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CONVERTING GENERAL NONLINEAR PROGRAMMING PROBLEMS TO SEPARABLE NONLINEAR PROGRAMMING PROBLEMS

by

Garth P. McCormick

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THE GEORGE WASHINGTON UNIVERSITY
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Program in Logistics

Abstract
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It is shown that with the introduction of additional variables and equality constraints most general nonlinear optimization problems can be converted to separable problems. Local solutions are preserved as well as first and second-order Kuhn-Tucker conditions.

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In the folklore of optimization theory it has been said that by means of tricks and transformations any nonlinear programming problem of the form

$$\underset{x \in E^n}{\text{minimize}} \quad f_0(x)$$

subject to

$$f_i(x) \le 0 \ , \ i = 1,\ldots,m$$

$$h_k(x) = 0 \ , \ k = 1,\ldots,p$$

can be converted to an "equivalent" separable problem of the form

$$\underset{y \in E^N}{\text{minimize}} \quad \Sigma_{j=1}^{N} F_{0j}(y_j)$$

subject to

$$\Sigma_{j=1}^{N} F_{ij}(y_j) \le 0 \ , \ i = 1,\ldots,M \ ,$$

$$\Sigma_{j=1}^{N} H_{kj}(y_j) = 0 \ , \ k = 1,\ldots,P \ .$$

There exist scattered results such as separation of a product form $f(x)g(x)$ or of a form $f(x)^{g(x)}$ (Wagner [1] , pp. 556-557), but there is to the author's knowledge no paper which states a systematic procedure for doing this, or shows the equivalence of the resulting problems.

The motivation for dealing only with separable problems is that it is simple to provide standard input formats for nonlinear programming algorithms which are separable, it is easy to provide canned computer programs to obtain the first and second derivatives of separable functions, it allows use of branch and bound methods for obtaining global solutions to nonconvex separable programming problems, and it allows the use of large scale separable programming codes.

The method for separating programming problems has two basic steps which are used repeatedly until they can no longer be applied.

(1) Replace any product term of the form

$$q_1(x)\ q_2(x)$$

by

$$y_1^2 - y_2^2$$

and add the constraints

$$q_1(x) = y_1 - y_2$$

$$q_2(x) = y_1 + y_2 .$$

One application of this step adds two variables and two equality constraints to the problem.

(2) Replace any term of the form $T[t(x)]$ by $T(y_1)$ and add the constraint

$$t(x) = y_1 .$$

One application of this step adds one variable and one equality constraint to the problem.

The class of nonlinear optimization problems which can be separated by application of the above steps is very broad. Let the word transformation mean a function of a single variable such as $e^{(\cdot)}$, $\sin(\cdot)$, $(\cdot)^a$ etc.

Definition. A factorable function is a function of n variables which is generated by first composing (adding or multiplying) functions of a single variable, transforming those functions, composing those, transforming ,... a finite number of times. For a factorable programming problem, one where all the functions are factorable, it is a simple matter to prove the following theorem.

Theorem. Repeated application of Steps (1) and (2) above will reduce any factorable programming problem to a separable programming problem.

A significant computable function of 2 variables which is not factorable is the gamma distribution function. That is

$$\int_{-\infty}^{a} \frac{(\mu k)^k}{\Gamma(k)} t^{k-1} e^{-k\mu t} dt$$

where μ, k are the variables is not a factorable function.

The 'equivalence' of problems created by application of Steps (1) and (2) can be established in two ways. The first is to state that local solutions of the final problem correspond to local solutions of the original one, and that all original local (and hence global) solutions are preserved. These facts are established by the following two lemmas.

Lemma:

(i) If $\bar{x}$ is a local minimizer

PROB(A) to: $\min_{x \in S} f_0(x)$ s.t. $q_1(x) \cdot q_2(x) + f_1(x) \leq 0$,

then $(\bar{x}, \bar{y}_1, \bar{y}_2)$ is a local minimizer

PROB(B) to: $\min_{(x \in S, y_1, y_2)} f_0(x)$ s.t. $y_1^2 - y_2^2 + f_1(x) \leq 0$,

$$q_1(x) = y_1 - y_2$$

$$q_2(x) = y_1 + y_2$$

where $\bar{y}_1 = [q_1(\bar{x}) + q_2(\bar{x})]/2, \bar{y}_2 = [q_2(\bar{x}) - q_1(\bar{x})]/2$.

(ii) If $(\bar{x}, \bar{y}_1, \bar{y}_2)$ is a local minimizer to Problem (B) then $\bar{x}$ is a local minimizer to Problem (A).

Proof:

(i) If $(\bar{x}, \bar{y}_1, \bar{y}_2)$ is not a local minimizer to (B) then there exists a sequence feasible to (B) such that $\{x^k, y_1^k, y_2^k\} \to (\bar{x}, \bar{y}_1, \bar{y}_2)$ with $f_0(x^k) < f_0(\bar{x})$ for all k . But this implies that each x^k is feasible to (A), $x^k \to \bar{x}$, and $f_0(x^k) < f(\bar{x})$. This contradicts the assumption that $\bar{x}$ is a local minimizer to (A).

(ii) The proof of part (ii) is similar to that above.

Q.E.D.

Lemma:

(i) If $\bar{x}$ is a local minimizer to

PROB(C) $\min_{x \in S} f_0(x)$ s.t. $T[t(x)] + f_1(x) \leq 0$,

then $(\bar{x}, \bar{y}_1)$ is a local minimizer to

PROB(D) $\min_{(x \in S, y_1)} f_0(x)$ s.t. $T(y_1) + f_1(x) \leq 0$, $t(x) = y_1$

where $\bar{y}_1 = T[t(\bar{x})]$.

(ii) If $(\bar{x},\bar{y}_1)$ is a local minimizer to Problem D, then $\bar{x}$ is a local minimizer to Problem C.

Proof:

The proof to this theorem is transparent.

The second way in which the problems are equivalent can be established by showing that points which are first and second-order Kuhn-Tucker points of the original problem correspond to first and second-order Kuhn-Tucker points of the separable problem and conversely. These facts can be proved using the following two lemmas.

Lemma:

(i) If $\bar{x}$ is a SOKTP for the problem

PROB($\bar{C}$) $\min_{x \in E^n} f_0(x)$ s.t. $f_i(x) \leq 0$, $i = 1,\ldots,m-1$,

$T[t(x)] + \beta(x) \leq 0$,

then $(\bar{x},\bar{y}_1)$,

where $\bar{y}_1 = t(\bar{x})$ is a SOKTP for the problem

PROB($\bar{D}$) $\min_{(x \in E^n, y_1)} f_0(x)$ s.t. $f_i(x) \leq 0$, $i = 1,\ldots,m-1$,

$T(y_1) + \beta(x) \leq 0$, $t(x) = y_1$.

(ii) If $(\bar{x}, \bar{y}_1)$ is a SOKTP for Problem $\bar{D}$,

then $\bar{x}$ is a SOKTP for Problem $\bar{C}$.

Proof:

The first-order Kuhn-Tucker conditions are

($\bar{C}$)

(1) $f_i(\bar{x}) \leq 0$, $i = 1,\ldots,m-1$ (2) $T[t(\bar{x})] + \beta(\bar{x}) \leq 0$

(3) $\bar{u}_i \geq 0$, $i = 1,\ldots m$ (4) $\bar{u}_i f_i(\bar{x}) = 0$, $i = 1,\ldots,m$

(5) $\underset{\sim}{0} = \nabla f_0(\bar{x}) + \Sigma_{i=1}^{m-1} \bar{u}_i \nabla f_i(\bar{x}) + \bar{u}_m [\nabla t(\bar{x}) \dot{T} [t(\bar{x})] + \nabla \beta(\bar{x})]$

(6) $f_i(\hat{x}) \leq 0$, $i = 1,\ldots,m-1$ (7) $T(\hat{y}_1) + \beta(\hat{x}) \leq 0$

$(\bar{D})$

(8) $t(\hat{x}) - \hat{y}_1 = 0$ (9) $\hat{u}_i \geq 0$, $i = 1,\ldots,m$

(10) $\hat{u}_i f_i(\hat{x}) = 0$, $i = 1, \ldots, m$

(11) $\underset{\sim}{0} = \nabla f_0(\hat{x}) + \Sigma_{i=1}^{m-1} \hat{u}_i \nabla f_i(\hat{x}) + \hat{u}_m \nabla \beta(\hat{x}) + \hat{w}_1 \nabla t(\hat{x})$

(12) $0 = \hat{u}_m \dot{T}(\hat{y}_1) + \hat{w}_1 (-1)$

Now if $(\bar{x},\bar{u})$ satisfies (1)-(5), then $(\bar{x},\bar{u},\bar{u}_m\{\dot{T}[t(\bar{x})]\})$ satisfies (6)-(12). If $(\hat{x},\hat{u},\hat{w}_1)$ satisfies (6)-(12), then $(\hat{x},\hat{u})$ satisfies (1)-(5). By Pennisi's theorem, all FOKTPS are SOKTPS.

Q.E.D.

Lemma:

(i) If $\bar{x}$ is a second-order Kuhn-Tucker point (SOKTP) for the problem

PROB$(\bar{A})$ $\underset{x\varepsilon E^n}{\text{Min}}$ $f_0(x)$ s.t. $f_i(x) \leq 0$, $i = 1,\ldots,m-1$,

$q_1(x) \cdot q_2(x) + \alpha(x) \leq 0$,

then $(\bar{x},\bar{y}_1,\bar{y}_2)$

where $\bar{y}_1 = [q_1(\bar{x})+q_2(\bar{x})]/2, \bar{y}_2 = [q_1(\bar{x})-q_2(\bar{x})]/2$ is a SOKTP for the problem

PROB$(\bar{B})$ $\underset{(x\varepsilon E^n,y_1,y_2)}{\text{Min}}$ $f_0(x)$ s.t. $f_i(x) \leq 0$, $i = 1,\ldots,m-1$,

$y_1^2 - y_2^2 + \alpha(x) \leq 0$, $q_1(x) = y_1 - y_2$

$q_2(x) = y_1 + y_2$.

(ii) If $(\bar{x},\bar{y}_1,\bar{y}_2)$ is a SOKTP for Problem $(\bar{B})$, then $\bar{x}$ is a SOKTP for Problem $(\bar{A})$.

Proof:

The proof is similar to that of the previous lemma.

The following example explains the steps and how they can be used to separate a problem which at first glance seems intractable.

Example:

$$\min_{(\mu,\sigma,p)} \Sigma_{i=1}^{r} \left\{ \beta_i - \frac{1}{\sigma\sqrt{2\pi}}\, p \int_{-\infty}^{\alpha_i} e^{\frac{-(\mu-\alpha)^2}{2\sigma^2}}\, d\alpha \right\}^2$$

This has three variables and no constraints. Replacing the transformed functions and rewritting the integral yields the equivalent problem

$$\min_{(\mu,\sigma,p,\{y_i\})} \Sigma_{i=1}^{r} (y_i)^2 \quad \text{subject to}$$

$$\beta_i - \frac{1}{\sqrt{2\pi}}\, p \int_{-\infty}^{(\alpha_i-\mu)/\sigma} e^{-t^2/2}\, dt = y_i$$

3 + r variables, r constraints for $i = 1,\ldots,r$.

Integral is of form $T[t(\mu,\sigma)]$ replacing this yields

$$\min_{(\mu,\sigma,p,\{y_i\},\{z_i\})} \Sigma_{i=1}^{r} (y_i)^2 \quad \text{subject to}$$

$$\beta_i - \frac{p}{\sqrt{2\pi}} \int_{-\infty}^{y_i} e^{-t^2/2} dt = y_i \ , \ i = 1,\ldots,r,$$

$$z_i = (\alpha_i-\mu)/\sigma, i=1,\ldots,r.$$

3 + 2r variables

2r constraints

Introducing new variables and constraints to get rid of product terms yields

$$\min_{(\mu,\sigma,p,\{y_i\},\{z_i\},\{u_i\},\{v_i\}, w_1,w_2)} \Sigma_{i=1}^{r} (y_i)^2$$

subject to

$$\beta_i - u_2^2 + v_i^2 = y_i \,, \qquad i = 1,\ldots,r$$

$$p = u_i - v_i \,, \qquad i = 1,\ldots,r$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_i} e^{-t^2/2}\,dt = u_i + v_i \,, \quad i = 1,\ldots,r$$

$$z_i = \alpha_i/\sigma - w_1^2 + w_2^2 \,, \qquad i = 1,\ldots,r$$

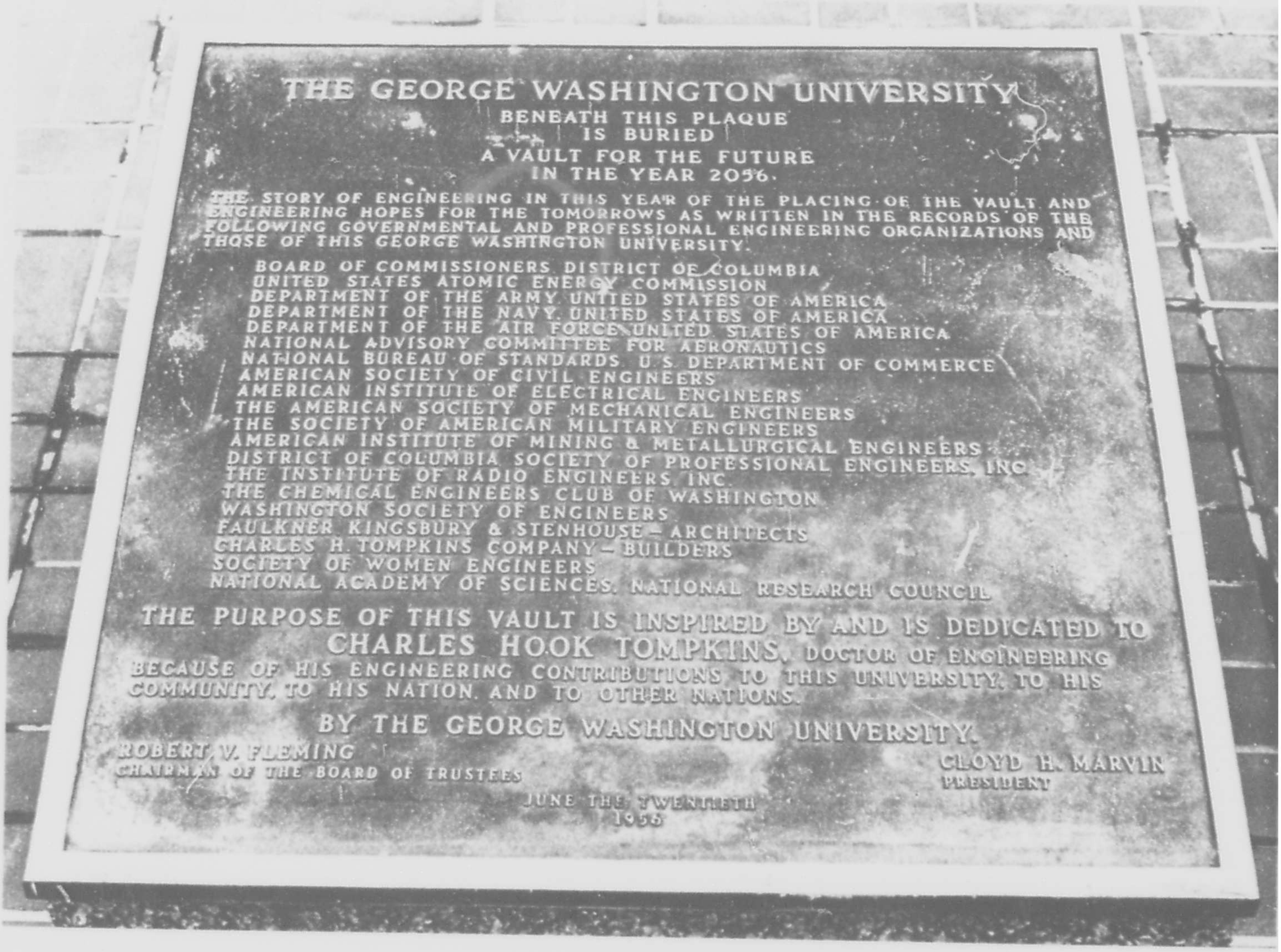
$$\mu = w_1 - w_2$$

$$\sigma^{-1} = w_1 + w_2 \,.$$

This separable problem has $4r + 5$ variables and $4r + 2$ equality constraints.

REFERENCE

[1] WAGNER, Harvey M. (1969). Principles of Operations Research, Prentice-Hall, Englewood Cliffs, N. J.



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