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RECURRENCE FORMULA FOR THE VENEZIANO MODEL N-POINT FUNCTIONS

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13. ABSTRACT

A recurrence formula is derived for a function which reduces to the Veneziano model $(n+3)$-point function. It is shown that the formula is equivalent to, but is more self-contained than, the Hopkinson and Plahte formula in that it does not require the prescription for the parameters involved.

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Recurrence Formula for the Veneziano Model *N*-Point Functions

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(Received 1 May 1972)

A recurrence formula is derived for a function which reduces to the Veneziano model $(n+3)$-point function. It is shown that the formula is equivalent to, but is more self-contained than, the Hopkinson and Plahte formula in that it does not require the prescription for the parameters involved.

The extension to the n-point functions of the Veneziano's four-point function was accomplished either by generalizing the integral representation for the beta function which comprises the essential ingredient of the four-point function[1,2] or by generating a recurrence formula for the n-point function.[3] In the latter approach, the authors attempted to justify the formula for arbitrary value of n after showing, through introduction of the integral representation for the beta function, that the formula produces the already known integral expressions for the cases of $n = 5, 6,$ and 7. The recurrence formula which has apparently been discovered on a heuristic basis is not necessarily very transparent, as the authors themselves admit it, especially in connection with the definition of their variables x'_{ij}.

Recently it was pointed out that the generalized Veneziano amplitudes may be regarded as the boundary values of a class of generalized hypergeometric functions that are Radon transforms of products of linear forms.[4] In a work by the present author which shows that the amplitudes possess a structure similar to that of the Lauricella's hypergeometric functions,[5] he has made an iterative use of a recurrence formula for the amplitude.[6]

The purpose of this note is to point out that the author's recurrence formula obtains itself in a quite natural manner such that for a special choice of the variables, it reproduces the formula proposed in Ref. 3 without requiring any prescription for the parameters involved therein.

To begin our discussion, let us consider a function V_n of variables w_{ij} defined as below:

$$\begin{aligned}
&V_n(\alpha_{01}, \alpha_{02}, \ldots, \alpha_{0n}; \alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n};\\
&\quad \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2,n-1}; \alpha_{31}, \ldots, \alpha_{3,n-2}; \ldots;\\
&\quad \alpha_{n-1,1}, \alpha_{n-1,2}; \alpha_{n1} \,|\, w_{02}, w_{03}, \ldots, w_{0n};\\
&\quad w_{13}, \ldots, w_{1n}; \ldots; w_{n-2,n})\\
&= \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} \Big\{ du_i\, u_i^{\alpha_{0i}-1}(1-u_i)^{\alpha_{1i}-1}\\
&\quad \times \prod_{k=2}^{i} \Big[1 - u_i\Big(w_{i-k,i} \prod_{j=0}^{k-2} u_{i-j-1}\Big)\Big]^{\alpha_{k,i-k+1}} \Big\}. \qquad (1)
\end{aligned}$$

When the parameters α_{ij} are regarded as functions of the momenta p_i, $i = 1, 2, \ldots, n+3$, of the external particles, it will be seen that V_n for $w_{ij} = 1$ can readily be related to the known integral representation for the $(n+3)$-point function.[7]

In carrying out the multiple integrations in Eq. (1), use has been made in Ref. 6 of the following recurrence formula:

$$\begin{aligned}
&V_n(\alpha_{01}, \ldots, \alpha_{0n}; \alpha_{11}, \ldots, \alpha_{1n}; \ldots; \alpha_{n1}\\
&\quad |\, w_{02}, \ldots, w_{0n}; w_{13}, \ldots, w_{1n}; \ldots; w_{n-2,n})\\
&= B(\alpha_{0n}, \alpha_{1n}) \sum \frac{(\alpha_{0n}, \beta_{n,n-1})}{(\alpha_{0n} + \alpha_{1n}, \beta_{n,n-1})}\\
&\quad \times \frac{(-\alpha_{2,n-1}, r_{2,n-1}) \cdots (-\alpha_{n1}, r_{n1})}{(1, r_{2,n-1}) \cdots (1, r_{n1})}\\
&\quad \times (w_{n-2,n})^{r_{2,n-1}} (w_{n-3,n})^{r_{3,n-2}} \ldots (w_{0n})^{r_{n1}}\\
&\quad \times V_{n-1}(\alpha_{01} + \beta_{n1}, \ldots, \alpha_{0,n-1} + \beta_{n,n-1}; \alpha_{11}, \ldots,\\
&\quad \alpha_{1,n-1}; \ldots; \alpha_{n-1,1} \,|\, w_{02}, \ldots, w_{0,n-1}; w_{13}, \ldots, w_{1,n-1};\\
&\quad \ldots; w_{n-3,n-1}). \qquad (2)
\end{aligned}$$

In Eq. (2) the summation is over the integers between 0 and ∞ of the indexes r_{ij}, B stands for the beta function, (α, r) under the summation symbol is written for $\Gamma(\alpha + r)/\Gamma(\alpha)$, and β_{pq} are given by

$$\beta_{pq} = \sum_{k=p-q+1}^{p} r_{k,p-k+1} \quad \text{for} \begin{cases} p = 2, 3, \ldots, n \\ q = 1, 2, \ldots, p-1. \end{cases} \qquad (3)$$

Our task in what follows is to show that when $w_{ij} = 1$, Eq. (2) reduces to the recurrence formula for the N-point function $B_N(x)$ in Ref. 3 as given below:

$$\begin{aligned}
B_N(x) = \sum_{k_{i,N-1}=0}^{\infty} &\left[\prod_{i=2}^{N-3} (-1)^{k_{i,N-1}} \binom{z_{i,N-1}}{k_{i,N-1}}\right]\\
&\times B_4\Big(x_{N-2,N-1}, x_{N-1,N} + \sum_{i=2}^{N-3} k_{i,N-1}\Big)\; B_{N-1}(x'), \qquad (4)
\end{aligned}$$

where

$$x_{ij} = -\alpha(s_{ij}) \quad \text{with} \quad s_{ij} = (p_i + p_{i+1} + \cdots + p_j)^2,$$
$$z_{ij} = x_{ij} - x_{i+1,j} - x_{i,j-1} + x_{i+1,j-1},$$

and x'_{ij} are defined according to certain rules (given in a tabular form) which will not be reproduced here. More noteworthy of the present formula is the fact that Eq. (2) is self-contained such that in contrast to Ref. 3, there is required no prescription for defining the parameters of the function V_{n-1} which corresponds to B_{N-1} of Eq. (4).

In order to achieve the above we have to establish the relation between our parameters α_{ij} and those of Ref. 3. For this purpose let us note first that the external lines are labeled $1, 2, \ldots, n+3$ both for the $(n+3)$-point function in Ref. 1 and for V_n in the present paper, while they are labeled $0, 1, \ldots, n+2$ for the $(n+3)$-point function B_{n+1} of Ref. 7. Further, we note that the integration variables $u_1, u_2, \ldots, u_n$ in Ref. 7 and the present paper may be made to correspond to $u_{12}, u_{13}, \ldots, u_{1,n+1}$ of Ref. 1. With this in mind one can compare Eq. (1) with the corresponding expression that follows from the representation for $B_{n+3}(x)$ of Ref. 1 through rearrangement of the integrand. Namely, by introducing $x_{ij} = -\bar{\alpha}_{ij} - 1$ from Ref. 1, where we write $\bar{\alpha}_{ij}$ for $\alpha_{ij} = \alpha(s_{ij})$ of Ref. 1, it becomes possible to express our α_{ij} in terms of $\bar{\alpha}_{km}$. If we further write $\xi_{ij} = -\bar{\alpha}_{ij}$ with $\xi_{ii} = 0$ and

$$\zeta_{ij} = \xi_{ij} - \xi_{i+1,j} - \xi_{i,j-1} + \xi_{i+1,j-1}, \qquad (5)$$

where ξ_{ij} and ζ_{ij} stand for x_{ij} and z_{ij}, respectively, of Ref. 3, it follows that

$$\left.\begin{aligned}\alpha_{0i} &= \xi_{i,i+1}\\ \alpha_{1i} &= \xi_{i+1,i+2}\end{aligned}\right\} \quad \text{for} \quad i = 1, 2, \ldots, n \tag{6}$$

and

$$\alpha_{ki} = \zeta_{i+1,i+k+1} \quad \text{for} \quad k = 2, 3, \ldots, n$$
$$\text{and } i = 1, 2, \ldots, n - k + 1. \tag{7}$$

That the integral in Eq. (1) reduces for $w_{ij} = 1$ to B_{n+1} of Ref. 7 can be seen from the observation that our $p_i, i = 1, 2, \ldots, n + 3$ correspond to $p_i, i = 0, 1, \ldots, n + 2$ of Ref. 7 and through specialization of the relation $\bar{\alpha}(s_{ij}) = \alpha' s_{ij} + \alpha_{ij}^0$ of Ref. 1 to the form $bs_{ij} + a$ as is done in Ref. 7.

Although we have connected our parameters to those of Ref. 3 in Eqs. (6) and (7), the precise correspondence between Eqs. (2) and (4) cannot be considered complete until the arrangement of ξ_{ij} in $B_N(\xi)$ of Eq. (4) is unambiguously established. [In Ref. 3 this arrangement has been left out unstated, which fact is responsible in part for requiring the somewhat troublesome rules for determining x' in $B_{N-1}(x')$ which should have really been unnecessary, as will be shown below.]

Let us suppose that the correspondence between V_n in this paper and B_N, for $N = n + 3$, of Ref. 3 is given by the following:

$$\begin{aligned}&V_n(\alpha_{01}, \ldots, \alpha_{0n}; \alpha_{11}, \ldots, \alpha_{1n}; \alpha_{21}, \ldots, \alpha_{2,n-1}; \ldots;\\ &\quad \alpha_{n1} | w_{02}, \ldots, w_{0n}; \ldots; w_{n-2,n})\\ &\quad \longleftrightarrow B_{n+3}(\xi_{12}, \xi_{23}, \ldots, \xi_{n+1,n+2}; \xi_{13}, \xi_{24}, \ldots,\\ &\quad \xi_{n,n+2}; \xi_{14}, \xi_{25}, \ldots, \xi_{n-1,n+2}; \ldots; \xi_{1,n+1}, \xi_{2,n+2}).\end{aligned} \tag{8}$$

Then the transition from V_n to B_{n+3} and vice versa can be effected on a firm basis by referring to Eqs. (6) and (7).

With the help of Eq. (8) we now can translate Eq. (2) into a formula which is given in terms of the function B_N:

$$\begin{aligned}&B_{n+3}(\xi_{12}, \xi_{23}, \ldots, \xi_{n+1,n+2}; \xi_{13}, \xi_{24}, \ldots,\\ &\quad \xi_{n,n+2}; \ldots; \xi_{1,n+1}, \xi_{2,n+2})\\ &= \sum \frac{(-\zeta_{n,n+2}, r_{2,n-1}) \cdots (-\zeta_{2,n+2}, r_{n1})}{(1, r_{2,n-1}) \cdots (1, r_{n1})}\\ &\quad \times B_4(\zeta_{n+1,n+2}, \xi_{1,n+1} + \beta_{n,n-1})\\ &\quad \times B_{n+2}(\xi_{12} + \beta_{n1}, \xi_{23}, \ldots, \xi_{n,n+1}; \xi_{13} + \beta_{n2},\\ &\quad \xi_{24}, \ldots, \xi_{n-1,n+1}; \ldots; \xi_{1n} + \beta_{n,n-1}, \xi_{2,n+1}).\end{aligned} \tag{9}$$

Note that we wrote B_4 for B, and use was made of the following relation in obtaining Eq. (9):

$$B(\alpha_{0n}, \alpha_{1n})\,(\alpha_{0n}, \beta_{n,n-1})/(\alpha_{0n} + \alpha_{1n}, \beta_{n,n-1}) = B(\alpha_{1n}, \alpha_{0n} + \beta_{n,n-1}).$$

That Eq. (9) is identical to Eq. (4) with $N = n + 3$, $x \to \xi$, and $z \to \zeta$, can be checked easily. This establishes, therefore, that the order in which x_{ij} appears in $B_N(x)$ of Eq. (4), which was not stated explicitly in Ref. 3, should be exactly as is displayed in B_{n+3} of Eq. (8)

We emphasize that the recurrence formula for the $(n + 3)$-point function, Eq. (9), as derived from Eq. (2) is complete as it stands and requires no rules for defining the parameters of the function B_{n+2}. In connection with the table for x'_{ij} of $B_{N-1}(x')$ in Ref. 3, we note that not all the entries in the table are actually needed for the recurrence formula. In fact, what is needed is only that portion of the table for $i = 1$, $j < N - 2$ and $i > 1$, $j \leq N - 2$ because, as may be seen from the arguments of B_{n+2} in Eq. (9), we require only ξ_{1j} for $j < N - 2 = n + 1$ and ξ_{ij} with $i > 1$ for $j \leq N - 2 = n + 1$. Moreover, there arises no need for including in the table the relation $x'_{1j} = x_{1j} + \sum_2^{j-1} k_{\alpha,N-1}$, for $j = N - 2$, unless we unnecessarily rewrite the argument $x_{N-1,N} + \sum_2^{N-3} k_{i,N-1}$ in B_4 of Eq. (4) as $x_{1,N-2} + \sum_2^{N-3} k_{i,N-1} = x'_{1,N-2}$. Finally, it is noted further that x'_{ij} for $i > 1$ and $j = N - 1$ should have not been included in the table since no such variables are actually involved in the recurrence formula for $B_N(x)$.

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