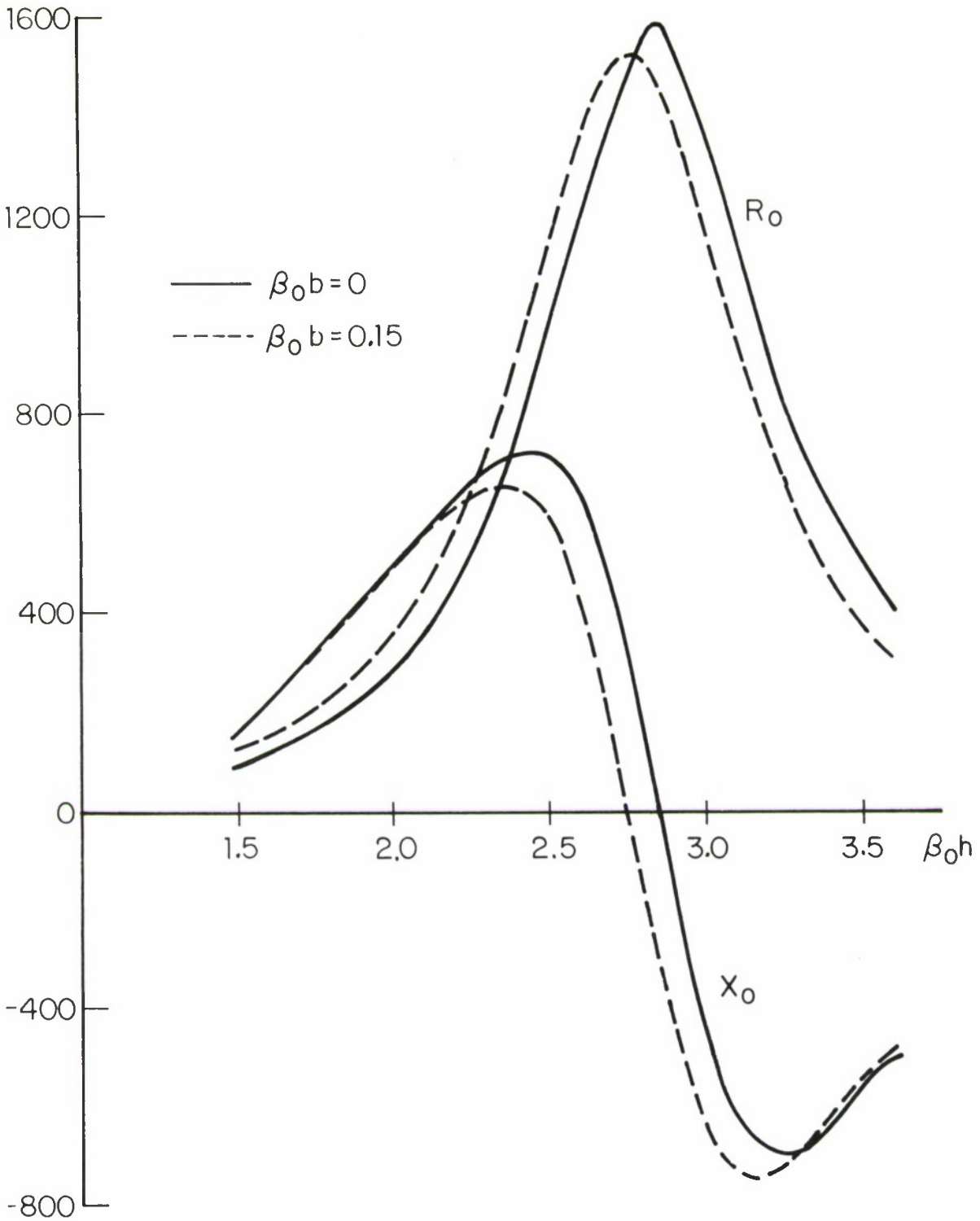
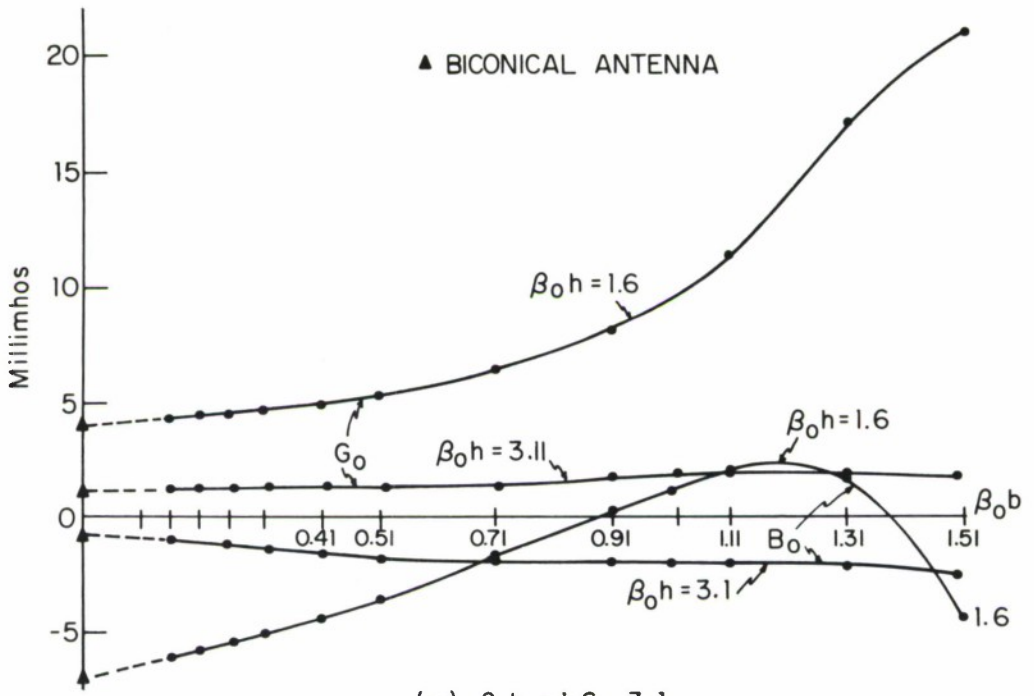
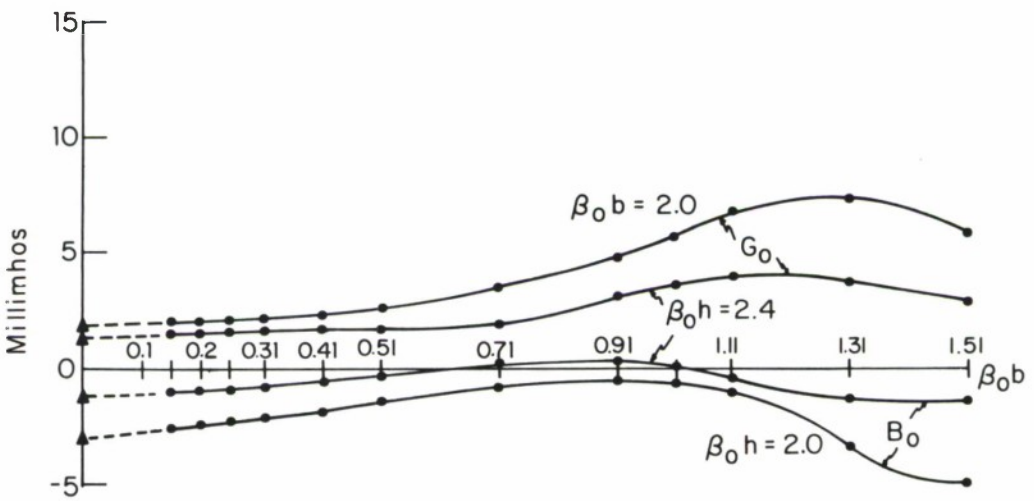


FIG. 2-10 INPUT IMPEDANCE NEAR ANTIRESONANCE OF CONICAL ANTENNA WITH SMALL CENTRAL SPHERE



(a) $\beta_0 h = 1.6, 3.1$



(b) $\beta_0 h = 2.0, 2.4$

FIG. 2-11 VARIATION OF INPUT ADMITTANCE FOR MODIFIED CONICAL ANTENNA WITH FIXED LENGTH

changes in the conductance but relatively large changes in the susceptance. It is also significant to note that all admittance curves can be extrapolated toward smaller $\beta_0 b$ and that they coincide with the values of the corresponding biconical antenna.

In Fig. 2-12, are shown the changes in antenna admittance with fixed total length (i. e., $\beta_0 a$ is fixed) of the radiating structure with the radius of the central sphere $\beta_0 b$ as the parameter. It is seen that drastic changes in performance are introduced by the central sphere. Again, as was observed in Fig. 2-11, the modified dipole with a small central sphere behaves essentially as the corresponding dipole with the same length.

With its driving point situated symmetrically at a distance from the center and an enlarged central volume, the modified dipole bears some resemblances to the sleeve dipole. One of the important properties of the sleeve dipole is its broad-band behavior. One would expect the same behavior from the modified dipole. This subject will not be pursued extensively here, since it is not involved in a fundamental check of the validity of the theoretical formulae. Furthermore, the broad-band properties cannot be investigated in general without extensive numerical computations of the input impedance for a range of antenna lengths and sphere radii. However, the fact that the modified conical dipole does possess broad-band properties can easily be seen by using the available tabulation of theoretical input impedance (Appendix A). Suppose the frequency for which $\beta_0 h = 1.6$, $\beta_0(h+2b) = 3.42$ and $\beta_0 b = 0.91$ is f_0 and the associated angular frequency is ω_0. For a variation in frequency from $\omega_0 - \Delta\omega_0$ to $\omega_0+\Delta\omega_0$, where $\Delta\omega_0 = \omega_0/4$, the electrical lengths have the following ranges:

$$\frac{3}{4}\omega_0 \leq \omega_0 \leq \frac{5}{4}\omega_0$$

$$1.2 \leq \beta_0 h \leq 2.0$$

$$2.6 \leq \beta_0(h+2b) \leq 4.2$$

$$0.7 \leq \beta_0 b \leq 1.1$$

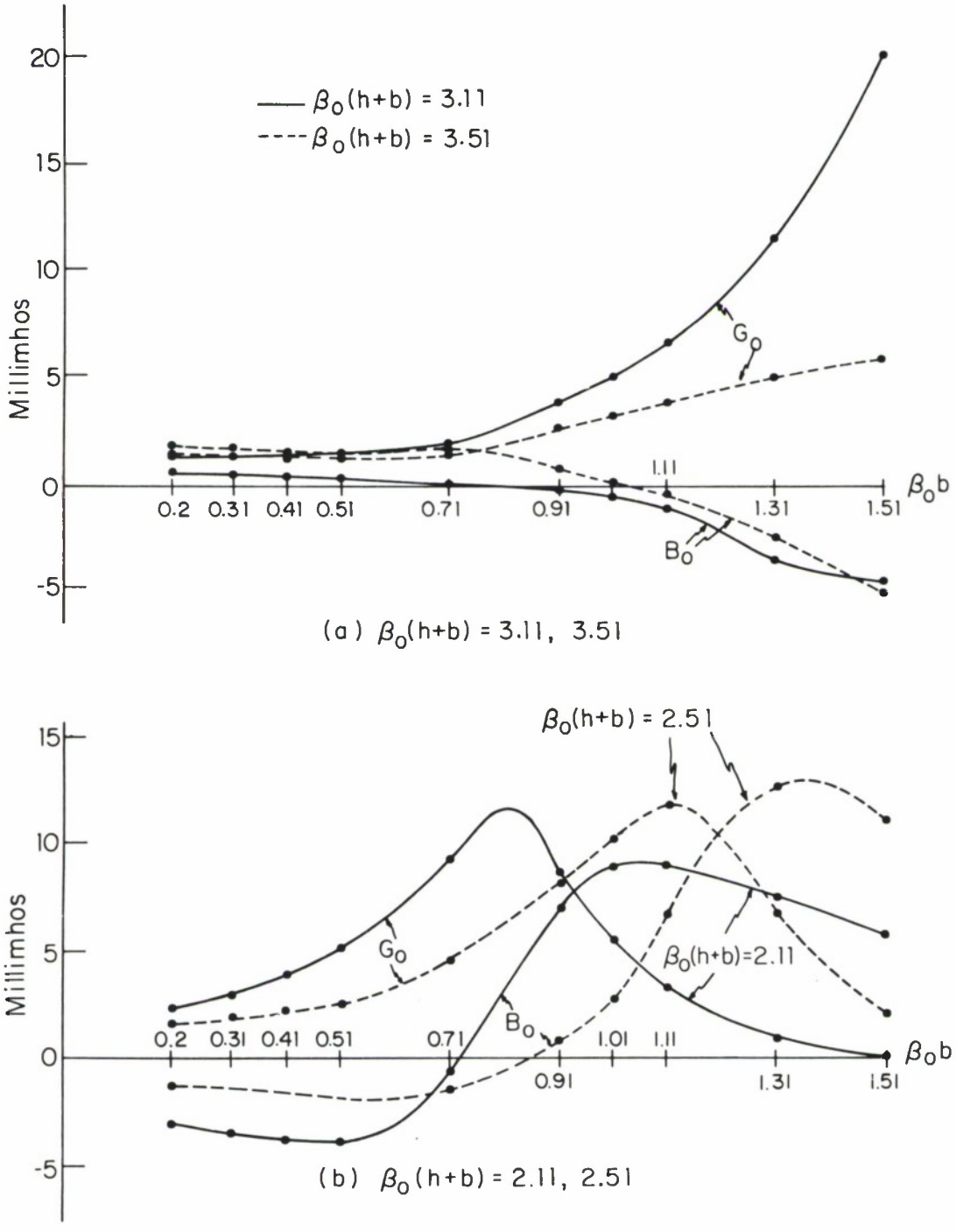


FIG. 2-12 VARIATION OF INPUT ADMITTANCE FOR MODIFIED CONICAL ANTENNA WITH FIXED TOTAL LENGTH.

Let the frequency response of the modified conical dipole be compared with corresponding biconical dipole with half lengths $\beta_0 h_1 = \beta_0 h$ or $\beta_0 h_2 = \beta_0 (h+2b)$. Let Z_1 and Z_2 denote the input impedance for antennas with half lengths $\beta_0 h_1$ and $\beta_0 h_2$ respectively. Tables (A-1) through (A-12) show the input impedance of the modified dipole and Fig. 2-3 shows the input impedance of the biconical antenna. It is seen that the reactance of the modified conical dipole is fairly constant near a mean value of -4 ohms over the range (i.e., from 25.3 ohms at $\beta_0 h = 2.0$ to -38.82 ohms at $\beta_0 h = 1.6$), whereas the corresponding reactance of the center-driven conical dipole (Fig. 2-3) varies widely over positive and negative values ($20 < X_1 < 480$, $-675 < X_2 < 650$). The resistance for the modified dipole varies within 30 per cent of a mean value near 111 ohms ($77 < R_0 < 145$) whereas R_1 ranges between 50 ohms and 475 ohms and R_2 varies over a range between 175 ohms and 1550 ohms. Clearly the frequency response of the modified conical dipole is very much better from the point of view of broad-band operation than the response of an unmodified dipole.

(B) Current Distribution

Due to the continuous change in the cross section of the conical antenna, the current density along the conical surface is proportional to the total current crossing the corresponding circular cross section with a constant whose value changes with the antenna length, i.e., $I_{tc} = 2\pi R \sin\theta \times I_{dc}$ where $b \leq R \leq a$. Correspondingly, for the central sphere one can also define $I_{ts}(\theta) = 2\pi b \sin\theta \, I_{ds}(\theta)$ where $\theta_0 + \theta_1 \leq \theta \leq \pi - \theta_0 - \theta_1$. The current density can be compared directly with experimental data while the total current along the antenna has the advantage of being readily compared with its corresponding cylindrical dipole as long as the cone angle is small.

Numerical results for the current distribution of antennas with the half cone angle $\theta_0 = 1.1^\circ$, antenna lengths $\beta_0 h = 1.6$ and 3.1 and central sphere with $\beta_0 b = 0$, 0.31, 0.51, 0.91 and 1.31 have been obtained and shown in Fig. 2-13 and Fig. 2-14.

(a) $\beta_0 h = 1.6$

Both total current distributions and the corresponding current density for $\beta_0 h = 1.6$ with various central spheres are shown in Fig. 2-13. The curves show components I''_{ct}/V^e_0, I'_{ct}/V^e_0 as a function of $X_h [= \beta_0(R-b)]$ and I''_{st}/V^e_0, I'_{st}/V^e_0 as a function of $X_b (= \beta_0 b\theta)$ with corresponding angle variations marked. It is to be noted that the current densities presented here are in milliamperes per volt per electrical length, so they differ from actual current density by a constant $\beta = \omega \sqrt{\mu\epsilon}$.

Both the real and imaginary part of the total current I_{tc} remain finite at the end and change their direction as the current passes around the edge of the conical antenna to the cap. The real part of the total current seems less affected by the size of the central sphere and remains much the same in shape except for a continuous increase in magnitude with an increase in the size of the central sphere. The real current behaves very much like a combination of terms made up of cosines and shifted cosines with half-angle arguments. The imaginary part of the current I'_{tc} is capacitive along the antenna for $\beta_0 b < 0.91$ but the magnitude continues to decrease with an increase in the size of central sphere. The range over which the I'_{tc} is capacitive is reduced and inductive currents begin to appear near the driving point. Although the total current on the surface of the sphere remains quite comparable in magnitude with that on the antenna, the magnitude of the corresponding current density decreases sharply with distance from the driving point. This is due to the fact that on the hemisphere, the surface of an incremental belt $(2\pi b \sin\theta)\,\Delta\theta$ increases with an increase in θ and current spreads out over a widening area with distance away from the driving point.

The real current I''_{ts}, which has the same value as I''_{tc} at the driving point, remains essentially constant for a small sphere (i. e., $\beta_0 b < 0.31$). As the sphere becomes larger, it can be approximated by trigonometric functions. The imaginary part I'_{ts}, like its counterpart I'_{tc}, changes significantly with variations in sphere radius. I'_{ts} becomes larger in magnitude and more capacitive with an increase in sphere radius.

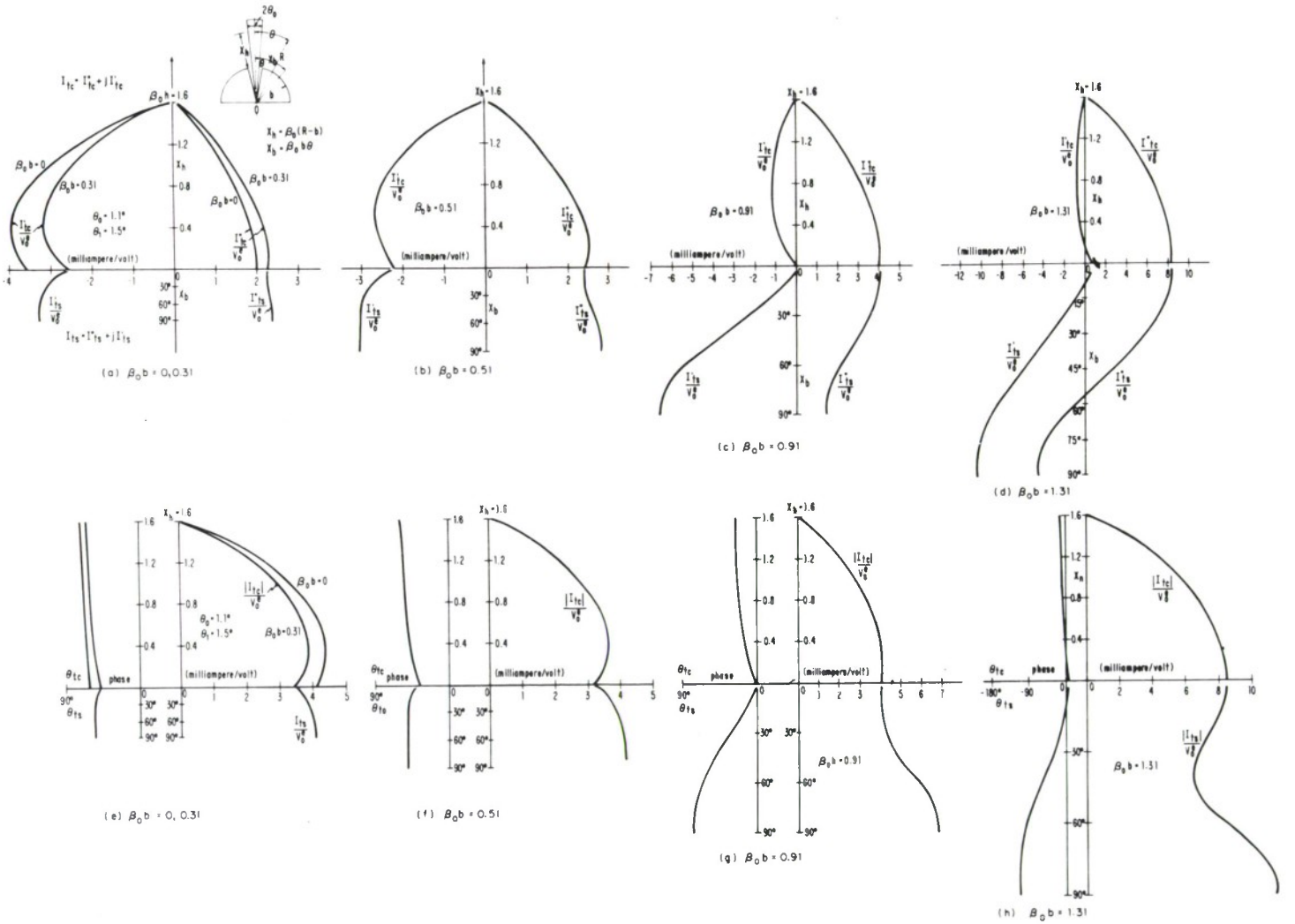


FIG. 2-13 CURRENT DISTRIBUTIONS OF MODIFIED CONICAL ANTENNA. $\beta_0 h = 1.6$

(b) $\beta_0 h = 3.1$

The distributions of total current for the antenna with $\beta_0 h = 3.1$ are shown in Fig. 2-14. The general shapes of the current distribution along the conical antenna seem less affected by the presence of the central sphere than when $\beta_0 h = 1.6$. In general, the real current I''_{tc} can still be approximated by a shifted cosine and a shifted cosine with half-angle argument, while the imaginary current I'_{tc} can still be approximated by trigonometric functions. Near the end of the antenna, the real part of the current is negative for small spheres. As b increases, this region with a small negative value decreases and the real current becomes more and more positive. The imaginary current I'_{tc} remains essentially capacitive over most of its range except near the driving point. The range over which the current is capacitive remains fairly constant except for a continuous increase in magnitude as the central sphere is made larger.

The current distribution on the sphere seems to have undergone a drastic change when compared with that for $\beta_0 h = 1.6$. Again, the behavior of I''_{ts} bears great resemblance to that with $\beta_0 h = 1.6$ but the rate of change with respect to the sphere radius seems to be approximately doubled (e.g., I''_{ts} begins to have a negative part for $\beta_0 b > 0.91$ at $\beta_0 h = 1.6$, whereas I''_{ts} has a negative part for a much smaller central sphere). The most affected part is the imaginary current I'_{ts} which has a small range of inductive current near the driving point and becomes capacitive rapidly over the rest of its range with an increase in the size of the central sphere. At $\beta_0 b = 1$, I'_{ts} begins to become less capacitive near the equatorial region of the sphere (i.e., $\theta = \pi/2$). As can be seen in Fig. 2-14c, I'_{tc} becomes inductive again near the $\theta = \pi/2$.

In general, the distribution of current along the thin conical antenna has much the same shape as along a simple dipole with enlarging central sphere. Yet, the current on the sphere is quite sensitive to the antenna lengths especially for larger spheres.

(C) Far-Field Pattern

Far-field patterns with a fixed central sphere and various antenna length are shown in Fig. 2-15. The size of the central sphere ranges from

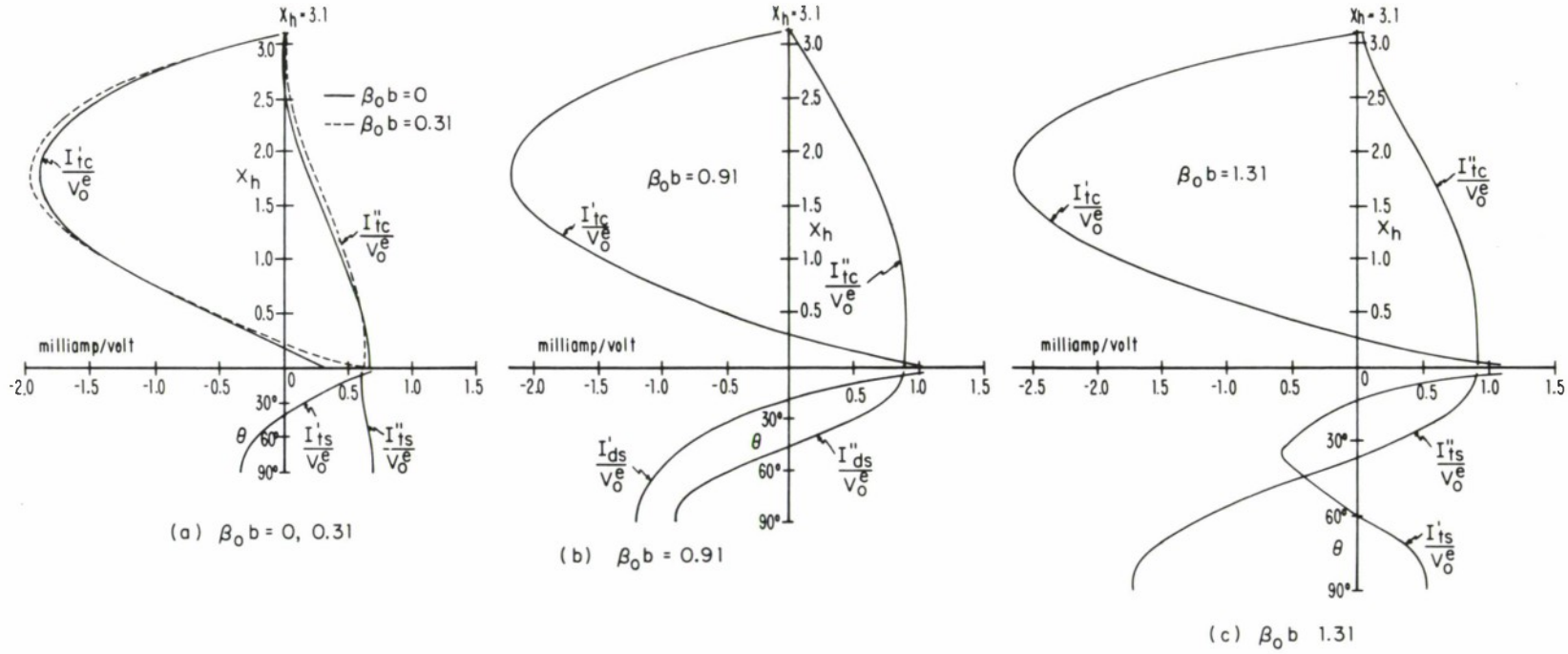


FIG. 2-14 CURRENT DISTRIBUTIONS MODIFIED CONICAL ANTENNA $\beta_0 h = 3.1$.

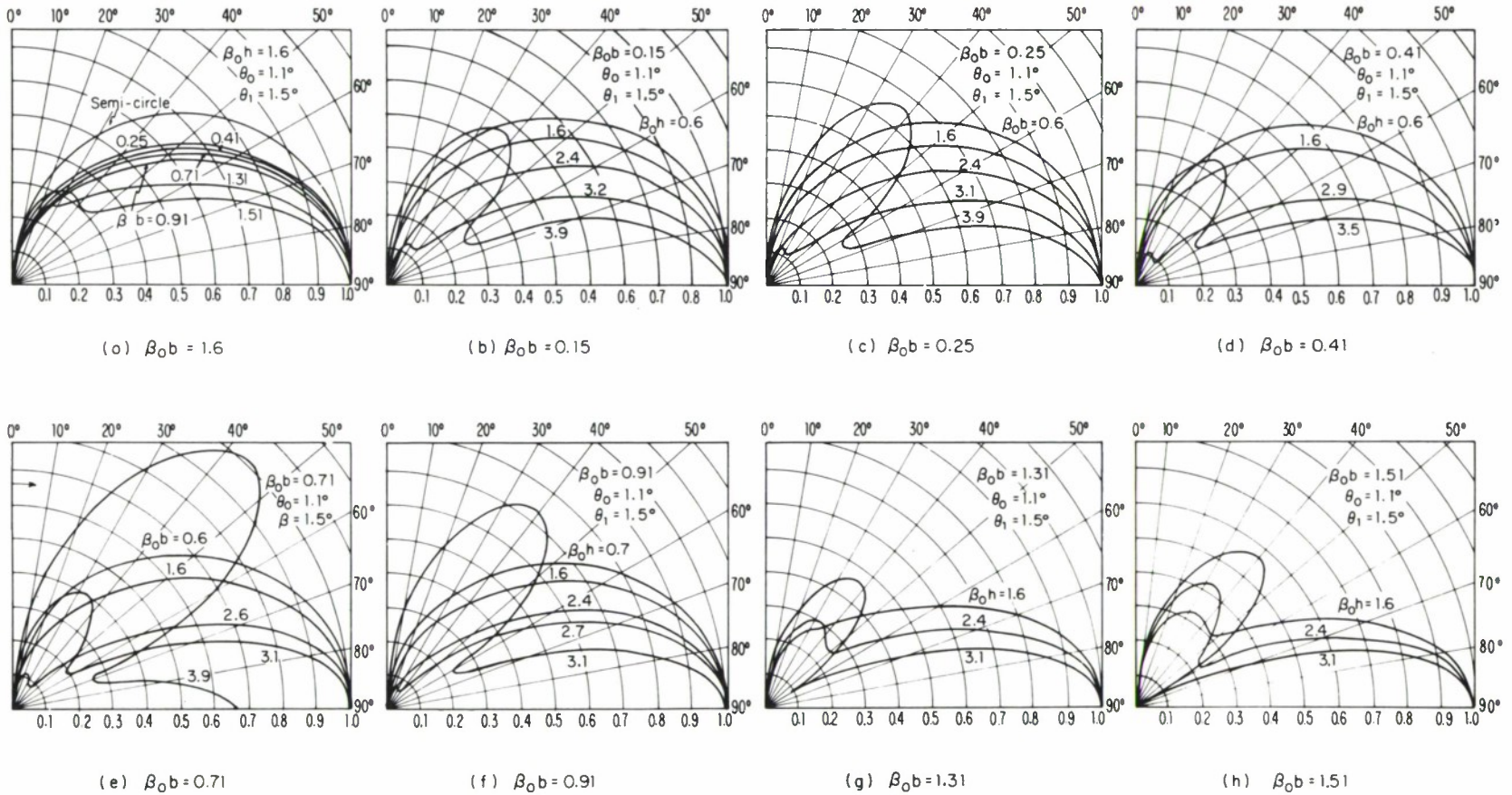


FIG. 2-15 FAR-FIELD PATTERN OF MODIFIED CONICAL ANTENNA

$\beta_0 b = 0.15$ to 1.51 and the antenna lengths range from $\beta_0 h = 1.6$ to 3.9.

Because of the properties of symmetry of this radiating structure (i.e., rotational symmetry and symmetry with respect to $\theta = \pi/2$ plane), it is necessary only to know the field patterns in the range $0^\circ \leq \theta \leq 90^\circ$. In order to facilitate the observation of the effect of the central sphere, all curves are normalized to have a maximum value one.

In general, the far-field pattern of the modified conical antenna behaves quite like a simple dipole for small central spheres. The presence of the central sphere effectively lengthens the antenna as compared to a simple dipole of length equal to the portion that protrudes from the sphere.

Note that the far-field patterns are more sensitive to the total length of the radiating structure than to the size of the central sphere.

In Fig. 2-15a, are drawn the field patterns of a quarter wave antenna with various central spheres. The effect of the central sphere can readily be seen in the continuous flattening of the field patterns which correspond in effect to those of a dipole with half length longer than quarter wavelength.

APPENDIX A

TABULATION OF THEORETICAL INPUT IMPEDANCES AND ADMITTANCES

The driving point characteristics (i. e., input impedances and admittances) have been computed for modified conical antennas with half cone angle $\theta_0 = 1.1^\circ$ and driving gap $\theta_1 = 1.5^\circ$. It covers the following ranges:

$$\beta_0 b = 0.15 \text{ to } 1.51$$

and

$$\beta_0 h = 0.6 \text{ to } 3.9$$

Table A-1 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.60	0.75	0.126 +j 3.850	8.48 -j259.33
0.70	0.85	0.254 +j 4.833	10.86 -j206.34
0.80	0.95	0.574 +j 6.313	14.28 -j157.00
0.90	1.05	1.303 +j 8.394	18.05 -j116.00
1.00	1.15	3.298 +j11.637	22.54 -j 79.54
1.10	1.25	9.806 +j15.980	27.90 -j 45.50
1.20	1.35	25.410 +j 9.732	34.30 -j 13.14
1.30	1.45	20.490 -j 8.294	41.93 +j 16.97
1.40	1.55	10.210 -j 9.713	51.40 +j 48.92
1.50	1.65	6.103 -j 7.740	62.82 +j 79.67
1.60	1.75	4.228 -j 6.097	76.80 +j110.75
1.70	1.85	3.228 -j 4.884	94.19 +j142.50
1.80	1.95	2.630 -j 3.971	115.90 +j175.00
1.90	2.05	2.270 -j 3.277	142.85 +j206.23
2.00	2.15	1.976 -j 2.675	178.70 +j241.80
2.10	2.25	1.805 -j 2.205	222.32 +j271.60
2.20	2.35	1.644 -j 1.764	282.70 +j303.00
2.30	2.45	1.553 -j 1.407	353.73 +j320.30
2.40	2.55	1.457 -j 1.050	452.20 +j325.30
2.50	2.65	1.409 -j 0.750	553.19 +j294.38
2.60	2.75	1.350 -j 0.431	672.10 +j214.60
2.70	2.85	1.318 -j 0.141	750.00 +j 80.10
2.80	2.95	1.298 +j 0.145	760.80 -j 85.20
2.90	3.05	1.290 +j 0.433	696.80 -j233.70
3.00	3.15	1.294 +j 0.727	587.50 -j329.70
3.10	3.25	1.313 +j 1.033	470.40 -j370.00
3.20	3.35	1.350 +j 1.359	367.90 -j370.40
3.30	3.45	1.410 +j 1.713	286.50 -j348.00
3.40	3.55	1.500 +j 2.061	230.81 -j317.24
3.50	3.65	1.642 +j 2.549	178.60 -j277.30
3.60	3.75	1.837 +j 2.999	148.56 -j242.48
3.70	3.85	2.186 +j 3.660	120.30 -j201.40
3.80	3.95	2.658 +j 4.274	104.92 -j168.74
3.90	4.05	3.605 +j 5.158	91.03 -j130.25

Table A-2 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.60	0.80	0.148 +j 3.972	2.36 -j251.40
0.70	0.90	0.310 +j 5.010	12.31 -j198.85
0.80	1.00	0.705 +j 6.555	16.20 -j150.80
0.90	1.10	1.577 +j 8.671	20.30 -j111.60
1.00	1.20	3.920 +j11.850	25.15 -j 76.00
1.10	1.30	11.056 +j15.340	30.92 -j 42.90
1.20	1.40	24.280 +j 7.262	37.81 -j 11.31
1.30	1.50	18.440 -j 7.750	46.09 +j 19.38
1.40	1.60	9.990 -j 8.849	56.92 +j 49.68
1.50	1.70	6.169 -j 7.232	68.27 +j 80.03
1.60	1.80	4.336 -j 5.773	83.19 +j110.75
1.70	1.90	3.331 -j 4.653	101.70 +j142.10
1.80	2.00	2.719 -j 3.792	124.90 +j174.16
1.90	2.10	2.356 -j 3.117	154.30 +j204.19
2.00	2.20	2.039 -j 2.546	191.60 +j239.30
2.10	2.30	1.867 -j 2.082	238.73 +j266.19
2.20	2.40	1.687 -j 1.657	301.70 +j296.30
2.30	2.50	1.598 -j 1.302	376.19 +j306.33
2.40	2.60	1.485 -j 0.951	477.60 +j305.60
2.50	2.70	1.440 -j 0.653	575.85 +j261.34
2.60	2.80	1.367 -j 0.338	689.70 +j170.40
2.70	2.90	1.329 -j 0.048	751.40 +j 26.96
2.80	3.00	1.304 +j 0.239	741.90 -j135.85
2.90	3.10	1.292 +j 0.527	663.60 -j270.95
3.00	3.20	1.292 +j 0.823	550.50 -j350.80
3.10	3.30	1.307 +j 1.133	436.80 -j378.70
3.20	3.40	1.340 +j 1.463	340.34 -j371.60
3.30	3.50	1.398 +j 1.823	264.85 -j345.40
3.40	3.60	1.493 +j 2.170	215.16 -j312.76
3.50	3.70	1.631 +j 2.676	166.03 -j272.50
3.70	3.90	2.195 +j 3.815	113.30 -j197.00
3.90	4.10	3.694 +j 5.332	87.79 -j126.70

Table A-3 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.60	0.85	0.180 +j 4.115	10.58 -j242.50
0.80	1.05	0.856 +j 6.784	18.30 -j145.10
0.90	1.15	1.884 +j 8.912	22.70 -j107.40
1.00	1.25	4.573 +j11.950	27.92 -j 72.98
1.10	1.35	12.870 +j14.430	34.10 -j 40.73
1.20	1.45	22.820 +j 5.433	41.48 -j 9.86
1.30	1.55	17.360 -j 6.465	50.58 +j 18.83
1.40	1.65	9.802 -j 8.037	61.00 +j 50.00
1.50	1.75	6.240 -j 6.738	74.00 +j 79.90
1.60	1.85	4.447 -j 5.450	89.88 +j110.15
1.70	1.95	3.437 -j 4.421	109.60 +j140.98
1.80	2.05	2.817 -j 3.611	134.20 +j172.40
1.90	2.15	2.440 -j 2.953	166.37 +j200.64
2.00	2.25	2.105 -j 2.416	205.00 +j235.30
2.10	2.35	1.936 -j 1.956	255.55 +j258.20
2.20	2.45	1.733 -j 1.548	320.90 +j286.70
2.30	2.55	1.649 -j 1.195	397.69 +j288.08
2.40	2.65	1.515 -j 0.852	501.30 +j281.90
2.50	2.75	1.476 -j 0.556	593.16 +j223.50
2.60	2.85	1.384 -j 0.243	700.90 +j123.00
2.70	2.95	1.341 +j 0.047	744.70 -j258.50
2.80	3.05	1.311 +j 0.334	716.30 -j182.20
2.90	3.15	1.294 +1 0.623	627.20 -j302.20
3.00	3.25	1.290 +j 0.922	513.20 -j366.70
3.10	3.35	1.301 +j 1.235	404.40 -j383.70
3.20	3.45	1.332 +j 1.570	314.30 -j370.40
3.30	3.55	1.387 +j 1.936	244.60 -j341.30
3.40	3.65		
3.50	3.75	1.623 +j 2.807	154.40 -j267.00
3.60	3.85		
3.70	3.95	2.210 +j 3.971	107.00 -j192.30
3.80	4.05	2.732 +j 4.573	96.28 -j161.17
3.90	4.15	3.796 +j 5.497	85.06 -j123.00

Table A-4 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.60	0.91	0.221 +j 4.274	12.06 -j233.40
0.70	1.01	0.452 +j 5.352	15.68 -j185.80
0.80	1.11	0.972 +j 6.878	20.14 -j142.50
0.90	1.21	2.149 +j 8.970	25.25 -j105.40
1.00	1.31	5.365 +j11.910	31.46 -j 69.82
1.10	1.41	12.940 +j13.100	38.17 -j 38.63
1.20	1.51	20.930 +j 3.925	46.15 -j 8.65
1.30	1.61	15.780 -j 5.855	55.70 +j 20.66
1.40	1.71	9.613 -j 7.116	67.20 +j 49.70
1.50	1.81	6.331 -j 6.158	81.17 +j 78.94
1.60	1.91	4.586 -j 5.065	98.24 +j108.48
1.70	2.01	3.570 -j 4.142	119.40 +j138.50
1.80	2.11	2.928 -j 3.393	145.79 +j168.90
1.90	2.21		
2.00	2.31	2.189 -j 2.258	221.30 +j228.30
2.10	2.41		
2.20	2.51	1.791 -j 1.417	343.40 +j271.65
2.30	2.61		
2.40	2.71	1.554 -j 0.733	526.48 +j248.16
2.50	2.81	1.528 -j 0.438	604.80 +j173.58
2.60	2.91	1.407 -j 0.128	704.89 +j 64.01
2.70	3.01	1.357 +j 0.161	726.47 -j 86.37
2.80	3.11	1.321 +j 0.449	678.50 -j230.67
2.90	3.21	1.298 +j 0.741	581.06 -j331.63
3.00	3.31	1.289 +j 1.042	469.09 -j379.29
3.10	3.41	1.296 +j 1.360	367.32 -j385.36
3.20	3.51	1.323 +j 1.700	285.01 -j366.34
3.30	3.61	1.377 +j 2.074	222.17 -j334.72
3.40	3.71	1.468 +j 2.491	175.56 -j297.95
3.50	3.81	1.629 +j 2.865	149.94 -j263.79
3.60	3.91	1.849 +j 3.398	123.56 -j227.10
3.70	4.01	2.237 +j 4.161	100.22 -j186.45
3.80	4.11	2.790 +j 4.736	92.33 -j156.75
3.90	4.21	3.934 +j 5.681	82.38 -j118.97

Table A-5 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.60	1.01	0.299 +j 4.507	14.64 -j220.90
0.70	1.11	0.603 +j 5.602	18.99 -j176.45
0.80	1.21	1.274 +j 7.122	24.34 -j136.50
0.90	1.31	2.730 +j 9.073	30.41 -j101.10
1.00	1.41	6.088 +j11.192	37.50 -j 68.95
1.10	1.51	12.710 +j10.770	45.81 -j 38.80
1.20	1.61	17.430 +j 3.117	55.60 -j 9.94
1.30	1.71	14.070 -j 4.232	65.14 +j 19.61
1.40	1.81	9.375 -j 5.675	78.06 +j 47.25
1.50	1.91	6.508 -j 5.209	93.65 +j 74.97
1.60	2.01	4.840 -j 4.421	112.64 +j102.90
1.70	2.11	3.814 -j 3.671	136.10 +j131.00
1.80	2.21	3.143 -j 3.024	165.24 +j158.98
1.90	2.31	2.745 -j 2.524	196.26 +j181.60
2.00	2.41	2.343 -j 1.992	247.74 +j210.61
2.10	2.51	2.217 -j 1.535	304.96 +j211.13
2.20	2.61	2.019 -j 1.173	370.39 +j215.18
2.30	2.71	1.747 -j 0.850	462.75 +j225.26
2.40	2.81	1.737 -j 0.533	526.13 +j161.51
2.50	2.91	1.528 -j 0.226	640.36 +j 94.48
2.60	3.01	1.450 +j 0.069	688.10 -j 32.50
2.70	3.11	1.388 +j 0.358	675.60 -j174.20
2.80	3.21	1.340 +j 0.648	604.84 -j292.29
2.90	3.31	1.307 +j 0.944	502.97 -j363.10
3.00	3.41	1.289 +j 1.251	399.48 -j387.73
3.10	3.51	1.288 +j 1.577	310.70 -j380.28
3.20	3.61	1.310 +j 1.929	240.95 -j354.80
3.30	3.71	1.361 +j 2.316	188.56 -j320.92
3.40	3.81	1.506 +j 2.610	165.84 -j287.48
3.50	3.91	1.613 +j 3.248	122.66 -j246.98
3.60	4.01	1.878 +j 3.629	112.46 -j217.34
3.70	4.11	2.263 +j 4.263	97.16 -j183.02
3.80	4.21	2.905 +j 4.976	87.50 -j149.89

Table A-6 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.60	1.11	0.385 +j 4.698	17.32 -j211.43
0.70	1.21	0.764 +j 5.784	22.43 -j169.94
0.80	1.31	1.575 +j 7.235	28.72 -j131.97
0.90	1.41	3.225 +j 8.926	35.81 -j 99.10
1.00	1.51	6.581 +j10.306	44.01 -j 68.93
1.10	1.61	11.845 +j 8.992	53.56 -j 40.66
1.20	1.71	14.788 +j 3.139	64.71 -j 13.74
1.30	1.81	12.550 -j 1.970	77.79 +j 12.22
1.40	1.91	9.238 -j 3.709	93.22 +j 37.42
1.50	2.01	7.060 -j 3.960	107.74 +j 60.43
1.60	2.11	5.131 -j 3.769	126.58 +j 92.97
1.70	2.21	4.095 -j 3.192	151.89 +j118.42
1.80	2.31	3.390 -j 2.649	183.14 +j143.12
1.90	2.41	2.940 -j 2.182	219.33 +j162.78
2.00	2.51	2.522 -j 1.723	270.30 +j184.70
2.10	2.61	2.273 -j 1.320	328.99 +j191.06
2.20	2.71	2.023 -j 0.971	401.83 +j192.85
2.30	2.81	1.847 -j 0.617	487.00 +j162.79
2.40	2.91	1.707 -j o.324	565.37 +j107.14
2.50	3.01	1.601 -j 0.045	624.02 +j 17.46
2.60	3.11	1.498 +j 0.271	646.41 -j116.94
2.70	3.21	1.421 +j 0.562	608.32 -j240.63
2.80	3.31	1.361 +j 0.855	526.76 -j331.00
2.90	3.41	1.316 +j 1.157	428.46 -j376.74
3.00	3.51	1.288 +j 1.474	336.22 -j384.63
3.10	3.61	1.280 +j 1.811	260.28 -j368.27
3.20	3.71	1.296 +j 2.177	201.93 -j339.14
3.30	3.81	1.346 +j 2.582	158.80 -j304.55
3.40	3.91	1.532 +j 3.120	126.82 -j258.23
3.50	4.01	1.618 +j 3.557	105.98 -j232.94
3.60	4.11	1.920 +j 4.144	92.05 -j198.67
3.70	4.21	2.390 +j 4.831	82.28 -j166.29
3.80	4.31	3.310 +j 5.499	80.35 -j133.49
3.90	4.41	4.470 +j 6.180	76.84 -j106.24

Table A-7 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.60	1.31	0.534 +j 4.946	21.59 -j199.85
0.70	1.41	1.006 +j 5.923	27.88 -j164.10
0.80	1.51	1.924 +j 7.099	35.56 -j131.23
0.90	1.61	3.498 +j 8.199	44.02 -j103.18
1.00	1.71	6.005 +j 8.710	53.66 -j 77.82
1.10	1.81	9.044 +j 7.630	64.59 -j 54.50
1.20	1.91	10.986 +j 4.681	77.04 -j 32.82
1.30	2.01	10.760 +j 1.485	91.22 -j 12.59
1.40	2.11	9.278 -j 0.542	107.41 +j 6.28
1.50	2.21	7.669 -j 1.446	125.91 +j 23.73
1.60	2.31	6.341 -j 1.706	147.06 +j 39.56
1.70	2.41	5.324 -j 1.655	171.27 +j 53.25
1.80	2.51	4.553 -j 1.470	198.89 +j 64.22
1.90	2.61	3.962 -j 1.230	230.20 +j 71.43
2.00	2.71	3.500 -j 0.970	265.30 +j 73.50
2.10	2.81	3.106 -j 0.726	305.26 +j 71.40
2.20	2.91	2.782 -j 0.445	350.47 +j 56.11
2.30	3.01	2.360 -j 0.178	493.38 +j 43.77
2.40	3.11	1.940 +j 0.089	514.39 -j 23.52
2.50	3.21	1.742 +j 0.395	545.98 -j123.80
2.60	3.31	1.591 +j 0.700	526.68 -j231.77
2.70	3.41	1.476 +j 1.010	461.37 -j315.77
2.80	3.51	1.382 +j 1.330	375.80 -j361.47
2.90	3.61	1.309 +j 1.663	292.20 -j371.30
3.00	3.71	1.258 +j 2.018	222.49 -j356.91
3.10	3.81	1.233 +j 2.401	169.30 -j329.60
3.20	3.91	1.242 +j 2.821	130.76 -j296.94
3.30	4.01	1.298 +j 3.289	103.92 -j263.23
3.40	4.11	1.469 +j 3.790	88.91 -j229.40
3.50	4.21	1.642 +j 4.410	74.13 -j199.15
3.60	4.31	2.129 +j 5.110	69.47 -j166.75
3.70	4.41	2.617 +j 5.821	64.25 -j142.90
3.80	4.51	3.690 +j 6.710	62.93 -j114.43
3.90	4.61	4.871 +j 7.392	62.16 -j 94.32

Table A-8 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.6	1.51	0.529 +j 5.053	20.507 -j195.74
0.7	1.61	0.985 +j 5.977	26.840 -j162.88
0.8	1.71	1.713 +j 6.922	33.690 -j136.12
0.9	1.81	2.861 +j 7.814	41.310 -j112.85
1.0	1.91	4.537 +j 8.402	49.760 -j 92.16
1.1	2.01	6.640 +j 8.258	59.137 -j 73.54
1.2	2.11	8.851 +j 7.046	69.490 -j 56.60
1.3	2.21	9.820 +j 4.988	80.950 -j 41.12
1.4	2.31	9.860 +j 2.839	93.660 -j 26.97
1.5	2.41	9.126 +j 1.197	107.726 -j 14.13
1.6	2.51	8.103 +j 0.173	123.360 -j 2.64
1.7	2.61	7.087 -j 0.366	140.720 +j 7.27
1.8	2.71	6.190 -j 0.597	160.050 +j 15.42
1.9	2.81	5.484 -j 0.638	179.920 +j 20.94
2.0	2.91	4.803 -j 0.564	205.370 +j 24.13
2.1	3.01	4.314 -j 0.440	229.449 +j 23.39
2.2	3.11	3.832 -j 0.250	259.860 +j 16.92
2.3	3.21	3.484 -j 0.068	286.960 +j 5.58
2.4	3.31	3.128 +j 0.172	318.740 -j 17.56
2.5	3.41	2.868 +j 0.386	342.462 -j 46.02
2.6	3.51	2.596 +j 0.657	362.030 -j 91.70
2.7	3.61	2.377 +j 0.919	365.950 -j141.49
2.8	3.71	2.185 +j 1.202	351.300 -j193.29
2.9	3.81	2.019 +j 1.502	318.780 -j237.26
3.0	3.91	1.880 +j 1.830	272.900 -j265.94
3.1	4.01	1.780 +j 2.148	228.750 -j275.84
3.2	4.11	1.702 +j 2.587	177.460 -j269.80
3.3	4.21	1.683 +j 2.966	144.699 -j255.10
3.4	4.31	1.737 +j 3.508	113.400 -j228.90
3.5	4.41	1.939 +j 4.005	97.930 -j202.28
3.6	4.51	2.142 +j 4.496	86.356 -j181.28
3.7	4.61	2.731 +j 5.041	83.080 -j153.36
3.8	4.71	3.273 +j 5.576	78.290 -j133.37
3.9	4.81	4.050 +j 6.271	72.670 -j122.50

Table A-9 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.6	1.61	0.477 +j 5.138	17.93 -j192.98
0.7	1.71	0.853 +j 6.006	23.19 -j163.22
0.8	1.81	1.457 +j 6.933	29.04 -j138.14
0.9	1.91	2.397 +j 7.857	35.53 -j116.44
1.0	2.01	3.785 +j 8.626	42.65 -j 97.22
1.1	2.11	5.650 +j 8.941	50.51 -j 79.93
1.2	2.21	7.764 +j 8.429	59.12 -j 64.18
1.3	2.31	9.559 +j 6.928	68.59 -j 49.71
1.4	2.41	10.442 +j 4.804	79.04 -j 36.36
1.5	2.51	10.315 +j 2.738	90.57 -j 24.04
1.6	2.61	9.530 +j 1.172	103.37 -j 12.71
1.7	2.71	8.500 +j 0.176	117.65 -j 24.41
1.8	2.81	7.470 -j 0.370	133.60 +j 6.61
1.9	2.91	6.540 -j 0.612	151.55 +j 14.19
2.0	3.01	5.744 -j 0.665	171.79 +j 19.90
2.1	3.11	5.067 -j 0.596	194.64 +j 22.91
2.2	3.21	4.494 -j 0.454	220.27 +j 22.25
2.3	3.31	4.004 -j 0.264	248.70 +j 16.41
2.4	3.41	3.581 -j 0.039	279.18 +j 3.01
2.5	3.51	3.215 +j 0.213	309.65 -j 20.47
2.6	3.61	2.897 +j 0.480	335.95 -j 55.68
2.7	3.71	2.618 +j 0.773	351.38 -j103.76
2.8	3.81	2.375 +j 1.083	348.62 -j158.93
2.9	3.91	2.166 +j 1.416	323.42 -j211.44
3.0	4.01	1.992 +j 1.777	279.60 -j249.31
3.1	4.11	1.857 +j 2.169	227.84 -j265.99
3.2	4.21	1.769 +j 2.592	179.66 -j263.25
3.3	4.31	1.738 +j 3.042	141.57 -j247.83
3.4	4.41	1.784 +j 3.528	114.12 -j225.73
3.5	4.51	1.901 +j 4.093	93.34 -j200.97
3.6	4.61	2.194 +j 4.657	82.79 -j175.73
3.7	4.71	2.624 +j 5.248	76.21 -j152.44
3.8	4.81	3.240 +j 5.723	74.91 -j132.32
3.9	4.91	4.040 +j 6.035	76.60 -j114.41

Table A-10 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.6	1.71	0.411 +j 5.248	14.84 -j189.38
0.7	1.81	0.729 +j 6.122	19.18 -j161.07
0.8	1.91	1.237 +j 7.071	24.00 -j137.22
0.9	2.01	2.031 +j 8.068	29.34 -j120.20
1.0	2.11	3.234 +j 9.024	35.19 -j104.32
1.1	2.21	4.958 +j 9.721	41.64 -j 81.64
1.2	2.31	7.180 +j 9.791	48.71 -j 66.42
1.3	2.41	9.534 +j 8.831	56.51 -j 52.29
1.4	2.51	11.298 +j 6.772	65.12 -j 39.03
1.5	2.61	11.889 +j 4.223	74.69 -j 26.53
1.6	2.71	11.374 +j 1.953	85.40 -j 14.67
1.7	2.81	10.249 +j 0.364	97.45 -j 3.46
1.8	2.91	8.960 -j 0.571	111.16 +j 7.09
1.9	3.01	7.746 -j 1.024	126.88 +j 16.77
2.0	3.11	6.688 -j 1.166	145.10 +j 25.29
2.1	3.21	5.795 -j 1.120	166.36 +j 32.15
2.2	3.31	5.046 -j 0.956	191.32 +j 36.25
2.3	3.41	4.416 -j 0.724	220.49 +j 36.16
2.4	3.51	3.886 -j 0.446	253.97 +j 29.12
2.5	3.61	3.435 -j 0.138	290.65 +j 11.67
2.6	3.71	3.049 +j 0.194	326.62 -j 20.81
2.7	3.81	2.720 +j 0.545	353.40 -j 70.81
2.8	3.91	2.441 +j 0.917	359.05 -j134.82
2.9	4.01	2.207 +j 1.309	335.26 -j198.80
3.0	4.11	2.019 +j 1.722	286.79 -j244.50
3.1	4.21	1.880 +j 2.157	229.69 -j263.51
3.2	4.31	1.795 +j 2.613	178.60 -j260.00
3.3	4.41	1.780 +j 3.091	139.91 -j242.95
3.4	4.51	1.829 +j 3.625	110.95 -j219.91
3.5	4.61	1.978 +j 4.263	89.59 -j193.02
3.6	4.71	2.245 +j 4.849	78.62 -j169.84
3.7	4.81	2.651 +j 5.399	73.28 -j149.25
3.8	4.91	3.216 +j 5.889	71.43 -j130.81
3.9	5.01	3.944 +j 6.253	72.08 -j114.38

Table A-11 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.6	1.91	0.291 +j 5.534	9.49 -j180.20
0.7	2.01	0.513 +j 6.469	12.31 -j154.08
0.8	2.11	0.889 +j 7.525	15.48 -j131.06
0.9	2.21	1.490 +j 8.730	19.01 -j111.31
1.0	2.31	2.471 +j10.082	22.94 -j 93.56
1.1	2.41	4.069 +j11.507	27.31 -j 77.24
1.2	2.51	6.607 +j12.712	32.19 -j 61.93
1.3	2.61	10.292 +j12.932	37.68 -j 47.34
1.4	2.71	14.496 +j10.967	43.88 -j 33.19
1.5	2.81	17.159 +j 6.499	50.97 -j 19.30
1.6	2.91	16.761 +j 1.554	59.15 -j 5.49
1.7	3.01	14.334 -j 1.753	68.74 +j 8.41
1.8	3.11	11.569 -j 3.246	80.13 +j 22.48
1.9	3.21	9.233 -j 3.611	93.94 +j 36.74
2.0	3.31	7.434 -j 3.427	110.95 +j 51.14
2.1	3.41	6.078 -j 3.000	132.29 +j 65.31
2.2	3.51	5.050 -j 2.478	159.59 +j 78.32
2.3	3.61	4.260 -j 1.929	194.82 +j 88.21
2.4	3.71	3.642 -j 1.382	240.01 +j 91.06
2.5	3.81	3.156 -j 0.849	295.51 +j 79.53
2.6	3.91	2.766 -j 0.329	356.55 +j 42.41
2.7	4.01	2.455 +j 0.209	404.41 -j 34.46
2.8	4.11	2.206 +j 0.661	415.89 -j124.69
2.9	4.21	2.049 +j 1.160	**369.60 -j209.21**
3.0	4.31	1.910 +j 1.622	304.09 -j258.19
3.1	4.41	1.823 +j 2.242	218.26 -j268.55
3.2	4.51	1.792 +j 2.824	160.21 -j252.48
3.3	4.61	1.820 +j 3.391	122.85 -j228.96
3.4	4.71	1.914 +j 3.957	99.09 -j204.80
3.5	4.81	2.086 +j 4.533	83.79 -j182.04
3.6	4.91	2.351 +j 5.117	74.14 -j161.36
3.7	5.01	2.727 +j 5.697	68.35 -j142.80
3.8	5.11	3.236 +j 6.254	65.26 -j126.12
3.9	5.21	3.901 +j 6.754	64.13 -j111.02

Table A-12 Theoretical data for the input admittance of modified dipole;

$\beta_0 h$	$\beta_0 a$	Y_0	Z_0
0.6	2.11	0.217 +j 5.854	6.319 -j170.58
0.7	2.21	0.393 +j 6.881	8.268 -j144.86
0.8	2.31	0.691 +j 8.084	10.492 -j122.80
0.9	2.41	1.202 +j 9.530	13.032 -j103.29
1.0	2.51	2.107 +j11.308	15.926 -j 85.46
1.1	2.61	3.771 +j13.488	19.230 -j 68.78
1.2	2.71	6.939 +j15.911	23.030 -j 52.81
1.3	2.81	12.828 +j17.406	27.440 -j 37.23
1.4	2.91	21.226 +j14.161	32.600 -j 21.75
1.5	3.01	25.196 +j 3.962	38.730 -j 6.09
1.6	3.11	20.705 -j 4.484	46.134 +j 9.99
1.7	3.21	14.673 -j 7.108	55.200 +j 26.74
1.8	3.31	10.401 -j 6.938	66.540 +j 44.38
1.9	3.41	7.682 -j 5.981	81.047 +j 63.10
2.0	3.51	5.926 -j 4.961	99.960 +j 82.92
2.1	3.61	4.745 -j 3.922	125.220 +j103.50
2.2	3.71	3.962 -j 3.029	159.768 +j123.58
2.3	3.81	3.315 -j 2.239	207.175 +j139.93
2.4	3.91	2.864 -j 1.497	274.258 +j143.32
2.5	4.01	2.515 -j 0.948	348.190 +j131.19
2.6	4.11	2.347 -j 0.35	416.81 +j 62.16
2.7	4.21	2.105 +j 0.248	468.453 -j 55.25
2.8	4.31	1.958 +j 0.870	426.473 -j189.59
2.9	4.41	1.858 +j 1.461	332.539 -j261.61
3.0	4.51	1.799 +j 2.015	246.564 -j276.09
3.1	4.61	1.782 +j 2.554	183.727 -j263.33
3.2	4.71	1.807 +j 3.090	141.020 -j241.18
3.3	4.81	1.877 +j 3.634	112.202 -j217.20
3.4	4.91	2.001 +j 4.191	92.781 -j194.31
3.5	5.01	2.189 +j 4.767	79.555 -j173.25
3.6	5.11	2.456 +j 5.361	70.621 -j154.17
3.7	5.21	2.823 +j 5.973	64.688 -j136.85
3.8	5.31	3.320 +j 6.594	60.914 -j120.98
3.9	5.41	3.984 +j 7.205	58.772 -j106.29

APPENDIX B. COMPUTER PROGRAMS FOR MODIFIED DIPOLES(1st set)

```
FORTRAN IV G LEVEL  18                 MAIN              DATE = 71110          18/16/39

              C        FORTRAN IV PROGRAM FOR THE MODIFIED CONICAL ANTENNA
              C
              C
              C        THIS PROGRAM SOLVES FOR THE COEFFICIENTS bk OF ELECTROMAGNETIC FIELD
              C     IN THE RADIATION REGION OF THE MODIFIED CONICAL ANTENNA BY FIRST SOLVING
              C     THE CHARACTERISTIC EQUATION Lv(θo)=0 WHICH GIVES A SET OF V'S FOR A FIXED
              C     CONE ANGLE θo AND THEN BY PROPERLY TRUNCATING THE INFININITE SET OF LINEAR
              C     ALGEBRAIC EQUATIONS.  THE ELECTROMAGNETIC FIELDS IN THE ANTENNA REGION,
              C     CURRENT DISTRIBUTIONS ALONG THE SURFACES OF THE RADIATING STRUCTURE AND
              C     THE INPUT CHARACTERISTICS ARE EXPRESSED IN TERMS OF bk'S TOGETHER WITH
              C     THE FRACTIONAL ORDER LEGENDRE FUNCTION AND BESSEL S FUNCTIONS.  THIS PROG
              C     IS VALID FOR HALF ANGLE  LESS THAN TWO DEGREES AND ANTENNA HEIGHT BETA*
              C     UP TO 5 AND CENTRAL SPHERE BETA*  UP TO 2.
              C
              C
0001                 EXTERNAL FCT
0002                 FKN(KK,Z)=KK*(KK+1)-Z*(Z+1)
0003                 COMPLEX V(50,50),A(50,53),SH(100),DSH(100),DTB(50),DSDS(100),CUNC(
                    120),CUND(20),CUNS(20),CUNDS(20),FARF(20),RIMG,ADM1,ADM2,ADM3,ADM4,
                    2ADM,RIM,CUNC1,CUNC2,CUNC3,CUNC4,CUNS1,CUNS2,CUNS3,CUNS4,CUN4
0004                 DIMENSION RT(75),FS(50,50),FI(50,50),PI(100),DPI(100),ITEST(60,50)
                    1,CNT(75),ASB(60),GMDMP(60),GMXJP(60),GPMXJ(60),PNP(110),DPNP(110),
                    2PNM(110),DPNM(110),PNL(60), DFNL(60), DTIB(60),GMXJP1(60),GPMXJI(6
                    30),BLTA(60),GYIST1(60,20),GYIST2(60,20),GMIOM(60,20),GMIOMP(60,20)
                    4,PK(60),P(60),CU3(60),CU4(60),GJGJP(60)
              C
              C
              C        READ INPUT DATA
              C             MAX=MAXIMUM VALUE OF K
              C             NMAX= MAXIMUM VALUE OF
              C             TA=HALF CONE ANGLE IN DEGREES
              C             BETA=DRIVING GAP IN DEGREES
              C
              C
0005               1 READ(5,2)MAX,NMAX,TA,BETA
0006               2 FORMAT(I5,I5,2(F10.4))
0007                 CALL ERRSET(207,2,1,1,DUMPER)
0008                 CALL ERRSET(210,2,1,1,DUMPER)
0009                 MAXX=(MAX+1)/2
0010                 M1=MAXX+1
0011                 M2=M1+1
0012                 MAI=MAX+1
0013                 D=1.E-07
0014                 PIC=3.14159265
0015                 TAR=PIC*TA/180
0016                 TARB=PIC*(TA+BETA)/180
0017                 TAB2=TARB/2
0018                 TA2=TAR/2
0019                 ITA=IFIX(TA)+1
0020                 DF=ITA-TA
0021                 CALL LEGFI(ITA,DF,100,PI,DPI)
0022                 ITAB=IFIX(TA+BETA)+1
0023                 DFB=ITAB-TA-BETA
0024                 ZETA=120.*PIC
0025                 ELTAB=ALOG(COTAN(TA2)/COTAN(TAB2))
0026                 ELTA=ALOG(COTAN(TA2))
0027                 ZC=120.*ELTA
0028                 CX=COS(TAR)
```

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FORTRAN IV G LEVEL  18                  MAIN                DATE = 71110          19/18/39

 0029                SX=SIN(TAR)
               C
               C
               C           SOLVE CHARACTERISTIC EQUATION L_V(θ_0)= 0
               C              RT= ROOTS OF L_V(θ_0)= 0
               C              DNT= ∂V/
               C
 0030                CALL RTNI(X,TAR,NMAX,F,DERF,FCT,0.2,1.E-06,17,IER,I,RT,DNT)
 0031                NMA=(IER-2)/2
 0032                DO 301 KM=1,NMA
 0033                ORD=RT(KM)
 0034                CALL LEGF(ITAR,DFB,ORD,0,PNP,DPNP,PNM,DPNM,PNL,DPNL)
 0035                RLTA(KM)=PNL(KM)
 0036            301 CONTINUE
               C
               C
               C           READ INPUT DATA
               C                HT= ANTENNA HEIGHTS IN ELECTRICAL LENGTH
               C                RS= CENTRAL SPHERE RADIUS IN ELECTRICAL LENGTH
               C                KMN= NUMBER OF EQUAL SEGMENTS ALONG THE ANTENNA AND THE SPHERE
               C                     FOR WHICH CURRENT DISTRIBUTIONS ARE COMPUTED.
               C
               C
 0037              3 READ(5,4)HT,RS,KMN
 0038                HT=HT+RS
 0039                WRITE(6,4)HT,RS,KMN
 0040              4 FORMAT(2F10.4,I5)
 0041                CT=-2.*TAN(HT-RS)/RLTA
 0042                CALL SPHR(...,HT,SH,DSH,...)
 0043                DO 300 KL=1,NMA
 0044                OLD=RT(KL)
 0045                CALL LEGF(ITA,DF,OLD,0,PNP,DPNP,PNM,DPNM,PNL,DPNL)
 0046                CALL GTJNM(OLD,HT,RS,KMN,CMOMP,GMXJP,GPMXJ,GMXJP1,GPMXJ1,GJOJF,GY1
                    2ST1,GY1ST2,GMIOM,GMICMP,GJICJ,IDEX,L)
 0047            300 CONTINUE
               C
               C           GENERATES MATRIX ELEMENTS
               C
 0048                DO 103 KR=1,MAX,2
 0049                IR=(KR+1)/2
 0050                ASBI(IR)=0.
 0051                DO 102 KK=KR,MAX,2
 0052                KK2=KK+4
 0053                IK=(KK+1)/2
 0054                ITEST(IR,IK)=0
 0055                FS(IR,IK)=0.
 0056                IDX=0
 0057                DO 10 IN=1,NMA
 0058                ITEST(IR,IK)=ITEST(IR,IK)+1
 0059                ORD=RT(IN)
 0060                P(IN)=(2.*ORD+1.)/(ORD*(ORD+1.))
 0061                PR(IN)=P(IN)*RLTA(IN)*DNT(IN)/(SX*SX*DPNL(IN))
 0062                FSP=P(IN)*DNT(IN)/(GMOMP(IN)*FKN(KR,ORD)*FKN(KK,ORD))
 0063                FS(IR,IK)=FS(IR,IK)+FSP
 0064                RATF=ABS(FSP/FS(IR,IK))
 0065                IF(KR.NE.KK)GO TO 17
 0066                IDX=1
 0067                ASPB=PR(IN)/(FKN(KR,ORD)*GMXJP1(IN))
```

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FORTRAN IV G LEVEL  18                  MAIN                DATE = 71110        18/18/39

 0068                ASB(IK)=ASB(IR)+ASPB
 0069                RATA=ABS(ASPB/ASB(IR))
 0070             17 IF(CKD-KK2)10,10,21
 0071             21 IF(IDX.LT.1)GO TO 25
 0072                IF(RATE.LT.1.E-04.AND.RATA.LT.1.E-04)GO TO 19
 0073                GO TO 10
 0074             25 IF(RATE.LT.1.E-04)GO TO 19
 0075             10 CONTINUE
 0076             19 FI(IR,IK)=-2.*PI(KR+1)*PI(KK+1)*FS(IR,IK)
 0077                FI(IK,IR)=FI(IR,IK)
 0078                IF(IR.EQ.KK)GO TO 51
 0079                V(IR,IK)=CT*PI(KR+1)*PI(KK+1)/(KR*KK*(KR+1)*(KK+1))
 0080                V(IK,IR)=V(IR,IK)
 0081                GO TO 101
 0082             51 V(IR,IR)=CT*PI(KR+1)**2/(KR*(KR+1))**2-2.*(1./HT+DSDS(KR+1))/((2.*
                    1KR+1)*KR*(KR+1))
 0083            101 A(IR,IK)=FI(IR,IK)+V(IR,IK)
 0084                A(IK,IR)=A(IR,IK)
 0085            102 CONTINUE
 0086                DTIB(IR)=PI(KR+1)*(-2/(ZC*COS(HT-RS)*KR*(KR+1))+ASB(IR)/(60*RLTAB)
                    1)
 0087                DTB(IR)=CMPLX(0.,-DTIB(IR))
 0088                A(IR,M1)=DTB(IR)
 0089            103 CONTINUE
 0090                CALL CSIMEQ(A,SO,MAXX,1)
 0091                WRITE(6,7)HT,RS
 0092              7 FORMAT(/////46H ***** ANSWER FOR COEFFICIENTS OF FIELD FOR HT=F10.4
                    1,1X7HAND RS=F10.4/5H     1,10X,5H REAL,20X,5H IMAG)
 0093                DO 8 I=1,MAXX
 0094                WRITE(6,9)I,A(I,M1)
 0095              9 FORMAT(I5,2(10X,E15.8))
 0096              8 CONTINUE
 0097                RIMG=(0.,1.)
 0098                ADM1=RIMG*TAN(HT-RS)/ZC
 0099                ADM2=(0.,0.)
 0100                DO 700 K1=1,MAXX
 0101                ADM2=ADM2+PI(2*K1)*A(K1,M1)/(2*K1*(2*K1-1))
 0102            700 CONTINUE
 0103                ADM2=120*ADM2/(ZC*COS(HT-RS))
 0104                ADM3=(0.,0.)
 0105                DO 701 K2=1,NMA
 0106                AD3=PK(K2)*GY1ST1(K2,1)
 0107                ADM3=ADM3+AD3
 0108                RAT=CABS(AD3/ADM3)
 0109                IF(RAT.LT.1.E-03)GO TO 901
 0110            701 CONTINUE
 0111            901 ADM3=-RIMG*ADM3/(120.*RLTAB)
 0112                ADM4=(0.,0.)
 0113                DO 703 K3=1,MAXX
 0114                K32=2*K3
 0115                AD4=(0.,0.)
 0116                DO 704 K4=1,NMA
 0117                CKD=RT(K4)
 0118                AD3=P(K4)*GM1CM(K4,1)*DNF(K4)/FKN((K32-1),CKD)
 0119                AD4=AD4+AD3
 0120                IF((CKD-4.).LT.K32)GO TO 704
 0121                RAT=ABS(AD3/AD4)
 0122                IF(RAT.LT.1.E-03)GO TO 902
```

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 0123          704 CONTINUE
 0124          902 ADM4=ADM4-A(K3,M1)*PI(2*K3)*AC4
 0125              RAT1=CABS(AC4*A(K3,M1)*PI(2*K3)/ADM4)
 0126              IF(RAT1.LT.1.E-02)GO TO 903
 0127          703 CONTINUE
 0128          903 ADM=2*(ADM1+ADM2+ADM3+ADM4)
 0129              RIM=1./ADM
 0130              AADM=CABS(ADM)
 0131              ARIM=1/AADM
 0132              TADM=ATAN2(AIMAG(ADM),REAL(ADM))*180/PIC
 0133              TRIM=-TADM
 0134              WRITE(6,16)
 0135           16 FORMAT(/////41H ***** INPUT ADMITTANCE AND IMPEDANCE ****/11H ADMIT
                   1TANCE,50X,11H IMPEDANCE )
 0136              WRITE(6,18)
 0137              WRITE(6,904)ADM,AADM,TADM,RIM,ARIM,TRIM
 0138              WRITE(6,27)ADM1,ADM2,ADM3,ADM4
 0139           27 FORMAT(2X,E15.8,2X,E15.8/)
 0140           18 FORMAT(2(2X,6H REAL ,10X,6H IMAG ,10X,11H MAGNITUDE ,8X,6H PHASE))
 0141          904 FORMAT(2(2X,E15.8),2X,E15.8,2X,F6.1,2(2X,E15.8),2X,E15.8,2X,F6.1)
 0142              WRITE(6,20)
              C         COMPUTES CURRENT DISTRIBUTIONS ALONG ANTENNA
 0143           20 FORMAT(////49H ** CURRENT DISTRIBUTION ALONG CONICAL,SURFACE **)
 0144              DIV=(HT-RS)/KMN
 0145              KMN1=KMN+1
 0146              CUNC(1)=0.5*ADM
 0147              CUND(1)=CUNC(1)/(2*PIC*RS*SX)
 0148              ACUNC=CABS(CUNC(1))
 0149              ACUND=CABS(CUND(1))
 0150              TCUNC=ATAN2(AIMAG(CUNC(1)),REAL(CUNC(1)))*180/PIC
 0151              TCUND=TCUNC
 0152              WRITE(6,904)CUNC(1),ACUNC,TCUNC,CUND(1),ACUND,TCUND
 0153              DO 905 K5=2,KMN1
 0154              HTC=RS+DIV*(K5-1)
 0155              CUNC1=-RIMG*(SIN(HTC-RS)-COS(HTC-RS)*TAN(HT-RS))/ZC
 0156              CUNC2=(0.,0.)
 0157              DO 800 K6=1,MAXX
 0158              CUNC2=CUNC2+PI(2*K6)*A(K6,M1)/(2*K6*(2*K6-1))
 0159          800 CONTINUE
 0160              CUNC2=CUNC2*COS(HTC-RS)/(COS(HT-RS)*RLTA)
 0161              CUNC3=(0.,0.)
 0162              IF(K5.EQ.KMN1)GO TO 666
 0163              DO 801 K7=1,MMA
 0164              AD3=PK(K7)*GY1ST1(K7,K5)
 0165              CUNC3=CUNC3+AD3
 0166              RAT=CABS(AD3/CUNC3)
 0167              IF(RAT.LT.1.E-03)GO TO 1001
 0168          801 CONTINUE
 0169         1001 CUNC3=-RIMG*CUNC3/(120*RLTAB)
 0170          666 CUNC4=(0.,0.)
 0171              DO 803 K8=1,MAXX
 0172              K82=2*K8
 0173              CUN4=(0.,0.)
 0174              DO 804 K9=1,MMA
 0175              CKD=ST(K9)
 0176              AD3=P(K9)*GM1CM(K9,K5)*CNT(K9)/PKN((K82-1),CKD)
 0177              CUN4=CUN4+AD3
 0178              IF((CKD-4.).LT.K82)GO TO 804
```

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FORTRAN IV G LEVEL  18                  MAIN              DATE = 71110          18/18/39

0179                RAT=CABS(ADD/CUN4)
0180                IF(RAT.LT.1.E-03)GO TO 1002
0181              804 CONTINUE
0182             1002 CUNC4=CUNC4+A(KB,M1)*PI(2*KB)*CUN4
0183                RAT1=CABS(CUN4*A(KB,M1)*PI(2*KB)/CUNC4)
0184                IF(RAT1.LT.1.E-03)GO TO 1003
0185              803 CONTINUE
0186             1003 CUNC(KB)=CUNC1+CUNC2+CUNC3+CUNC4
0187                CUND(KB)=CUNC(KB)/(2*PI*RIC*SX)
0188                ACUNC=CABS(CUNC(KB))
0189                ACUND=CABS(CUND(KB))
0190                TCUNC=ATAN2(AIMAG(CUNC(KB)),REAL(CUNC(KB)))*180/PIC
0191                TCUND=TCUNC
0192                WRITE(6,16)
0193                WRITE(6,904)CUNC(KB),ACUNC,TCUNC,CUND(KB),ACURC,TCUND
0194                WRITE(6,22)KB,CUNC1,CUNC2,CUNC3,CUNC4
0195               22 FORMAT(2X,I5/(2X,E15.8,2X,E15.8/))
0196              905 CONTINUE
                C          COMPUTES CURRENT DISTRIBUTIONS ON THE SPHERE
0197                WRITE(6,23)
0198               23 FORMAT(////31H ** CURRENT DISTRIBUTION ALONG SPHERICAL SURFACE **)
0199                CUNS1=RIMG*TAN(HT-RS)/ZC
0200                CUNS2=(0.,0.)
0201                DO 501 KM1=1,MAXX
0202                CUNS2=CUNS2+(I**(2*KM1))*A(KM1,M1)/(2*KM1*(2*KM1-1))
0203              501 CONTINUE
0204                CUNS2=CUNS2/(COS(HT-RS)*RLTA)
0205                DO 502 KM2=1,NMA
0206                CU3(KM2)=RK(KM2)*CMPLX(ST(KM2,1)/(SX*SX*DPNL(KM2)))
0207                CU4(KM2)=P(KM2)*GMLDP(KM2,1)*DN1(KM2)/(SX*SX*DPNL(KM2))
0208              502 CONTINUE
0209                DO 503 KM3=1,10
0210                IF(KM3.EQ.10)GO TO 504
0211                ITABS=ITAG+10*(KM3-1)
0212                TARBS=PIC*(ITABS-DFB)/180
0213                DFBS=DFB
0214                GO TO 505
0215              504 ITABS=95
0216                DFBS=DFB
0217                TARBS=PIC*(ITABS-DFBS)/180
0218              505 DO 506 KM4=1,NMA
0219                CKD=RT(KM4)
0220                CALL LEGF(ITABS,DFBS,CKD,D,PNP,DPNP,PNN,DPNN,PNL,DPNL)
0221              506 CONTINUE
0222                CUNS3=(0.,0.)
0223                DO 507 KM5=1,NMA
0224                ADD=CU3(KM5)*SIN(TARBS)**2*DPNL(KM5)
0225                CUNS3=CUNS3+ADD
0226                RAT=CABS(ADD/CUNS3)
0227                IF(RAT.LT.1.E-03)GO TO 508
0228              507 CONTINUE
0229              508 CUNS3=-RIMG*CUNS3/(120*RLTAB)
0230                CUNS4=(0.,0.)
0231                DO 509 KM6=1,MAXX
0232                KM62=2*KM6
0233                CUS4=(0.,0.)
0234                DO 510 KM7=1,NMA
0235                DPD=RT(KM7)
```

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FORTRAN IV G LEVEL  18                  MAIN                    DATE = 71110          18/18/39

0236                AD3=CUN(KM7)*(SIN(TARBS)**2)*DPNL(KM7)/FKN((KM2-1),DPD)
0237                CUS4=CUS4+AD3
0238                IF((DFE-6),LT.12*KM6)GO TO 510
0239                RAT=ABS(AD3/CUS4)
0240                IF(RAT).LT.1.E-30)GO TO 511
0241              510 CONTINUE
0242              511 CUNS4=CUNS4-CUN*A(KM6,M1)*PI(2*KM6)
0243                RAT1=CABS(CUS*A(KM6,M1)*PI(2*KM6)/CUNS4)
0244                IF(RAT1.LT.1.E-03)GO TO 512
0245              509 CONTINUE
0246              512 CUNS(KM3)=CUNS1+CUNS2+CUNS3+CUNS4
0247                CUNDS(KM3)=CUNS(KM3)/(2.*PEC*RS*SIN(TARBS))
0248                ACUNS=CABS(CUNS(KM3))
0249                ACUNDS=CABS(CUNDS(KM3))
0250                TACUS=ATAN2(AIMAG(CUNS(KM3)),REAL(CUNS(KM3)))*180/PIC
0251                TACUD=TACUS
0252                WRITE(6,19)
0253                WRITE(6,904)CUNS(KM2),ACUNS,TACUS,CUNDS(KM3),ACUNDS,TACUD
0254                WRITE(6,24)KM3,TARBS,CUNS1,CUNS2,CUNS3,CUNS4
0255             24 FORMAT(2X,I5,2X,E15.8/(2X,E15.8,2X,E15.8/))
0256              503 CONTINUE
0257                ITAF=-3
0258                WRITE(6,26)
0259             26 FORMAT(////10H ** FAR FIELD **)
0260                DO 201 KF=1,18
0261                ITAF=ITAF+5
0262                RITAF=ITAF*PIC/180
0263                DFF=0.
0264                CALL LEGF1(ITAF,DFF,100,PI,DPI)
0265                FARF(KF)=(0.,0.)
0266                KT=-1
0267                DO 202 KFF=1,13
0268                KT=-KT
0269                FARF(KF)=FARF(KF)+A(KFF,M1)*DPI(2*KFF)*KT/((2*KFF*(2*KFF-1)))*HT*SH
                   1(2*KFF))
0270              202 CONTINUE
0271                FARF(KF)=SIN(RITAF)*FARF(KF)/(REAL(CUNC(1)))
0272                ARARF=CABS(FARF(KF))
0273                TFARF=ATAN2(AIMAG(FARF(KF)),REAL(FARF(KF)))*180/PIC
0274                WRITE(6,203)ITAF,FARF(KF),ARARF,TFARF
0275              201 CONTINUE
0276              203 FORMAT(2X,I5,2X,E15.8,2(2X,E15.8),2X,F10.4)
0277                CALL LEGF1(ITA,DF,100,PI,DPI)
0278                RRST=0.
0279                DO 678 IRST=1,13
0280                DOWN=CABS(HF*SF(2*IRST))**2*(2**(2*IRST-1))*(2*IRST)*(4*IRST-1)
0281                RRST=RRST+CABS(A(IRST,M1))**2/DOWN
0282              678 CONTINUE
0283                RRST=1./(RRST*60.)
0284                WRITE(6,679)RRST
0285              679 FORMAT(2X,E15.8)
0286                GO TO 3
0287                $ENDS SUBCHK
0288                AT 1
0289                END
```

```
FORTRAN IV G LEVEL  16                  CTJNM                   DATE = 71110          18/19/39

0001                  SUBROUTINE CTJNM(ORD,HT,RS,BMN,GMOMP,GMXJP,GPMXJ,GMXJP1,GPMXJ1,GJO
                     1JP,GYIST1,GYIST2,GM1OM,GM1OMP,GJ1OJ,IDEX,IAMI)
                C
                C
                C        THIS SUBROUTINE GENERATES ALL FRACTIONAL ORDER BESSEL'S FUNCTION FOR
                C    MODIFIED CONICAL ANTENNA.
                C    IF IAMI=1,THE RADIUS OF CENTRAL SPHERE IS GREATER THAN ZERO.
                C    IF IAMI=2, IT IS BICONICAL ANTENNA.
                C        ORD= ORDER OF BESSEL'S FUNCTION
                C        HT = ANTENNA HEIGTH
                C        RS=RADIUS OF CENTRAL SPHERE IN ELECTRICAL LENGTH
                C        GMOMP = M(HT)/M'(HT)
                C        GMXJP = M(HT)*J'(HT)
                C        GPMXJ = M'(HT)*J(HT)
                C        GMXJP1= M(HT)*J(RS)
                C        GPMXJ1= M'(HT)*J(RS)
                C        GJOJP = J(HT)/J'(HT)
                C        GM1OMP=M(RS)/M'(HT)
                C        GJ1OJ=J(RS)/J(HT)
                C        GYIST1=(J(RS)*N(HT)-M(RS)*J(HT))/(M'(HT)*J'(RS))
                C        GYIST2=(M'(HT)*J(RS)-J'(HT)*M(RS))/(M'(HT)*J'(RS))
                C
0002                  DIMENSION GTJ(60),DGTJ(60),GTN(60),DGTN(60),BJ(120),DBJ(120),BY(12
                     10),DBY(120),GMOMP(60),GMXJP(60),GPMXJ(60),GTM(60),DGTM(60),UTR(5),
                     2UTH(5),VTR(5),VTH(5),GMXJP1(60),GPMXJ1(60),GJOJP(60),GYIST1(60,20)
                     3,GYIST2(60,20),GM1OM(60,20),GM1OMP(60,20),UTRR(5),VTRR(5),GTN1(60)
                     4,GTJ1(60),GTN1(60),DGTN1(60),GJ1OJ(60,20)
0003                  C=1.E-05
0004                  ITET=0
0005                  IDEX=0
0006                  K=(IFIX(ORD+C)+1)/2
0007                  ORD=ORD+0.5
0008                  KB=IFIX(ORD)+1
0009                  PIC=3.14159265
0010                  STB=SQRT(PIC/(2.*RS))
0011                  STBU=SQRT(PIC*RS/2.)
0012                  STHD=SQRT(PIC/(2.*HT))
0013                  STHU=SQRT(PIC*HT/2)
0014                  SXR=RS/ORD
0015                  SXH=HT/ORD
0016                  IF(SXR.GE.1..OR.SXR.GE.1.)GO TO 14
0017                  ALPHR=ALOG((1.+SQRT(1.-SXR**2))/SXR)
0018                  ALPHH=ALOG((1.+SQRT(1.-SXH**2))/SXH)
0019                  TXR=SQRT(1-SXR*SXR)
0020                  TXH=SQRT(1-SXH*SXH)
0021                  CXR=1/TXR
0022                  CXH=1/TXH
0023                  TEST1=ALPHR
0024                  TEST2=ALPHR-ALPHH
0025                  TEST3=TEST2/2
0026                  AU=80/ORD
0027                  IF(TEST1-AU)14,15,15
0028               14 CALL BESJ(RS,ORD,BJ,DBJ,1.E-07,120)
0029                  CALL BESY(RS,ORD,BY,DBY,1.E-07)
0030                  DC=STB*(0.5*BY(KB)*RS*DBY(KB))
0031                  BA=STB*(0.5*BJ(KB)+RS*DBJ(KB))
0032                  GTM1(K)=-DC*STBU*BJ(KB)/BA*STBU*BY(KB)
0033                  GTJ(K)=STBU*BJ(KB)
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FORTRAN IV G LEVEL  19                  GTJNM                   DATE = 71110          18/18/39

0034                CALL BESY(HT,2*0,BY,DBY,1.E-07)
0035                CALL BESJ(HT,2*0,BJ,DBJ,1.E-07,IER)
0036                GTJ(K)=*TMP*BJ(KB)
0037                DGTJ(K)=ST*P*(0.5*BJ(KB)+HT*DBJ(KB))
0038                GTN(K)=*TMP*BY(KB)
0039                DGTN(K)=ST*P*(0.5*BY(KB)+HT*DBY(KB))
0040                GTM(K)=-DE*GTJ(K)/GTN(K)
0041                DGTM(K)=-DE*DGTJ(K)/GTN(K)
0042                GMJMP(K)=(-DE*BJ(K)-BM*GTM(K))/(-DE*DGTJ(K)+BM*DGTN(K))
0043                GMXJP(K)=GTM(K)*DGTJ(K)
0044                GPMXJ(K)=DGTM(K)*DGTJ(K)
0045                GMXJP1(K)=GTM(K)*BM
0046                GPMXJ1(K)=DGTM(K)*BM
0047                GJDJP(K)=GTJ(K)/DGTJ(K)
0048                GM1DM(K,1)=GTM1(K)/GTM(K)
0049                GJ1DJ(K,1)=GTJ1(K)/GTJ(K)
0050                GM1UMP(K,1)=GTM1(K)/DGTM(K)
0051                GY1ST1(K,1)=(GTJ1(K)*GTM(K)-GTM1(K)*GTJ(K))/GMXJP1(K)
0052                GY1ST2(K,1)=(DGTM(K)*GTJ1(K)-DGTJ(K)*GTM1(K))/GPMXJ1(K)
0053                GO TO 101
0054             15 CXR2=CXR*CXR
0055                CXH2=CXH*CXH
0056                ITET=1
0057                UTR(1)=CXR*(3.-5.*CXR2)/24
0058                UTH(1)=CXH*(3.-5.*CXH2)/24
0059                UTR(2)=CXR2*(81.+CXR2*(-462+385*CXR2))/1152
0060                UTH(2)=CXH2*(81.+CXH2*(-462+385*CXH2))/1152
0061                UTR(3)=CXR*CXR2*(30375+CXR2*(-36960+CXR2*(765765-425425*CXR2)))/4
                   114720
0062                UTH(3)=CXH*CXH2*(30375+CXH2*(-36960+CXH2*(765765-425425*CXH2)))/4
                   114720
0063                VTR(1)=CXR*(-9+7*CXR2)/24
0064                VTH(1)=CXH*(-9+7*CXH2)/24
0065                VTR(2)=CXR2*(-135+CXR2*(594-455*CXR2))/1152
0066                VTH(2)=CXH2*(-135+CXH2*(594-455*CXH2))/1152
0067                VTR(3)=CXR*CXR2*(-425425+CXR2*(451737+CXR2*(-883575+475475*CXR2)))
                   1/414720
0068                VTH(3)=CXH*CXH2*(-425425+CXH2*(451737+CXH2*(-883575+475475*CXH2)))
                   1/414720
0069                DEJR=1.
0070                DEYR=1.
0071                DEJPR=1.
0072                DEYPR=1.
0073                DEJH=1.
0074                DEYH=1.
0075                DEJPH=1.
0076                DEYPH=1.
0077                JT=1
0078                DO 16 IP=1,3
0079                JT=-JT
0080                CB=CRD**IP
0081                DEJR=DEJR+UTR(IP)/CB
0082                DEJH=DEJH+UTH(IP)/CB
0083                DEYR=DEYR+JT*UTR(IP)/CB
0084                DEYH=DEYH+JT*UTH(IP)/CB
0085                DEJPR=DEJPR+VTR(IP)/CB
0086                DEJPH=DEJPH+VTH(IP)/CB
0087                DEYPR=DEYPR+JT*VTR(IP)/CB
```

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FORTRAN IV G LEVEL  18                  GTJNM               DATE = 71110          10/18/39

0088                DEYPH=DEYPH+JT*VTH(IP)/CB
0089             19 CONTINUE
0090                GMOMP(K)=HT/(0.5*HT*SINH(ALPHH)*DEJPH/DEJH)
0091                GJOJP(K)=GMOMP(K)
0092             30 A=(0.5*DEJH+HT*SINH(ALPHH)*DEJPH)
0093                B=(0.5*DEJR+RS*SINH(ALPHR)*DEJPR)
0094                DBR=(0.5-RS*SINH(ALPHR)*DEYPR/DEYR)/(0.5+RS*SINH(ALPHR)*DEJPR/DEJR
                   1)
0095                DBH=(0.5-HT*SINH(ALPHH)*DEYPH/DEYH)/(0.5+HT*SINH(ALPHH)*DEJPH/DEJH
                   1)
0096                EXPNT=(ALPHR-ALPHH)-(TANH(ALPHR)-TANH(ALPHH))
0097                EXPNT=EXPNT*ORD
0098                Y2=SQRT(RS/HT)
0099                Y3=SQRT(TANH(ALPHH)/TANH(ALPHR))
0100                Z1=DEJR*(1.-1./DBR)/DEJH
0101                GM1OM(K,1)=Y2*Y3*EXP(-EXPNT)*Z1
0102                GJ1OJ(K,1)=GM1OM(K,1)
0103                GM1OMP(K,1)=GM1OM(K,1)*HT*DEJH/A
0104                GY1ST1(K,1)=RS/(0.5-RS*SINH(ALPHR)*DEYPR/DEYR)
0105                GY1ST2(K,1)=GY1ST1(K,1)
0106                IF(TEST2.GT.AU)GO TO 32
0107                GMXJP(K)=-DEYR*A*(-DBR*EXP(2.*EXPNT)*DEJH/DEJR+DEYH/DEYR)/(2.*ORD*
                   1TANH(ALPHH))
0108                GPMXJ(K)=-DEYR*A*(-DBR*EXP(2.*EXPNT)*DEJH/DEJR+DEYH*DBH/DEYR)/(2.*
                   1ORD*TANH(ALPHH))
0109                GO TO 45
0110             32 IDEX=1
0111                GMXJP(K)=1.E+70
0112                GPMXJ(K)=1.E+70
0113             45 IF(TEST3.GT.AU)GO TO 33
0114                GMXJP1(K)=-DEYR*B*SQRT(HT/(RS*TANH(ALPHH)*TANH(ALPHR)))*(-DBR*EXP(
                   1EXPNT)*DEJH/DEJR+EXP(-EXPNT)*DEYH/DEYR)/(2.*ORD)
0115                GPMXJ1(K)=-DEYR*B*A*(-DBR*EXP(EXPNT)*DEJH/DEJR+EXP(-EXPNT)*DBH*DEY
                   1H/DEYR)/(2.*SQRT(HT*RS*TANH(ALPHH)*TANH(ALPHR))*ORD*DEJH)
0116                GO TO 101
0117             33 IDEX=2
0118                GMXJP1(K)=-1.E+70
0119                GPMXJ1(K)=-1.E+70
0120            101 IF(IAMI.EQ.1) GO TO 108
0121                DIV=HT/KMN
0122                RSR=0.
0123                GO TO 109
0124            108 DIV=(HT-RS)/KMN
0125                RSR=RS
0126            109 KMN1=KMN+1
0127                DO 103 KTG=2,KMN1
0128                RSR=RSR+DIV
0129                ST3R=SQRT(PIC/(2.*RSR))
0130                ST3UR=SQRT(PIC*RSR/2)
0131                SXRR=RSR/ORD
0132                IF(SXR.GE.1..OR.SXRR.GE.1.)GO TO 114
0133                ALPHRR=ALOG((1.+SQRT(1.-SXRR**2))/SXRR)
0134                TXRR=SQRT(1-SXRR*SXRR)
0135                CXRR=1/TXRR
0136                TEST1=ALPHRR
0137                TEST2=ALPHRR-ALPHH
0138                TEST3=TEST2/2
0139                IF(ORD.GT.30.)GO TO 115
```

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FORTRAN IV G LEVEL  18                  GTJNM                DATE = 71110          18/19/39

0140                IF(TEST1-AU)114,115,115
0141            114 CALL BESJ(RSR,ORD,BJ,DBJ,1.E-07,IER)
0142                CALL BESY(RSR,ORD,BY,DBY,1.E-07)
0143                GTM1(K)=-DB*STBUR*BJ(KB)/EN+STBUR*BY(KB)
0144                GTJ1(K)=STBUR*BJ(KB)
0145                GTN1(K)=STBUR*BY(KB)
0146                DGTN1(K)=STBR*(0.5-BY(KB)+RSR*DBY(KB))
0147                IF(ITET.NE.1)GO TO 31
0148                CALL BESY(HT,ORD,BY,DBY,1.E-07)
0149                CALL BESJ(HT,ORD,BJ,DBJ,1.E-07,IER)
0150                GTJ(K)=STHU*BJ(KB)
0151                DGTJ(K)=STHD*(0.5*BJ(KB)+HT*DBJ(KB))
0152                GTN(K)=STHU*BY(KB)
0153                DGTN(K)=STHD*(0.5*BY(KB)+HT*DBY(KB))
0154                GTM(K)=-DB*GTJ(K)/BR+GTN(K)
0155                DGTM(K)=-DB*DGTJ(K)/BR+DGTN(K)
0156             31 GM1OM(K,KTG)=GTM1(K)/GTM(K)
0157                GJ1OJ(K,KTG)=GTJ1(K)/GTJ(K)
0158                GM1OMP(K,KTG)=GTM1(K)/DGTM(K)
0159                GY1ST1(K,KTG)=-(GTN1(K)*GTJ(K))*(1.-(GTM(K)/GTN1(K))*(GTJ1(K)/GTJ(
                   1K)))/GMXJP1(K)
0160                GY1ST2(K,KTG)=-(GTN1(K)*DGTJ(K))*(1.-(DGTM(K)/GTN1(K))*(GTJ1(K)/DG
                   1TJ(K)))/GPMXJ1(K)
0161                GO TO 103
0162            115 CXRR2=CXRR*CXRR
0163                UTRR(1)=CXRR*(3.-5.*(XRR2)/24
0164                UTRR(2)=CXRR2*(21.+CXRR2*(-462+385*CXRR2))/1152
0165                UTRR(3)=CXRR*CXRR2*(30375+CXRR2*(-360603+CXRR2*(765765-425425*CXRR
                   12)))/414720
0166                VTRR(1)=CXRR*(-9+7*CXRR2)/24
0167                VTRR(2)=CXRR2*(-135+CXRR2*(594-455*CXRR2))/1152
0168                VTRR(3)=CXRR*CXRR2*(-425425+CXRR2*(451737+CXRR2*(-363375+475475*CX
                   1RR2)))/414720
0169                DEJRR=1.
0170                DEJPRR=1.
0171                DEYRR=1.
0172                DEYPRR=1.
0173                JT=1
0174                DO 119 IPP=1,3
0175                JT=-JT
0176                DB=ORD**IPP
0177                DEJRR=DEJRR+UTRR(IPP)/DB
0178                DEYRR=DEYRR+JT*UTRR(IPP)/DB
0179                DEJPRR=DEJPRR+VTRR(IPP)/DB
0180                DEYPRR=DEYPRR+JT*VTRR(IPP)/DB
0181            119 CONTINUE
0182                BR=(0.5-RS*SINH(ALPHR)*DEYPR/DEYR)
0183                EXPNT=(ALPHRR-ALPHH)-(TANH(ALPHRR)-TANH(ALPHH))
0184                EXPNT=EXPNT*ORD
0185                EXPNTB=(ALPHR-ALPHRR)-(TANH(ALPHR)-TANH(ALPHRR))
0186                EXPNTB=EXPNTB*ORD
0187                Y2=SQRT(RSR/HT)
0188                Y3=SQRT(TANH(ALPHH)/TANH(ALPHRR))
0189                Z1=(DEJRR/DEJH-DEJR*DEYRR*EXP(-2.*EXPNTB)/(DEJH*DEYR*DBR))
0190                GM1OM(K,KTG)=Y2*Y3*EXP(-EXPNT)*Z1
0191                GJ1OJ(K,KTG)=GM1OM(K,KTG)
0192                GM1OMP(K,KTG)=GM1OM(K,KTG)*HT*DEJH/A
0193                GY1ST1(K,KTG)=SQRT(RSR*RS)*SQRT(TANH(ALPHR)/TANH(ALPHRR))*DEYRR*EX
```

```
FORTRAN IV G LEVEL  18                  GTJNM           DATE = 71110        10/18/39

                1P(-EXPNTB)/(DR*LEVR)
0194              GYIST2(K,KTG)=GYIST1(K,KTG)
0195           103 CONTINUE
0196              ORD=ORD-0.5
0197              RETURN
0198              END
```

```
FORTRAN IV G LEVEL  18                  CSIMEQ          DATE = 71110        10/18/39

0001              SUBROUTINE CSIMEQ (A,M,N,N1)
                C
                C
                C       THIS SUBROUTINE SOLVES SET OF N LINEAR ALGEBRAIC EQUATIONS.
                C
                C
0002              COMPLEX Z,X,A(M,3)
0003               NPN1 = N+ N1
0004               DO 9 I = 1,N
0005              Z1=0.
0006               DO 2 J = I,N
0007              X1=CABS(A(J,I))
0008              IF (Z1-X1) 1,2,2
0009            1  Z1= X1
0010               I1 = J
0011            2  CONTINUE
0012              IF (Z1) 3,11,3
0013            3  Z = A(I1,I)
0014               A(I1,I) = A(I,I)
0015               IP1 = I + 1
0016               DO 4 L = IP1,NPN1
0017               X = A(I1,L)
0018               A(I1,L) = A(I,L)
0019            4  A(I,L) = X/Z
0020               DO 8 J = 1,N
0021               IF (J-I) 5, 8, 5
0022            5 IF (REAL(A(J,I)) .EQ. 0. .AND. AIMAG(A(J,I)).EQ. 0.) GO TO 8
0023               Z = -A(J,I)
0024              DO 7 L= IP1, NPN1
0025            7  A(J,L) = A(J,L) + Z * A(I,L)
0026            8  CONTINUE
0027            9  CONTINUE
0028           10  RETURN
0029           11 WRITE (6,12)
0030           12 FORMAT       (31H COEFFICIENT MATRIX IS SINGULAR )
0031               RETURN
0032              END
```

```
FORTRAN IV G LEVEL  18                  RTNI                DATE = 71110          18/18/39

0001                  SUBROUTINE RTNI(X,TA,NMAX,F,DERF,FCT,XST,EPS,IEND,IER,I,RT,DNT)
                C
                C
                C        THIS SUBROUTINE SOLVES THE CHARACTERISTIC EQUATION L_V(θ_0)=0 FOR A FIX-
                C     ED CONE ANGLE.  IT PROVIDES BOTH ROOTS OF L_V(θ_0)=0 AND DERIVATIVE OF WITH
                C     RESPECT TO COSθ
                C
                C
0002                  DIMENSION RT(50),DNT(50)
0003                  TA2=TA/2
0004                  TOLF=100.*EPS                                                  RTN03870
0005                  DO 10 N1=1,NMAX,2
0006                  X=XST                                                          RTN03940
0007                  TOL=X                                                          RTN03950
0008                  CALL FCT(TA,TOL,N1,F,DERF,FF,I)
0009                  DO 6 JJ=1,IEND
0010                  IF(F)1,7,1                                                     RTN03920
0011                1 IF(DERF)2,8,2                                                  RTN03960
0012                2 DX=F/DERF                                                      RTN03980
0013                  X=X-DX                                                         RTN03990
0014                  TOL=X                                                          RTN04000
0015                  CALL FCT(TA,TOL,N1,F,DERF,FF,I)
0016                  TOL=EPS                                                        RTN04040
0017                  A=ABS(X)                                                       RTN04050
0018                  IF(A-1.)4,4,3                                                  RTN04060
0019                3 TOL=TOL*A                                                      RTN04070
0020                4 IF(ABS(DX)-TOL)5,5,6                                           RTN04080
0021                5 IF(ABS(F)-TOLF)7,7,6                                           RTN04090
0022                6 CONTINUE                                                       RTN04100
0023                7 K=(N1+1)/2
0024                  RT(K)=X+N1
0025                  DNT(K)=-2*COS(TA2)=COS(TA2)*FF/DERF
0026                  IF((2.-X).LT.0.001)GO TO 12
0027                  IF(X.GT.1.8)GO TO 11
0028                  XST=IFIX(X*10000.)/10000.
0029                  GO TO 10
0030               11 XST=1.999000
0031               10 CONTINUE
0032                  IER=NMAX
0033                  RETURN
0034               12 IER=N1
0035                  RETURN
0036                8 IER=N1
0037                  RETURN
0038                  END
```

```
FORTRAN IV G LEVEL  18                  FCN                 DATE = 71110          18/18/39

0001                  FUNCTION FCN(X,XN)
0002                  T=.5*X
0003                  Y=1.57079633*XN
0004                  FCN=.6366198*(PSIG(XN+1.)+ALOG(SIN(T))+.5772156649)-TAN(Y)
0005                  FCN=-FCN
0006                  RETURN
0007                  END
```

FORTRAN IV G LEVEL 18 LEGF DATE = 71110 18/18/39

```
      SUBROUTINE LEGF(ITA,DF,XN,D,PNP,DPNP,PNM,DPNM,PNL,DPNL)
C
C
C         THIS SUBROUTINE GENERATES VALUES FOR FRACTIONAL ORDER LEGENDRE FUNC-
C     TIONS.
C         ANGLE=IFIX(θ+1)-DF
C         XN=ORDER OF LEGENDRE FUNCTION
C         PNP=Pν(μ)
C         DPNP=P'ν(μ).
C         PNM=Pν(-μ).
C         DPNM=P'ν(-μ).
C         PNL=PNP-PNM
C         DPNL=DPNP-DPNM.
C
C
      DIMENSION TERM1(50),TERM2(50),SUM1(50),SUM2(50),PNP(100),PNM(100),
     1DPNP(100),DPNM(100),PNL(50),DPNL(50)
      PIC=3.14159265
      X=(ITA-DF)*PIC/180
      CXP=COS(X)
      CXM=-CXP
      K=(IFIX(XN)+1)/2
      C=5.E-05
      N=IFIX(XN+C)
      FN=XN-FLOAT(N)
      NN1=N+1
      X0=FN+1.
      X1=X0+1.
    2 CALL TERMS(X,FN,D,TERM1,TERM2,I)
      PNP(2)=0.
      PNP(3)=0.
      DO 30 II=1,I
      PNP(2)=PNP(2)+TERM1(II)
      PNP(3)=PNP(3)+TERM2(II)
   30 CONTINUE
   60 CALL SUMS(FN,I,SUM1,SUM2)
    4 SNO=SIN(X0*PIC)
      SN1=SIN(X1*PIC)
      SU1=0.
      SU2=0.
      DO 50 L=2,I
      SU1=SU1+SNO*(TERM1(L)*SUM1(L))/PIC
      SU2=SU2+SN1*(TERM2(L)*SUM2(L))/PIC
   50 CONTINUE
      PNM(2)=(1.-SNO*FCN(X,X0))*PNP(2)+SU1
      PNM(3)=(1.-SN1*FCN(X,X1))*PNP(3)+SU2
      PNP(1)=((2.*X0+1.)*CXP*PNP(2)-(X0+1.)*PNP(3))/X0
      PNM(1)=((2.*X0+1.)*CXM*PNM(2)-(X0+1.)*PNM(3))/X0
      CXP2=CXP*CXP-1
      CXM2=CXM*CXM-1
      DPNP(1)=X0*(PNP(2)-CXP*PNP(1))/CXP2
      DPNM(1)=X0*(PNM(2)-CXM*PNM(1))/CXM2
      DPNP(2)=X0*(CXP*PNP(2)-PNP(1))/CXP2
      DPNM(2)=X0*(CXM*PNM(2)-PNM(1))/CXM2
      DPNP(3)=X1*(CXP*PNP(3)-PNP(2))/CXP2
      DPNM(3)=X1*(CXM*PNM(3)-PNM(2))/CXM2
      IF(NN1-4)100,5,5
    5 DO 40 KK=4,NN1
```

```
FORTRAN IV G LEVEL  18                  LEGF                DATE = 71110          18/18/39

 0044                KK1=KK-1
 0045                KK2=KK-2
 0046                KK3=KK-3
 0047                XX=XO+FLOAT(KK3)
 0048                XXD=XO+KK2
 0049                PNP(KK)=((2.*XX+1.)*CXP*PNP(KK1)-XX*PNP(KK2))/(XX+1)
 0050                PNM(KK)=((2.*XX+1.)*CXM*PNM(KK1)-XX*PNM(KK2))/(XX+1)
 0051                DPNP(KK)=XXD*(CXP*PNP(KK)-PNP(KK1))/CXP2
 0052                DPNM(KK)=XXD*(CXM*PNM(KK)-PNM(KK1))/CXM2
 0053             40 CONTINUE
 0054            100 PNL(K)=PNP(NN1)-PNM(NN1)
 0055                DPNL(K)=DPNP(NN1)+DPNM(NN1)
 0056                PNL(K)=0.5*PNL(K)
 0057                DPNL(K)=0.5*DPNL(K)
 0058                RETURN
 0059                END
```

```
FORTRAN IV G LEVEL  18                  SPYNX               DATE = 71110          18/18/39

 0001                FUNCTION SPYNX(N,X)
                C
                C
                C         THIS SUBROUTINE GENERATES VALUES FOR SPHERICAL FUNCTIONS OF THE
                C    SECOND KIND.
                C
                C
 0002                IF (X .GT. 0.) GO TO 1
 0003                SPYNX=0.
 0004                WRITE (6,2)
 0005              2 FORMAT (78H SIR, SOME THING WRONG WITH YOU ... YOU KNOW THAT THE A
                    1RGUMENT SHOULD NOT BE 0)
 0006                RETURN
 0007              1 IF (N .EQ. 1) GO TO 3
 0008                A=-COS(X)/X
 0009                IF (N .NE. 0) GO TO 3
 0010                SPYNX=A
 0011                RETURN
 0012              3 B=-(COS(X)+X*SIN(X))/X/X
 0013                IF (N .NE. 1) GO TO 4
 0014                SPYNX=B
 0015                RETURN
 0016              4 U=1.
 0017                DO 5 I=2,N
 0018                U=U+2.
 0019                C=U*B/X-A
 0020                A=B
 0021              5 B=C
 0022                SPYNX=B
 0023                RETURN
 0024                END
```

```
FORTRAN IV G LEVEL 18                    BESY                    DATE = 71110          19/16/39

0001                SUBROUTINE BESY(X,XN,BY,DBY,D)
                C
                C
                C   THIS SUBROUTINE GENERATES FRACTIONAL BESSEL'S FUNCTION OF THE SECOND KIND
                C   FOR ARGUMENT LESS THAN 5
                C
                C
0002                DIMENSION BY(50),DBY(50),BYT1(50),BYT2(50),BJT1(50),BJT2(50),BJP(1
                   10),BJM(10),BYT1(50),BYT2(50)
0003                PI=3.14159265
0004                C=1.D-08
0005                N=IDINT(XN+C)
0006                N1=N+1
0007                FN=XN-FLOAT(N)
0008                IF(X.GE.5.0)GO TO 100
0009                CALL BYTERM(X,FN,D,BYT1,BYT2,MM)
0010                CALL BJTERY(X,FN,D,BJT1,BJT2,MM)
0011                BJP(1)=0.
0012                BJP(2)=0.
0013                BJM(1)=0.
0014                BJM(2)=0.
0015                XTFN=(X/2.)**FN
0016                XTFN1=(X/2.)**(1.+FN)
0017                DO 400 IK=1,MM
0018                BJP(1)=BJP(1)+BJT1(IK)
0019                BJP(2)=BJP(2)+BJT2(IK)
0020              400 CONTINUE
0021                BJP(1)=XTFN*BJP(1)
0022                BJP(2)=XTFN1*BJP(2)
0023                DO 401 IH=1,MM
0024                BJM(1)=BJM(1)+BYT1(IH)
0025                BJM(2)=BJM(2)+BYT2(IH)
0026              401 CONTINUE
0027                BJM(1)=BJM(1)/XTFN
0028                BJM(2)=BJM(2)/XTFN1
0029                BY(1)=(BJP(1)*DCOS(FN*PI)-BJM(1))/SIN(FN*PI)
0030                BY(2)=(BJP(2)*DCOS((1.+FN)*PI)-BJM(2))/SIN((1.+FN)*PI)
0031                DBY(1)=-BY(2)+FN*BY(1)/X
0032                DBY(2)=BY(1)-(1.+FN)*BY(2)/X
0033                IF(N1.LE.2) GO TO 101
0034                DO 200 M=3,N1
0035                CRD1=FLOAT(M-1)+FN
0036                CRD2=CRD1-1.
0037                BY(M)=2.*CRD2*BY(M-1)/X-BY(M-2)
0038                DBY(M)=BY(M-1)-CRD1*BY(M)/X
0039              200 CONTINUE
0040              101 RETURN
0041              100 CALL BESYC(X,XN,BY,DBY,D,IER)
0042                RETURN
0043                END
```

```
FORTRAN IV G LEVEL  18                  BESYG                   DATE = 71117          18/18/39

0001                  SUBROUTINE BESYG(X,XN,BY,DBY,D,IER)
                      C
                      C
                      C         THIS SUBROUTINE GENERATES VALUES FOR FRACTIONAL ORDER BESSEL S FUNCT-
                      C    IONS OF THE SECOND KIND WITH ARGUMENT GREATER THAN 5.
                      C
                      C
0002                  DIMENSION BY(50),DBY(50),BYE1(50),BYE2(50)
0003                  PI=3.14159265
0004                  C=1.E-05
0005                  N=IFIX(XN+C)
0006                  N1=N+1
0007                  FN=XN-FLOAT(N)
0008                  CALL BYTERG(X,FN,BYE1,BYE2,MA)
0009                  XEI=1./(2*X)
0010                  XEI2=XEI**2
0011                  BY11=1.
0012                  BY12=BYE1(2)*XEI
0013                  BY21=1.
0014                  BY22=BYE2(2)*XEI
0015                  JT=1
0016                  DO 400 IK=1,MA
0017                  JT=-JT
0018                  IK2=2*IK
0019                  BY11=BY11+JT*BYE1(IK2+1)*XEI2**IK
0020                  BY12=BY12+JT*BYE1(IK2+2)*XEI*XEI2**IK
0021                  BY21=BY21+JT*BYE2(IK2+1)*XEI2**IK
0022                  BY22=BY22+JT*BYE2(IK2+2)*XEI*XEI2**IK
0023              400 CONTINUE
0024                  AG1=X-0.25*(2*FN+1)*PI
0025                  AG2=X-0.25*(2*(1+FN)+1)*PI
0026                  BY(1)=(SIN(AG1)*BY11+COS(AG1)*BY12)*SQRT(2./(PI*X))
0027                  BY(2)=(SIN(AG2)*BY21+COS(AG2)*BY22)*SQRT(2./(PI*X))
0028                  DBY(1)=-BY(2)+FN*BY(1)/X
0029                  DBY(2)=BY(1)-(1.+FN)*BY(2)/X
0030                  IF(N1.LE.2) GO TO 101
0031                  DO 200 M=3,N1
0032                  ORD1=FLOAT(M-1)+FN
0033                  ORD2=ORD1-1.
0034                  BY(M)=2.*ORD2*BY(M-1)/X-BY(M-2)
0035                  DBY(M)=BY(M-1)-ORD1*BY(M)/X
0036              200 CONTINUE
0037              101 RETURN
0038                  END


FORTRAN IV G LEVEL  18                  DPSI                    DATE = 71110          18/18/39

0001                  FUNCTION DPSI(Z)
0002                  X=Z-1.5
0003                  DPSI=0.93480200+X*(-0.82878804+X*(0.70450294+X*(-0.5795609 4+X*(0.4
                     16426845+X*(-0.35206372+X*(0.26484460+X*(-0.27630530+0.21785378*X))
                     1)))))
0004                  RETURN
0005                  END


FORTRAN IV G LEVEL  18                  PSI                     DATE = 71110          18/18/39

0001                  FUNCTION PSI(Z)
0002                  X=Z-1.5
0003                  PSI=0.36489964E-01+X*(0.93480092+X*(-0.41439837+X*(0.23492277+X*(-
                     10.14479804+X*(0.91159502E-01+X*(-0.59422363E-01+X*(0.49350996E-01-
                     10.32681592E-01*X)))))))
0004                  RETURN
0005                  END
```

```
FORTRAN IV G LEVEL  18                  BESJG           DATE = 71110        18/18/39

0001            SUBROUTINE BESJG(X,XN,BJT)
                C
                C
                C       THIS SUBROUTINE GENERATES VALUES FOR FRACTIONAL ORDER BESSEL'S FUNCT-
                C   IONS OF THE FIRST KIND WITH ARGUMENT GREATER THAN 5.
                C
                C
0002            DIMENSION BJE1(50)
0003            PI=3.14159265
0004            C=1.E-05
0005            N=IFIX(XN+C)
0006            N1=N+1
0007            EN=XN-FLOAT(N)
0008            CALL BJTERG(X,EN,BJE1,MA)
0009            XEI=1./(2*X)
0010            XEI2=XEI**2
0011            BJ11=1.
0012            BJ12=BJE1(2)*XEI
0013            JT=1
0014            DO 400 IK=1,MA
0015            JT=-JT
0016            IK2=2*IK
0017            BJ11=BJ11+JT*BJE1(IK2+1)*XEI2**IK
0018            BJ12=BJ12+JT*BJE1(IK2+2)*XEI*XEI2**IK
0019          400 CONTINUE
0020            AG=X-0.25*(2.*EN+1)*PI
0021            BJT=(COS(AG)*BJ11-SIN(AG)*BJ12)*SQRT(2./(PI*X))
0022            RETURN
```

```
FORTRAN IV G LEVEL  18                  PSIG            DATE = 71110        18/18/39

0001              FUNCTION PSIG(X)
0002              IF (X-2.) 10,11,12
0003           10 PSIG=PSI(X)
0004              RETURN
0005           11 PSIG=-0.4227843351
0006              RETURN
0007           12 I=IFIX(X)-1
0008              Z=X-FLOAT(I)
0009              PSIG=PSI(Z)+SUMP(Z-1.,I)
0010              RETURN
0011              END
```

```
FORTRAN IV G LEVEL  18                  DPSIG           DATE = 71110        18/18/39

0001              FUNCTION DPSIG(X)
0002              IF(X-2.) 10,11,12
0003           10 DPSIG=DPSI(X)
0004              RETURN
0005           11 DPSIG=0.6449340668
0006              RETURN
0007           12 I=IFIX(X)-1
0008              Z=X-FLOAT(I)
0009              DPSIG=DPSI(Z)-SUMPP(Z-1.,I)
0010              RETURN
0011              END
```

```
FORTRAN IV G LEVEL  18                  BYTERG              DATE = 71110          18/18/39

0001                SUBROUTINE BYTERG(X,FN,BYE1,BYE2,MA)
0002                DIMENSION BYE1(50),BYE2(50)
0003                MA=1+10/ALOG(X)
0004                BYE1(1)=1.
0005                BYE2(1)=1.
0006                MA2=MA*2+2
0007                DO 100 I=2,MA2
0008                I1=I-1
0009                I2=I-2
0010                XI1=FLOAT(I1)
0011                XI2=FLOAT(I2)
0012                BYE1(I)=BYE1(I1)*(0.5+FN-XI1)*(0.5+FN+XI2)/XI1
0013                BYE2(I)=BYE2(I1)*(1.5+FN-XI1)*(1.5+FN+XI2)/XI1
0014            100 CONTINUE
0015                RETURN
0016                END
```

```
FORTRAN IV G LEVEL  18                  BJTERM              DATE = 71110          18/18/39

0001                SUBROUTINE BJTERM(X,FN,C,BT,MN)
0002                DIMENSION BT(50)
0003                MJ=IFIX(1.6*X+10.)
0004                CALL GAMMA(1.+FN,GX,IER)
0005                XE2=(X/2.)**2
0006                JT=1
0007                BT(1)=1./GX
0008                DO 20 MN=2,MJ
0009                JT=-JT
0010                MN1=MN-1
0011                XMN=FLOAT(MN)
0012                XMN1=FLOAT(MN1)
0013                BT(MN)=JT*ABS(BT(MN1))*XE2/(XMN1*(XMN1+FN))
0014                IF(ABS(BT(MN)).LE.C) GO TO 10
0015             20 CONTINUE
0016             10 RETURN
0017                END
```

```
FORTRAN IV G LEVEL  18                  BJTERG              DATE = 71110          18/18/39

0001                SUBROUTINE BJTERG(X,FN,BJE1,MA)
0002                DIMENSION BJE1(50)
0003                MA=1+10/ALOG(X)
0004                BJE1(1)=1.
0005                MA2=MA*2+2
0006                DO 100 I=2,MA2
0007                I1=I-1
0008                I2=I-2
0009                XI1=FLOAT(I1)
0010                XI2=FLOAT(I2)
0011                BJE1(I)=BJE1(I1)*(0.5+FN-XI1)*(0.5+FN+XI2)/XI1
0012            100 CONTINUE
0013                RETURN
0014                END
```

```
FORTRAN IV G LEVEL  18                  FACK              DATE = 71110          18/18/39

 0001                FUNCTION FACK(K)
 0002                FACK=1.
 0003                DO 13 J=1,K
 0004                XJ=J
 0005             13 FACK=FACK*XJ
 0006                RETURN
 0007                END


FORTRAN IV G LEVEL  18                  FACKP             DATE = 71110          18/18/39

 0001                FUNCTION FACKP(X,K)
 0002                FACKP=1.
 0003                IF(K.LE.1) GO TO 12
 0004                K1=K-1
 0005                DO 11 I=1,K1
 0006                XI=I
 0007                FACKP=FACKP*(X**2-XI**2)
 0008             11 CONTINUE
 0009             12 FACKP=FACKP*X*(X+K)
 0010                RETURN
 0011                END


FORTRAN IV G LEVEL  18                  FCTO              DATE = 71110          18/18/39

 0001                FUNCTION FCTO(TA,X,N1)
 0002                XN=X+N1
 0003                Y=1.57079633*(XN+1.)
 0004                A=TA/2
 0005                FCTO=TAN(Y)-0.63661978*(ALOG(SIN(A))+0.57721566+PSIG(XN+1.))
 0006                RETURN
 0007                END


FORTRAN IV G LEVEL  18                  DFCTO             DATE = 71110          18/18/39

 0001                FUNCTION DFCTO(TA,X,N1)
 0002                XN=X+N1
 0003                Y=1.57079633*(X+1.)
 0004                A=TA/2
 0005                DFCTO=-0.63661978*DPSIC(XN+1.)+1.57079633/COS(Y)**2
 0006                RETURN
 0007                END


FORTRAN IV G LEVEL  18                  SUMP              DATE = 71110          18/18/39

 0001                FUNCTION SUMP(X,K)
 0002                SUMP=0.
 0003                DO 13 J=1,K
 0004                XJ=J
 0005             13 SUMP=SUMP+1./(X+XJ)
 0006                RETURN
 0007                END
```

```
FORTRAN IV G LEVEL  18                  SUMS                DATE = 71110          18/18/39

0001                SUBROUTINE SUMS(FN,I,SUM1,SUM2)
0002                DIMENSION SUM1(50),SUM2(50)
0003                X0=1.+FN
0004                X1=X0+I.
0005                SU41(1)=0.
0006                SUM2(1)=0.
0007                DO 20 J=2,I
0008                J1=J-1
0009                J2=J-2
0010                FJ1=FLOAT(J1)
0011                FJ2=FLOAT(J2)
0012                SUM1(J)=SUM1(J1)-2./FJ1+1./(X0+FJ1)-1./(X0-FJ2)
0013                SUM2(J)=SUM2(J1)-2./FJ1+1./(X1+FJ1)-1./(X1-FJ2)
0014             20 CONTINUE
0015                RETURN
0016                END

FORTRAN IV G LEVEL  18                  SUMM                DATE = 71110          18/18/39

0001                FUNCTION SUMM(X,K)
0002                SUMM=0.
0003                K1=K+1
0004                DO 13 J=1,K1
0005                J1=J-1
0006                XJ1=J1
0007             13 SUMM=SUMM+1./(X-XJ1)
0008                RETURN
0009                END

FORTRAN IV G LEVEL  18                  SUMNS               DATE = 71110          18/18/39

0001                SUBROUTINE SUMNS(TA,X,N1,I,SUMN,SUMND)
0002                DIMENSION SUMN(50),SUMND(50)
0003                PIC=3.14159265
0004                XN=X+N1
0005                FC=FCTC(TA,X,N1)
0006                DFC=DFCTC(TA,X,N1)
0007                DO 20 M=1,I
0008                M1=M-1
0009                SUMN(M)=(FC*PIC+2*SUM(M)-SUMP(XN,M)+SUMM(XN,M1))
0010                SUMND(M)=(SUMM(XN,M1)+SUMP(XN,M))*SUMN(M)+(DFC*PIC+SUMPP(XN,M)-SUM
                   1MM(XN,M1))
0011             20 CONTINUE
0012                RETURN
0013                END

FORTRAN IV G LEVEL  18                  SUM                 DATE = 71110          18/18/39

0001                FUNCTION SUM(K)
0002                SUM=0.
0003                DO 13 J=1,K
0004                XJ=J
0005             13 SUM=SUM+1./XJ
0006                RETURN
0007                END
```

```
FORTRAN IV G LEVEL  18                  BF0                 DATE = 71110          18/18/39

 0001                FUNCTION BF0(X)
 0002                IF (X-3.) 1,1,2
 0003              1 Y=X*X/9.
 0004                BF0=1.+Y*(-2.2499997+Y*(1.2656208+Y*(-.3163866+Y*(.0444479+Y*(-.00
                    139444+.00021*Y)))))
 0005                RETURN
 0006              2 Z=3./X
 0007                F0=.79788456+Z*(-.00000077+Z*(-.00552740+Z*(-.00009512+Z*(.00137237
                    1+Z*(-.00072805+.00014476)))))
 0008                T0=X-.78539816+Z*(-.04166397+Z*(-.00003954+Z*(.00262573+Z*(-.00054
                    1125+Z*(-.00029333+.00013558*Z)))))
 0009                BF0=F0*COS(T0)/SQRT(X)
 0010                RETURN
 0011                END
```

```
FORTRAN IV G LEVEL  18                  BF1                 DATE = 71110          18/18/39

 0001                FUNCTION BF1(X)
 0002                IF (X-3.) 1,1,2
 0003              1 Y=X*X/9.
 0004                BF1=X*(.5+Y*(-.56249985+Y*(.21093573+Y*(-.03954289+Y*(.00443319+Y*
                    1(-.00031761+Y*.00011109)))))
 0005                RETURN
 0006              2 Z=3./X
 0007                F1=.79788456+Z*(.00000156+Z*(.01659667+Z*(.00017105+Z*(-.00249511+
                    1Z*(.00113653-.00020033*Z)))))
 0008                T1=X-2.35619449+Z*(.12499612+Z*(.00005650+Z*(-.00637879+Z*(.00074348
                    1+Z*(.00079824-.00029166)))))
 0009                BF1=F1*COS(T1)/SQRT(X)
 0010                RETURN
 0011                END
```

```
FORTRAN IV G LEVEL  18                  SUMMM               DATE = 71110          18/18/39

 0001                FUNCTION SUMMM(X,K)
 0002                SUMM=0.
 0003                K1=K+1
 0004                DO 13 J=1,K1
 0005                J1=J-1
 0006                XJ1=J1
 0007             13 SUMM=SUMM+1./(X-XJ1)**2
 0008                RETURN
 0009                END
```

```
FORTRAN IV G LEVEL  18                  SUMPP               DATE = 71110          18/18/39

 0001                FUNCTION SUMPP(X,K)
 0002                SUMPP=0
 0003                DO 10 J=1,K
 0004                XI=J
 0005             10 SUMPP=SUMPP+1./(X+XI)**2
 0006                RETURN
 0007                END
```

```
FORTRAN IV G LEVEL  18                  GAMMA                   DATE = 71110          18/18/39

0001                  SUBROUTINE GAMMA(XX,GX,IER)
0002                  IF(XX-57.)6,6,4
0003                4 IER=2
0004                  GX=1.E+75
0005                  RETURN
0006                6 X=XX
0007                  ERR=1.0E-06
0008                  IER=0
0009                  GX=1.
0010                  IF(X-2.0)50,50,15
0011               10 IF(X-2.0)110,110,15
0012               15 X=X-1.0
0013                  GX=GX*X
0014                  GO TO 10
0015               50 IF(X-1.0)60,120,110
0016               60 IF(X-ERR)62,62,80
0017               62 Y=FLOAT(INT(X))-X
0018                  IF(ABS(Y)-ERR)130,130,64
0019               64 IF(1.0-Y-ERR)130,130,70
0020               70 IF(X-1.0)80,80,110
0021               80 GX=GX/X
0022                  X=X+1.
0023                  GO TO 70
0024              110 Y=X-1.0
0025                  GY=1.0+Y*(-0.5771017e+Y*(0.9858539+Y*(-0.8764218+Y*0.8328212+Y*
                     1(-0.5684729+Y*(0.2548205+Y*(-0.0514993)))))))
0026                  GX=GX*GY
0027              120 RETURN
0028              130 IER=1
0029                  RETURN
0030                  END
```

```
FORTRAN IV G LEVEL  18                  FCT                     DATE = 71110          18/18/39

0001                  SUBROUTINE FCT(TA,X,N1,F,DERF,FP,I)
0002                  DIMENSION TERMN(50),SUMN(50),SUMND(50)
0003                  PI=3.14159265
0004                  FP=0.
0005                  CALL TERMNS(TA,X,N1,TERMN,I)
0006                  CALL SUMNS(TA,X,N1,I,SUMN,SUMND)
0007                  F=FCTC(TA,X,N1)
0008                  DERF=DFCTC(TA,X,N1)
0009                  DO 40 M=1,I
0010                  F=F+TERMN(M)*SUMN(M)/PI
0011                  DERF=DERF+TERMN(M)*SUMND(M)/PI
0012                  FP=FP-(TERMN(M)*(M*SUMN(M)-1))
0013               40 CONTINUE
0014                  FP=(1.+FP)/PI
0015               50 RETURN
0016                  END
```

```
FORTRAN IV G LEVEL  18                  TERMS               DATE = 71110          18/18/39

 0001               SUBROUTINE TERMS(X,FN,D,TERM1,TERM2,I)
 0002               DIMENSION TERM1(50),TERM2(50)
 0003               T=.5*X
 0004               S=SIN(T)**2
 0005               X0=1.+FN
 0006               X1=X0+1.
 0007               TERM1(1)=1.
 0008               TERM2(1)=1.
 0009               JT=1
 0010               DO 10 I=2,50
 0011               JT=-JT
 0012               I1=I-1
 0013               TERM1(I)=JT*S**I1*FACKP(X0,I1)/FACK(I1)**2
 0014               TERM2(I)=JT*S**I1*FACKP(X1,I1)/FACK(I1)**2
 0015               IF(ABS(TERM2(I))-D)9,10,10
 0016            10 CONTINUE
 0017               I=50
 0018               RETURN
 0019             9 I=I
 0020               RETURN
 0021               END
```

```
FORTRAN IV G LEVEL  18                  TERMNS              DATE = 71110          18/18/39

 0001               SUBROUTINE TERMNS(TA,X,N1,TERMN,I)
 0002               DIMENSION TERMN(50)
 0003               XN=X+N1
 0004               T=0.5*TA
 0005               S=SIN(T)**2
 0006               JT=1
 0007               DO 10 I=1,50
 0008               JT=-JT
 0009               TERMN(I)=JT*S**I*FACKP(XN,I)/FACK(I)**2
 0010               IF(ABS(TERMN(I))-1.E-07)9,10,10
 0011            10 CONTINUE
 0012               I=50
 0013               RETURN
 0014             9 I=I
 0015               RETURN
 0016               END
```

```
FORTRAN IV G LEVEL  18                  FCN                 DATE = 71110          18/18/39

 0001               FUNCTION FCN(X,XN)
 0002               T=.5*X
 0003               Y=1.57079633*XN
 0004               FCN=.63661978*(PSIG(XN+1.)+ALOG(SIN(T))+.5772156649)-TAN(Y)
 0005               FCN=-FCN
 0006               RETURN
 0007               END
```

```
FORTRAN IV G LEVEL  18                  BESJ                DATE = 71110        18/18/39

0001                  SUBROUTINE BESJ(X,XN,BJ,CBJ,C,IER)
                C
                C
                C        THIS SUBROUTINE GENERATES VALUES FOR FRACTIONAL ORDER BESSEL'S FUNCT-
                C   TIONS OF THE FIRST KIND.
                C
                C
0002                  DIMENSION BJ(100),CBJ(100),CT(50),BJE1(50)
0003                  PI=3.14159265
0004                  C=5.E-05
0005                  N=IFIX(XN+C)
0006                  N1=N+1
0007                  N2=N+2
0008                  FN=XN-FLOAT(N)
0009                20 IF(X)30,30,29
0010                30 IER=2                                                    BESC0670
0011                   RETURN                                                   BESC0680
0012                29 IF(X-15.)32,32,34
0013                32 NTEST=40.+10.*X-X** 2/3                                  BESC0500
0014                   GO TO 36
0015                34 NTEST=90.+X/2.                                           BESC0520
0016                36 IF(N1-NTEST)40,38,38
0017                38 IER=4                                                    BESC0540
0018                   RETURN                                                   BESC0550
0019                40 IF(X.GE.6.0) GO TO 405
0020                   CALL BJTERM(X,FN,0,BJT,MA)
0021                   BJT=0.
0022                   DO 40 IK=1,MA
0023                   BJT=BJT+BJ(IK)
0024               400 CONTINUE
0025                   BJT=(X/2.)**FN*BJT
0026                   GO TO 35
0027               405 CALL BESJG(X,FN,BJT)
0028                35 BPREV=0.
0029                70 MMAX=NTEST
0030                   MZERO=N+10
0031               100 DO 190 M=MZERO,MMAX,5
0032                   FM1=1.0E-70
0033                   FM=.0                                                    BESC0770
0034               130 M2=M-2                                                   BESC0830
0035                   M1=M-1
0036               210 DO 180 K=1,M2
0037                   MK=M-K                                                   BESC0850
0038                   MK1=MK-1
0039                   XMK=FLOAT(MK)+FN
0040                   BJ(MK)=2.*XMK*FM1/X-FM
0041                   FM=FM1                                                   BESC0870
0042               180 FM1=BJ(MK)
0043                   BJ(1)=2.*(1.+FN)*FM1/X-FM
0044                   ALPHAT=BJ(1)/BJT
0045                   DO 301 JI=1,M2
0046                   BJ(JI)=BJ(JI)/ALPHAT
0047               301 CONTINUE
0048                   IF(ABS(BJ(N1)-BPREV)-ABS(C*BJ(N1)))201,201,190
0049               190 BPREV=BJ(N1)
0050                   IER=3
0051                   GO TO 203
0052               201 IER=0
```

```
FORTRAN IV G LEVEL  18                  BESJ              DATE = 71110          18/18/39

0053          200 DBJ(1)=-BJ(2)+FN*BJ(1)/X
0054              IF(N1-2)500,401,401
0055          401 DO 420 LL=2,M2
0056              LL1=LL-1
0057              DRD1=FLOAT(LL1)+FN
0058              DBJ(LL)=BJ(LL1)-DRD1*BJ(LL)/X
0059          420 CONTINUE
0060          500 RETURN
0061              END


FORTRAN IV G LEVEL  18                  TDSUM             DATE = 71110          18/18/39

0001              SUBROUTINE TDSUM(ITA,DF,M,ZDSU)
                  C
                  C
                  C        THIS SUBROUTINE GENERATES VALUES FOR C_FK.
                  C        HALF CONE ANGLE=IFIX(ITA+1)-DF
                  C
                  C
0002              DIMENSION PI(100),DPI(100)
0003              DOUBLE PRECISION TER(200),DS(200),TA,TA2,SX2,DSU,PIC
0004              PIC=0.31415926D+01
0005              TA=(ITA-DF)*PIC/180
0006              TA2=TA/2
0007              M1=M+1
0008              MI1=M+1
0009              M2=M*2+1
0010              SX2=DSIN(TA2)**2
0011              TER(1)=0.1D+01
0012              JT=1
0013              DO 20 I=2,M1
0014              JT=-JT
0015              TER(I)=JT*DABS(TER(I-1))*(M-I+2)*(M+I-1)*SX2/((I-1)*(I-1))
0016           20 CONTINUE
0017              DO 40 II=MI1,M2
0018              TER(II)=0.D+01
0019           40 CONTINUE
0020              DS(1)=0.1D+01
0021              DO 10 II=2,M2
0022              DS(II)=0.D+01
0023              DO 11 I2=1,II
0024              DS(II)=DS(II)+TER(I2)*TER(II-I2+1)/11
0025           11 CONTINUE
0026           10 CONTINUE
0027              DSU=0.D+01
0028              DO 30 KK=1,M2
0029              DSU=DSU+DS(KK)
0030           30 CONTINUE
0031              CALL LEGFI(ITA,DF,M1,PI,DPI)
0032              ZDSU=2*(-DSIN(TA)**2*PI(M1)*DPI(M1)+2.*M*(M+1)*SX2*DSU)
0033              RETURN
0034              END
```

```
FORTRAN IV G LEVEL  18                  SPJNX           DATE = 71110        18/18/39

0001                  SUBROUTINE SPJNX(L,X,SB)
                C
                C
                C       THIS SUBROUTINE GENERATES SPHERICAL BESSEL'S FUNCTIONS OF THE FIRST
                C   KIND.
                C
                C
0002                  DIMENSION SB(L)
0003                  DO 20 NN=1,L
0004                  N=NN-1
0005                  IPX=0
0006                  PIC=3.14159265
0007                  TEST1=X-PIC
0008                  TEST2=X-2*PIC
0009               30 IF(X.NE.0.)GO TO 1
0010                  IF (N .EQ. 0) GO TO 2
0011                  SB(NN)=0.
0012                  GO TO 20
0013                2 SB(NN)=1.
0014                  GO TO 20
0015                1 IF (N .GE. 2 .AND. X .LT. 1.) GO TO 10
0016                  IF (N .EQ. 1) GO TO 3
0017                  A=SIN(X)/X
0018                  IF (N .NE. 0) GO TO 4
0019                  SB(NN)=A
0020                  GO TO 20
0021                4 IF (X .GE. FLOAT(N)) GO TO 3
0022                  IF(ABS(TEST1).LE.1.E-05.OR.ABS(TEST2).LE.1.E-05)GO TO 45
0023                  GO TO 46
0024               45 A=(SIN(X)-X*COS(X))/X/X
0025                  IPX=1
0026               46 N1= 3.+(0.5+EXP(-FLOAT(N-2)/4.))*X
0027                  M=N+N1
0028                  U=2*M+5
0029                  FN=1.E-30
0030                  FN1=0.
0031                  N1=N1+1
0032                  DO 5 I=1,N1
0033                  U=U-2.
0034                  FIN=U*FN/X-FN1
0035                  FN1=FN
0036                5 FN=FIN
0037                  SB(NN)=FN
0038                  IF(IPX.NE.1)GO TO 55
0039                  N=N-1
0040               55 DO 6 I=1 ,N
0041                  U=U-2.
0042                  FIN=U*FN/X-FN1
0043                  FN1=FN
0044                6 FN=FIN
0045                  SB(NN)=SB(NN)*A/FN
0046                  GO TO 20
0047                3 B=(SIN(X) -X*COS(X))/X/X
0048                  IF (N .NE. 1) GO TO 7
0049                  SB(NN)=B
0050                  GO TO 20
0051                7 U=1.
0052                  DO 8 I=2,N
```

```
FORTRAN IV G LEVEL  18                  SPJNX                   DATE = 71110        18/18/39

0053                U=U+2.
0054                FN1=U*B/X-A
0055                A=B
0056              8 B=FN1
0057                SB(NN)=B
0058                GO TO 20
0059             10 A=1.
0060                B=1.
0061                X2=-X*X/2.
0062                B1=0.
0063                B2=N*N+1
0064             11 B1=B1+1.
0065                B2=B2+2.
0066                B=B*X2/B1/B2
0067                A=A+B
0068                IF (ABS(B/A) .GE. 1.E-6) GO TO 11
0069                B=1.
0070                DO 12 I=1,N
0071                B=B+2.
0072             12 A=A/B
0073                SB(NN)=A*X**N
0074             20 CONTINUE
0075                RETURN
0076                END
```

```
FORTRAN IV G LEVEL  18                  BJTERY                  DATE = 71110        18/18/39

0001                SUBROUTINE BJTERY(X,FN,B,BJT1,BJT2,MN)
0002                DIMENSION BJT1(50),BJT2(50)
0003                MJ=IFIX(1.6*X+10.)
0004                CALL GAMMA(1.+FN,GX,IER)
0005                XE2=(X/2.)**2
0006                JT=1
0007                BJT1(1)=1./GX
0008                BJT2(1)=1./(GX*(1.+FN))
0009                DO 20 MN=2,MJ
0010                JT=-JT
0011                MN1=MN-1
0012                MN2=MN-2
0013                XMN=FLOAT(MN)
0014                XMN1=FLOAT(MN1)
0015                XMN2=FLOAT(MN2)
0016                BJT1(MN)=JT*ABS(BJT1(MN1))*XE2/(XMN1*(XMN1+FN))
0017                BJT2(MN)=JT*ABS(BJT2(MN1))*XE2/(XMN1*(XMN1+FN))
0018                IF(ABS(BJT1(MN)).LE.0.AND.ABS(BJT2(MN)).LE.0) GO TO 10
0019             20 CONTINUE
0020             10 RETURN
0021                END
```

```
FORTRAN IV G LEVEL  18                  LEGFI                DATE = 71110        18/18/39

0001                  SUBROUTINE LEGFI(NX,DF,N,PI,DPI)
0002                  DIMENSION SB(6),PI(50),DPI(50)
0003                  PIC=3.14159265
0004                  C=PIC/180
0005                  IDX=0
0006                  IF(NX.EQ.180.OR.NX.EQ.0)GO TO 305
0007                  IF(NX-90)2,3,1
0008                1 IDX=1
0009                  NX=180-NX
0010                  X=(NX+DF)*C
0011                  GO TO 3
0012                2 X=(NX-DF)*C
0013                  IF(ABS(X).LE.1.E-03)GO TO 305
0014                3 CX=COS(X)
0015                  X2=X*X
0016                  PI(1)=1.
0017                  DPI(1)=0.
0018                  PI(2)=CX
0019                  DPI(2)=1.
0020                  DO 30 L=3,6
0021                  L1=L-1
0022                  L2=L-2
0023                  FL1=FLOAT(L1)
0024                  FL2=FLOAT(L2)
0025                  PI(L)=((2.*FL2+1.)*CX*PI(L1)-FL2*PI(L2))/(FL2+1.)
0026                  DPI(L)=L1*(CX*PI(L)-PI(L1))/(CX**2-1.)
0027               30 CONTINUE
0028                  IF(N.LE.6)GO TO 103
0029                  CALL SPJAX(4,X,SB)
0030                  PH2=SB(2)/(4.*SB(1))
0031                  PH3=SB(3)/(24.*SB(1))
0032                  PH4=SB(4)/(192.*SB(1))
0033                  IF(ABS(X)-0.02)13,13,14
0034               13 A0=SQRT(1.+X2*(1./6.+7.*X2/360.))
0035                  GO TO 15
0036               14 A0=SQRT(X/SIN(X))
0037               15 A1=-.5*A0*PH2
0038                  A2=A0*3.*(-.5*PH2+.375*PH2**2)
0039                  A3=A0*15.*(-.5*PH4+.75*PH2*PH3-.3125*PH2**3)
0040                  DO 40 M=7,N
0041                  M1=M-1
0042                  M2=M-2
0043                  FM1=FLOAT(M1)
0044                  FM2=FLOAT(M2)
0045                  XN=FLOAT(M-1)
0046                  XN=XN+0.5
0047                  XX=XN*X
0048                  PI(M)=BFO(XX)*(A0-A2/XN**2-4.*A3/(X*XN**4))+BF1(XX)*(A1/XN+2.*A2/(
                     1X*XN**3)+A3*(8./XX**2-1.)/XN**3)
0049                  DPI(M)=M1*(CX*PI(M)-PI(M1))/(CX**2-1.)
0050               40 CONTINUE
0051              101 IF(IDX.NE.1)GO TO 103
0052                  JT=1
0053                  KT=-1
0054                  NX=180-NX
0055                  DO 102 M=1,N
0056                  JT=-JT
0057                  KT=-KT
```

```
FORTRAN IV G LEVEL  18                  LEGPI              DATE = 71110          18/18/39

0058                PI(K)=KT*PI(K)
0059                DPI(K)=JT*DPI(K)
0060              102 CONTINUE
0061                GO TO 103
0062              305 DO 304 IL=1,N
0063                PI(IL)=1.
0064              304 CONTINUE
0065                DPI(1)=0.
0066                DPI(2)=1.
0067                DPI(3)=3.
0068                DO 303 ID=4,N
0069                DPI(ID)=(ID-1)*(DPI(ID-1)-PI(ID))/(ID-3)
0070              303 CONTINUE
0071                IF(NX.NE.180)GO TO 103
0072                JS=1
0073                KS=-1
0074                DO 306 KP=1,N
0075                JS=-JS
0076                KS=-KS
0077                PI(KP)=KS*PI(KP)
0078                DPI(KP)=JS*DPI(KP)
0079              306 CONTINUE
0080              103 RETURN
0081                END
```

```
FORTRAN IV G LEVEL  18                  BYTERM             DATE = 71110          18/18/39

0001                SUBROUTINE BYTERM(X,FN,D,BYT1,BYT2,MM)
0002                DIMENSION BYT1(50),BYT2(50)
0003                MJ=1FIX(10.+SQRT(1.+3.*X*X))
0004                CALL GAMMA(1.-FN,GX,IER)
0005                XE2=(X/2.)**2
0006                JT=1
0007                BYT1(1)=1./GX
0008                BYT2(1)=-FN/GX
0009                DO 20 MM=2,MJ
0010                JT=-JT
0011                MM1=MM-1
0012                MM2=MM-2
0013                XMM=FLOAT(MM)
0014                XMM1=FLOAT(MM1)
0015                XMM2=FLOAT(MM2)
0016                BYT1(MM)=JT*ABS(BYT1(MM1))*XE2/(XMM1*(XMM1-FN))
0017                BYT2(MM)=JT*ABS(BYT2(MM1))*XE2/(XMM1*ABS(XMM2-FN))
0018                IF(ABS(BYT1(MM)).LE.D.AND.ABS(BYT2(MM)).LE.D) GO TO 10
0019              20 CONTINUE
0020              10 RETURN
0021                END
```

```
FORTRAN IV G LEVEL  18                  SPHNX               DATE = 71110          18/18/39

0001                  SUBROUTINE SPHNX(L,X,SH,DSH,DSCS)
                C       THIS SUBROUTINE GENERATES VALUES FOR HANKEL'S FUNCTIONS.
0002                  COMPLEX SH(50),DSH(50),DSCS(50)
0003                  DIMENSION SB(50),SY(50),VTS(5),UTS(5)
0004                  CALL SPJAX(L,X,SB)
0005                  CALL SPYAX(L,X,SY)
0006                  DO 20 NN=1,L
0007                  SH(NN)=CMPLX(SB(NN),-SY(NN))
0008                20 CONTINUE
0009                  DSH(1)=-SH(2)
0010                  DO 30 LL=2,L
0011                  DSH(LL)=SH(LL-1)-FLOAT(LL)*SH(LL)/X
0012                30 CONTINUE
0013                  IF(L.LE.27)GO TO 31
0014                  DO 100 KP=28,L
0015                  ORD=KP-0.5
0016                  HS=X
0017                  SXS=HS/ORD
0018                  ALPHS=ALOG((1.+SQRT(1.-SXS**2))/SXS)
0019                  TXS=SQRT(1.-SXS*SXS)
0020                  CXS=1./TXS
0021                  CXS2=CXS*CXS
0022                  EXPN=ORD*(ALPHS-TANH(ALPHS))
0023                  UTS(1)=CXS*(3.-5.*CXS2)/24
0024                  UTS(2)=CXS2*(81.+CXS2*(-462+385*CXS2))/1152
0025                  UTS(3)=CXS*CXS2*(30375+CXS2*(-369603+CXS2*(765765-425425*CXS2)))/4
                     114720
0026                  VTS(1)=CXS*(-9+7*CXS2)/24
0027                  VTS(2)=CXS2*(-135+55*CXS2*(54-33*CXS2))/1152
0028                  VTS(3)=CXS*CXS2*(-425425+CXS2*(451737+CXS2*(-883575+475475*CXS2)))
                     1/414720
0029                  DEJS=1.
0030                  DEYS=1.
0031                  DEJPS=1.
0032                  DEYPS=1.
0033                  JT=1
0034                  DO 19 IP=1,3
0035                  JT=-JT
0036                  DB=ORD**IP
0037                  DEJS=DEJS+UTS(IP)/DB
0038                  DEYS=DEYS+JT*UTS(IP)/DB
0039                  DEJPS=DEJPS+VTS(IP)/DB
0040                  DEYPS=DEYPS+JT*VTS(IP)/DB
0041                19 CONTINUE
0042                  RIM=0.5*EXP(-2.*EXPN)*(DEJPS/DEYPS+DEJS/DEYS)
0043                  DSCS(KP)=CMPLX(1.,RIM)
0044                  DSOS(KP)=0.5/HS+SINH(ALPHS)*DEYPS*DSCS(KP)/DEYS
0045                  DSOS(KP)=-DSCS(KP)
0046                100 CONTINUE
0047                 31 DO 40 IK=1,L
0048                  DSOS(IK)=DSH(IK)/SH(IK)
0049                 40 CONTINUE
0050                  RETURN
0051                  END
```

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ABSTRACT

The 'modified dipole' has its origin in the consideration of the general properties of a aatallite antenna which baars great ressmblance to a dipole modified to incorporata at the center a conducting volume which is used to radiate electromagnetic waves and to house a power supply and radio frequancy generators, etc. The object of this research is to pursue a theoretical and experimental exploration of the effects induced by the presence of the conducting volume on the antenna performance, i. e., input characteristics, current distribution along the surfaces of the entire radiating structure and radiation properties.

In Volume I a mathematical model consiating of a perfectly conducting sphere from which project the ends of a thin biconical antenna is chosen to simulate the actual sphere-centered thin dipole. The conical antenna is driven at its junction with the sphere by a rotationally symmetric electric field maintained across the gap by a biconical transmission line excited by the TEM mode. The attractive features of this model include the fact that it has surfaces that permit a simple specification of boundary conditions and, hence, a rigorous formulation for the electromagnetic fields and a shape such that its properties should come reasonably close to those of a modified cylindrical antenna as the cone angle becomes quite small.

The measurements of both input admittances and current distributions on modified dipoles (with either conical or cylindrical antenna projecting from the sphere) are also presented in Volume I. Comparisons were also made between modified conical and cylindrical antennas with the same sphere radii and antenna heights. The radius of the cylindrical antenna is the same as the smaller end of the cone. The fact that the admittance curves for modified cylindrical and conical antennas involve only slight shifts suggests that by introducing an equivalent antenna length that is a little longer than the actual physical length of the conical antenna a good approximation is obtained for the cylindrical antenna.

An infinite set of algebraic equations was solved numerically in Volume II for small cone angles. Comparisons were made between the modified conical antenna and its limiting biconical antenna which provides both an extrapolatory numerical check for the modified conical antenna with shrinking central sphere and an understanding of the underlying physical phenomena. Theoretical and experimental results are in very good agreement.

KEY WORDS

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Cylindrical antennas
Input admittances
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