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A NOTE ON HOMOGENEOUS PROCESSES WITH INDEPENDENT INCREMENTS

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iii

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13. ABSTRACT

A class of stochastic processes, X(t) is characterized based in the property that the conditional mean and variance of X(t), given $X(t_1) = y$, for some $0 < t < t_1$, are linear functions of y. Two particular cases resulting in Wiener and Poisson processes are discussed.

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1. INTRODUCTION.

Let $X(t)$, $t \geq 0$, be a homogeneous stochastic process with independent increments. Fix $t_1 > 0$. Then, given $X(t_1) = y$, the process $X(t)$, for $0 \leq t \leq t_1$, is called a tied-down process with end point equals y. Suppose $X(t)$ is a Poisson process, then the conditional distribution of $X(t)$ given $X(t_1) = y$, for all $0 \leq t \leq t_1$, is binomial with parameter $(y, t/t_1)$, and therefore the conditional expectation and variance of $X(t)$ given $X(t_1) = y$ are linear functions of y. Suppose $X(t)$ is a Wiener process, then the conditional distribution of $X(t)$ given $X(t_1) = y$, for all $0 \leq t \leq t_1$, is normal with parameter $((t/t_1)y, \sigma^2 t(1 - t/t_1))$, and hence the conditional expectation of $X(t)$ given $X(t_1) = y$ is a linear function of y and the conditional variance does not depend upon y.

In this note, we shall characterize a class of stochastic processes based on the property that the conditional mean and variance of $X(t)$, given $X(t_1) = y$, for some $0 < t < t_1$, are linear functions of y. It will be proved that if $E(X(t) \mid X(t_1) = y) = \alpha_0 + \alpha_1 y$, then 1) $Var(X(t) \mid X(t_1) = y) =$ constant a.e. if and only if $X(t) = W(t) + \mu t$, where $W(t)$ is a Wiener process and μ is a real constant,

(2) $\mathrm{Var}(X(t) \mid X(t_1) = y) = \beta_0 + \beta_1 y$ $(\beta_1 \neq 0)$ if and only if $X(t) = cY(t) - \nu t$, where $Y(t)$ is a Poisson process and ν and c are real constants. To avoid trivial cases, we shall assume that $X(t)$ is not a degenerate process. Also all stochastic processes $X(t)$, $t \geq 0$, considered in this note are assumed to be homogeneous, second-order and with independent increments.

For a recent survey of the results on characterizations of stochastic processes see the paper [1] by Lukacs.

2. THE RESULT.

We need the following two lemmas.

LEMMA 1. *If* $E(X(t) \mid X(t_1) = y) = \alpha_0 + \alpha_1 y$, *then* $\alpha_0 = 0$ *and* $\alpha_1 = t/t_1$, *for all* $0 \leq t \leq t_1 < \infty$.

PROOF. Since $X(t)$ is a homogeneous, second-order process with independent increments, it follows that $\mu(t) = E(X(t)) = \mu t$, $\sigma^2(t) = \mathrm{Var}(X(t)) = \sigma^2 t$, for all $t \geq 0$, and $\rho(t_1, t_2) =$ Corr. Coeff. $(X(t_1), X(t_2)) = \min(t_1, t_2)/\sqrt{t_1}\sqrt{t_2}$, for all $t_1 > 0$ and $t_2 > 0$, where $\mu = E(X(1))$ and $\sigma^2 = \mathrm{Var}(X(1))$.

Therefore

$$\alpha_1 = \rho(t, t_1)\, \sigma(t)/\sigma(t_1) = t/t_1,$$

and

$$\alpha_0 = \mu(t) - \alpha_1 \mu(t_1) = 0.$$

LEMMA 2. *Let* $g(s, t) = E(e^{isX(t)})$ *be the characteristic function of* $X(t)$ *and* $\gamma(s) = g(s, 1)$. *Then*

$$E(X(t_0)e^{isX(t_1)}) = -it_0 g^{t_1 - 1}(s)\, g'(s)$$

and

$$E(X^2(t_0)e^{isX(t_1)}) = -t_0(t_0 - 1)g^{t_1 - 2}(s)(g'(s))^2 - t_0 g^{t_1 - 1}(s)\, g''(s),$$

for all $0 < t_0 < t_1$ *and real* s.

PROOF. Because $X(t)$ is a homogeneous, second-order process with independent increments, $g(s, t) = g^t(s)$ and the second partial derivative of $g(s, t)$ w.r.t. s exists for all t and s. It then follows that

$$E(X(t)e^{isX(t)}) = -itg^{t - 1}(s)\, g'(s)$$

and

$$E(X^2(t)e^{isX(t)}) = -t(t - 1)g^{t - 2}(s)(g'(s))^2 - t\, g^{t - 1}(s)\, g''(s),$$

for all real s and $t > 0$.

Then Lemma 2 follows from the fact that

$$E(X^k(t_0)e^{isX(t_1)}) = E(X^k(t_0)e^{isX(t_0)})\, g(s, t_1 - t_0)$$

for all k, $0 \leq t_0 \leq t_1$ and real s.

We now state and prove the main result of this note. For more detail of the proof, especially the last part, see the proof of theorem 2.1 in [2] by this author.

THEOREM. *Let* $0 < t_0 < t_1$. *Then the necessary and sufficient condition that*

(1) $$E(X(t_0) \mid X(t_1) = y) = \alpha_0 + \alpha_1 y$$

and

(2) $$\mathrm{Var}(X(t_0) \mid X(t_1) = y) = \beta_0 + \beta_1 y \quad \text{a.e.},$$

where α_0, α_1, β_0 *and* β_1 *are constants w.r.t.* y, *is that*

1) $X(t) = W(t) + \mu t$, *if* $\beta_1 = 0$, *where* $W(t)$ *is a Wiener process and* μ *is a real constant,*

2) $X(t) = cY(t) - \nu t$, *if* $\beta_1 \neq 0$, *where* $Y(t)$ *is a Poisson process and* ν *and* c *are real constants.*

PROOF. The sufficient condition can be verified by a straightforward calculation.

To prove the necessary condition. Suppose the conditions (1) and (2) hold. Then by lemma 1, the conditions (1) and (2) imply

(3) $$E(X^2(t_0)e^{isX(t_1)}) - (t_0/t_1)^2E(X^2(t_1)e^{isX(t_1)}) = \beta_0E(e^{isX(t_1)}) + \beta_1E(X(t_1)e^{isX(t_1)})$$

for all real s.

By lemma 2, equation (3) is equivalent to

(4) $$t_0(1 - t_0/t_1)\{g^{t_1 - 2}(s)(g'(s))^2 - g^{t_1 - 1}(s)\, g''(s)\} = \beta_0 g(s) - i\beta_1 t_1 g^{t_1 - 1}(s)\, g'(s),$$

for all real s.

Without loss of generality, we may assume that $g(s) \neq 0$ for all s. And we rewrite equation (4) in the form

(5) $$\frac{d}{ds}\Big(g'(s)/g(s)\Big) = -B_0 + iB_1\Big(g'(s)/g(s)\Big),$$

where $B_0 = \beta_0/(t_0(1 - t_0/t_1))$ and $B_1 = \beta_1 t_1/(t_0(1 - t_0/t_1))$.

Because the second derivative of $g(s)$ exists and does not vanish for the s in a neighborhood N of the origin and $g'(s)/g(s)$ is independent of t_0 and t_1, B_0 and B_1 are independent of t_0 and t_1.

In addition, it is easy to check that if $\beta_1 = 0$, then $B_0 > 0$.

The solution of equation (4) is, if $\beta_1 = 0$,

$$g(s) = \exp\left\{ i\mu s - \frac{1}{2}\sigma^2 s^2 \right\},$$

where μ is a real sonctant and $\sigma^2 = B_0 > 0$, and if $\beta_1 \neq 0$,

$$g(s) = \exp\left\{ - i\nu s + \lambda(e^{ics} - 1)\right\},$$

where λ is a positive real constant independent of t and s, and $\nu = B_0/B_1$, $c = B_1$.

Therefore, the characteristic function of $X(t)$ is, if $\beta_1 = 0$,

$$g(s,t) = \exp\left\{ i\mu ts - \frac{1}{2}\sigma^2 ts^2 \right\},$$

and, if $\beta_1 = 0$,

$$g(s, t) = \exp\left\{- i\nu ts + \lambda t(e^{ics} - 1) \right\}.$$

This completes our proof of the theorem.

The following two corollaries of the theorem are characterizations of the Wiener and the Poisson processes.

COROLLARY 1. _If_ $E(X(t)) = 0$ _for some_ $t > 0$. _Then the necessary and sufficient condition that_ $X(t)$ _is a Wiener process is that_

$E(X(t_0) \mid X(t_1) = y)$ *is a linear function of* y *and* $Var(X(t_0) \mid X(t_1) = y)$ *is constant* a.e. *for some* $0 < t_0 < t_1$.

COROLLARY 2. *The necessary and sufficient condition that* $X(t)$ *is a Poisson process is that* $E(X(t_0) \mid X(t_1) = y)$ *is a linear function of* y *and* $Var(X(t_0) \mid X(t_1) = y) = (t_0/t_1)(1 - t_0/t_1)y$ a.e. *for some* $0 < t_0 < t_1$.

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