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A STOCHASTIC SURVIVABILITY MODEL

FOR

COMPARISON OF COMBAT ORGANIZATIONAL STRUCTURES

by

Dennis Harry Long

Thesis Advisor: J. D. Esary

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A Stochastic Survivability Model for Comparison

of Combat Organizational Structures

by

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13. ABSTRACT

A model is developed which relates the structure of a combat unit to its expected effectiveness during a time interval. Sensitivity to variations of structure is incorporated into the model by adapting effectiveness concepts from reliability theory, and by constructing the attrition process as a continuous parameter Markov chain. The model is highly sensitive to the judgment of the user, due to parametric flexibility both in the attrition process, and effectiveness measurement. Two solution methods are given. Examples are employed to illustrate the model's use, and proposals for refinement are offered.

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I. INTRODUCTION

The purpose of any operational model is to aid a decision-maker by relating his possible alternatives to their consequent results, thus increasing the knowledge upon which his decision is based. Therefore a model, in addition to validly representing the process or system involved, must be capable of displaying the contrast, in terms of meaningful criteria, of even subtle variations of choice. Sensitivity to variables outside the area of the current decision is not particularly valuable, and in fact may produce detrimental complication. In short, a model should ideally be a reflection of the problem of its user. It is for this reason that distinct decision problems often require the use of different models of a single system.

This dependence of model upon problem is nowhere more apparent than in the study of combat dynamics. As an example a single battle situation may be modeled using game theory if the decision involves strategy, or using search theory if alternative fire distributions are being studied, or using Lanchester theory if weapon characteristics or tactical movements are being considered. Each modeling approach has been found useful for certain problem types but useless for others.

The purpose of this thesis is to propose an alternate modeling viewpoint for those problems in which the structure of a combat unit must be related to its effectiveness. The term structure is meant to denote the personnel and equipment of the unit, the roles of these elements, as well as the channels employed to receive and transmit information and instructions. As an example, a decision-maker who must choose for field test a small number of prototype tank killer teams from all the infinite organizational possibilities, faces a structural decision problem, and might profit from the model to be proposed.

The importance of structure derives from the fact that when a leader or communicator becomes a casualty, his removal from the structure produces twin transformations in both the effectiveness and vulnerability of the unit. These transformations are determined at least in part by the casualty's position in the organization. As an illustration consider a rifle squad in defense whose squad leader has just become the first casualty. His function is assumed by a fire team leader, who must now receive and transmit orders, direct the fire of his squad and search his sector for targets. The effectiveness of the squad is reduced not only because one fewer weapon is firing but because a less highly trained leader is in charge. The vulnerability has increased for the team leader because he now must increase his exposure in order to direct the whole squad and observe its whole sector. The vulnerability of the whole squad is perhaps increased because it is being directed with less expertise.

Thus, to know the condition of a unit it is not enough to know the number of casualties without knowing who they are. Perhaps the strongest testimony to this viewpoint is the universal military principle that enemy leaders and radio operators are the targets of highest priority in any type of engagement, and the simultaneous effort to conceal visual signs of their identity. If the validity of this approach has been justified the next step is to outline the model requirements.

The decision-maker will be assumed to have available to him a facility for military judgment which will include his own assessment of unit and individual vulnerability, and his own criteria for measuring unit effectiveness on the basis of its surviving structure. The mission of the model will be to

1) provide a framework in which the decision-maker's assessment of threat can be specifically quantified,
2) translate the threat assessment into a stochastic description of the behavior of the candidate organization,
3) provide a framework for expressing the decision-maker's effectiveness criteria as an algebraic function of the unit condition,
4) derive a stochastic description of effectiveness of the candidate organization, using the effectiveness function.

Fortunately, the complete process of model construction is unnecessary, since there already exists an established field of study which treats the concept of structure. The field is stochastic reliability theory, which has historically developed in answer to the need for relating the design of complex systems of mechanical and electronic components to system effectiveness. The comparison of a system of components to a unit of fighting men may seem rash. However, through a process of adaptation, a model will be derived which may be considered more consistent while still taking advantage of reliability concepts.

Additional reliance on the theory of continuous parameter Markov processes will permit treatment of significant interdependencies in the attrition process.

The resultant model is one with capacity for immediate application, as well as promising opportunities for improvement and refinement.

II. RELIABILITY CONCEPTS

Reliability theory deals with collections of interdependent components. These collections are called systems. A component is any entity which can be described at any time as either functioning or failed, depending upon its ability to perform its assigned mission. Components are generally assumed to be in one of these two conditions. No partially failed conditions are considered.

The system which is discussed in this thesis is an engaged combat unit whose components are included in one or more of the following categories:

1) personnel,

2) weapons and equipment,

3) channels of information and communication.

The number of components in the unit is represented by the variable n called the <u>order</u> of the unit. Each component is identified with an integer in the set $\{1,\ldots,n\}$, called its <u>index</u>.

The <u>performance random variable</u> of the ith component is defined as follows:

$$X_i(t) = \begin{cases} 0 & \text{if the ith component is in the failed condition at time t,} \\ 1 & \text{if the ith component is in the functioning condition at time t.} \end{cases}$$

The <u>component reliability</u> of the ith component is defined as follows:

$$r_i(t) = P[X_i(t) = 1].$$

Since there are n components of interest, it is an obvious extension to define the <u>performance random vector</u> as follows:

$$\underline{X}(t) = (X_1(t),\ldots,X_n(t)).$$

The outcome space of $\underline{X}(t)$ is the set of all binary n-vectors. Call this set Ω_n. This set has N members where $N = 2^n$.

It is often necessary to partition the outcome space Ω_n into m subsets, $a_1,\ldots,a_m$, where $1 \leq m \leq N$, and define a <u>partition function</u> by assigning a unique real number b_i to all vectors in each subset a_i, so that

$$a(\underline{x}) = b_i \quad \text{if and only if} \quad \underline{x} \in a_i.$$

Each vector is a member of exactly one of the subsets.

The first partition function of interest is the <u>state function</u> $s(\underline{x})$, which is constructed by partitioning Ω_n into N subsets, each containing one vector. The value assigned to each one-member subset is one of the integers $1,\ldots,N$. Therefore

$$s(\underline{x}) = j \quad \text{if and only if} \quad \underline{x} \in a_i \text{ and } b_i = j.$$

It is assumed here that in all cases

$$s(\underline{1}) = 1 \quad \text{and}$$
$$s(\underline{0}) = N \quad \text{where}$$
$$\underline{1} = (1,\ldots,1) \quad \text{and}$$
$$\underline{0} = (0,\ldots,0).$$

The composition of the state function with the performance random vector results in definition of the state random variable

$$S(t) = s(\underline{X}(t)).$$

The common practice in reliability theory is to consider the entire system to be either in the functioning or failed condition at any time. Predictably the system's condition is described by a binary random variable. However, when the system is a combat unit, such a measure is too coarse to be of value. Still, since this binary concept is of value in the adaptation process, it is introduced below.

The structure function $\phi(\underline{x})$ of a system of order n is a partition function constructed by forming two subsets:

1) a_1, the set of all vectors $\underline{x}$ in Ω_n, such that if $\underline{X}(t) = \underline{x}$, then the system is functioning,

2) a_2, the set of all vectors $\underline{y}$ in Ω_n, such that if $X(t) = \underline{y}$, the system is in the failed condition.

Consistent with the use of binary indicators,

$$\begin{aligned} \phi(\underline{x}) &= 1(=b_1) \quad \text{if and only if} \quad \underline{x} \in a_1 \\ &= 0(=b_2) \quad \text{if and only if} \quad \underline{x} \in a_2. \end{aligned}$$

Composing the structure function with the performance random vector results in definition of the structure random variable

$$\Phi(t) = \phi(\underline{X}(t)).$$

The <u>system</u> <u>reliability</u> is defined to be

$$R(t) = P[\Phi(t) = 1].$$

If more than one system is being considered then it is necessary to place a subscript on the structure function, structure random variable and system reliability.

A system is called <u>coherent</u> if its structure function satisfies three conditions:

1) $\phi(\underline{0}) = 0,$

2) $\phi(\underline{1}) = 1,$

3) If $\underline{x} \leq \underline{y}$, then $\phi(\underline{x}) \leq \phi(\underline{y})$.

Essentially a coherent system is one which fails if all its components fail, functions if all its components function, and whose condition is not impaired by improving the condition of any of its components.

A component is called <u>irrelevant</u> if its condition never has any effect on the value of the structure function.

Components are assumed to be of two varieties. <u>Renewable components</u> alternate between the functioning and failed conditions. For example, tactical radios are communication channels whose functioning is interrupted by atmospheric disturbance or jamming. <u>Nonrenewable</u> <u>components</u> never return to the functioning condition after entering the failed condition. Personnel are nonrenewable components of a unit.

A common graphic method of illustrating the relationship of component performance to system performance is the use of the block diagram. Examples of these diagrams are shown in

Figures 1-4. The method is derived from the common practice of representing electrical circuits. Each block corresponds to the like indexed component of the system. A potential flow is imagined, attempting to pass through the diagram along the connecting arcs. The flow is able to pass through a component block only if the component is functioning. If the flow is able to traverse the diagram over any path then the system is functioning and its structure function equals one. Irrelevant components are represented by broken blocks totally unconnected to the system.

Figure 1 shows the block diagram for the system of three components which functions if and only if all three components are functioning. This is called a series system. Figure 2 shows a system which fails if and only if all its components fail. This is called a parallel system. Figure 3 shows a system which functions if and only if at least two of the components are functioning. This is called a two-out-of-three system. As in this case, a single component may be represented by more than one block, with the understanding that all blocks with the same index are always in the same condition. Figure 4 shows a system with components one and two in parallel, with component three irrelevant. It should be remembered that the block diagram shows only structural relationship, and implies nothing concerning actual location.

A final concept, not unique to reliability, but useful here, is that of Laplace transformation. Given a function

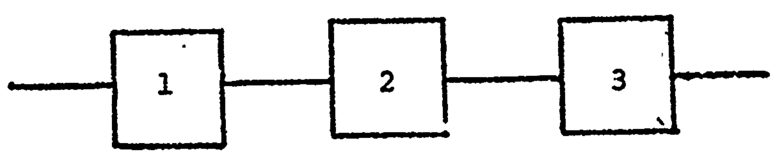

Figure 1. Series System

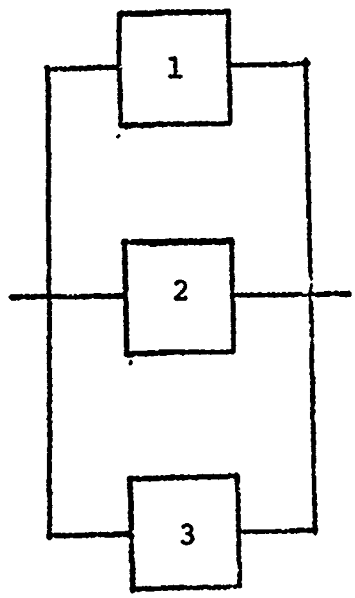

Figure 2. Parallel System

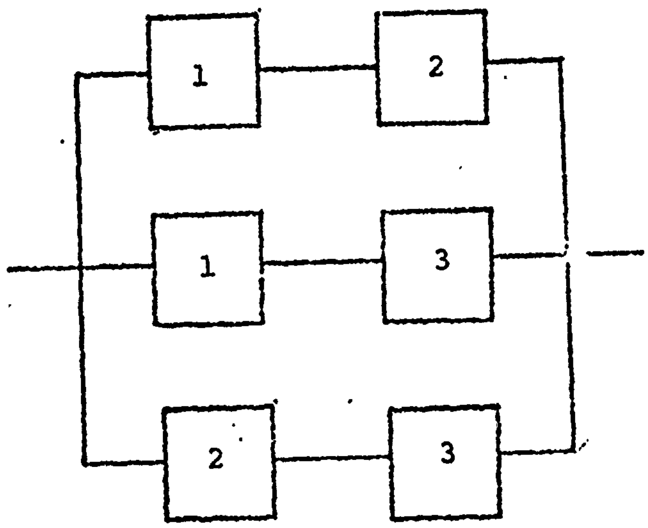

Figure 3. Two-Out-of-Three System

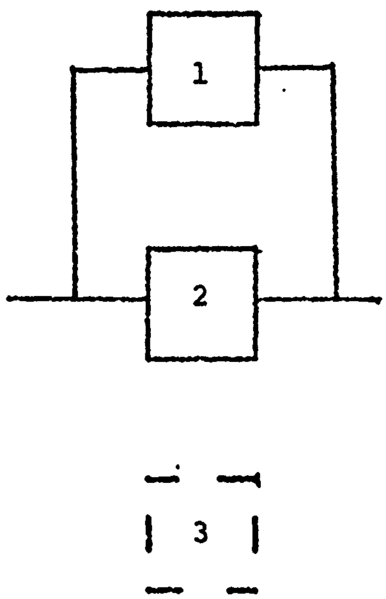

Figure 4. Parallel System with Irrelevant Component

c(t), the Laplace transform of c(t) is defined as follows:

$$L\{c(t)\} = \tilde{c}(s) = \int_0^\infty \exp(-st)\, c(t)\, dt,$$

a function of s. The function c(t) is called the inverse of the Laplace transform $\tilde{c}(s)$.

III. MEASUREMENT OF EFFECTIVENESS

In order to make any description of unit performance useful to a decision-maker, it is necessary to translate that description into some measure of effectiveness or goal attainment. Such a measure is useful only if it is consistent with the decision-maker's standards of value, and is in a form sufficiently simple to permit reasonable comparison of competing alternatives. The method developed here, of constructing such a measure begins with the designation by the user of a relatively simple instantaneous indicator function based upon his intuitive value judgments. This function then serves as a basis for derivation of a one dimensional deterministic function, which measures unit quality over a variable time interval.

The actual effectiveness of an engaged unit at a point in time is a very complex function of numerous variables, including friendly and enemy force levels, and many immeasurable physical and psychological factors. The simplifying assumption is made here that effectiveness depends only upon the vector of component performance and the length of time elapsed since the beginning of the engagement. The enemy performance is not considered directly, but the designation of an effectiveness function should reflect consideration of a hostile enemy environment, based on a previously determined scenario. The friendly unit is always assumed to have all components functioning at time zero.

The decision-maker's first step is to define a partition function on Ω_n. The set is partitioned into subsets a_i, $i=1,\ldots,m$, where vectors in the same subset are judged to represent equal effectiveness unit conditions. Vectors in different subsets represent outcomes which are not equally effective. This partition is performed without reference to time. To each subset a_i is assigned a dimensionless real number g_i, called its absolute effectiveness level. The function thus defined is called the <u>absolute effectiveness function</u> $g(x)$ where

$$g(\underline{x}) = g_i \quad \text{if and only if} \quad \underline{x} \in a_i,$$

and the partition is constructed so that

$$g_1 < g_2 < \cdots < g_m.$$

It is assumed without serious limitation that

$$(3\text{-}1) \qquad g(\underline{0}) = 0,$$

$$(3\text{-}2) \qquad g(\underline{1}) = g_m,$$

$$(3\text{-}3) \qquad \text{if} \quad \underline{x} \le \underline{y} \quad \text{then} \quad g(\underline{x}) \le g(\underline{y}).$$

These assumptions merely require that a completely destroyed unit be considered completely ineffective, that a completely undamaged unit be considered maximally effective, and that loss of a component never improve effectiveness.

Since the vector members of Ω_n and the integers in the set $\{1,\ldots,N\}$ are placed in one-to-one correspondence by the state function $s(\underline{x})$, it is possible to designate a companion function to $g(\underline{x})$ called the <u>alternate effectiveness function</u> defined on $\{1,\ldots,N\}$ such that

$$d(j) = g_i \quad \text{if and only if} \quad s(\underline{x}) = j \text{ and } \underline{x} \in a_i.$$

It follows that

$$g(\underline{x}) = d(s(\underline{x})).$$

This function is useful in later calculations.

The construction of the actual effectiveness function is of extreme importance to the successful employment of the model. The user should take special care to assure that the absolute effectiveness levels truly represent his judgment, not only with respect to ordinal relationships, but to ratio relationships as well.

A unit's real effectiveness depends upon how long it has been engaged. One factor to be considered is the change in enemy situation due to losses or reinforcement as provided in the scenario. Another consideration is that in missions of fixed length, such as delaying actions, casualties are less damaging, the later they occur. To allow for such trends, the user's next step is to define a <u>time variance function</u>, f(t), and a resultant <u>actual effectiveness</u> function,

$$h(\underline{x},t) = g(\underline{x})\ f(t).$$

If in a specific application the variance effect is not considered, then

$$f(t) = 1, \ t \geq 0$$

and therefore

$$h(\underline{x}, t) = g(\underline{x}), \ t \geq 0.$$

The function of $g(\underline{x})$ can be expressed in another equivalent manner which shows the close relationship with reliability theory. Esary, Proschan, and Walkup [Ref. 3] point out that a function such as $g(\underline{x})$, having a finite range can be written as

$$(3\text{-}4) \qquad g(\underline{x}) = (g_2-g_1)\ \phi_1(\underline{x}) + (g_3-g_2)\ \phi_2(\underline{x}) + \ldots + (g_m-g_{m-1})\ \phi_{m-1}(\underline{x})$$

where the functions $\phi_i(\underline{x})$ are defined such that

$$(3\text{-}5) \qquad \phi_i(\underline{x}) = 0 \quad \text{if and only if} \quad g(\underline{x}) \leq g_i$$

$$(3\text{-}6) \qquad \phi_i(\underline{x}) = 1 \quad \text{if and only if} \quad g(\underline{x}) > g_i.$$

It is now shown that each of the functions $\phi_i(\underline{x})$ is the structure function of a unique binary coherent system composed of the components of the unit, in which at least one of the components is not irrelevant. Furthermore the systems are related in such a way that no system is functioning if any lower indexed system has failed, and no system is failed if any higher indexed system is functioning.

<u>Theorem 1</u>

The functions $\phi_i(\underline{x})$ in (3-4) are structure functions of coherent systems of the n components of the parent system.

<u>Proof</u>

It must be shown that for $i=1,\ldots,m$,

1) $\phi_i(\underline{0}) = 0$,

2) $\phi_i(\underline{1}) = 1$,

3) if $\underline{x} \leq \underline{y}$ then $\phi_i(\underline{x}) \leq \phi_i(\underline{y})$.

The proof is by contradiction. The above three points are each in turn assumed false, and a result, contradicting one of the assumptions made in defining $g(\underline{x})$, is shown to follow.

1) Assume that $\phi_i(\underline{0}) \neq 0$ for some i. Then it must be that $\phi_i(\underline{0}) = 1$, but then from (3-4)

$$g(\underline{0}) \geq (g_{i+1}-g_i) > 0$$

contradicting (3-1). Thus $\phi_i(\underline{0}) = 0$, for $i = 1,\ldots,m-1$.

2) Assume that $\phi_i(\underline{1}) \neq 1$ for some i. Then it must be that $\phi_i(\underline{1}) = 0$, but then from (3-4)

$$g(\underline{1}) < (g_2-g_1) + \ldots + (g_m-g_{m-1}) = g_m, \text{ or}$$

$$g(\underline{1}) < g_m$$

contradicting (3-2). Thus $\phi_i(\underline{1}) = 1$, for $i=1,\ldots,m-1$.

3) Assume that there exist vectors $\underline{x}$ and $\underline{y}$ in Ω_n, such that $\underline{x} \leq \underline{y}$ and that for some i, $\phi_i(\underline{x}) > \phi_i(\underline{y})$. Since the function is binary, $\phi_i(\underline{x}) = 1$, and $\phi_i(\underline{y}) = 0$. By (3-6), $\phi_i(\underline{x}) = 1$ implies that $g(\underline{x}) > g_i$. By 3-5), $\phi_i(\underline{y}) = 0$ implies that $g(\underline{y}) \leq g_i$. Therefore

$$g(\underline{y}) < g(\underline{x}) \quad \text{and} \quad \underline{x} \leq \underline{y}$$

contradicting (3-3). Thus if $\underline{x} \leq \underline{y}$, then $\phi_i(\underline{x}) \leq \phi_i(\underline{y})$, for $i = 1,\ldots, m-1.$ ||

<u>Theorem 2</u>

The functions $\phi_i(\underline{x})$, $i = 1,\ldots, m-1$, are related as follows:

1) if $\phi_k(\underline{x}) = 1$, then $\phi_j(\underline{x}) = 1$, for $j = 1,\ldots, k-1$,

2) if $\phi_k(\underline{x}) = 0$, then $\phi_j(\underline{x}) = 0$, for $j = k + 1,\ldots, m-1$.

Proof

The proof is again by contradiction.

1) Let $\phi_k(\underline{x}) = 1$ for some vector $\underline{x}$. If $k=1$ then the result is trivially true. If $k > 1$, assume that $\phi_j(\underline{x}) = 0$ for some $j < k$. By (3-6), $\phi_k(\underline{x}) = 1$ implies that $g(\underline{x}) > g_k$. By (3-5), $\phi_j(\underline{x}) = 0$ implies that $g(\underline{x}) \leq g_j$. Since $g_j < g_k$, it follows that

$$g(\underline{x}) < g_k < g(\underline{x}),$$

an obvious contradiction. Therefore $\phi_j(\underline{x}) = 1$ for $j = 1,\ldots,k-1$.

2) Let $\phi_k(\underline{x}) = 0$ for some vector $\underline{x}$. If $k = m-1$, the result holds trivially. If $k < (m-1)$, assume that $\phi_j(\underline{x}) = 1$ for some $j > k$. By (3-5), $\phi_k(\underline{x}) = 0$ implies that $g(\underline{x}) \leq g_k$. By (3-6), $\phi_j(\underline{x}) = 1$ implies that $g(\underline{x}) > g_j$. Since $g_j > g_k$, it follows that

$$g(\underline{x}) < g_j < g(\underline{x}),$$

an obvious contradiction. Therefore $\phi_j(\underline{x}) = 0$, for $j = k + 1,\ldots, m-1$.||

Theorem 3

For any vector $\underline{x}$, $g(x) = g_i$, if and only if $\phi_i(\underline{x}) = 0$, and $\phi_{i-1}(\underline{x}) = 1$.

Proof

1) Let $g(\underline{x}) = g_i$ for some $\underline{x}$. Then by (3-5), $g(\underline{x}) = g_i \leq g_i$ implies that $\phi_i(\underline{x}) = 0$. By (3-6), $g(\underline{x}) = g_i > g_{i-1}$ implies that $\phi_{i-1}(\underline{x}) = 1$.

2) Let $\phi_i(\underline{x}) = 0$, and $\phi_{i-1}(\underline{x}) = 1$ for some $\underline{x}$. By (3-5) $\phi_i(\underline{x}) = 0$ implies that $g(\underline{x}) \leq g_i$. By (3-6), $\phi_{i-1}(\underline{x}) = 1$ implies that $g(\underline{x}) > g_{i-1}$. Therefore it follows that

$$g_{i-1} < g(\underline{x}) \leq g_i,$$

but the only value in the range of the function which satisfies the inequality is g_i. Therefore $g(\underline{x}) = g_i$.||

The ultimate importance of this method of expressing $g(\underline{x})$ is that the study of a very complicated multivalued absolute effectiveness function can be simplified to the study of a linear combination of simple dependent binary structure functions of the type commonly treated in classic reliability.

It follows that the actual effectiveness function can be written as follows:

$$h(\underline{x},t) = f(t) \{(g_2-g_1) \phi_1(\underline{x}) + \ldots + (g_m-g_{m-1}) \phi_{m-1}(\underline{x})\}.$$

Thus far the functions introduced are deterministic with domain Ω_n. Since Ω_n is also the outcome space of the performance random vector, $\underline{X}(t)$, each of the functions can be composed with this vector to define analogous random variables. The absolute effectiveness random variable is defined by the relation

$$G(t) = g(\underline{X}(t)),$$

or equivalently

$$G(t) = (g_2-g_1)\ \Phi_1(t) + \ldots + (g_m-g_{m-1})\ \Phi_{m-1}(t),$$

where

$$\Phi_i(t) = \phi_i(\underline{X}(t)),\ i = 1,\ldots, m-1.$$

The <u>actual effectiveness random variable</u> is defined by the relation

$$H(t) = f(t)G(t).$$

These effectiveness random variables have the advantage that they are defined through a process which assures their consistency with the decision-maker's standards. Their primary limitation is that they represent only instantaneous effectiveness. That is for every nonnegative value of t there is defined a distinct random variable H(t). In Figure 5, the bold horizontal line depicts a possible curve of outcomes of H(t), where f(t) is assumed to equal one for all nonnegative t.

Since combat engagements are extended over indefinite time periods, it is desirable to aggregate the effectiveness curve into a variable which measures accumulated effectiveness over an interval. To fill this need the <u>value random variable</u> is defined by the relation

$$V(t) = \int_0^t H(s)\ ds = \int_0^t G(s)\ f(s)\ ds.$$

In Figure 5, the value random variable is represented by the shaded area below the curve of H(t).

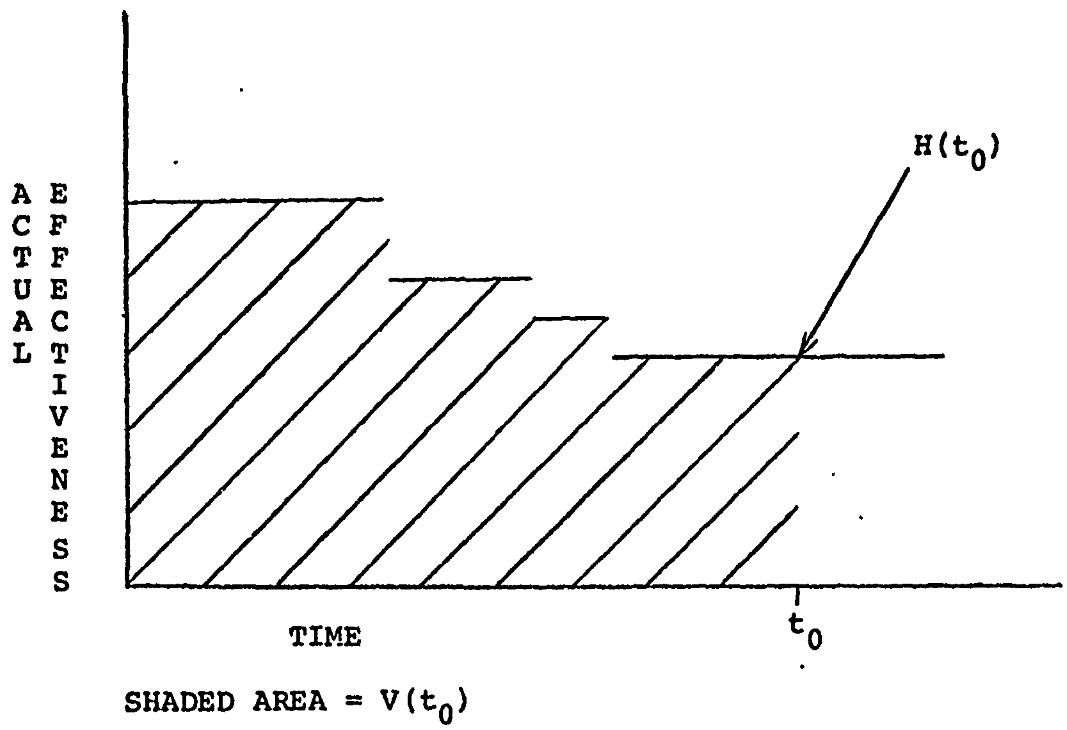

Figure 5. Example Outcome of Actual Effectiveness Random Variable and Value Random Variable

The value random variable is a measure of the type desired, but in general it will have a complicated distribution, too unwieldy to be of practical use in decision making. A relatively simple measure of effectiveness, which still preserves many advantages of the value random variable, is the <u>quality function</u> defined by the relation

$$q(t) = E[V(t)],$$

the mean or expected value of the value random variable.

To compute the quality function in a specific case, note that

$$\begin{aligned} q(t) &= E\left[\int_0^t H(u)\, du\right] \\ &= \int_0^t E[H(u)]\, du \\ &= \int_0^t f(u)\, E[G(u)]\, du, \end{aligned}$$

where the interchange of expectation and integration in the second line is justified by application of Fubini's Theorem, since H(u) is a family of nonnegative random variables. The validity of this interchange is established by Royden [Ref. 12].

There are two approaches for calculating q(t). The first arises from the fact that

$$E[G(u)] = \sum_{\underline{X}} g(\underline{x})\ P[\underline{X}(u) = \underline{x}\ |\underline{X}(0) = \underline{1}]$$

$$= \sum_{i=1}^{N} d(i)\ p_{1i}(u),$$

where

$$p_{1i}\ (u) = P[S(u) = i \mid S(0) = 1]$$

is called the interval probability function from state one to state i. Thus the first result is that

$$q(t) = \int_{o}^{t} f(u) \left[\sum_{i=1}^{N} d(i)\ p_{1i}(u) \right] du$$

$$= \sum_{i=1}^{N} d(i) \left[\int_{o}^{t} f(u)\ p_{1i}(u)\ du \right]$$

(3-7) $$= \sum_{i=1}^{N} d(i)\ I(i,t)$$

where

(3-8) $$I(i,t) = \int_{o}^{t} f(u)\ p_{1i}(u)\ du.$$

The second approach follows from relation (3-4),

$$E[G(u)] = E\left[\sum_{j=1}^{m-1} (g_{j+1}-g_j)\ \Phi_j(u)\right]$$

$$= \sum_{j=1}^{m-1} (g_{j+1}-g_j)\ E[\Phi_j(u)]$$

$$= \sum_{j=1}^{m-1} (g_{j+1}-g_j)\ P[\Phi_j(u) = 1]$$

$$= \sum_{j=1}^{m-1} (g_{j+1}-g_j)\ R_j(u),$$

where $R_j(u)$ is the system reliability of the jth associated system, computed on the basis of all components functioning at time zero. And the second result is that

$$q(t) = \int_0^t f(u)\left[\sum_{j=1}^{m-1} (g_{j+1}-g_j)\ R_j(u)\right]du$$

$$= \sum_{j=1}^{m-1} (g_{j+1}-g_j)\left[\int_0^t f(u)\ R_j(u)\ du\right]$$

$$(3\text{-}9) \qquad = \sum_{j=1}^{m-1} (g_{j+1}-g_j)\ J(j,t)$$

where

$$(3\text{-}10) \qquad J(j,t) = \int_0^t f(u)\ R_j(u)\ du.$$

Now since the functions f(t) and g($\underline{x}$) or d(i) are assumed already defined by the decision-maker, all that is required for calculation of the quality function is either set of functions $\{p_{11}(u), \ldots, p_{1n}(u)\}$, or $\{R_1(u), \ldots, R_{m-1}(u)\}$. The determination of the functions is treated in Section IV.

The quality function is the primary measure of effectiveness proposed here. It is a one dimensional deterministic function, yet it is based upon consideration of the randomness of unit performance. Since it is derived from functions defined by the decision-maker, it is consistent with his implicit judgment. And since it is a function of time, it displays sensitivity to the length of engagement.

In order to illustrate the use of the quality function in predicting unit effectiveness, this section will be closed with an example.

Example 1

Consider a tank killer team consisting of the following components:

COMPONENT	INDEX
Team Leader	1
Gunner	2
Recoilless Rifle	3

It is assumed that the unit is engaged in a large scale battle as part of a large force. The performance random vector for this unit, $\underline{X}(t)$, has an outcome space Ω_3, consisting of the eight binary 3-vectors. Let the state function be defined as follows:

$$s\{(1,1,1)\} = 1 \qquad s\{(1,0,0)\} = 5$$
$$s\{(1,0,1)\} = 2 \qquad s\{(0,1,0)\} = 6$$
$$s\{(0,1,1)\} = 3 \qquad s\{(0,0,1)\} = 7$$
$$s\{(1,1,0)\} = 4 \qquad s\{(0,0,0)\} = 8$$

The decision-maker is now faced with the problem of constructing the absolute effectiveness function $g(\underline{x})$. Suppose the hypothetical team is considered to have two missions. At any time the weapon is functioning, the primary mission is to detect enemy tanks and bring them under fire. If the weapon is in the failed condition, the secondary mission is to detect enemy tanks and direct the fire of adjacent teams against them. From experience, field test, or some other information source, the user determines that when both personnel and the weapon are intact the team is capable of bringing effective fire on eight tanks per hour. He decides to base his absolute effectiveness function on this fire capability, and his first step is to set $g\{(1,1,1)\} = 8$. He feels that in most units the leader and gunner will have equal ability in firing the weapon. Therefore when either member becomes a casualty when the weapon is functioning, the decrease in effectiveness is about the same. The user estimates the reduced capability to be about five tanks per hour, and defines $g\{(0,1,1)\} = g\{(1,0,1)\} = 5$. When the weapon fails, the team performs the secondary fire direction mission. The user determines that both members together can detect and bring fire on as many as three tanks per hour. However, if the leader alone survives, he has a capability of two tanks per hour, while the gunner alone has a capability of only one per hour because he has less experience and training than the leader in directing fire. In the two cases where both members are casualties the capability is, of course, zero. Thus the decision-maker defines his absolute effectiveness as follows:

$$g\{(1,1,1)\} = d(1) = g_6 = 8$$
$$g\{(1,0,1)\} = d(2) = g_5 = 5$$
$$g\{(0,1,1)\} = d(3) = g_5 = 5$$
$$g\{(1,1,0)\} = d(4) = g_4 = 3$$
$$g\{(1,0,0)\} = d(5) = g_3 = 2$$
$$g\{(0,1,0)\} = d(6) = g_2 = 1$$
$$g\{(0,0,1)\} = d(7) = g_1 = 0$$
$$g\{(0,0,0)\} = d(8) = g_1 = 0$$

The only requirement to be met in practice is that the resulting function faithfully reflect the true values of the decision-maker. Additionally, when several units are being compared, a separate function must be defined for each one, and however they are constructed, it must be the case that states of different units adjudged by the user to be equally effective have the same value of $g(\underline{x})$.

As indicated in (3-4), $g(\underline{x})$ can be expressed as follows: $g(\underline{x}) = (1-0)\ \phi_1(\underline{x}) + (2-1)\ \phi_2(\underline{x}) + (3-2)\ \phi_3(\underline{x}) + (5-3)\ \phi_4(\underline{x}) + (8-5)\ \phi_5(\underline{x})$ where the structure functions $\phi_i(\underline{x})$ are defined in the table below.

$\underline{x}$	$\phi_1(\underline{x})$	$\phi_2(\underline{x})$	$\phi_3(\underline{x})$	$\phi_4(\underline{x})$	$\phi_5(\underline{x})$
(1,1,1)	1	1	1	1	1
(1,0,1)	1	1	1	1	0
(0,1,1)	1	1	1	1	0
(1,1,0)	1	1	1	0	0
(1,0,0)	1	1	0	0	0
(0,1,0)	1	0	0	0	0
(0,0,1)	0	0	0	0	0
(0,0,0)	0	0	0	0	0

The block diagrams of coherent systems associated with these structure functions are shown in Figure 6.

The decision-maker also wishes to define a time variance function for the tank killer team which enhances the actual effectiveness of every state the longer the engagement lasts. He feels that enemy losses and confusion make friendly firepower more valuable later in the battle. He consequently defines

$$f(t) = 2-e^{-t}.$$

Thus $f(0) = 1$ and at the outset $h(\underline{x},0) = g(\underline{x})$, and after one hour $h(\underline{x},1) = (1.633)g(\underline{x})$.

Assume the following conditional distribution for $S(t)$ is given.

$$p_{11}(t) = e^{-7t} \qquad p_{15}(t) = e^{-3t}-e^{-4t}$$

$$p_{12}(t) = e^{-6t}-e^{-7t} \qquad p_{16}(t) = e^{-2t}-e^{-3t}$$

$$p_{13}(t) = e^{-5t}-e^{-6t} \qquad p_{17}(t) = e^{-t}-e^{-2t}$$

$$p_{14}(t) = e^{-4t}-e^{-5t} \qquad p_{18}(t) = 1-e^{-t}$$

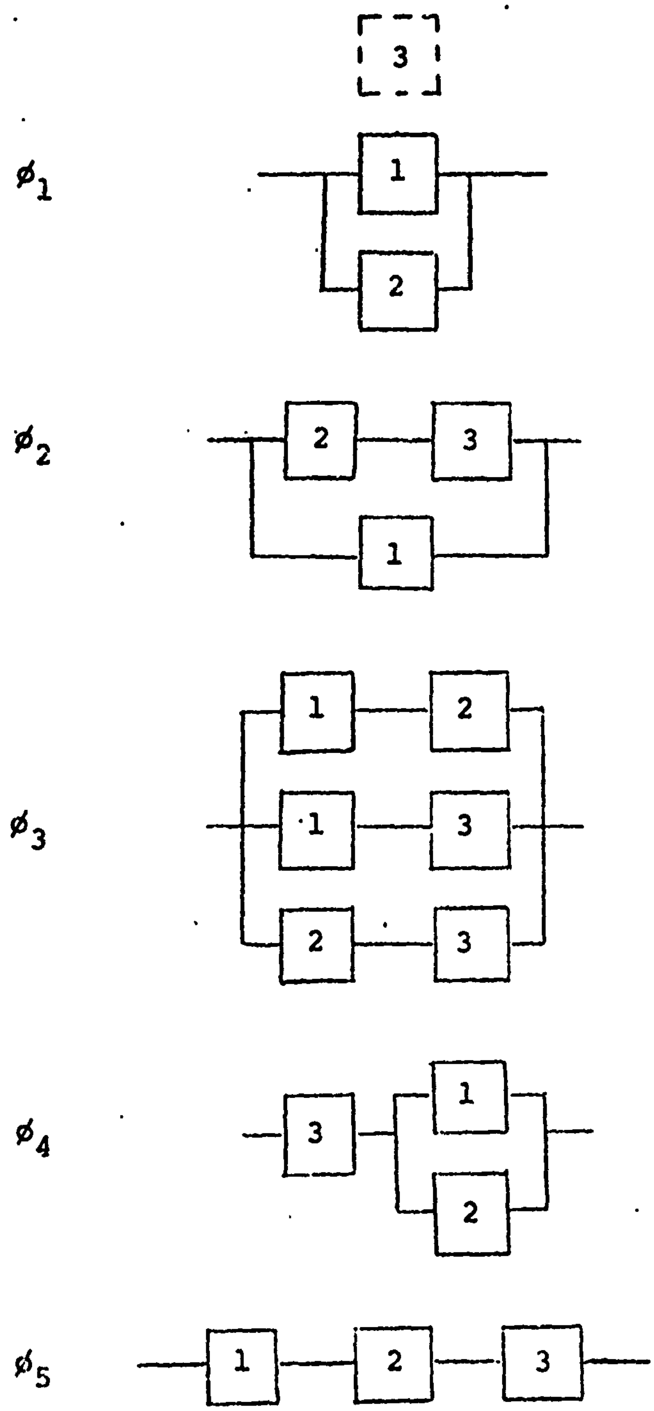

Figure 6. Coherent Systems Associated with the Absolute Effectiveness Function of Example 1.

By (3-8),

$$I(1,t) = \frac{9}{56} - \frac{2}{7} e^{-7t} \quad \frac{1}{8} e^{-8t}$$

$$I(2,t) = \frac{5}{168} - \frac{1}{3} e^{-6t} + \frac{3}{7} e^{-7t} - \frac{1}{8} e^{-8t}$$

$$I(3,t) = \frac{3}{70} - \frac{2}{5} e^{-5t} + \frac{1}{2} e^{-6t} - \frac{1}{8} e^{-7t}$$

$$I(4,t) = \frac{1}{15} - \frac{1}{2} e^{-4t} + \frac{3}{5} e^{-5t} - \frac{1}{6} e^{-6t}$$

$$I(5,t) = \frac{7}{60} - \frac{2}{3} e^{-3t} + \frac{3}{4} e^{-4t} - \frac{1}{5} e^{-5t}$$

$$I(6,t) = \frac{1}{4} - e^{-2t} + e^{-3t} - \frac{1}{4} e^{-4t}$$

$$I(7,t) = \frac{5}{6} - 2e^{-t} + \frac{3}{2} e^{-2t} - \frac{1}{3} e^{-3t}$$

$$I(8,t) = 2t - \frac{5}{2} + 3e^{-t} - \frac{1}{2} e^{-2t} .$$

And by (3-7),

$$\begin{aligned} q(t) &= \sum_{i=1}^{8} d(i)\; I(i,t) \\ &= 8\, I(1,t) + 5\, I(2,t) + 5\, I(3,t) \\ &\quad +3\, I(4,t) + 2\, I(5,t) + 1\, I(6,t) \\ &\quad +0\, I(7,t) + 0\, I(8,t) \\ &= \frac{653}{280} + \frac{3}{8} e^{-8t} - \frac{6}{7} e^{-7t} + \frac{1}{3} e^{-6t} \\ &\quad - \frac{3}{5} e^{-5t} - \frac{1}{4} e^{-4t} - \frac{1}{3} e^{-3t} - e^{-2t} . \end{aligned}$$

Now the given interval probabilities are equivalent to the following reliability functions of the associated systems in Figure 6:

$$R_1(t) = e^{-7t}$$

$$R_2(t) = e^{-5t}$$

$$R_3(t) = e^{-4t}$$

$$R_4(t) = e^{-3t}$$

$$R_5(t) = e^{-2t}$$

By (3-10),

$$J(1,t) = \frac{9}{56} - \frac{2}{7} e^{-7t} + \frac{1}{8} e^{-8t}$$

$$J(2,t) = \frac{7}{30} - \frac{2}{5} e^{-5t} + \frac{1}{6} e^{-6t}$$

$$J(3,t) = \frac{3}{10} - \frac{1}{2} e^{-4t} + \frac{1}{5} e^{-5t}$$

$$J(4,t) = \frac{5}{12} - \frac{2}{3} e^{-3t} + \frac{1}{4} e^{-4t}$$

$$J(5,t) = \frac{2}{3} - \quad e^{-2t} + \frac{1}{3} e^{-3t} .$$

And by (3-9),

$$q(t) = \sum_{j=1}^{5} (g_{j+1}-g_j)\ J(j,t)$$

$$= 3\ J(1,t) + 2\ J(2,t) + 1\ J(3,t)$$
$$+ 1\ J(4,t) + 1\ J(5,t)$$

$$= \frac{653}{280} + \frac{3}{8} e^{-8t} - \frac{6}{7} e^{-7t} + \frac{1}{3} e^{-5t}$$
$$- \frac{3}{5} e^{-5t} - \frac{1}{4} e^{-4t} - \frac{1}{3} e^{-3t} - e^{-2t} ,$$

which agrees with the previously computed result.||

IV. UNIT PERFORMANCE MODEL

The quality function, q(t), has already been introduced as a measure of effectiveness. The next step is the introduction of a model to translate the user's judgment and intuition about combat dynamics into a description of the stochastic performance process of the proposed unit structure. The primary requirements for the model are that it be sensitive to structural alternatives, and that it provide results in a form suitable for input to the quality function. There are two alternative forms of performance information which can be used in calculation of q(t). If expression (3-9) is used, the inputs required are the reliability functions of the associated coherent systems. If the calculation is performed according to expression (3-7), the information needed is the set of interval probability functions $\{p_{11}(t),...,p_{1N}(t)\}$. This second type of information is the type of performance description which is most easily determined employing the model proposed in this section.

The unit to be modeled is assumed to be a complex structure which includes, as components, personnel, equipment, and channels of information. The assumption will be made that each of the components is either functioning or failed. At the opening of an engagement all components are considered to be functioning. Some components such as personnel are nonrenewable and do not return to the functioning condition once they leave it. Other components are renewable and can return to the functioning condition repeatedly. For instance the information channel between a squad leader and a team leader which consists of voice communication can be interrupted by battle noise, then restored when the noise subsides. Therefore an engagement can be looked upon as a series of events in time which change the state of the unit, through attrition, interruption, restoration or some combination of these effects. The state of the unit remains unaltered between these events.

One approach commonly used to portray this process is to consider the performance of each component to be probabilistically independent. This assumption is not acceptable for this application since it severely limits the consideration of dependence in the attrition process.

In order to integrate structural sensitivity into the model, it is necessary that the interevent times and the nature of each event be considered to obey random distributions which depend upon the surviving structure. The following proposal for accomplishing this is derived in part from the work of Marshall and Olkin [Ref. 9].

The engaged unit is subject to a hostile environment which consists of the delivery over time of one or more types of lethal projectiles, including nuclear weapons, artillery, and small arms fire. Two assumptions are made about these fires.

1) The delivery of each type of projectile constitutes an independent time homogeneous Poisson process, having interarrival times distributed exponentially with expected value $\frac{1}{\tau_k}$, where k is the index of the weapon type.

2) For each unit state and each weapon type the damage caused by each round obeys a probability mass function: $P_k(j|i) = P$[Projectile of type k causes transition to state j | unit is in state i].

From these assumptions it follows that when the unit is in state i, the distribution of arrivals of projectiles of type k, which send the unit into state j, is Poisson having exponential interarrival times with expected value $\frac{1}{P_k(j|i)\tau_k}$.

Now in general the unit in state i can be subject to transition into state j as a result of several threats. If there are z types of projectiles which can cause the transition, then the distribution of all such events is the superposition of z independent Poisson distributions, which is itself Poisson, having exponential interarrival times with mean $\frac{1}{P_1(j|i)\tau_1 + \ldots + P_z(j|i)\tau_z}$.

In addition to attrition events there are also interruption and restoration events which describe alterations in condition of the renewable components. It is assumed that for each state i all functioning renewable components have an exponential time to failure with mean $\frac{1}{\theta_{ij}}$, where j is the state which the component's failure causes the unit to enter. Likewise, if state i has a failed renewable component, then $\frac{1}{\theta_{ik}}$ is the mean of the exponential time to restoration and consequent transition into state k.

Let T_{ij} be defined as the time interval from the unit's entry into state i until an event occurs causing transition into state j. T_{ij} is then the time until an attrition event, or an interruption-restoration event of the proper type occurs. This is again a superposition of independent Poisson processes resulting in a Poisson distribution having exponential interarrival times with expected value,

$$E[T_{ij}] = \frac{1}{P_1(j|i)\tau_1 + \ldots + P_z(j|i)\tau_z + \theta_{ij}} = \frac{1}{\lambda_{ij}},$$

and survival function

$$P[T_{ij} > t] = e^{-(\lambda_{ij})t}.$$

In general the unit in state i faces threats which can cause transition to several different states. The outcome is determined by the type of the earliest state-changing event to occur. Let T_α by defined as the length of the time interval between transition α-1 and transition α, so that

$$T_\alpha = \min\{T_{11}, T_{12}, \ldots, T_{iN}\},$$

if state i was entered at transition α-1. If transition from state i to state j is not possible, then T_{ij} is considered to be infinite, with $\lambda_{ij} = 0$. Howard [Ref. 6] refers to this type of model as a "competitive process" since it is equivalent to a process in which, upon entry into state i, the random variables T_{ij} are sampled and, in effect, compete for the earliest outcome.

The survival function of T_α is derived as follows:

$$P[T_\alpha > t] = P[T_{i1} > t,\ T_{i2} > t, \ldots, T_{iN} > t]$$

$$= P[T_{i1} > t]\ \ P[T_{i2} > t] \ldots P[T_{iN} > t]$$

$$= e^{-(\lambda_{i1} + \lambda_{i2} + \ldots + \lambda_{iN})t}$$

Thus T_α is distributed exponentially with mean,

$$E[T_\alpha] = \frac{1}{\lambda_{i1} + \lambda_{i2} + \ldots + \lambda_{iN}} = \frac{1}{\lambda_i} .$$

Thus the original assumptions of Poisson arrival of projectiles, and exponential interruption-restoration times, imply that the times between state-changing events are also exponentially distributed with mean $\frac{1}{\lambda_i}$.

The validity of these assumptions must be considered in any application of the model. The assumption of Poisson incidence of fires is characteristic of relative stability in delivery rates. This distribution would not be consistent with scenarios which included large-scale enemy reinforcement or temporary massing of fires.

The exponential distribution of the interevent times, T_α, implies the following result:

$$P[T_\alpha > \ t + s | T_\alpha > t] = \frac{e^{-(\lambda_i)(t+s)}}{e^{-\lambda_i t}} = e^{-\lambda_i s}$$

$$= P[T_\alpha > s] .$$

This result is referred to as the "memoryless" property or, in reliability theory, as the "no wear" property. The effect of the property in this model is that a unit's probability of successfully remaining in its current state for one additional minute, is completely independent of the amount of time already spent in the state. This property fails to account for such factors as fatigue or depletion of supplies. However in any case where the effect of these factors is insignificant compared with the actual attrition and interruption, the "no wear" property may prove to be acceptable.

The major consequence of the exponential assumptions is that the Markov property holds. That is, for $t \geq u \geq 0$,

$$P[S(t) = i|S(v),\ u \geq v \geq 0] = P[S(t) = i|S(u)],$$

or the distribution of future $S(t)$ depends only upon the current state, and not at all upon earlier information. The disadvantage of this property is that it does not permit consideration of such phenomena as momentum and collapse of situation. In applications where such effects are considered important, alteration would be required.

The fortunate consequence of the Markov property is that it qualifies the process, $\{S(t),\ t \geq 0\}$, as a continuous parameter Markov chain, a type of process which possesses a well developed method of solution.

The decision-maker using this model is presumed to have in mind a specific unit structure and a specific engagement scenario. From the scenario he determines the variety of projectile types to which the unit is vulnerable along with rates of fire τ_k. Then for each state i he must specify the mass functions $P_k(j|i)$ for each type threat k. These mass functions are the primary ingredient of the model's sensitivity to structure. They should be based on such variables as each individual's level of training, the capacity in which he is serving due to attrition of key personnel, the degree to which he must expose himself to hostile fire as a result of his current position, and the current condition of channels of communication. The specification of the mass functions will be the most difficult part of parameterizing the model. But it is indispensible if worthwhile investigation of unit structure is desired.

The fact that the Markov property holds leads to the derivation of two sets of simultaneous first order differential equations which can be solved for all the functions $p_{ij}(t)$. These two sets of equations are known as Kolmogorov's forward equations and backward equations. The actual derivation is not presented here but is given in Ross [Ref. 11]. Before presenting these equations, several notational concepts are introduced.

There are N^2 distinct functions $p_{ij}(t)$, which appear in the functional matrix

$$P(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1N}(t) \\ \vdots & \ddots & \vdots \\ p_{N1}(t) & \cdots & p_{NN}(t) \end{bmatrix},$$

where the function $p_{ij}(t)$ appears in the ith row and jth column. For any $t \geq 0$ the rows of $p(t)$ add to one. Since

$$p_{ij}(0) = 1 \quad i=j$$
$$= 0 \quad i \neq j,$$

it follows that

$$P(0) = I = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & 1 & 0 \\ 0 & \dots & & 0 & 1 \end{bmatrix} .$$

The functional matrix $P'(t)$ is defined as follows:

$$P'(t) = \begin{bmatrix} p'_{11}(t) & \dots & p'_{1N}(t) \\ \cdot & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p'_{N1}(t) & \dots & p'_{NN}(t) \end{bmatrix} ,$$

where

$$p'_{ij}(t) = \frac{d\ p_{ij}(t)}{dt} .$$

Using the parameters already defined in this section, the constant matrix Λ is defined as follows:

$$\Lambda = \begin{bmatrix} 0 & \lambda_{12} & \dots & \lambda_{1N} \\ \lambda_{21} & 0 & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \lambda_{N1} & & \dots & 0 \end{bmatrix} ,$$

where λ_{ij} is the reciprocal of the mean of T_{ij}, and λ_{ij} appears ith row and jth column. Since no transitions are considered from any state into itself $\lambda_{ii} = 0$, for $i = 1, \dots, N$.

The constant matrix B is defined as follows:

$$B = \begin{bmatrix} -\lambda_1 & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & -\lambda_2 & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \lambda_{N1} & & & -\lambda_N \end{bmatrix} ,$$

where $\lambda_i = \sum_{j=1}^{N} \lambda_{ij}$, the reciprocal mean of the interevent time when the unit is in state i.

Kolmogorov's equations are stated in matrix form below. The backward equations are given by

(4-1) $$P'(t) = B\,P(t) ,$$

with initial condition $P(0) = I$. The forward equations are given by

(4-2) $$P'(t) = P(t)\,B,$$

with initial condition $P(0) = I$. The operation indicated on the right of each expression is matrix multiplication. The fact that $B\,P(t) = P(t)B$ is of special interest since matrix multiplication is not, in general, commutative. The solution of the process will consist of solving the backward equations using Laplace transforms.

The Laplace transform of any functional matrix A(t), is indicated by

$$L\{A(t)\} = \tilde{A}(s) ,$$

which is simply the matrix of Laplace transforms displayed in the same order as in A(t). For any functional matrix A(t),

$$(4\text{-}3) \qquad L\{A'(t)\} = \tilde{A}'(s) = s\tilde{A}(s) - A(0),$$

an equation proved by Kreyszig [Ref. 8]. Also f : any functional matrix A(t) and any constant matrix C,

$$(4\text{-}4) \qquad L\{C\,A(t)\} = C\,\tilde{A}(s)$$

$$(4\text{-}5) \qquad L\{A(t)C\} = \tilde{A}(s)C$$

If Laplace transforms are taken of both sides of (4-1) the result is

$$L\{P'(t)\} = L\{B\,P(t)\}\ ,$$

which by (4-3) and (4-4) is equivalent to

$$s\tilde{P}(t) - P(0) = B.\tilde{P}(t)\ ,$$

or since $P(0) = I$,

$$s\tilde{P}(t) - B\,\tilde{P}(t) = I,$$

or

$$[sI - B]\,\tilde{P}(t) = I.$$

Premultiplying both sides by the inverse of [sI - B] gives,

$$\tilde{P}(t) = [sI - B]^{-1},$$

which gives the solution for L{P(t)} in terms of the constant matrix B. The solution is complete when the transform is inverted to give P(t).

Howard [Ref. 6] shows that

$$L\{P(t)\} = [sI - B]^{-1},$$

if and only if

$$P(t) = e^{Bt}\ ,$$

where

$$e^{Bt} = I + Bt + \frac{B^2}{2!}t^2 + \frac{B^3}{3!}t^3 + \ldots ,$$

is called the matrix exponential function. This expression may be used to calculate approximations to the elements of P(t), but will not provide exact solutions. Such a solution can be arrived at by finding the inverse of [sI - B] and using the method of partial fractions to translate the result into invertible Laplace transforms.

Example 2

Consider again the tank killer team of Example 1, with the same three components, all considered nonrenewable, and the same state function $s(\underline{x})$.

The scenario of interest involves two types of external threat. The first threat consists of small arms projectiles, delivered with a Poisson rate of fire τ_1 = 50/hour. The second threat is artillery fire with rate τ_2 = 10/hour.

The decision-maker now determines the approximate probability mass function for each state-threat combination.

For state one and the small arms threat, the user first notes a single round cannot cause more than one casualty. He therefore defines $P_1(5|1) = P_1(6|1) = P_1(7|1) = P_1(8|1) = 0$, since these transitions would entail multiple attrition. He feels that the gunner is about 50% more vulnerable to small arms than the leader because of his firing position, and the fact that the backblast of his weapon pinpoints his location. The weapon is considered only half as vulnerable as the leader because its small presented profile. The user thus estimates the following mass function:

$$P_1(1|1) = .94 \qquad P_1(5|1) = 0$$

$$P_1(2|1) = .03 \qquad P_1(6|1) = 0$$

$$P_1(3|1) = .02 \qquad P_1(7|1) = 0$$

$$P_1(4|1) = .01 \qquad P_1(8|1) = 0$$

where the actual magnitudes may be based on field data, simulation, or intuition.

For state one and the artillery threat multiple attrition is possible. The user feels that each artillery round has a capability to destroy all three components and defines $P_2(8|1) = .1$. The gunner is considered more vulnerable to artillery fire for identical reasons as cited under small arms. The decision-maker defines this mass function as follows:

$$P_2(1|1) = .60 \qquad P_2(5|1) = .1$$

$$P_2(2|1) = .05 \qquad P_2(6|1) = 0$$

$$P_2(3|1) = .1 \qquad P_2(7|1) = 0$$

$$P_2(4|1) = .05 \qquad P_2(8|1) = .1 \; .$$

The probability of transition into state seven is considered to be zero, since a round placed well enough to eliminate both personnel would be expected to destroy the weapon as well. Transition to state six is similarly considered impossible. The actual magnitudes are again based on the user's judgment.

Since there are no renewable components the parameters λ_{ij} can now be computed:

$$\lambda_{11} = 0 \quad \text{(trivial transitions are ignored)}$$

$$\lambda_{12} = P_1(2|1)\tau_1 + P_2(2|1)\tau_2$$

$$= (.03)(50) + (.05)(10)$$

$$= 2$$

$$\lambda_{13} = P_1(3|1)\tau_1 + P_2(3|1)\tau_2$$

$$= (.02)(50) + (.1)(10)$$

$$= 2$$

$$\lambda_{14} = P_1(4|1)\tau_1 + P_2(4|1)\tau_2$$

$$= (.01)(50) + (.05)(10)$$

$$= 1$$

$$\lambda_{15} = P_1(5|1)\tau_1 + P_2(5|1)\tau_2$$

$$= (0)(50) + (.1)(10)$$

$$= 1$$

$$\lambda_{16} = P_1(6|1)\tau_1 + P_2(6|1)\tau_2$$

$$= 0$$

$$\lambda_{17} = P_1(7|1)\tau_1 + P_2(7|1)\tau_2$$

$$= 0$$

$$\lambda_{18} = P_1(8|1)\tau_1 + P_2(8|1)\tau_2$$

$$= 0 + (.1)(10)$$

$$= 1$$

The construction of the full set of mass functions results in the constant matrix Λ shown in Figure 7, and the functional matrix [sI - B] in Figure 8.

The method of solution is to invert [sI - B] by the method of cofactors using the relation

$$\tilde{p}_{ij}(s) = \frac{(-1)^{i+j}}{|sI - B|} M_{ji}(s) .$$

where $|sI - B|$ is the determinant of the matrix [sI - B], and $M_{ji}(s)$ is the determinant derived from $|sI - B|$ by deleting the jth row and ith column.

The advantage of the cofactor method is that it permits the elements of $[sI - B]^{-1}$ to be computed individually saving time in cases where only a portion of the inverse is needed.

The results are as follows:

$$\tilde{p}_{11}(s) = \frac{1}{s + 7} = L\{e^{-7t}\},$$

$$\tilde{p}_{12}(s) = \frac{2}{(s+7)(s+5)} = \frac{1}{s+5} - \frac{1}{s+7}$$

$$= L\{e^{-5t} - e^{-7t}\},$$

$$\tilde{p}_{13}(s) = \frac{2}{(s+7)(s+6)} = \frac{2}{s+6} - \frac{2}{s+7}$$

$$= L\{2e^{-6t} - 2e^{-7t}\},$$

$$\tilde{p}_{14}(s) = \frac{1}{(s+7)(s+6)} = \frac{1}{s+6} - \frac{1}{s+7}$$

$$= L\{e^{-6t} - e^{-7t}\},$$

$$
\begin{bmatrix}
0 & 2 & 2 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Figure 7. Constant Matrix Λ

$$\begin{bmatrix} s+7 & -2 & -2 & -1 & -1 & 0 & 0 & -1 \\ 0 & s+5 & 0 & 0 & -2 & -2 & 0 & -1 \\ 0 & 0 & s+6 & 0 & -1 & 0 & -3 & -2 \\ 0 & 0 & 0 & s+6 & 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & 0 & s+4 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & s+4 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & s+4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s \end{bmatrix}$$

Figure 8. Functional Matrix [sI - B]

$$\tilde{p}_{15}(s) = \frac{1}{(s+7)(s+4)} + \frac{4}{(s+7)(s+5)(s+4)} + \frac{2}{(s+7)(s+6)(s+4)}$$

$$= \frac{1}{s+7} - \frac{1}{s+6} - \frac{2}{s+5} + \frac{2}{s+4}$$

$$= L\{e^{-7t} - e^{-6t} - 2e^{-5t} + 2e^{-4t}\},$$

$$\tilde{p}_{16}(s) = \frac{4}{(s+7)(s+5)(s+4)} + \frac{1}{(s+7)(s+6)(s+4)}$$

$$= \frac{1}{s+7} - \frac{\frac{1}{2}}{s+6} - \frac{2}{s+5} + \frac{\frac{3}{2}}{s+4}$$

$$= L\{e^{-7t} - \frac{1}{2}e^{-6t} - 2e^{-5t} + \frac{3}{2}e^{-4t}\},$$

$$\tilde{p}_{17}(s) = \frac{9}{(s+7)(s+6)(s+4)}$$

$$= \frac{3}{s+7} - \frac{\frac{9}{2}}{s+6} + \frac{\frac{3}{2}}{s+4}$$

$$= L\{3e^{-7t} - \frac{9}{2}e^{-6t} + \frac{3}{2}e^{-4t}\},$$

$$\tilde{p}_{18}(s) = \frac{48}{(s+7)(s+6)(s+4)s} + \frac{32}{(s+7)(s+5)(s+4)s} + \frac{6}{(s+7)(s+6)s}$$

$$+ \frac{2}{(s+7)(s+5)s} + \frac{4}{(s+7)(s+4)s} + \frac{1}{(s+7)s}$$

$$= \frac{1}{s} - \frac{2}{s+7} + \frac{3}{s+6} + \frac{3}{s+5} - \frac{5}{s+4}$$

$$= L\{1-2e^{-7t} + 3e^{-6t} + 3e^{-5t} - 5e^{-4t}\}.$$

As required $\sum_{j=1}^{8} p_{1j}(t) = 1,\ t \geq 0.$

The interval probabilities $\{p_{11}(t), \ldots, p_{18}(t)\}$ are now known and the quality function can be computed for any functions $f(t)$, and $g(\underline{x})$ or $d(i)$.||

It should be noted that the example was particularly simplified by the fact that the matrix [sI - B] had no non-zero entries below the diagonal. The reason for this is that all three components were considered to be nonrenewable. Thus when the unit left any state it could never return to it. And by arranging the states in order so that the earlier states corresponded to the higher rows, the triangular form was assured. The resulting denominator of all the expressions $\tilde{p}_{ij}(s)$ was of the form

$$|sI - B| = \prod_{i=1}^{8} (s+\lambda_i)$$

which is already in the factored form necessary for the partial fraction expansion.

If however at least one of the components had been renewable, there would have been at least one positive entry below the diagonal and the denominator of the expressions $\tilde{p}_{ij}(s)$ would have been of the form,

$$|sI - B| = \prod_{i=1}^{8} (s+\lambda_i) - K(s)$$

where K(s) is a polynomial function of s of degree less than eight. Therefore $|sI - B|$ is a polynomial of degree eight whose roots and factors are unknown in general. This computational problem is formidable since the degree of polynomials to be solved will equal $N = 2^n$ (where n is number of system components), and there is no exact solution for polynomial equations of degree greater than four. There are however computer routines which perform approximation procedures, which may prove to be of value in employing this model.

Finally, a rough measure of effectiveness may be computed from q(t) without inverting the Laplace transforms when $f(t) = 1,\ t \geq 0$. Let the <u>quality index</u> be defined as follows:

$$Q = q(\infty) = \sum_{j=1}^{N} d(j) \int_{o}^{\infty} p_{ij}(u)du$$

$$= \sum_{j=1}^{N} d(j) \int_{o}^{\infty} e^{-0}\ p_{ij}(u)du$$

$$= \sum_{j=1}^{N} d(j)\ \ \tilde{p}_{ij}(0)$$

Thus Q is the expected aggregate value over an infinite time period. Of course, any realistically defined unit will reach a state with zero effectiveness in finite time, so that Q has a finite value. However this measure does not differentiate between units whose quality functions are thinly distributed over the real time line and those whose effectiveness is heavily concentrated in short intervals of reasonable engagement duration.

V. CONCLUSION

The purpose of this thesis has been to outline a proposed modeling viewpoint. It has not been advanced as a finished product. It is hoped that the concepts introduced will be considered as a worthwhile point of departure for additional research, which could result in establishment of a valuable decision tool for choosing among alternative unit structures.

The most critical unsolved problem is the computational difficulty involved in solving the model. In addition to the inversion of Laplace transforms, the current solution necessitates the evaluation of large order determinants, which can become unwieldy even with electronic computers. Thus the current span of application is limited to small organizations. Research to extend this span should concentrate on finding simplified computational procedures for exact solution, or alternatively, determining efficient and accurate approximation methods. The use of the infinite series definition of the matrix exponential function as an approximation method was introduced in Section IV. Unfortunately, this expression requires sequential multiplication of large order matrices, an operation which is as computationally explosive as evaluation of determinants.

Current research in reliability theory may offer a valuable approximation method, particularly since all of the random processes and measures of effectiveness were shown to be expressible as linear functions of variables related to reliability of coherent systems. Esary and Proschan [Ref. 4] show the usefulness of reliability bounds and approximations based on the concepts of minimal cut sets and minimal path sets of components. Adaptation of this method to the available type of data inputs may prove useful to the current model.

A possible exact solution method using flow graph analysis is proposed by Howard [Ref. 6], but at present it appears to be subject to an even greater computational burden than the current method.

In order to enhance the accuracy of the model it may be desirable to alter the assumption of exponentially distributed interevent times in favor of some other distribution. This would change the process to one of the semi-Markov variety. These processes are more difficult to solve in general, but offer greater flexibility in definition. Ross [Ref. 11] and Howard [Ref. 6] consider several solution methods.

A highly desirable addition to the model would be a provision for random threat. Rather than assuming constant Poisson rates of enemy fire, such a refinement would allow rates to vary randomly, thus representing periods of non-engagement as well as change of situation during engagements. The analogous reliability concepts of random wear and random shock have been studied by Gaver [Ref. 5] and Reynolds and Savage [Ref. 10].

Despite its capacity for refinement, the proposed model in present form does fill many of the needs for the study of structure. It offers the decision-maker a model which uses his own judgment, in an intuitively clear process of parameterization, in order to compute a functional measure of effectiveness based on fundamental functions which he, himself, defines. Indeed, with the exception of clearly stated assumptions made in building the model, and the purely mathematical solution process, the model in use is completely established by the user, with whom ultimate decision responsibility rests.

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