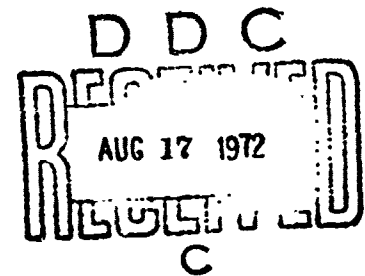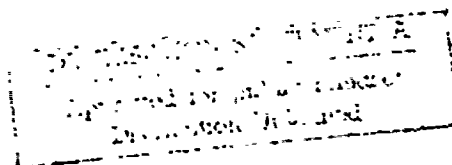TECHNICAL REPORT

No. 35

JUNE 1972

TRANSFORMATIONS FOR RANGE INSTRUMENT
COORDINATES AND THEIR TRACKING SPACE
SINGULARITY REGIONS

Special Projects Section
Mathematical Services Branch
Analysis & Computation Division
NATIONAL RANGE OPERATIONS DIRECTORATE
WHITE SANDS MISSILE RANGE, NEW MEXICO

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
ANALYSIS AND COMPUTATION DIVISION NATIONAL RANGE OPERATIONS DIRECTORATE WHITE SANDS MISSILE RANGE, NM 88002	UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE

TRANSFORMATIONS FOR RANGE INSTRUMENT COORDINATES AND THEIR TRACKING SPACE
SINGULARITY REGIONS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

SPECIAL REPORT

5. AUTHOR(S) *(First name, middle initial, last name)*

PAPPAS, JAMES S.
BOOKHOUT, GLENN A., CAPT (DR)

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
JUNE 1972	25	0

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. DISTRIBUTION STATEMENT

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

The non-linear transformations relating rectangular coordinates to range instrumentation coordinates for conventional radar, phased array radar, AME (Angle Measuring Equipment) etc., are considered. The transformations are linearly related through Jacobian matrices at the derivative (rate) and first differential or discrete difference (residuals or linearizing about a nominal) level. State variables near the singularity regions of these matrices can cause numerical computer problems for the unwary analyst.

DD FORM
, NOV 65 **1473** REPLACES DD FORM 1473, 1 JAN 64, WHICH IS
OBSOLETE FOR ARMY USE.

1a

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
STATE SPACE						
DYNAMICS						
INERTIAL MEASUREMENTS						
RANGE TESTING						

TECHNICAL REPORT

NO. 35

Transformations for Range Instrument

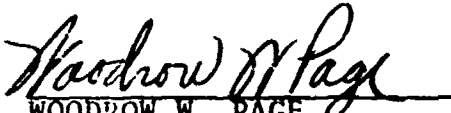Coordinates and Their Tracking Space

Singularity Regions

June 1972

PREPARED BY: J. S. PAPPAS
Supv Physical Scientist

CAPT (Dr) G. A. BOOKHOUT
ADP Officer

REVIEWED BY: WOODROW W. PAGE
Chief, Math Services Br

APPROVED BY: W. A. McCOOL
Acting Chief, Anal & Cmpt Div

ic

CONTENTS

ABSTRACT

The non-linear transformations relating rectangular coordinates to range instrumentation coordinates for conventional radar, phased array radar, AME (Angle Measuring Equipment) etc., are considered. The transformations are linearly related through Jacobian matrices at the derivative (rate) and first differential or discrete difference (residuals or linearizing about a nominal) level. State variables near the singularity regions of these matrices can cause numerical computer problems for the unwary analyst.

INTRODUCTION

The development of operational computer programs from math-ware to software presents many opportunities for bad output and expensive reruns. Systematic procedures for obtaining computable math-ware from the physical problems of range testing are being developed in the state-space frame-work. A deeper understanding of the math-ware will enhance the analysts' ability to make judgments on the practical implications of the quantitative answers provided by the digital computer.

Range flight testing and trajectory state-vector estimation normally requires as a minimum, the estimation of a nine dimensional state vector, three coordinates at each of the three levels: position, velocity, and acceleration. The transformations from rectangular to spherical polar coordinates (FPS-16 radars) or sine-space coordinates (phased-array radars) are non-linear. However at the velocity level the transformations are achieved via the Jacobian matrix as the connection matrix.

In most statistical data processing procedures for range instrument measurements one computes residuals or error vectors between actual measurements and estimated measurements and actual states and estimated states.

These error vectors normally imply discrete differences or approximations to differentials of vectors. The Jacobian matrix provides the transformation between differentials of rectangular coordinates and spherical-polar (radar coordinates) or any other set of generalized coordinates, for example sine-space coordinates occurring in phased-array radar systems. Conventional least squares polynomial filtering, or Kalman filters in which one linearizes about a nominal trajectory or linearizes about the current state estimates can lead the unwary analyst into computational problems of ill-conditional matrices, or the scalar analog of inverting trigonometric functions when the variables are near singularities.

Quite often one estimates state variables at the rectangular coordinate level, for example three position, three velocity, and three acceleration coordinates. At some point in the computations one transforms back to

instrument coordinates to compute instrument errors. The computation of a matrix of variances and covariances between the variables implies two variance matrices one a function of the rectangular coordinates, the second a function of the generalized coordinates and such that they are related through a congruent transformation on the Jacobian matrix.

If one performs statistical analysis and computations on data originating from these instrumentation systems he should be wary of ill-conditioned matrices or the equivalent problem of solutions appearing unstable or meaningless.

The greatest insight into the values of the variables where computational problems can occur comes from a vector-matrix analysis of the geometry.

It is perhaps well-known that singularities of FPS-16 type radar coordinates occur near 90° elevation angles. Perhaps it is not so well known that sine-space (phased-array radars) singularities occur near 45° angles. Angle measuring equipment (AME) that measures direction cosines has singularities at zero elevation angles.

The mathematics of these relations are derived and discussed below.

COORDINATES AND JACOBIAN MATRICES

The non-linear relationships between the rectangular coordinates (x, y, z) - or when convenient (x_1, x_2, x_3) - and five other sets of covering coordinates are developed. At the position level the functional relationships are as always non-linear. At the differential level the coordinates are connected by a Jacobian matrix, hence matrix analysis tools are useful, or the tools of <u>linear transformation theory</u>.

Consider five sets of covering coordinates as indicated by the many variables shown in Figure 1.
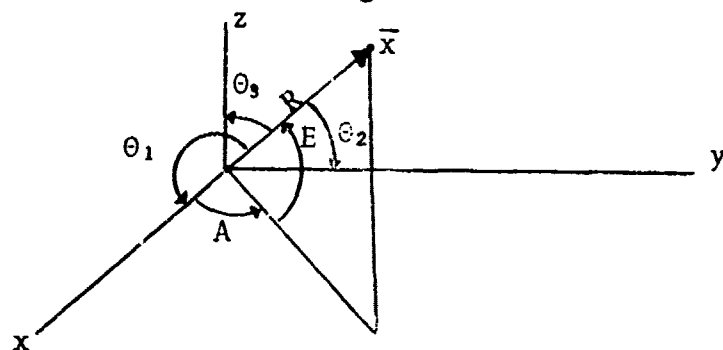

FIGURE 1 - Covering Coordinates

The point designated by the vector $\bar{x}$ has three independent coordinates x_i $i=1,2,3$, non-linearly related to three other independent coordinates (except at singularities) q_i $i=1,2,3$

$$x_1 = x_1(q_1, q_2, q_3)$$

$$x_2 = x_2(q_1, q_2, q_3) \tag{1}$$

$$x_3 = x_3(q_1, q_2, q_3)$$

for five sets of q_i's.

Designate the five systems as

a. Rectangular cartesian coordinates (x,y,z) or (x_1, x_2, x_3).

b. Spherical polar coordinates (r, Λ, E) FPS-16 type radar. Optics type coordinates without range r.

c. Range and two direction cosines $(r, \cos \Theta_1, \cos \Theta_2)$ the unit sight line vector

$$\frac{\langle x}{\langle xx \rangle^{1/2}} = \langle s = (s_1, s_2, s_3) \tag{2}$$

is a function of only two of the three s_i independent coordinates since

$$1 = s_1^2 + s_2^2 + s_3^2 . \tag{3}$$

The rectangular coordinates, s_i take on many equivalent forms, for example

$$s_1 = \cos \Theta_1 = CECA = etc. \tag{4}$$

Range equipment such as AME, (angle measuring electronic equipment) outputs direction cosines (minus range).

d. Sine-space coordinates $(r, \sin\phi_1, \sin\phi_3)$.

These coordinates occur in phased array radars. If one redefines the direction angles Θ_i of Figure 1 in terms of complements, that is

$$\phi_i = 90° - \Theta_i \tag{5}$$

then

$$s_1 = \cos\Theta_1 = \sin\phi_1 \tag{6}$$

$$s_3 = \cos\Theta_3 = \sin\phi_3 \tag{7}$$

a number of phased-array radar studies appear to define angles α and β such that

$$\phi_2 = -\alpha$$

$$\phi_3 = -\beta \tag{8}$$

a number of publications related to $\alpha_1\beta$ trackers as filter etc. exist.

    e. The final set of independent covering coordinates are range and two of the direction angles say

$$(r, \phi_1, \phi_3).$$

    The differentials of the scalar functions of Equation 1 are

$$dx^1 = \left\langle \frac{\partial x^1}{q} \; dq \right\rangle \tag{9}$$

$$dx^2 = \left\langle \frac{\partial x^2}{q} \; dq \right\rangle$$

$$dx^3 = \left\langle \frac{\partial x^3}{q} \; dq \right\rangle$$

and

$$\left\langle \frac{\partial x^i}{\partial q} \equiv \left(\frac{\partial x^i}{\partial q_1}, \; \frac{\partial x^i}{\partial q_2}, \; \frac{\partial x^i}{\partial q_3} \right) \right. \tag{10}$$

or vector wise

$$\begin{vmatrix} dx^1 \\ dx^2 \\ dx^3 \end{vmatrix} = J \begin{vmatrix} dq_1 \\ dq_2 \\ dq_3 \end{vmatrix} \tag{11}$$

5

where the Jacobian matrix of (11) has the three gradient vectors of (9) as its row-space

$$
\underset{3x3}{J} = \begin{bmatrix} \left\langle \dfrac{\partial x^1}{\partial q} \right\rangle \\[1em] \left\langle \dfrac{\partial x^2}{\partial q} \right\rangle \\[1em] \left\langle \dfrac{\partial x^3}{\partial q} \right\rangle \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x^i}{\partial q_j} \end{bmatrix} \tag{12}
$$

The inverse of J exists when the three gradient vectors are linearly independent. Likewise when the vectors are linearly dependent the matrix is singular and the values of the generalized coordinates for which singularities occur are called poles. For example with spherical polar coordinates (r,A,E) at the pole (thinking of the earth) elevation angle E (or latitude) equals $90°$.

It is well known that the values of the variables that make the determinant equal to zero are the singularities that is

$$
\det J = 0. \tag{13}
$$

where

$$
J^{-1} = adj\ J\ (\det J) \tag{14}
$$

and adj J is adjoint of J or transpose of matrix of co-factors.

It is also of interest to express the inverse as

$$
J^{-1} = (J^T J)^{-1}\ J^T \tag{15}
$$

since the computational aspects of inversion is reduced to the inversion of a symmetric matrix (an easier task-in general).

6

It is also conceptually worthwhile to consider the symmetric matrix $J^T J$ and JJ^T which in general are not equal. These "Grammian" or "Metric-Matrices" yield insight into orthogonality, obliqueness etc. of row and column-space vectors. No further structure properties are considered in this report.

The notation J_{km}

$$k, m = 1, 2, \ldots 5$$

is used to designated the coordinate system for example J_{12} is the Jacobian matrix connecting systems 1 and 2

$$\begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} = J_{12} \begin{pmatrix} dr \\ dA \\ dE \end{pmatrix} . \tag{16}$$

In the following tables some of the Jacobians are expressed as functions of the rectangular coordinates some of the generalized coordinates and some for both.

Clearly if one divides Equation (16) by Δt and passes to the limit by velocity relations result, that is

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} \tag{17}$$

where the example of (16) yields

$$\dot{q} = (\dot{r}, \dot{A}, \dot{E}). \tag{18}$$

7

Variance matrix relations are also obtained by transposing Equation (11)

$$dx = \langle dq J^T \tag{19}$$

and computing the dyadic product

$$dx \rangle \langle dx = J dq \rangle \langle dq J^T \tag{20}$$

In a-priori flight testing one has a sequence j of past tests and

$$\xi(dx \rangle \langle dx) = \frac{1}{j_{max}} \sum_{j=1}^{j_{max}} dx \rangle \langle dx = \sum_{3 \times 3} \tag{21}$$

or

$$\Sigma_{xx} = J \Sigma_{qq} J^T \tag{22}$$

where ε is expectation operator.

Sample variance estimates instead of past or population sequences yield similar equation forms. The Jacobians are evaluated about a nominal, or known estimated point. Equation (22) expresses the well known variance connections via congruent transformations.

In order to further simplify the typing and the results observe that the unit magnitude "sight-line" vector is

$$\frac{\langle x}{\langle x x \rangle^{1/2}} = \langle s = (s_1, s_2, s_3) \tag{23}$$

where

$$\langle s s \rangle = 1 \tag{24}$$

8

or

$$s_2 = \sqrt{1 - s_1^2 - s_3^2} \qquad (25)$$

and

$$s_3 = \sqrt{1 - s_1^2 - s_2^2} \qquad (26)$$

The rectangular coordinates called "direction cosines" of the sight-line vector $\lessdot$ s has many equivalent forms, some of which are

$$\begin{pmatrix} s^1 \\ s^2 \\ s^3 \end{pmatrix} = \begin{pmatrix} CECA \\ CEsA \\ sE \end{pmatrix} = \begin{pmatrix} c\theta_1 \\ s_2 \\ c\theta_3 \end{pmatrix} = \begin{pmatrix} c\theta_1 \\ c\theta_2 \\ s^3 \end{pmatrix} \qquad (27)$$

where the square root terms of Equation (25) and (26) are used.

The following matrices are developed

$$
\begin{array}{cccc}
J_{12} & & & \\
& J_{23} & & \\
J_{13} & & J_{34} & \\
& J_{24} & & J_{45} \\
J_{14} & & J_{35} & \\
& J_{25} & & \\
J_{15} & & &
\end{array}
$$

Consider the singularities (J_{14}) in transforming from sine-space to rectangular coordinates. The singularities occur when

$$s^2\phi_1 + s^2\phi_3 = 1$$

or when

9

$$\phi_1 + \phi_3 = 90°$$

thus we see a continum of singularities as opposed to a single point ($E=90°$) for conventional radar or spherical polar coordinates.

One trajectory region to watch out for ill-conditioned matrices is when $\phi_1=\phi_2=45°$.

The analyst can study the singularity regions as applicable to his particular problem. The position vectors between station sites are not included in this report but will be developed in a later report.

CONCLUSIONS

The geometrically related computational problems presented here are only part of the relations occurring in range data processing. To be sure, the position vectors between sites and the relations between generalized coordinates measured at the two different sites must be considered. Error propagation studies for a total instrumentation system will be developed and draw-upon some of the relations herein.

TABLE 1

[1,2]

$$X = rcEcA$$

$$Y = rcEsA$$

$$Z = rsE$$

$$J_{12} = \begin{bmatrix} cEcA & -rcEsA & -rSEcA \\ cESA & rCEcA & -rSESA \\ -SE & 0 & rCE \end{bmatrix}$$

$$J_{12}^{T}J_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2c^2E & 0 \\ 0 & 0 & r^2 \end{bmatrix}$$

$$J_{12}^{-1} = \begin{bmatrix} CECA & CESA & SE \\ -SA/RCE & CA/rCE & 0 \\ -SECA/r & -SESA/r & CE/r \end{bmatrix}$$

$$\det J_{12} = r^2CE$$

TABLE II

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} rc\theta_1 \\ rc\theta_2 \\ rs_3 \end{pmatrix} \quad , \quad s_3 = (1 - c^2\theta_1 - c^2\theta_2)^{1/2}$$

$$r = (X^2 + Y^2 + Z^2)^{1/2}$$

$$J_{13} = \begin{pmatrix} c\theta_1 & r & 0 \\ c\theta_2 & 0 & r \\ s_3 & \dfrac{-rc\theta_1}{s_3} & \dfrac{-rc\theta_2}{s_3} \end{pmatrix}$$

$$J_{13} = \begin{pmatrix} X/r & r & 0 \\ Y/r & 0 & r \\ Z/r & \dfrac{-Xr}{Z} & \dfrac{-Yr}{Z} \end{pmatrix}$$

$$J_{13}J_{13}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2\left(1 + \dfrac{X^2}{Z^2}\right) & \dfrac{XY\, r^2}{Z^2} \\ 0 & \dfrac{XY\, r^2}{Z^2} & r^2\left(1 + \dfrac{Y^2}{Z^2}\right) \end{pmatrix}$$

$$J_{13}^{-1} = \begin{pmatrix} \dfrac{r^2 X}{Z} & \dfrac{r^2 Y}{Z} & r^2 \\ Z + \dfrac{Y^2}{Z} & \dfrac{-XY}{Z} & -X \\ \dfrac{-XY}{Z} & Z + \dfrac{X^2}{Z} & -Y \end{pmatrix} \quad \dfrac{1}{\det J_{13}}$$

$$\det J_{13} = \dfrac{r^2}{Z}$$

13

TABLE III

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}' = \begin{pmatrix} rs\dot{\phi}_1 \\ rs_2 \\ rs\phi_3 \end{pmatrix} \quad , \quad s_2 = (1-s^2\phi_1 - s^2\phi_3)^{1/2}$$

$$J_{14} = \begin{pmatrix} s\phi_1 & r & 0 \\ s_2 & \dfrac{-rs\phi_1}{s_2} & \dfrac{-rs\phi_3}{s_2} \\ s\phi_3 & 0 & r \end{pmatrix}$$

$$= \begin{pmatrix} X/r & r & 0 \\ Y/r & -Xr/Y & -Zr/Y \\ Z/r & 0 & r \end{pmatrix}$$

$$J_{14}^{T}J_{14} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2(1+X^2/Y^2) & XZr^2/Y^2 \\ 0 & XZY^2r^2 & r^2(1+Z^2/Y^2) \end{pmatrix}$$

$$J_{14}^{-1} = \begin{pmatrix} -r^2X/Y & -r^2 & -r^2Z/Y \\ -Y-Z^2/Y & X & XY/Y \\ \dfrac{-XY}{Y} & Z & \dfrac{-X^2}{Y} -Y \end{pmatrix} \; \dfrac{1}{\det J_{14}}$$

$$\det J_{14} = -\dfrac{r^2}{Y}$$

14

TABLE IV

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} rs\phi_1 \\ rs_2 \\ rs\phi_3 \end{pmatrix}, \quad s_2 = (1-s^2\phi_1 - s^2\phi_2)^{1/2}$$

$$\phi_i = 90° - \theta_i$$

$$J_{15} = \begin{pmatrix} s\phi_1 & rc\phi_1 & 0 \\ s_2 & \dfrac{-rs\phi_1 c\phi_1}{s^2} & \dfrac{-rs\phi_3 c\phi_3}{s^2} \\ s\phi_3 & 0 & rc\phi_3 \end{pmatrix}$$

$$J_{15} = \begin{pmatrix} \dfrac{X}{r} & r\left(1-\dfrac{X^2}{r^2}\right)^{1/2} & 0 \\ \dfrac{Y}{r} & -\dfrac{rX}{Y}\left(1-\dfrac{X^2}{r^2}\right)^{1/2} & -\dfrac{rZ}{Y}\left(1-\dfrac{Z^2}{r^2}\right)^{1/2} \\ \dfrac{Z}{r} & 0 & r\left(1-\dfrac{Z^2}{r^2}\right)^{1/2} \end{pmatrix}$$

$$J_{15}^T J_{15} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r\left(1-\dfrac{X^2}{r^2}\right)\left(1+\dfrac{X^2}{Y^2}\right) & \dfrac{r^2 XZ}{Y}\left(1-\dfrac{X^2}{r^2}\right)^{1/2}\left(1-\dfrac{Z^2}{r^2}\right)^{1/2} \\ 0 & \dfrac{r^2 XZ}{Y}\left(1-\dfrac{X^2}{r^2}\right)^{1/2}\left(1-\dfrac{Z^2}{r^2}\right)^{1/2} & r^2\left(1-\dfrac{Z^2}{r^2}\right)\left(1+\dfrac{Z^2}{Y^2}\right) \end{pmatrix}$$

$$Ad_j J_{15} = \begin{pmatrix} \dfrac{-r^2 X}{Y}\left(1-\dfrac{X^2}{r^2}\right)^{1/2}\left(1-\dfrac{Z^2}{r^2}\right)^{1/2} & -r^2\left(1-\dfrac{X^2}{r^2}\right)\left(1-\dfrac{Z^2}{r^2}\right)^{1/2} & \dfrac{-r^2 Z}{Y}\left(1-\dfrac{X^2}{r^2}\right)^{1/2}\left(1-\dfrac{Z^2}{r^2}\right) \\ -\left(Y^2+Z^2\right)\left(1-\dfrac{Z^2}{r^2}\right)^{1/2}\dfrac{1}{Y} & X\left(1-\dfrac{Z^2}{r^2}\right)^{1/2} & \dfrac{XZ}{Y}\left(1-\dfrac{Z^2}{r^2}\right)^{1/2} \\ \dfrac{ZX}{Y}\left(1-\dfrac{X^2}{r^2}\right)^{1/2} & -Z\left(1-\dfrac{X^2}{r^2}\right)^{1/2} & \dfrac{-(X^2+Y^2)\left(1-\dfrac{X^2}{r^2}\right)^{1/2}}{Y} \end{pmatrix}$$

$$\det J = \dfrac{-r}{Y}(Y^2+Z^2)^{1/2}(X^2+Y^2)^{1/2}$$

15

TABLE V

$$r = r'$$

$$A = TAN^{-1} \frac{c\theta_2}{c\theta_1}$$

$$E = S^{-1}\sqrt{1 - c^2\theta_1 - c^2\theta_2}$$

$$J_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\dfrac{c\theta_2}{c^2\theta_1 + c^2\theta_2} & \dfrac{c\theta_1}{c^2\theta_1 + c^2\theta_2} \\ 0 & \dfrac{-2c\theta_1}{(c^2\theta_1+c^2\theta_2)^{1/2}} & \dfrac{-2c0_2}{(c^2\theta_1+c^2\theta_2)^{1/2}} \end{bmatrix}$$

$$\text{if } c\theta_1 \neq 0 \,\&\, c\theta_2 \neq 0$$

$$\det J_{23} = 2(c^2\theta_1 + c^2\theta_2)^{1/2}$$

$$J_{23}^{T}J_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{4c^2\theta_1(c^2\theta_1+c^2\theta_2)+ c^2\theta_2}{(c^2\theta_1+c^2\theta_2)^2} & \dfrac{c\theta_1 c\theta_2\left[(c^2\theta_1+c^2\theta_2)4-1\right]}{(c^2\theta_1+c^2\theta_2)^2} \\ 0 & \dfrac{c\theta_1 c\theta_2[4(c^2\theta_1+c^2\theta_2)-1]}{(c^2\theta_1+c^2\theta_2)^2} & \dfrac{c^2\theta_1}{(c^2\theta_1+c^2\theta_2)^2} + \dfrac{4c^2\theta_2}{c^2\theta_1+c^2\theta_2} \end{bmatrix}$$

$$\text{ADJ } J_{23} = \begin{bmatrix} 2(c^2\theta_1+c^2\theta_2)^{1/2} & 0 & 0 \\ 0 & -\dfrac{2c\theta_2}{(c^2\theta_1+c^2\theta_2)^{1/2}} & \dfrac{c\theta_1}{c^2\theta_1+c^2\theta_2} \\ 0 & -\dfrac{2c\theta_1}{(c^2\theta_1+ {}^2\theta_2)^{1/2}} & \dfrac{c\theta_2}{c^2\theta_1+c^2\theta_2} \end{bmatrix}$$

16

TABLE VI

$R = R$

$A = TAN^{-1} \dfrac{(1-s^2\phi_1 - s^2\phi_3)^{1/2}}{s\phi_1}$

$E = S^{-1}(s\phi_3)$

$$J_{24} = \begin{bmatrix} 1 & 0 & 0 \\[2ex] 0 & -(1-s^2\phi_1 - s^2\phi_3)^{-1/2} & \dfrac{s\phi_1 s\phi_3 (1-s^2\phi_1 - s^2\phi_3)^{-1/2}}{s^2\phi_3 - 1} \\[3ex] 0 & 0 & \dfrac{1}{(1-s^2\phi_3)^{1/2}} \end{bmatrix}$$

if $s\phi_3 \neq \pm 1$

$$J_{24}^T J_{24} = \begin{pmatrix} 1 & 0 & 0 \\[2ex] 0 & (1-s^2\phi_1 - s^2\phi_3)^{-1} & \dfrac{s\phi_1 s\phi_3 (1-s^2\phi_1 - s^2\phi_3)}{1-s^2\phi_3} \\[3ex] 0 & \dfrac{s\phi_1 s\phi_3 (1-s^2\phi_1 - s^2\phi_3)}{1-s^2\phi_3} & \dfrac{1}{(1-s^2\phi_3)} \dfrac{[1+s^2\phi_1 + s^2\phi_3]}{(1-s^2\phi_1 - s^2\phi_3)} \end{pmatrix}$$

$$ADJ\ J_{24} = \begin{bmatrix} 1 & 0 & 0 \\[2ex] 0 & (1-s^2\phi_3)^{-1/2} & \dfrac{s\phi_1 s\phi_3 (1-s^2\phi_1 - s^2\phi_3)^{-1/2}}{1-s^2\phi_3} \\[3ex] 0 & 0 & -(1-s^2\phi_1 - s^2\phi_3)^{-1/2} \end{bmatrix}$$

17

TABLE VII

$$r = r'$$

$$A = TAN^{-1} \frac{(1-s^2\phi_1-s^2\phi_3)^{1/2}}{s\phi_1}$$

$$E = \phi_3$$

$$J_{25} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j_{22} & 2 \\ 0 & 0 & j_{33} \end{pmatrix}$$

$$j_{22} = \frac{\left[\dfrac{c\phi_1}{s^2\phi_1(1-s^2\phi_1-s^2\phi_3)^{1/2}} + (1-s^2\phi_1-s^2\phi_3)^{1/2} \right]}{s^2\phi_3 - 1}$$

$$j_{33} = \frac{s\phi_1 s\phi_3 c\phi_3}{(s^2\phi_3-1)(1-s^2\phi_1-s^2\phi_3)^{1/2}}$$

$$J_{25}^T J_{25} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & j_{22}^2 & 2j_{22} \\ 0 & 2j_{22} & 1+4 \end{bmatrix}$$

$$ADj J_{25} = \begin{bmatrix} j_{22} & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & j_{22} \end{bmatrix}$$

$$\det J_{25} = j_{22}j_{33}$$

18

TABLE VIII

$R = R'$

$c\theta_1 = s\phi_1$

$c\theta_2 = 1-s^2\phi_1-s^2\phi_2$

$$J_{34} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1-s^2\phi_1-s^2\phi_2)^{-1/2}s\phi_1 & -(1-s^2\phi_1-s^2\phi_2)^{-1/2}s\psi_2 \end{bmatrix}$$

$$J_{34}^T J_{34} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+s^2\phi_1/(1-s^2\phi_1-s^2\phi_2) & \dfrac{s\phi_1 s\phi_2}{(1-s^2\phi_1-s^2\phi_2)} \\ 0 & \dfrac{s\phi_1 s\phi_2}{(1-s^2\phi_1-s^2\phi_2)} & \dfrac{s^2\phi_2}{(1-s^2\phi_1-s^2\phi_2)} \end{bmatrix}$$

$$J_{34}^{-1} = -\frac{(1-s^2\phi_1-s^2\phi_2)^{1/2}}{s\phi_2} \begin{bmatrix} \dfrac{s\phi_2}{(1-s^2\phi_1-s^2\phi_2)^{12}} & 0 & 0 \\ 0 & \dfrac{s\phi_2}{(1-s^2\phi_1-s^2\phi_2)^{1/2}} & 0 \\ 0 & \dfrac{s\phi_1}{(1-s^2\phi_1-s^2\phi_2)^{1/2}} & 1 \end{bmatrix}$$

19

TABLE IX

$$J_{35} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & 0 \\ 0 & -\dfrac{s\phi_1 c\phi_1}{(1-s^2\phi_1-s^2\phi_2)^{1/2}} & \dfrac{s\phi_2 c\phi_2}{(1-s^2\phi_1-s^2\phi_2)^{1/2}} \end{bmatrix}$$

$$J_{35}^T J_{35} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^2\phi_1(1+s^2\phi_1/(1-s^2\phi_1-s^2\phi_2)) & \dfrac{s\phi_1 c\phi_1 s\phi_2 c\phi_2}{1-s^2\phi_1-s^2\phi_2} \\ 0 & \dfrac{s\phi_1 c\phi_1 s\phi_2 c\phi_2}{1-s^2\phi_1-s^2\phi_2} & \dfrac{s^2\phi_2 c^2\phi_2}{1-s^2\phi_1-s^2\phi_2} \end{bmatrix}$$

$$J_{35}^{-1} = -\dfrac{(1-s^2\phi_1-s^2\phi_2)}{c\phi_1 s\phi_2 c\phi_2}$$

$$\begin{bmatrix} -\dfrac{c\phi_1 s\phi_2 c\phi_2}{(1-s^2\phi_1-s^2\phi_2)^{1/2}} & 0 & 0 \\ 0 & \dfrac{-s\phi_2 c\phi_2}{(1-s^2\phi_1-s^2\phi_2)^{1/2}} & 0 \\ 0 & \dfrac{s\phi_1 c\phi_1}{(1-s^2\phi_1-s^2\phi_2)^{1/2}} & c\phi_1 \end{bmatrix}$$

TABLE X

Let $(q_1, q_2, q_3) = (r, s\phi_1, s\phi_3)$

$(p_1, p_2, p_3) = (r, \phi_1, \phi_3)$

$q_1 = p_1$

$q_2 = \sin p_2$

$q_3 = \sin p_3$

$$J_{45} = q \Big\rangle\!\!\Big\langle \frac{\partial}{\partial p} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & 0 \\ 0 & 0 & c\phi_3 \end{pmatrix}$$

$$J_{45}^T J_{45} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^2\phi_1 & 0 \\ 0 & 0 & c^2\phi_3 \end{pmatrix}$$

$$J_{45}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c\phi_1} & 0 \\ 0 & 0 & \frac{1}{c\phi_3} \end{pmatrix}$$

$\det J_{45} = \cos\phi_1\cos\phi_3$

Technical Report No. 35, "Transformations for Range Instrument Coordinates and Their Tracking Space Singularity Regions," UNCLASSIFIED, National Range Operations Directorate, White'Sands Missile Range, New Mexico 88002, June 1972.

White Sands Missile Range, NM (STEWS)	NUMBER of COPIES
National Range Operations Directorate	
Analysis & Computation Division	1
Mathematical Services Branch	50
(ATTN: Mr. J. S. Pappas)	
Technical Library	3
Defense Documentation Center	20
Cameron Station	
Alexandria, Virginia 22314	