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The Design of a Multi-item, Multi-ship, Multi-echelon Repair System

by

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NONTECHNICAL SUMMARY

The design of a multi-item, multi-ship, multi-echelon repair system is considered. The model assumes an overall budget constraint is imposed, the budget being required to fund the number of spares of each type of item on board each ship, and also to fund the repair expenditures over all the repair echelons. The inputs into the model are the marginal costs of reducing the repair times at each echelon for each item, the costs of spares, the failure rates for each item on each ship, and the likelihoods that the repair of a failure of a given item on a given ship will necessitate work at a given echelon (e.g., local, tender, depot). The model then determines the optimal provisioning in terms of the number of spares for each item on board each ship, and the average repair times at each echelon, by item, so as to minimize, within the budget constraint, an arbitrary weighting of the average deficiencies, i.e. backorders, over all items and ships. It is assumed that the failures follow the Poisson distribution and that an infinite number of repair channels are available at each echelon, where the average repair time at the echelon depends upon the individual item. The model does not require any assumptions about the form of the repair distributions. It should be pointed out that the model allows one of the repair echelons to actually be a resupply option on a one-for-one basis, so that it could be used for resupply decisions as well.

The computation method is tractable for large number of items and ships due to a decomposition which is used and also due to the Lagrangian approach utilized. A numerical example is included containing two items, two ships and three repair echelons. In the particular example considered, there are twelve decision variables to be determined simultaneously, namely the number of spares of each type of item to provision on board each ship, and the average repair time for each item at each echelon.

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1. INTRODUCTION

Consider a multi-item, multi-ship, multi-echelon repair system, which, for a given item, used at a given ship with several repair echelons, is diagrammed in Figure 1 below.

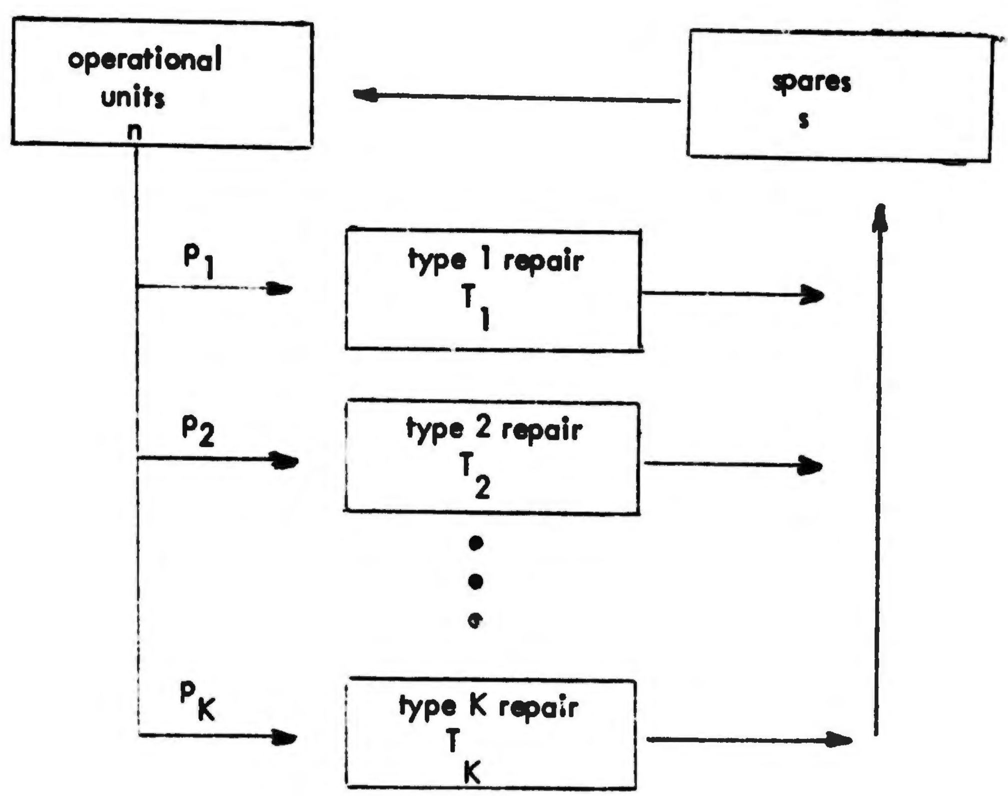

Figure 1. Diagram of a Multi-echelon Repair System for One Item on One Ship.

When fully operative, the ship employs n units of the item, designated operational units. In addition to these n, s spares are retained in a pool, to be used to replace operational units, when they fail.

When a unit fails, one of the (operative) spares in the pool is used to replace the inoperative unit. If the pool of spares is exhausted, then a shortage occurs, and the number of operational units decreases below n. If a shortage exists, the number of shortages is reduced (the number of operational units is increased) when an operative spare is delivered to the pool, at which point it is installed in the system.

Given that an operational unit has failed on board the ship, it must undergo one of the K types of repair, ρ_k being the conditional probability that a failure on a given ship will necessitate the k^{th} type (repair). The expected k^{th} echelon repair time is denoted T_k, where repair time is the time elapsing from the point of breakdown of the unit to the point at which the unit is returned to the pool of spares.

The problem is to determine the number of spares and the expected k^{th} echelon repair times for each k, for each item on board each ship. The object is, roughly, to minimize weighted expected shortages (over all items and ships) subject to a budget constraint covering total expenditures on spares and repair times.

The next section describes the model for a given item at a given location. Section 3 then explicitly defines the optimization model. Section 4 gives an interpretation for the use of the model. Section 5 notes some related literature.

The next five sections are devoted to the model. Section 11 provides a numerical example and a discussion.

2. DESCRIPTION OF THE MODEL FOR A GIVEN ITEM AND SHIP

We center attention on one item at one location. Let $N^F(t)$ denote the number of failures (of operational units) which have occurred during $[0,t]$ for $t \geq 0$. We call $\{N^F(t);\ t \geq 0\}$ the failure process and assume it is a Poisson process (e.g., [8]) with rate $\lambda\ (\lambda > 0)$.

The design parameters under control are s, the number of spares, and the sequence $\{T_k;\ k = 1, 2, \ldots, K\}$ of expected type k repair times.

For precision, assume that, for each repair type k, there is a single design parameter under control, denoted by T_k, and that $0 < T_k < \infty$.

Let $X_{kn}(T_k)$ denote the repair time for the n^{th} unit to undergo type k repair, given T_k, for each $k(= 1, 2, \ldots, K)$ and $n (= 1, 2, \ldots)$. We assume

$$\{X_{11}(T_1), X_{12}(T_1), \ldots, X_{21}(T_2), X_{22}(T_2), \ldots, X_{K1}(T_K), X_{K2}(T_K), \ldots\}$$

are mutually independent random variables for all specifications of $\{T_k\}$, and that, for each k,

$$\{X_{k1}(T_k),\ X_{k2}(T_k),\ \dots\}$$

are identically distributed random variables, with distribution function F_k; i.e.,

$$F_k(x; T_k) \equiv P\{X_{kn}(T_k) \leq x\} \qquad \text{for each } k \text{ and } n .$$

We also assume that, for each k, the first moment of X exists, is unique and equals T_k; i.e.,

$$\int_{x=0}^{\infty} x \, dF_k(x; T_k) = T_k \qquad \text{for } 0 < T_k < \infty .$$

Thus, the expected type k repair times are under control. We suppress notational dependence on $\{T_k\}$ in the remainder of this section, as its specification remains fixed.

Let $Y = Y(\{T_k\})$ denote the repair time for the n^{th} failure (the n^{th} operational unit to fail and thereby require repair), for each n, (given the specification $\{T_k\}$). We assume $Y_1, Y_2, \ldots$ are independent, identically distributed random variables with distribution function F given by

$$F(x) \equiv \sum_{k=1}^{K} p_k \, F_k(x, T_k) \; .$$

That is, given a failure, the type of repair required is k with probability p_k , for each k.

Thus, (given T_k) the expected repair time for the n^{th} failure is independent of n and equals

$$T \equiv \sum_{k=1}^{K} p_k T_k \; .$$

Let $N^R(t)$ denote the total number of units in repair (of any type) at time $t \geq 0$. Let $N^S(t)$ denote the number of shortages at time $t \geq 0$. Then

$$N^S(t) = \begin{cases} 0 & \text{if } N^R(t) \leq s \\ N^R(t) - s & \text{otherwise.} \end{cases}$$

In short, $N^S(t) = [N^R(t) - s]^+$.

Let $S(t)$ $(= S(t; s, T_k \cdot))$ $\equiv E\, N^S(t)$, the expected number of shortages at time $t \geq 0$. If it exists, let

$$S(s, T_k) \equiv \lim_{t\to\infty} S(t) .$$

Let $c^S(s)$ denote the cost of retaining s spares in the system and $c_k^R(T_k)$ the cost of setting the expected type k repair time equal to T_k, for each k.

3. THE OPTIMIZATION PROBLEM OVER ALL ITEMS AND SHIPS

In this section, in contrast to the last, we indicate the notational dependence on the item i and location j under consideration. For example $c_{ij}^S(s_{ij})$ denotes the cost of retaining s_{ij} spares of item i on ship j, for each i and j. The optimization problem can then be stated as: find s_{ij} and T_{ijk} for each $i, j,$ and k to

(1.1) $$\min \sum_{ij} S_{ij} (s_{ij}, T_{ij1}, \ldots, T_{ijK})$$

(1.2) s.t. $$\sum_{ij} c_{ij}^S (S_{ij}) + \sum_{ijk} c_{ijk}^R (T_{ijk}) \leq b ,$$

where w_{ij} is a weight, assigned in relation to the importance of a shortage of item i at location j, and b is a real number denoting the budget, which applies to the total expenditures on spares and expected repair times. For notational convenience, we assume hereafter that $w_{ij} = 1$ for all i and j. The adjustments are easily made in the more general case.

Our approach to solving (1) is a Lagrangian one. That is, (1) is of the form: find x in R^n to

$$(P_\beta) \qquad \min f(x) \quad \text{s.t.} \quad g(x) \leq \beta$$

where R, and (1) corresponds to (P_b). We say x^* is undominated if x^* is optimal for (P_β) for some β. We form the Lagrangian $L(x,u) \equiv f(x) - u\left[g(x) - \beta\right]$ and employ the following:

Lemma 1. (Everett [2]). If x^* minimizes $L(\cdot,u)$, (where $u \geq 0$), then x^* is undominated. In fact, x^* is optimal for (P_β) for the case $\beta = g(x^*)$.

Thus, in particular, if x^* minimizes $L(\cdot,u)$ for some u and $g(x^*) = b$, then x^* is optimal for (1). The problem is to find an appropriate multiplier, if one exists.

Our approach is that of Fox and Landi [3]. A sequence $u_1 < u_2 < \ldots < u_N$ of positive multipliers is specified. For each i, $L(x,u_i)$ is minimized, yielding $x(u_i)$, which is optimal for (P_β), where $\beta = g(x(u_i))$. Important computational advantages accrue from this approach in this case, as discussed in [3]. The sequence of multipliers is selected so that

$$g(x(u_N)) \le b \le g(x(u_1)) .$$

Since $f(x(u_1)) \le f(x(u_2)) \le \ldots \le f(x(u_N))$ and $g(x(u_1)) \ge g(x(u_2)) \ge \ldots \ge g(x(u_N))$ we pick the smallest index i such that $g(x(u_i)) \le b$. If $g(x(u_i)) < b$, then $x(u_i)$ may not be optimal for (1). In fact, as is well known [2], there may be no multiplier u such that $x(u)$ is optimal for (1). Thus, we do not guarantee that we can find an optimal solution to (1) with our approach. However, the sequence $x(u_1), x(u_2), \ldots, x(u_N)$ provides a sequence of undominated solutions; i.e., optimal solutions to problems of the form (1) with different budget levels. In practice, such information is often more useful than just the single optimal solution to (1).

Note that the Lagrangian for (1) is separable by item and location. That is, let

$$L_{ij}(s_{ij}, T_{ij1}, \ldots, T_{ijK}, u) \equiv S_{ij}(s_{ij}, T_{ij1}, \ldots, T_{ijK}) + uc^S_{ij}(s_{ij}) + u\sum_k c^R_{ijk}(T_{ijk}).$$

Then we have

$$L(x,u) = \sum_{ij} L_{ij}(s_{ij}, T_{ij1}, \ldots, T_{ijK}, u) - ub.$$

Thus, given u, if we minimize each term separately, we minimize the sum.

This justifies concentrating on a single item at a single location in the remaining sections. That is, we shall consider the problem: for fixed u, find $s, T_1, \ldots, T_K$ to

$$\text{(2)} \qquad \min \; L(s, T_1, \ldots, T_K, u)$$

where $L(s, T_1, \ldots, T_K, u) \equiv S(s, T_1, \ldots, T_K) + uc^S(s) + u\sum_k c_k^R(T_k)$.

4. INTERPRETATION OF THE MODEL

4.1 Repair Times As Response Times

Repair time, for a given repair type, consists, in practice, of the sum of transportation time to the repair facility, time until physical repair begins, time to complete physical repair, and time to transport back to the pool of spares. Therefore, some readers may want to substitute "response time" for "repair time".

4.2 Repair Times Are i.i.d. Random Variables

One interpretation for this is that each type of repair consists of an infinite number of repair facilities, modelled as an infinite server queue. Here $\{X_{k1}(T_k), X_{k2}(T_k), \ldots\}$ are i.i.d. random variables, which would not be the case if there were a finite number of repair facilities.

Another, approximate, interpretation is that there are a large number of repair facilities for each repair type, and that the behavior of each such system is approximated by one with an infinite number of repair facilities.

A final interpretation here comes by contending that each type of repair applies not to just one item, with separate facilities for every different item, but to many items. With this in mind, and with only a finite number of facilities for each type of repair, one gai. s little by modelling a separate finite server queue for each item, because the major competition for servers comes not from other units of the same item requiring the same type of repair, but from other units of different items requiring the same type of repair. An exact way of modeling this is to keep track simultaneously of the number of units of each item which are awaiting and/or undergoing type k repair, for each k. Many practitioners would argue that the cost of using such a model would outweigh the benefits to be derived from its use. That is, some would argue that, with current technology, multi-item models intended for implementation should be restricted to being separable by item. One such approximate model would come from first analyzing a finite server model, over all items, and determining the expected time (including waiting plus service time) in the system for each item and then using these derived times in our "infinite server" model. The approximations come by assuming (i) the repair times are independent random variables, and (ii) the cost of achieving a given prescription of expected repair times is separable by item.

4.3 A Repair Type As A Supply Function

4.3.1 A Repair Type As A Joint Repair And Supply Function

We may interpret one of the repair types, say type k, as a central repair facility which maintains spares of each item to reduce the response time to requests for central repair. That is, when a unit fails and it requires central repair, then the central repair facility is notified that the unit is being sent for repair. Rather than waiting for the unit to arrive, to then repair it and send it back, the central facility immediately sends one of its spares, if it has one available. If none are available, the next unassigned one coming out of repair is sent.

In this case, expected "repair time" consists of expected time to notify the central facility, time to locate an unassigned unit there, and time to transport it back to the pool of spares. The time to locate an unassigned unit at the central facility is negligible if operative spares are available there, but may consist of as much as the time required to transport the item from the initial location to the central facility, repair the unit there, and transport it back to the pool of spares (at the initial location).

This line of thought is due to Sherbrooke [9], who used it in his approach to determining optimal numbers of spares for a two echelon supply system.

4.3.2 A Repair Type As Solely A Supply Function

Here, we interpret one of the repair types, say type k, as new procurement. That is, type k failure is irreparable, and repair consists of buying a new unit. In this case, expected type k repair time consists of the expected time to order, receive, and transport a new unit to the pool of spares.

4.3.3 One For One Replenishment Required

In any case where one of the repair types is interpreted as having a supply function, we are making the assumption that a (continuous review) $(s-1, s)$ policy is being followed, rather than, for example an (s,S) policy, where $s \leq S-2$. That is, a one for one replenishment policy must be followed, and batch orders are not allowed.

4.4 Failures Generated By A Poisson Process

We assumed that, for each item at each location, $\{N(t);\ t \geq 0\}$ is a stationary Poisson process with rate λ ($\lambda > 0$). That is, the time between failures follows a negative exponential distribution with mean $1/\lambda$, which is independent of time and the number of operational units. In particular, λ can depend on n, the maximum number of operational units, but not on the actual number of operational units, which may be less than n.

This assumption contrasts with a common assumption that the mean time between failures does depend on the number of operational units. This latter situation occurs when, for example, a unit can fail only if it is operating (installed as one of the operational units) and if, given that a unit is operating, the time until it fails is a random variable which (a) is independent of the time until failure for all other operational units and (b) follows a negative exponential distribution.

However, there are several ways of giving plausibility to our assumption. The first is to assume that whenever a shortage occurs, an emergency "loan" is effected, whereby an operative spare is immediately made available to eliminate the shortage, and when an operative spare is delivered to the pool, it is returned to the emergency borrower. With this assumption, the number of operational units always remains at n, and, therefore, it is irrelevant whether or not the mean time between failures depends on the number of operational units. With this interpretation, our approach is to minimize the expected number of units "on loan" subject to a budget constraint.

A second way is to assert that the mean time between failures does not depend on the number of operational units, but on, say, the "level of activity" of the system. For example, if the operational units are installed on aircraft, and the mean time between failures for this particular item correlates better with the number of flying hours of the fleet, rather than the number of operationally ready aircraft, then our approach gains plausibility. This interpretation is still approximate, because when there are no operational units, there should be no failures. In practice, this circumstance hopefully happens very rarely, in which case, this effect should be negligible.

4.5 The Cost Functions

4.5.1 Equivalent Stationary Cost

Recall that for a given item at a given location, $c^S(s)$ denotes the cost of retaining s spares in the system and $c_k^R(T_k)$ is the cost of the specification T_k as the expected type k repair time, for each k. One reasonable way of determining these costs, in practice, is to use what we call equivalent stationary cost, called equivalent average cost by some authors (e.g., [13]).

By equivalent stationary cost of an activity, we mean that amount, which spent evenly over the infinite horizon $[0,\infty)$ is equivalent to the actual expenditures incurred by that activity over $[0,\infty)$. For example, if expenditures occur periodically and the cost of capital is r per dollar per period, and c_n denotes the expenditure incurred at the beginning of period n for $n = 0, 1, \ldots,$ then the present value is $v \equiv \sum_{n=0}^{\infty} \alpha^n c_n$, where $\alpha = 1/(1+r)$, and the equivalent stationary cost is $v(1-\alpha) = rv/(1+r)$. Thus, if the stationary cost $v(1-\alpha)$ were expended at the beginning of period n, for each n, the present value of this cost stream would also be v.

The appropriate selection of equivalent stationary cost when there is no cost of capital is the Césaro mean $\lim_{n\to\infty} \sum_{i=0}^{n} c_n/n$ if it exists.

Consider $c^S(s)$. At first glance, one might suggest that it be zero. This would occur if we assume (i) the cost of retaining s spares consists of merely initial procurement of the spares and (ii) there is no cost of capital. Taking into account new procurement required by irreparable failures would only add a fixed amount, which would not depend on s, to $c^S(s)$. Thus, adding spares would appear to have no additional cost. However, in practical economic problems, there is always a nonzero cost of capital. In addition, obsolescence would require periodic procurement of s units of the replacement item.

5. SOME RELATED LITERATURE

The approach taken there is closely related to that presented by Sherbrooke [9]. The similarities are that his is a multi-item, multi-location model with several operational units; and it has several repair types (only two are mentioned, but the generalization is immediate), a general repair distribution, an infinite number of repair facilities, and control over the number of spares on location. The differences are that the failure process is compound Poisson, there is control over spares kept for one of the repair types (a higher echelon of supply), and there is no control over the repair times.

Related models, which focus on characterization rather than optimization, are summarized by Barlow and Proschan [1, Chap. 5]. In these models, failures occur only when a unit is operational. One model has one operational unit; general failure times; exponential repair times; and a finite number of repair facilities. A similar model, with exponential failure times, is discussed by Natarajan [7]. Another model, due to Takacs [11], has several operational units; exponential failure times; general repair times; and one repair facility.

Jacobson [5] applies marginal analysis and partial enumeration in his study of a multi-item, single location model with several operational units; a Poisson failure process; one repair type; identical exponential repair times for each item; a finite number of repair facilities; and control over the spares and the number of repair facilities.

6. EVALUATION OF EXPECTED SHORTAGES

As discussed in Section 3, we need only consider a fixed item and location in this section.

Lemma 2. $S(s, \{T_k\})$ exists and equals $B(s,T) \equiv \sum_{x=s+1}^{\infty} (x-s)\, p(x; \lambda T)$,

where

$$p(x; \lambda T) \equiv \frac{e^{-\lambda T}(\lambda T)^x}{x!} \qquad \text{for } x = 0, 1, \ldots,$$

$$T \equiv \sum_k p_k T_k \text{ , the expected repair time,}$$

and λ is the rate of the (Poisson) failure process.

Remark. This result is used by Sherbrooke [9]. $S(s, \{T_k\})$ depends only on s and t; hence, the new notation $B(s,T)$.

Proof: Because the failure process $\{N^F(t);\ t \geq 0\}$ is a stationary Poisson process and repair times are i.i.d. random variables, the distribution function of $N^R(t)$, the number of units in repair at time $t \geq 0$, is Poisson (e.g., [10, p. 18]). As $t \to \infty$, the probability that there are x units in repair at time t, approaches

$$p(x; \lambda T) \equiv e^{-\lambda T}(\lambda T)^x / x! \ .$$

The result follows immediately on recalling the definition of $S(s, \{T_k\})$. ||

7. APPROACH TO THE OPTIMIZATION PROBLEM

Our approach to solving (2) is to decompose it into two optimization problems.

The first is find $T_1, T_2, \ldots, T_K$ to

(3.1) $\min \sum_k c_k^R(T_k)$

(3.2) s.t. $\sum_k p_k T_k = T$ and

(3.3) $0 < T_k < \infty$ for each k.

Define $c^R(T)$ to be the optimal value of (3.1), for the value of T specified in (3.2). We want to be able to solve (3) for each T (and for each item and location). We do this in Section 8 for two special cases in which $c^R(T)$ can be written down in a simple closed form expression.

The second optimization problem is: find s and T to

(4) $\min L(s,T,u)$

where

$$L(s,T,u) \equiv B(s,T) + uc^S(s) + uc^R(T) .$$

The advantage of (4) over (2) is that here there are only two variables (for each item and location) rather than the $K+1$ which (2) has.

We now show that by solving (3) and (4), we solve (2).

<u>Lemma 3</u>. If for fixed u, s^* and T^* solve (4) and $T_1^*, \ldots, T_K^*$ solves (3) with T^* specified in (3.2), then $s^*, T_1^*, \ldots, T_K^*$ solves (2).

<u>Proof</u>: Pick arbitrary s and $T_1, \ldots, T_K$ $(s \in I \equiv \{0, 1, \ldots\}$ and $0 < T_k < \infty$ for each k). Let $T \equiv \sum_k p_k T_k$. Then

$$S(s^*, T_1^*, \ldots, T_K^*) = B(s^*, T^*) \qquad \left[\text{by Lemma 2 and since } T^* = \sum_k p_k T_k^*\right]$$

$$\leq B(s,T) \qquad [(s^*, T^*) \text{ solves } (4)]$$

$$= S(s, T_1, \ldots, T_K) \qquad [\text{by Lemma 2}]$$

which completes the result. ||

8. EVALUATION OF $c^R(T)$

We supply two examples of cases where $c^R(T)$ can be written down in closed form, in fact in either the form a/T or $ae^{-T/r}$.

8.1 Case One. $c_k(T_k) = c_k/T_k$ for each k

Suppose $c_k(T_k) = c_k/T_k$ for each k. We can write $c^R(T)$ in closed form in this case as follows. We form the Lagrangian (for fixed T)

$$L(T_1, T_2, \ldots, T_K, u) \equiv \sum_k c_k/T_k + u\left[\sum_k p_k T_k - T\right]$$

and calculate

(5) $$\frac{\partial}{\partial T_k} L(T_1, \ldots, T_K, u) = -\frac{c_k}{T_k^2} + up_k$$

and $$\frac{\partial^2}{\partial T^2} L(T_1, \ldots, T_K, u) = \frac{2c_k}{T_k^3} > 0$$

so, since the cross partials vanish, L is strictly convex, for fixed u. Therefore, for fixed u, we can minimize L by setting (5) equal to zero for each k, yielding

(6) $$T_k^*(u) = \sqrt{\frac{c_k}{up_k}} \quad .$$

Then, for fixed u, we have

$$\sum_k p_k T_k^*(u) = u^{-\frac{1}{2}} \sum_k (p_k c_k)^{\frac{1}{2}} \quad ,$$

which, when set equal to T, yields

$$u^* = \left[\sum_k (p_k c_k)^{\frac{1}{2}} / T \right]^2 \quad ,$$

which, when substituted in (6), yields (for fixed T)

$$T_k^* = \frac{T\sqrt{c_k/p_k}}{\sum_j (p_j c_j)^{\frac{1}{2}}} ,$$

so that,

$$\sum_k c_k/T_k^* = \left[\sum_k (p_k c_k)^{\frac{1}{2}}\right]^2/T .$$

Thus, in this case

$$c^R(T) = a/T ,$$

where $$a \equiv \left[\sum_k (p_k c_k)^{\frac{1}{2}}\right]^2 .$$

That is, if production functions for the repair times for each type of repair are of the form a/T, then the production function for the expected repair time is of the same form.

8.2 Case Two. $c_k(T_k) = c_k e^{-T_k/r_k}$ for each k

Following the approach of the last section, we can write $c^R(T)$ in closed form in this case, too.

Here

$$\frac{\partial}{\partial T_k} L(T_1, \ldots, T_k, u) = -(c_k/r_k)\, e^{-T_k/r_k} + u p_k$$

which yields

$$T_k^*(u) = r_k \ln\left(\frac{c_k}{u p_k r_k}\right)$$

so that

(7)
$$\sum_k c_k(T_k^*(u)) = u \sum_k (p_k r_k) \quad .$$

Picking u so that

$$\sum_k p_k T_k^*(u) = T$$

yields

$$u^* = \exp\left\{\left[\sum_k p_k r_k \ln\left(\frac{c_k}{p_k r_k}\right)\right] - T\right\} \Big/ \left\{\sum_k p_k r_k\right\} ,$$

which, when substituted into (7), yields

$$c^T(T) = a\, e^{-T/r} \quad ,$$

where

$$r \equiv \sum_k p_k r_k$$

and

$$a \equiv r \exp \left\{ \left[\sum_k p_k r_k \ln \left(\frac{c_k}{p_k r_k} \right) \right] / r \right\} \quad .$$

Thus, the form of the production function is again conserved.

9. MINIMIZING $L(s,T,u)$ FOR FIXED u

9.1 Preliminaries

Lemma: $B(s,T)$ is

1° for fixed T, strictly decreasing and discretely convex in s, and

2° for fixed s, continuously differentiable, strictly increasing, and strictly convex in T.

Proof: Recall that $p(x;\lambda T) \equiv e^{-\lambda T}(\lambda T)^x / x!$ for $x = 0,1,\ldots$.

Let, for $s = 1,2,\ldots,$

(8) $$\Delta_1 B(s,T) \equiv B(s,T) - B(s-1,T) \quad .$$

Then, by direct computation

(9) $$\Delta_1 B(s,T) = -\sum_{x=s}^{\infty} p(x;\lambda T) < 0, \quad \text{and}$$

$$\Delta_{11} B(s,T) \equiv \Delta_1(\Delta_1 B(s,T)) = p(s-1;\lambda T) > 0 ,$$

from which 1° follows.

Again, by direct computation, we see that

(10) $$\frac{\partial}{\partial T} p(x;\lambda T) = \lambda\left[p(x-1;\lambda T) - p(x;\lambda T)\right]$$

so that

(11) $$\frac{\partial B(s,T)}{\partial T} = \lambda \sum_{x=s}^{\infty} p(x;\lambda T) > 0$$

and

$$\frac{\partial^2}{\partial T^2} B(s,t) = \lambda^2 p(s-1;\lambda T) > 0 ,$$

from which 2° follows. ||

9.2 Coordinate Direction Minimization

Assume in the sequel that

1* $c^S(s)$ is (discretely) convex in s, and

2* $c^R(T)$ is continuously differentiable and convex in T.

Also assume, unless indicated otherwise, that $s \in \{0, 1, \ldots\}$ and $T, u \in (0, \infty)$.

Lemma 5.

1° $L(s,T,u)$ is, for fixed T and u, (discretely) convex in s,

2° $L(s,T,u)$ is, for fixed s and u, continuously (partially) differentiable and strictl, convex in T,

3° $\frac{\partial}{\partial T} L(s,T,u)$ is, for fixed T and u, strictly decreasing in s, and

4° $\Delta_1 L(s,T,u)$ is, for fixed s and u, strictly decreasing in T.

Proof: 1° and 2° follow directly from 1*,2*, and Lemma 4, since,

(12) $$L(s,T,u) \equiv B(s,T) + uc^S(s) + uc^R(T) \quad ,$$

and $u \geq 0$.

3° and 4°: By (9),(10), (11), (12), and direct calculation,

$$\Delta_1 \frac{\partial}{\partial T} L(s,T,u) = \frac{\partial}{\partial T} \Delta_1 L(s,T,u) = -\lambda p(s-1; \lambda T) \quad \text{if} \quad s \geq 1 ,$$

which is strictly negative. ||

Let $s^*(T,u)$ denote the set of points in $\{0,1,...\}$ which minimize $L(\cdot,T,u)$ for fixed T and u, and $T^*(s,u)$ the set of points in $(0,\infty)$ which minimize $L(s,\cdot,u)$ for fixed s and u. For example, if $s^\circ \in s^*(T,u)$, then

$$L(s^o, T, u) \le L(s, T, u) \qquad \text{for all} \quad s.$$

Assume hereafter, for convenience, that $s^*(T,u)$ and $T^*(s,u)$ are nonempty. This poses no hardship in practice.

<u>Lemma 6.</u>

1° $s^o \in s^*(T,u)$ and $s^o \ge 1$ if and only if

2°

(13) $$\Delta_1 L(s^o, T, u) \le 0$$

and

(14) $$\Delta_1 L(s^o + 1, T, u) \ge 0 \; ;$$

3° $0 \in s^*(T,u)$

if and only if

4° (14) holds for $s^o = 0$; and

5° $T^o \in T^*(s,u)$

if and only if

6°

(15) $$\frac{\partial}{\partial T} L(s, T^o, u) = 0 \; .$$

Remark: Thus, by 2° of Lemma 5, $T^*(s,u)$ consists of one element, for each s and u, which, for convenience, we also denote by $T^*(s,u)$.

Proof: 1° iff 2°: (13) and (14) are clearly necessary conditions. They are sufficient, by 1° of Lemma 5.

3° iff 4°: Here, $\Delta_1 L(0,T,u)$ is undefined, prompting the special case, otherwise the same as above.

5° iff 6°: (15) is clearly necessary. It is sufficient, by 2° of Lemma 5. ||

We say (s°,T°) is u-optimal if $L(s^\circ,T^\circ,u) \leq L(s,T,u)$ for all s and T; i.e., (s°,T°) minimizes $L(\cdot, \cdot,u)$ for fixed u.

Lemma 7. (Necessary conditions). If

1° (s°,T°) is u-optimal, then

2° $s^\circ \in s^*(T^\circ,u)$ and $T^\circ \in T^*(s^\circ,u)$.

Proof: This follows by contradiction. For example, if $s^\circ \notin s^*(T^\circ,u)$, there is some $s \in s^*(T^\circ,u)$ such that $L(s,T^\circ,u) < L(s^\circ,T^\circ,u)$ which contradicts 1°. ||

Recall that our approach starts with a sequence $\{u_i\}$ of multipliers and then finds an associated sequence of undominated solutions $\{(s^i,T^i)\}$ (where (s^i,T^i) is u_i-optimal for each i). Thus, (13) - (15) must be satisfied by such solutions. However, it is important to note that (13) - (15) are not sufficient conditions. Related difficulties were pointed out by Veinott [12]. Miller [6] defines "discrete convexity" so that the conditions are sufficient when $L(\cdot,\cdot,u)$ is (discretely) convex for fixed u. However, his conditions are not satisfied in our case. In fact, in the appendix, we provide a multiplier u and two solutions (s_1,T_1) and (s_2,T_2) which satisfy (13) - (15) (e.g., s_1 minimizes $L(\cdot,T_1,u)$ for fixed T_1 and u) yet $L(s_1,T_1,u) \neq L(s_2,T_2,u)$. In addition, $s_2 = s_1 - 1$. Therefore, if $s^o \in s^*(T^o,u)$ and $T^o \in T^*(s^o,u)$ (so that (s^o,T^o) is a fixed point of the cyclic coordinate ascent method (e.g., [14]), then it is <u>not</u> necessarily true that $L(s^o,T^o,u) \leq L(s,T,u)$ for $|s-s^o| \leq 1$ and all T. Furthermore, as shown in the appendix, for each positive number of spares, say n, there is a multiplier u_n, and T_n^1 and T_n^2, such that (n,T_n^1) and $(n-1,T_n^2)$ satisfy (13) - (15). In other words, the difficulty is pervasive, and cannot be eliminated by imposing an upper or lower bound on the feasible number of spares.

Nevertheless, conditions (13) - (15), although only necessary, prove to be very useful. We shall use them to derive upper and lower bounds on u-optimal solutions in certain cases. However, we first supply the following.

<u>Lemma 8.</u> If

1^o $\quad s_1 < s_2$, $T_1 \in T^*(s_1,u)$, and $T_2 \in T^*(s_2,u)$,

then

2^o $\quad T_1 < T_2$.

Similarly, if

3° $T_1 < T_2$, $s_1 \in s^*(T_1, u)$, and $s_2 \in s^*(T_2, u)$,

then

4° $s_1 \le s_2$.

<u>Remark</u>: This result is useful when using our algorithm of Section 3, which for each item, sweeps through the sequence $\{u_n\}$ of multipliers, finding u_n-optimal solutions for each n.

<u>Proof:</u> 1° implies 2°: Suppose $T_2 \le T_1$. Then

$$
\begin{aligned}
0 &= \frac{\partial}{\partial T} L(s_2, T_2, u) && [\text{by (15)}] \\
&< \frac{\partial}{\partial T} L(s_1, T_2, u) && [\text{by } 3^\circ \text{ of Lemma 5}] \\
&\le \frac{\partial}{\partial T} L(s_1, T_1, u) && [\text{by } 2^\circ \text{ of Lemma 5}] \\
&= 0 && [\text{by (15)}] \quad ,
\end{aligned}
$$

a contradiction.

3° implies 4°: Similarly, suppose $s_1 > s_2$ so that $s_1 \ge s_2 + 1 \ge 1$.

Then,

$$
\begin{aligned}
0 &\le \Delta_1 L(s_2 + 1, T_2, u) && [\text{by (14)}] \\
&< \Delta_1 L(s_2 + 1, T_1, u) && [\text{by } 4^\circ \text{ of Lemma 5}] \\
&\le \Delta_1 L(s_1, T_1, u) && [\text{by } 1^\circ \text{ of Lemma 5 and } s_1 \ge s_2 + 1] \\
&\le 0 && [\text{by (13)}]
\end{aligned}
$$

a contradiction. ||

For an algorithm which will minimize $L(s,T,u)$ for fixed u, we offer the following partial enumeration scheme, with the assumption that there exists lower and upper limits, say s^d and s^u, on s^*; i.e., $s^d \leq s \leq s^u$.

1° Set $s = s^d$.

2° Set $T = T^*(s,u)$.

3° Store $L(s,T,u)$ if it is the smallest yet found.

4° If $s = s^u$, stop.

5° Replace s by $s+1$ and repeat from 2°.

Any of several one dimensional optimization schemes (e.g., [13]) may be used in step 2°.

It is clear that, given the lower and upper limits on feasible values of s, the algorithm will, for fixed u, find a u-optimal solution (subject to the constraints on s). In the next section, for specific cases, we show how to derive bounds not only on s but on T for any u-optimal solution (s,T), thereby allowing the algorithm to be used with no loss in optimality.

10. QUALITATIVE RESULTS. SPECIAL CASES.

In the next two subsections, we examine two special cases, under the assumption

$$1^{**} \quad c^S(s) = c \cdot s ,$$

where $c > 0$.

For this assumption, we have the following.

<u>Lemma 9.</u> If

1° (s°, T°) is u-optimal, then

2°

$$\lambda \sum_{x=s^\circ}^{\infty} p(x; \lambda T^\circ) + u \frac{\partial}{\partial T} c^R(T^\circ) = 0 \quad , \tag{16}$$

3° if $uc \geq 1$, then $s^\circ = 0$, and

4° if $uc < 1$, then

$$\sum_{x=s^\circ}^{\infty} p(x; \lambda T^\circ) \geq uc \tag{17}$$

and

$$\sum_{x=s^\circ+1}^{\infty} p(x; \lambda T^\circ) \leq uc \quad . \tag{18}$$

<u>Proof:</u> (16) is simply (15), which must hold by Lemmas 6 and 7.

3°: By 1^{**}, (14) becomes (18), which holds, since

$$\sum_{x=s^\circ+1}^{\infty} p(x; \lambda T^\circ) < 1 \leq uc \quad .$$

That is, $L(0, T^\circ, u) < L(1, T^\circ, u)$.

4°: (13) and (14) have become (17) and (18) here, which, by Lemmas 6 and 7, must hold if $s^\circ \geq 1$. If $s^\circ = 0$, then (18) must hold by Lemma 6, and (17) holds, since $\sum_{x=0}^{\infty} p(x;\lambda T^\circ) = 1 > uc$. ||

10.1 Case One. $c^R(T) = a/T$.

Recall from section 8.1 the circumstances under which this assumption is justified. Here

$$L(s,T,u) = B(s,t) + ucs + ua/T \quad ,$$

and (16) becomes

$$\text{(19)} \qquad \sum_{x=s^\circ}^{\infty} p(x;\lambda T^\circ) = ua/(\lambda T^2) \quad .$$

Theorem 1. In this case, if

1° (s°, T°) is u-optimal, then

2° if $uc \geq 1$, then $s^\circ = 0$ and $T^\circ = T^d$,

and

3° if $uc < 1$ then

$$T^d \leq T^\circ \leq T^u \qquad \text{and} \qquad s^d \leq s^\circ \leq s^u \quad ,$$

where

(20) $$T^u \equiv \sqrt{a/\lambda c} \quad ,$$

(21) $$T^d \equiv \sqrt{ua/\lambda} \quad = \quad \sqrt{uc} \quad T^u \quad ,$$

s^u is the smallest s which satisfies

(22) $$\sum_{x=s^u+1}^{\infty} p(x;\lambda T^u) < uc \quad ,$$

and s^d is the largest s which satisfies

(23) $$\sum_{x=s^d}^{\infty} p(x;\lambda T^d) > uc \quad .$$

Remark: These bounds allow the algorithm of Section 9.2 to be used, with no loss in optimality.

Proof: 2^o: By Lemma 9, $s^o = 0$. $T^o = T^d$ follows from (19) and (21).

3^o: $T^o \leq T^u$: Here,

$$ua/(\lambda T^2) = \sum_{x=s^o}^{\infty} p(x;\lambda T^o) \qquad \text{[by (19)]}$$

$$\geq uc \qquad \text{[by (17)]}$$

from which the result follows.

$T^d \leq T^o$: Here,

$$ua/(\lambda T)^2 = \sum_{x=s^o}^{\infty} p(x;\lambda T^o) \leq 1$$

which yields the result.

$s^o \leq s^u$: Suppose $s^o > s^u$, which implies $s^o \geq s^u + 1 \geq 1$. First observe that, by (10), for $s \geq 1$,

$$(24) \qquad \frac{\partial}{\partial T}\left[\sum_{x=s}^{\infty} p(x;\lambda T)\right] = \lambda p(s-1,\lambda T) > 0 \quad .$$

Then

$$uc \leq \sum_{x=s^o}^{\infty} p(x;\lambda T^o) \qquad \text{[by (17)]}$$

$$\leq \sum_{x=s^o}^{\infty} p(x;\lambda T^u) \qquad \text{[by (24), } s^o \geq 1 \text{ and } T^o \leq T^u\text{]}$$

$$\leq \sum_{x=s^u+1}^{\infty} p(x;\lambda T^u) \qquad \text{[by } s^o \geq s^u+1\text{]}$$

$$< uc \qquad \text{[by (22)]} ,$$

a contradiction.

$s^d \le s^o$: Similarly, suppose $s^d > s^o$, so that $s^d \ge s^o+1 \ge 1$.

$$uc \ge \sum_{x=s^o+1}^{\infty} p(x;\lambda T^o) \qquad [\text{by } (18)]$$

$$\ge \sum_{x=s^o+1}^{\infty} p(x;\lambda T^d) \qquad [\text{by } (24),\ s^o+1 \ge 1, \text{ and } T^d \le T^o]$$

$$\ge \sum_{x=s^d}^{\infty} p(x;\lambda T^d) \qquad [\text{by } s^d \ge s^o+1]$$

$$> uc, \qquad [\text{by } (23)]$$

a contradiction. ||

10.2 Case Two. $c^R(T) = ae^{-T/r}$.

Recall from section 8.2 the justification for this case. Here

$$L(s,T,u) = B(s,T) + ucs + uae^{-T/r},$$

and (16) becomes

$$(25) \qquad \sum_{x=s^o}^{\infty} p(x;\lambda T^o) = uae^{-T/r}/(r\lambda).$$

Theorem 2. In this case, if

1° (s^o, T^o) is u-optimal,

then

2^o if $uc \geq 1$, then $s^o = 0$ and $T^o = T^d$,

and

3^o if $uc < 1$, then $T^d \leq T^o \leq T^u$ and $s^d \leq s^o \leq s^u$,

where

(26) $$T^u \equiv r \ln(a/(r\lambda c)) \ ,$$

(27) $$T^d \equiv r \ln(ua/(r\lambda)) = r \ln(uc) + T^u \ ,$$

and s^d and s^u are given as in (22) and (23) .

Remark: For convenience, we use the same notation for our bounds, although their values differ from those of case one.

Proof: The proof follows that of Theorem 1, and is therefore omitted. ||

11. A NUMERICAL EXAMPLE

Next we illustrate the results obtained in the previous sections by solving a two item, two ship problem. Let s_1 be the number of spares for item A on board ship #1, s_2 denote the number of spares of item A on ship #2, and s_3 denote the number of spares of item B on ship #2. (For ease of presentation it has been assumed that ship #1 does not carry item A). In addition, let T_{ik} be the mean repair time for the k^{th} echelon of repair for the i^{th} item. In this example there are three types of repair: k=1 refers to local repair, k=2 refers to repair at another facility, and k=3 refers to resupply on a one-for-one basis. We will assume in this example that the cost functions satisfy $c_i^S(s_i) = c_i^S \cdot s_i$ and $c_{ik}^R(T_{ik}) = c_{ik}^R / T_{ik}$. Thus, our problem is to determine s_i and T_{ik} to

(28.1) $$\min \sum_{i=1}^{3} B(s_i, T_i)$$

i.e., the total backorders over both items and both ships subject to

(28.2) $$T_i = \sum_{k=1}^{3} T_{ik}\, p_{ik} \quad ,$$

(28.3) $$\sum_{i=1}^{3} \left[c_i^S \cdot s_i + \sum_{k=1}^{3} c_{ik}^R / T_{ik} \right] \leq b \quad ,$$

where $B(s_i, T_i)$ is the expected number of shortages for item i, b is the available budget, and p_{ik} is the probability that the k^{th} type of repair will be needed when a unit of item i fails. Note that in this problem there are twelve decision variables (s_i, T_{ik}) altogether. The values used for the parameters p_{ik}, c^R_{ik}, c^S_i, and λ_i are given in Table 1. As discussed in Section 3, our solution procedure automatically constructs the solutions for a sequence of budgets b. However, for the sake of illustration we will consi 'er specifically the case $b=115$.

Rather than solving (28) directly, Section 3 shows how a Lagrangian approach could be used which would allow the problem to be decomposed by item and location. Then Section 7 shows how this latter problem could be further decomposed into two optimization problems: the first problem computes (for $i=1,2,3$)

$$(29) \qquad c^R_i(T) = \min \sum_{k=1}^{3} c^R_{ik}/T_{ik}$$

subject to (28.2); and the second problem determines s_i and T_i (for $i=1,2,3$) to minimize the Lagrangian function

$$(30) \qquad L_i(s_i, T_i, u) = B(s_i, T_i) - u \cdot c^S_i \cdot s_i - u \; c^R_i(T_i)$$

for different values of the multiplier u.

In this example the repair cost functions satisfy $c^R_{ik}(T_{ik}) = c^k_{ik}/T_{ik}$. Section 8.1 shows how to solve problem (29) with these functions and demonstrates that $c^R_i(T_i) = a_i/T_i$,

$$(31) \qquad a_i = \left[\sum_{k=1}^{3} (p_{ik} \, c^R_{ik})^{\frac{1}{2}} \right]^2 \quad .$$

Table 1: Parameter Values

Interpretation Of Subscript	p_{ik} k=1	p_{ik} k=2	p_{ik} k=3	c_{ik}^R k=1	c_{ik}^R k=2	c_{ik}^R k=3	c_i^S	λ_i
Item A, Ship #1	.20	.40	.40	20	40	40	1	5
Item A, Ship #2	.50	.25	.25	3.2	6.4	57.6	2	5
Item B, Ship #2	.50	.25	.25	24	48	432	3	1

The parameter values in Table 1 imply that using the derived formulae of (31) $a_1=100$, $a_2=40$, $a_3=300$. Hence, $c_1(T) = 100/T$, $c_2(T) = 40/T$, $c_3(T)=300/T$.

Theorem 1 of Section 10.1 provides the conditions of optimality for minimizing the Lagrangian function (30). Note that the conditions in this theorem uniquely specify the u-optimal solution for the case $u \cdot c_i^S \geq 1$, but only provide upper and lower bounds on s_i and T_i for the case $u \cdot c_i^S < 1$. In this latter case it is necessary to use the search algorithm given in Section 9.2 to compute the u-optimal solutions. Typical results obtained with this algorithm are discussed next.

Tables 2-4 provide the u-optimal values of s_i and T_i, as a function of u, for each item. Table 2 considers the item A on ship #1, where $\lambda=5$, $a_1=100$, and $c_1^S=1$; Table 3 considers item A on ship #2, where $\lambda_2=5$, $a_2=40$, and $c_2^S=2$;* and Table 4 considers item B on ship #2, where $\lambda_3=1$, $a_3=300$, and $c_3^S=3$. In each of these tables, the first column provides the multiplier u; the second column gives the corresponding minimum value of the Lagrangian (30); the third the fourth columns give the u-optimal solutions s_i and T_i respectively; the fifth column computes the cost of the solution, $c_i^S \cdot s_i + a_i/T_i$; the sixth column gives the expected number of backorders $B(s_i,T_i)$; and the remaining columns provide the upper and lower bounds on s_i and T_i which are implied by Theorem 1. For the cases in which the upper and lower bounds do not coincide, it was necessary to use the search algorithm of Section 9.2 to compute the u-optimal solutions given in the third and fourth column.

---

* Note: To illustrate the generality of the approach, the marginal cost of providing spares of item A was assumed to be different on ship 1 than on ship 2. In a real application, the marginal cost would most likely be identical.

Tables 2-4 can be interpreted in the following way. Each of these tables provides the solution of the single item, single ship problem with a budget constraint limited to that item. For example, consider the first row in Table 2. Here the multiplier $\upsilon=1$ is associated with the cost 22.36. According to Lemmas 1 and 3, the solution $s_1=0$ and $T_1=4.7$ would be optimal if the repair budget for that item were equal to 22.36. The other rows in these tables may be interpreted in the same way.

Next consider the original problem (28), which is a two item, two ship problem with a joint budget constraint. The total costs and expected shortages for this problem are equal to the sum of the costs and expected shortages for each individual item and ship. Thus Table 5 is constructed by summing the cost and backorder columns in Tables 2-4. Table 5 may be interpreted in the following way. Consider the first row in which the multiplier $\upsilon=1$ is associated with the cost 53.82 and expected shortages 53.82. It follows from Lemmas 1 and 3 that 53.82 is the minimum expected number of shortages in problem (28), subject to a budget $b=53.82$. The corresponding optimal values of s_i and T_i for each item are given in the first row of Tables 2-4. Similarly, the second row in Table 5 implies that 39.07 is the minimum expected number of shortages subject to a budget $b=68.64$. Thus Tables 2-5 provide a sequence of undominated solutions; i.e. optimal solutions to problems of the form (28) with different budget levels. Suppose the actual budget is $b=115$. Table 5 shows that the multiplier $\upsilon=.36$ has a cost of 111.71, which is the largest cost less than 115 in this table. Thus Tables 2-3 imply that an approximate solution of problem (28) with $b=115$

Table 2: Solution Of The Lagrangian Problem For Item A on Ship #1
i.e. an Item with $\lambda_1 = 5$, $a_1 = 100$, and $c_1^S = 1$

u	$L_1(s_1, T_1, u)$	s_1	T_1	$c_1^S \cdot s_1 + a_1/T_1$	$B(s_1, T_1)$	Bounds On s_1 Low	Bounds On s_1 High	Bounds On T_1 Low	Bounds On T_1 High
1.00	44.72	0	4.47	22.36	22.36	0	0	4.47	4.47
.96	43.31	14	4.44	36.53	8.24	14	14	4.38	4.47
.92	41.81	16	4.45	38.48	6.41	15	16	4.29	4.47
.88	40.26	17	4.44	39.51	5.49	16	17	4.20	4.47
.84	38.67	17	4.38	39.86	5.19	16	18	4.10	4.47
.80	37.05	18	4.39	40.76	4.44	16	18	4.00	4.47
.76	35.41	19	4.43	41.59	3.80	16	19	3.90	4.47
.72	33.74	19	4.36	41.91	3.56	16	19	3.80	4.47
.68	32.06	19	4.30	42.25	3.33	16	20	3.69	4.47
.64	30.35	20	4.35	42.98	2.84	16	21	3.58	4.47
.60	28.62	20	4.29	43.31	2.64	16	21	3.46	4.47
.56	26.88	21	4.35	43.99	2.24	16	21	3.35	4.47
.52	25.11	21	4.29	44.33	2.06	16	22	3.23	4.47
.48	23.33	21	4.22	44.69	1.88	16	22	3.10	4.47
.44	21.52	22	4.29	45.34	1.58	15	23	2.97	4.47
.40	19.70	22	4.22	45.71	1.42	15	23	2.83	4.47
.36	17.86	23	4.28	46.36	1.17	15	24	2.68	4.47
.32	16.00	23	4.21	46.78	1.03	14	24	2.53	4.47
.28	14.12	24	4.26	47.46	.83	14	25	2.37	4.47
.24	12.21	24	4.18	47.94	.70	13	26	2.19	4.47
.20	10.28	25	4.22	48.70	.54	13	26	2.00	4.47
.16	8.32	25	4.11	49.32	.43	12	27	1.79	4.47
.12	6.33	26	4.12	50.26	.30	11	28	1.55	4.47
.08	4.30	27	4.10	51.42	.18	10	29	1.27	4.47
.04	2.21	28	3.98	53.11	.08	8	31	.89	4.47

Table 3: Solution Of The Lagrangian Problem For Item A On Ship #2
i.e. an item with $\lambda_2 = 5$, $a_2 = 40$, and $c_2^S = 2$

u	$L_2(s_2,T_2,u)$	s_2	T_2	$c_2^S \cdot s_2 + a_2/T_2$	$B(s_2,T_2)$	Bounds On s_2 Low	Bounds On s_2 High	Bounds On T_2 Low	Bounds On T_2 High
1.00	28.28	0	2.83	14.14	14.14	0	0	2.83	2.83
.96	27.71	0	2.77	14.43	13.86	0	0	2.77	2.77
.92	27.13	0	2.71	14.74	13.57	0	0	2.71	2.71
.88	26.53	0	2.65	15.08	13.27	0	0	2.65	2.65
.84	25.92	0	2.59	15.43	12.96	0	0	2.59	2.59
.80	25.30	0	2.53	15.81	12.65	0	0	2.53	2.53
.76	24.66	0	2.47	16.22	12.33	0	0	2.47	2.47
.72	24.00	0	2.40	16.67	12.00	0	0	2.40	2.40
.68	23.32	0	2.33	17.15	11.66	0	0	2.33	2.33
.64	22.63	0	2.26	17.68	11.31	0	0	2.26	2.26
.60	21.91	0	2.19	18.26	10.95	0	0	2.19	2.19
.56	21.17	0	2.12	18.90	10.58	0	0	2.12	2.12
.52	20.40	0	2.04	19.61	10.20	0	0	2.04	2.04
.48	19.44	5	1.99	30.10	4.99	5	5	1.96	2.00
.44	18.18	6	1.95	32.49	3.89	6	6	1.88	2.00
.40	16.83	7	1.94	34.61	2.98	6	7	1.79	2.00
.36	15.41	8	1.95	36.51	2.27	7	8	1.70	2.00
.32	13.94	8	1.88	37.25	2.02	7	9	1.60	2.00
.28	12.41	9	1.91	38.95	1.50	7	9	1.50	2.00
.24	10.84	9	1.84	39.77	1.29	7	10	1.39	2.00
.20	9.20	10	1.87	41.40	.92	7	11	1.27	2.00
.16	7.53	10	1.78	42.42	.74	7	11	1.13	2.00
.12	5.79	11	1.80	44.19	.48	6	12	.98	2.00
.08	3.98	12	1.80	46.21	.28	6	13	.80	2.00
.04	2.08	13	1.75	48.92	.13	5	15	.57	2.00

Table 4: Solution Of The Lagrangian Problem For Item B on Ship #2
i.e. an Item with $\lambda_3 = 1$, $a_3 = 300$, and $c_3^S = 3$

u	$L_3(s_3, T_3, u)$	s_3	T_3	$c_3^S \cdot s_3 + a_3/T_3$	$B(s_3, T_3)$	Bounds On s_3 Low	Bounds On s_3 High	Bounds On T_3 Low	Bounds On T_3 High
1.00	34.64	0	17.32	17.32	17.32	0	0	17.32	17.32
.96	33.94	0	16.97	17.68	16.97	0	0	16.97	16.97
.92	33.23	0	16.61	18.06	16.61	0	0	16.61	16.61
.88	32.50	0	16.25	18.46	16.25	0	0	16.25	16.25
.84	31.75	0	15.88	18.90	15.88	0	0	15.88	15.88
.80	30.98	0	15.49	19.37	15.49	0	0	15.49	15.49
.76	30.20	0	15.10	19.87	15.10	0	0	15.10	15.10
.72	29.39	0	14.70	20.41	14.70	0	0	14.70	14.70
.68	28.57	0	14.28	21.00	14.28	0	0	14.28	14.28
.64	27.71	0	13.86	21.65	13.86	0	0	13.86	13.86
.60	26.83	0	13.42	22.36	13.42	0	0	13.42	13.42
.56	25.92	0	12.96	23.15	12.96	0	0	12.96	12.96
.52	24.98	0	12.49	24.02	12.49	0	0	12.49	12.49
.48	24.00	0	12.00	25.00	12.00	0	0	12.00	12.00
.44	22.98	0	11.49	26.11	11.49	0	0	11.49	11.49
.40	21.91	0	10.95	27.39	10.95	0	0	10.95	10.95
.36	20.79	0	10.39	28.87	10.39	0	0	10.39	10.39
.32	19.44	5	9.95	45.15	4.99	5	5	9.80	10.00
.28	17.52	7	9.87	51.39	3.13	6	7	9.17	10.00
.24	15.41	8	9.75	54.76	2.27	7	8	8.49	10.00
.20	13.18	9	9.72	57.86	1.61	7	9	7.75	10.00
.16	10.84	9	9.19	59.65	1.29	7	10	6.93	10.00
.12	8.37	10	9.14	62.83	.83	7	11	6.00	10.00
.08	5.79	11	9.01	66.29	.48	6	12	4.90	10.00
.04	3.05	12	8.62	70.81	.21	6	14	3.46	10.00

Table 5: Undominated Solutions Of The Two Item, Two Ship— 3 Echelon Repair System

u	$\sum_i (c_i^S \cdot s_i + a_i/T_i)$	$\sum_i B(s_i, T_i)$
1.00	53.82	53.82
.96	68.64	39.07
.92	71.28	36.58
.88	73.05	35.01
.84	74.19	34.02
.80	75.94	32.58
.76	77.68	31.23
.72	78.99	30.26
.68	80.40	29.28
.64	82.31	28.01
.60	83.93	27.01
.56	86.04	25.78
.52	87.96	24.75
.48	99.79	18.87
.44	103.94	16.95
.40	107.71	15.36
.36	111.74	13.83
.32	129.18	8.04
.28	137.79	5.46
.24	142.47	4.26
.20	147.96	3.07
.16	151.40	2.46
.12	157.28	1.61
.08	163.92	.95
.04	172.84	.43

is given by $s_1=23$, $T_1=4.28$, $s_2=8$, $T_2=1.95$, $s_3=0$, and $T_3=10.39$. In other words, there should be 23 spares of item A on ship#1, 8 spares of item A on ship#2, and no spares of item B on ship#2. In addition, the average repair time for item A, failing on ship #1, should be set at 4.28 days. Similarly, the average repair time for item B, failing on ship #2, should be 10.39 days. Improved approximate solutions can be obtained by repeating the algorithm with a sequence of multipliers in the range $[.32, .36]$.

Once the optimal values of T_1, T_2, and T_3 have been obtained, the corresponding values for T_{ik} can be computed with

$$T_{ik} = \frac{T_i (c_{ik}^R / p_{ik})^{\frac{1}{2}}}{\sum_{j=1}^{3} (p_{ij} c_{ij}^R)^{\frac{1}{2}}} ,$$

which was derived in Section 8.1. We used this formula to compute T_{ik} from T_i for the case b=115, using the values of c_{ik}^R and p_{ik} given in Table 1, and the resulting values for T_{ik} are given in Table 6. Note that this solution allocates the largest portion of the budget to the first item, as this is where the greatest reduction in shortages per dollar can be achieved. As indicated at the beginning of this paper, it is straightforward to minimize the weighted sum of expected shortages, instead of the unweighted sum (all weights are one) in (28.1), where the weights are input parameters. Note also that this algorithm could easily handle a problem with much larger numbers of items, facilities, and repair types.

Table 6: Approximate Solutions For $b=115$

Item i	s_i	T_i	k=1	k=2	k=3	$B(s_i, T_i)$	$c_i^S \cdot s_i + a_i/T_i$
Item A, Ship 1	23	4.28	4.28	4.28	4.28	1.17	46.36
Item A, Ship 2	8	1.95	.78	1.56	4.68	2.27	36.51
Item B, Ship 2	0	10.39	4.16	8.31	24.94	10.39	28.87

Note that for item A, failing on ship #2, the local repair time should be .78 days, whereas the repair turnaround time at the next higher echelon (including transit to and from the ship) should be 1.56 days. Note this is much lower than that for item A on ship #1, or for item B on ship #2, since it was assumed the costs to lower the repair times for failing item A's on board ship #2 was much lower than for the other two cases (see Table 1).

It is also of interest to note that this repair design and spare stock mix gives rise to average backorders of 1.17 units for item A at ship #1, 12.27 units for item A at ship #2, and 10.39 units for item B at ship #2. Thus the total minimum average backorders for item A and B at ships #1 and #2, given a total budget of 115, is 13.83 units. Also note the 108.74 is distributed so that 46.36 of it is used to support item A on ship #1, 36.51 of it supports item A on ship #2, and 28.87 supports item B on ship #2.

12. APPENDIX

Here, as indicated in Section 9.2, we provide a sequence $\{u_n; n \geq 1\}$ of multipliers such that for each positive number of spares, say n, there exist repair times T_n^1 and T_n^2 such that both (n, T_n^1) and $(n-1, T_n^2)$ satisfy (13)-(15) for u_n. Thus, solutions to (13)-(15) are not unique. Furthermore, we provide a numerical example of a multiplier u and two solutions (s_1, T_1) and (s_1, T_1) which satisfy (13)-(15) yet $L(s_1, T_1, u) \neq L(s_2, T_2, u)$. Thus, solutions to (13)-(15) do not necessarily yield the same objective function values.

Consider the case where

1** $\quad c^S(s) = c \cdot s \qquad (c > 0)$, and

2** $\quad c^R(T) = a/T \qquad (a > 0)$.

Recall from Theorem 1 that:

$$(20) \qquad T^u \equiv \sqrt{a/\lambda c} \quad .$$

For convenience, we reproduce (13)-(15), wherein in the present case, are:

$$(17) \qquad \sum_{x=s^o}^{\infty} p(x; \lambda T^o) \geq uc \quad ,$$

$$(18) \qquad \sum_{x=s^o+1}^{\infty} p(x; \lambda T^o) \leq uc, \quad \text{and}$$

$$(19)\qquad \sum_{x=s^o}^{\infty} p(x;\lambda T^o) = ua/(\lambda T^2) .$$

Theorem 3. There exists sequences $\{u_n; n \geq 1\}$ and $\{T_n, n \geq 1\}$ such that both (n, T^u) and $(n-1, T_u)$ satisfy (17)-(19) for $u = u_n$, for each n.

Proof: Let

$$u_n = c^{-1} \sum_{x=n}^{\infty} e^{-\lambda T^u} (\lambda T^u)^x / x! \qquad \text{for} \quad n = 1, 2, \ldots$$

By direct checking (n, T^u) satisfy (17)-(19), (for $u = u_n$) for each n; (17) being satisfied with equality. Let $T_n = T^*(n-1, u_n)$, so that by Lemma 8, $T_n < T^u$. Thus, $(n-1, T_n)$ satisfies (19) which becomes

$$\begin{aligned} \sum_{x=n-1}^{\infty} p(x;\lambda T_n) &= ua/(\lambda T_n^2) \\ &> ua/[\lambda (T^u)^2] \qquad [\text{since } T_n < T^u] \\ &= u_n c, \qquad [\text{by (20)}] \end{aligned}$$

which implies that $(n-1, T_n)$ satisfies (17).

Now

$$\begin{aligned} \sum_{x=n}^{\infty} p(x;\lambda T_n) &< \sum_{x=n}^{\infty} p(x;\lambda T^u) \qquad [\text{by (24) and } T_n < T^u] \\ &= uc, \qquad [(n, T^u) \text{ satisfies (17) with equality}] \end{aligned}$$

which implies that $(n-1, T_n)$ satisfies (18). ||

We finally provide a numerical illustration. Set $a = \lambda = c = 1$ and pick $n = 1$, so that $u_n = u_1 = (e-1)/e \cong 0.63$, $T^U = 1$, $n-1 = 0$, $T_1 = u_1^{\frac{1}{2}}$, $L(n, T^U, u_n) = 1 + u_1 \cong 1.63$, and $L(n-1, T_n, u_n) = 2T_1 \cong 1.59$. Thus, solutions of the necessary conditions (13)-(15) do not necessarily yield the same objective function values.

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