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COMPARISON OF A DETERMINISTIC AND A STOCHASTIC MODEL FOR THE PROBABILITY OF WINNING IN A TWO-SIDED COMBAT SITUATION

Woo Young Lee, et al

Naval Postgraduate School
Monterey, California

September 1972

NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

Comparison of a Deterministic and a
Stochastic Model for the Probability of
Winning in a Two-sided Combat Situation

by

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September 1972

43

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

COMPARISON OF A DETERMINISTIC AND A STOCHASTIC MODEL FOR THE PROBABILITY OF WINNING IN A TWO-SIDED COMBAT SITUATION

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)
Master's Thesis; September, 1972

5. AUTHOR(S) (First name, middle initial, last name)

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6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1972	43	9

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

Combat attrition can be modelled as either a deterministic or a stochastic process. In this thesis the forecasts of combat outcomes generated by deterministic and stochastic Lanchaster-type models for combat between two homogeneous forces are compared. Using stochastic formulations, the probability of winning is studied and contrasted with a deterministic win criterion for some idealized combat situations. Force ratios required to assure victory in an attack are studied from this standpoint.

I

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

UNCLASSIFIED

Security Classification

A-31408

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
COMBAT ATTRITION COMPARISON						

II

DD FORM 1 NOV 65 1473 (BACK)

S/N 0101-807-6821

UNCLASSIFIED

Security Classification

A-31409

Comparison of a Deterministic and a Stochastic Models
for the Probability of Winning in a
Two-sided Combat Situation

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Submitted in partial fulfillment of the
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MASTER OF SCIENCE IN OPERATIONS RESEARCH

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I. INTRODUCTION

A. REVIEW OF LANCHESTER THEORY OF COMBAT

F. W. Lanchester (1868-1946) did pioneering work in military operations research by attempting to quantitatively justify VonClausewitz's principle of concentration. His work led him to postulate a simple deterministic model for the mutual attrition of two homogeneous forces. This pioneering work first appeared in a series of articles in British Journal Engineering during 1914. Since that time deterministic formulations have been vastly extended. The attrition process has also been viewed from the stochastic standpoint.

During World War II, B. Q. Koopman extended Lanchester's original results and also suggested a reformulation of the problem in stochastic form. This stochastic attrition process has been approximately termed by Koopman as the Lanchester stochastic process. Other workers have subsequently employed a stochastic analysis of combat.

Among these various works, Brackney in his paper, "The Dynamics of Military Combat," ([3]), categorized the types of combat situation into nine separate, clearly defined situations.

Subsequent workers (including Smith [8] whose work is reviewed below) have used Brackney's scheme for classifying combat situations.

Analogous to the deterministic solution of Lanchester's equation, Koopman (whose work is reported in Morse and Kimball [5]) developed the probability that there will be some survivors on one side when the other side is anihilated (i.e. the probability of winning) when the casualty rate on each side is proportional to the product of the number of firers and the number of targets. This, of course, corresponds to the deterministic "linear-law" attrition process.

Later, Brown [6] developed the probability of winning corresponding to a deterministic "square-law" attrition process (i.e. casualty rate on each side proportional to the number of enemy firers). Brown also gives some approximations for computing the probability of winning. Smith [8] gave the solution for the remaining case ("mixed law").

In this thesis results from the stochastic and deterministic Lanchester models are contrasted, much in the same spirit as George H. Weiss did in [9].

B. STUDY OBJECTIVES

The term "Lanchester theory of combat" is commonly taken to refer to a set of theories which attempt to explain changes in force levels due to combat attrition in terms of weapon system performance characteristics, force levels, and composition. There are two general approaches to modeling the combat attrition process:

(1) deterministic formulation which takes the form of a system of first order differential equations

(2) Stochastic formulation which views the casyaktt process as a Markov process.

In this thesis some idealized models of combat between two homogeneous forces are considered. The forecasts of victory from stochastic and deterministic models are compared. Numerical computations of the probability of winning are performed according to theoretical results which have appeared in the literature.

Numerical computations of probability of winning were done by the IBM 360/67 computer at the Naval Postgraduate School. The computational results will be discussed by considering graphs and tables for various cases.

Similarities and differences between the deterministic and the stochastic models will be discussed. Based upon the computational results obtained, the tactic "Don't attack an enemy position without at least a three-to-one initial force ratio" will be evaluated

The model parameters selected for the computations presented in this thesis were chosen to be in agreement with the military experience of the authors. Although rather idealized situations were studied in order to preserve analytic tractability of the models, many of the quantitative results discussed in this thesis apply in more complex situations.

II. COMBAT BETWEEN TWO HOMOGENEOUS FORCES

Brackney [3] considered nine types of combat situations. These nine situations correspond to all the ways of combining three basic attrition-rate schemes for each side. Three basic situations are shown below for combat between homogeneous X- and Y-forces.

$$\frac{dx}{dt} = -\alpha \; ; \quad \frac{dy}{dt} = -\beta \tag{1}$$

$$\frac{dx}{dt} = -\alpha y \; ; \quad \frac{dy}{dt} = -\beta x \tag{2}$$

$$\frac{dx}{dt} = -\alpha xy ; \quad \frac{dy}{dt} = -\beta x \tag{3}$$

where x and y are the numbers of surviving forces at time t on Red and Blue side respectively, α and β are the constant of proportionality (attrition-rate coefficients) and are equal to the effectiveness per unit force for each side.

We shall call (1) the linear law, (2) the square law, (3) the mixed law. The reason for the name "linear," "square" and "mixed" law is explained by Lanchester [2].

For mathematical convenience, only the case of fighting to finish, that is when one side is anihilated , is considered in this thesis.

A. LINEAR LAW ATTRITION ON BOTH SIDES

1. Assumptions

(1) Two forces attack each other. Each unit (man) on each side is within range of all units of the other side.

(2) Units on each side are identical but the units on one side may have a different attrition rate than the opposing units.

(3) Each firing unit is aware only of the general area in which enemy forces are located and fires into this area without knowledge of consequence of this fire.

(4) Fire from surviving unit is distributed uniformly over the area in which enemy are located.

2. Deterministic Model

The two sides are designated as "Red" and "Blue".

x_o, y_o = initial force levels.

$x(t)$, $y(t)$ = force levels on the side of "Red" and "Blue" respectively at time t.

Then the Lanchester equations are:

$$\frac{dx}{dt} = -\alpha xy \quad ; \quad \frac{dy}{dt} = -\beta xy$$

The solution with time eliminated is

$$\beta\{x_o - x(t)\} = \alpha\{y_o - y(t)\}$$

Since we are considering the fight to finish only, we could think the probability of winning for the Blue can be represented as follows:

$$P(B,y_o) = \begin{cases} 1 & \text{if } \alpha/\beta \geq \dfrac{x_o}{y_o} \\ 0 & \text{if } \alpha/\beta \leq \dfrac{x_o}{y_o} \end{cases}$$

where $P(B,y_o)$ is the probability of Blue win with initial force level y_o. This probability can be charged as

$$P(B,y_c) = \begin{cases} 1 & \text{if } 1 \geq \dfrac{\beta x_o}{\alpha y_o} \\ 0 & \text{if } 1 \leq \dfrac{\beta x_o}{\alpha y_o} \end{cases}$$

Let us hence forth refer to $\frac{\beta x_o}{\alpha y_o}$ as the normalized force ratio.

3. <u>Stochastic Model</u>

Define $P(0,n,M,N)$ as the probability that there will be n survivors on the Blue side when the Red side is anihilated.

$P(B,N)$ the probability of Blue win with initial force level N

where M,N = initial numbers of units

m,n = number of surviving units on the side of "Red" and "Blue" respectively at time t.

Then

$$P(0,n,M,N) = \left(\frac{\alpha}{\alpha+\beta}\right) P(0,n,M-1,N) + \left(\frac{\beta}{\alpha+\beta}\right) P(0,n,M,N-1)$$

for notational conveneince, let $p = \frac{\alpha}{\alpha+\beta}$, $q = \frac{\alpha}{\alpha+\beta}$ and $P(0,n,M,N) = F(M,N)$

$$F(M,N) = pF(M-1,N) + qF(n,N-1)$$

with $F(0,n) = 1$

$F(0,n) = 0$

otherwise solution (can be verified by substitution) is

$$F(M,N) = \binom{M+N-n-1}{M-1} p^M q^{N-n} = P(0,n,M,N)$$

Hence

$$P(B,N) = \sum_{n=1}^{N} P(0,n,M,N)$$

$$= \sum_{n=1}^{N} \binom{M+N-n-1}{M-1} p^N q^{N-m}$$

let $\quad j = N-n$

then $\quad n = 1$ implies $j = N-1$

$\quad n = N$ implies $j = 0$

Then

$$P(B,N) = \sum_{j=0}^{N-1} \binom{M+j-1}{M-1} p^M q^j$$

or

$$P(B,N) = \left(\frac{\alpha}{\alpha+\beta}\right)^n \sum_{j=0}^{N-1} \binom{M+j-1}{j} \left(\frac{\beta}{\alpha+\beta}\right)^j$$

B. SQUARE LAW ATTRITION OR BOTH SIDES

1. Assumptions

(1) as in Case A

(2) as in Case A

(3) Each firing unit is sufficiently well aware of the location and condition of all enemy units so that when a target is killed, fire may be immediately shifted to a new target

(4) Fire is uniformly distributed over surviving units.

2. Deterministic Model

The Lanchester-type equations are

$$\frac{dx}{dt} = -\alpha y$$

$$\frac{dy}{dt} = -\beta x$$

with solution

$$\beta\{x_o - x^2(t)\} = \alpha\{y_o - y^2(t)\}$$

Then the probability of winning for Blue can be represented as:

$$P(B, y_o) = \begin{cases} 1 & \text{if} \quad \alpha/\beta \geq \dfrac{x_o^2}{y_o^2} \\ 0 & \text{if} \quad \alpha/\beta \leq \dfrac{x_o^2}{y_o^2} \end{cases}$$

This can be changed as

$$P(B, y_o) = \begin{cases} 1 & \text{if} \quad 1 \geq \sqrt{\beta/\alpha}\,\left(\dfrac{x_o}{y_o}\right) \\ 0 & \text{if} \quad 1 \leq \sqrt{\beta/\alpha}\,\left(\dfrac{x_o}{y_o}\right) \end{cases}$$

We refer to $\sqrt{\beta/\alpha}\,\left(\frac{x_o}{y_o}\right)$ as the normalized force ratio and $\frac{\beta}{\alpha}\,\frac{x_o^L}{y_o^L}$ as the squared normalized force ratio for square-law attrition.

3. <u>Stochastic Model</u>

$$P(O,n,M,N) = \left(\frac{\alpha N}{\alpha N+\beta M}\right) P(O,n,M-1,N) + \left(\frac{\beta M}{\alpha N+\beta M}\right) P(O,n,M,N-1)$$

let $\quad F(M,N) = P(O,n,M,N)$

Then solving for

$$F(M,N) = \left(\frac{\alpha N}{\alpha N+\beta M}\right) F(M-1,N) + \left(\frac{\beta M}{\alpha N+\beta M}\right) F(M,N-1)$$

with $F(0,n) = 1$

$F(0,N) = 0$

otherwise we can get [6]

$$P(0,n,M,N) = \left(\frac{\alpha}{\beta}\right)^M \sum_{k=n}^{N} \frac{(-1)^{N-k}\, k^{M+N-n-1}\, \Gamma\left(\frac{\alpha k}{\beta} + 1\right)}{\Gamma(N-k+1)\, \Gamma\left(M + \frac{\alpha k}{\beta} + 1\right)\, \Gamma(k-n+1)}$$

and

$$P(B,N) = \sum_{k=0}^{N} \frac{(-1)^{N-k}\, k^{M+N}\, \Gamma\left(\frac{\alpha k}{\beta} + 1\right)}{(N-k)!\, k!\, \Gamma\left(M + \frac{\alpha k}{\beta} + 1\right)}$$

C. THE MIXED CASE

1. Assumption

(1) As in Case A

(2) As in Case B

(3) As in Case A for Blue side and Case B for Red side

2. Deterministic Model

The Lanchester-type equations are:

$$\frac{dx}{dt} = -\alpha x y$$

$$\frac{dy}{dt} = -\beta y$$

with solution

$$2\beta \{x_o - x_o(t)\} = \alpha \{y_o^2 - y^2(t)\}$$

Then the probability of winning for Blue can be represented as:

$$P(E,y_o) = \begin{cases} 1 & \text{if } \frac{\alpha}{2\beta} \geq \frac{x_o}{y_o^2} \\ 0 & \text{if } \frac{\alpha}{2\beta} \leq \frac{x_o}{y_o^2} \end{cases}$$

This can be changed as

$$\mathcal{P}(B,y_o) = \begin{cases} 1 & \text{if } 1 \geq \dfrac{2\beta x_o}{y_o^{\,2}} \\ 0 & \text{if } 1 \leq \dfrac{2\beta x_o}{y_o^{\,2}} \end{cases}$$

Define $\dfrac{2\beta x_o}{y_o^{\,2}}$ as normalized force ratio for mixed case.

3. Stochastic Model

$$P(O,n,M,N,) = \left(\frac{\alpha N}{\alpha N+\beta}\right)P(O,n,M-1,N) + \left(\frac{\beta}{2N+\beta}\right)P(O,n,M,N-1)$$

after solving this

$$P(O,n,M,N) = \sum_{\ell=n}^{N} \left(\frac{\alpha\ell}{\alpha\ell+\beta}\right)^{M} \frac{(-1)^{N-\ell}\ \ell^{N-n-1}\cdot n}{\Gamma(N-\ +1)\ \Gamma\ (\ell-n+1)}$$

and

$$P(B,N) = \sum_{\ell=0}^{N} \frac{(-1)^{N-\ell}\ \ell^{n+N-1}}{(\ell+\frac{\beta}{\alpha})^{M}\ \Gamma\ (N-\ell+1)\ \Gamma\ (\ell)}$$

III. COMPUTATIONAL RESULTS

A. LINEAR LAW

A digital computer program was developed to perform the rather tedious computation of the probability of winning for various levels of model parameters.

Plots of probability of winning for Blue versus normalized force ratio are given on pages 19-24.

In order to clarify the probability shift, each graph in Figures 1 to 5 are drawn for different initial force level of Blue forces. In Figure 6 the quantity M+N is held constant. In Figures 1 to 5 the results of using different values of the attrition-rate coefficient can be seen. The straight line of each graph represents the probability of winning for deterministic law.

As we can see in these figures, increasing the number of combatants causes the probability of winning as computed from the stochastic model to approach the deterministic win criterion (i.e. x wins when $\beta x_o > \alpha y_o$).

It is reasonable to use the deterministic model to study combat between two forces with large numbers of combatants. This is because the mathematical convenience obtained from using the deterministic model is much greater than the possible errors between the two models.

One really interesting result is that conditions to win for both deterministic and stochastic models turn out to be very similar. In case of deterministic model if the normalized force ratio is less than 1, the probability of winning for Blue is 1 all the time. This implies Blue side is always sure to win. But in case of stochastic model, when the normalized force ratio is less than 1, there is still a possibility to loose the battle. Especially when the number of Blue combatants are less than 40, the difference between deterministic is critical. An extreme case is the example from Figure 1 when N = 2 and M = 1. In this case we can see the probability of winning for Blue is equal to 1 for deterministic case. But from the stochastic model the winning chance is only 3/4. This means the Red side still has 1/4 chance of winning.

When the initial number of Blue combatants is less than 40, the critical part for the normalized force ratio is between 0.5 and 1.5. This implies that for the combat, in which the relatively small number of combatants are participating and the normalized forces ratios are .5 apart from unity, using the stochastic model is reasonable. But for the case when the large number of combatants are participating or normalized force ratio are large enough, using the deterministic model is more realistic.

Another interesting result is a probability shift apparently caused the attrition-rate coefficient difference. In Figures 2 to 5 when the normalized force ratio is unity the probability of winning is different from .5. Before performing these computations it was anticipated that the probability of winning would be .5. But these numerical results showed that this guess was not true. Also, the probability difference from 0.5 becomes smaller as the initial number of combatants becomes larger. The attrition-rate coefficients influence the probability of winning more than increasing the initial number of combatants.

Figure 6 also indicates when larger number of forces are involved, the probability that the larger force will win approach unity. The probabilistic form is preferable to the simple differential equation, however it is not clear that the additional difficulty of solution is consistent with the improved realism in view of the other known variable of actual combat.

It seems appropriate to examine the often quoted maxim of military tactics that one should not attack without at least a three-to-one force ratio. From Figures 1 to 5, the region where the normalized force ratio is less than 1/3, the probability of winning is greater than .99. This implies Blue side is more than 99% sure to win. But it is natural result because the Blue side has 3 times greater "combat power" than the Red side.

Hence we may investigate another way when

$$\alpha N > \beta M \quad \text{Blue win}$$

if N = 3M the winning condition become

$$3\alpha M \geq \beta M$$

This can be written

$$\frac{\beta}{\alpha} \leq 3$$

To analyze this condition we made Table I. From column 6 of Table I we can see the probability of winning is inversely proportional to force ratio and approach to 0.5 when the number of combatants for Blue side are increasing. This implies in linear law battle if $\frac{\beta}{\alpha} = 3$, then the 3 to 1 force ratio attack gives the Blue side at least draw. Column 8 shows the winning chance of the Blue side is very small because the winning condition $\frac{\beta}{\alpha} \leq 3$ is not satisfied.

B. SQUARE LAW

The same procedure for the analysis using "linear-law" attrition has been done for square-law attrition except the abcissa of the graph of square law is the square of normalized force ratio.

The computational results are very similar to linear law case. To avoid the same citation twice, we try to analyze the square law case by comparing the difference from linear law case.

One difference is the probability shift which cause from the different value of the ratio of attrition rate. In the case of square law the attrition-rate coefficient influence less the probability of winning than in the case of linear-law attrition.

To investigate the most usual situation in combat, where one antagonist attacks another; we can develop condition of winning by considering the practical forces ratios.

Figure 1

Linear Law: $\alpha = 0.05$, $\beta = 0.05$

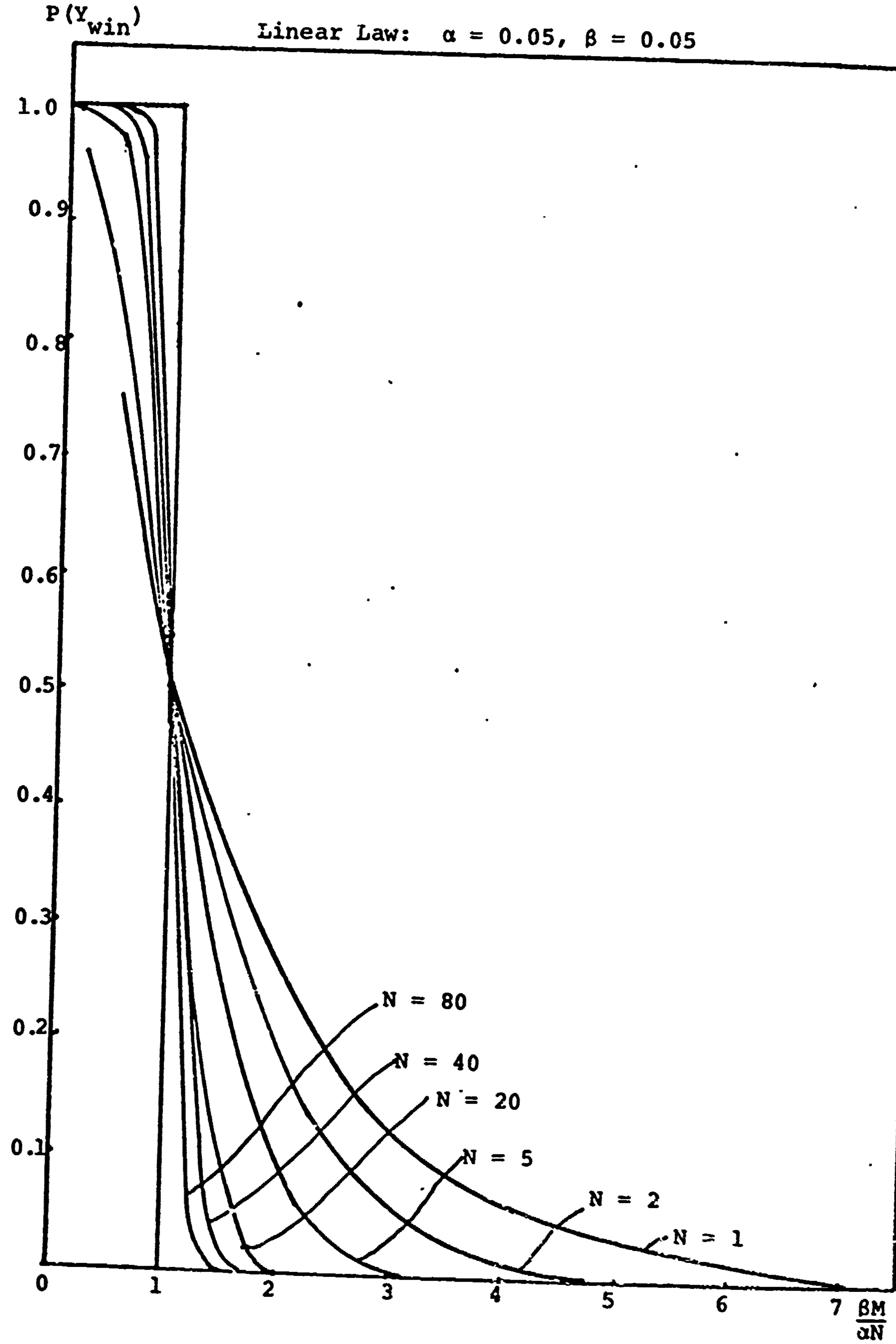


Figure 2

Linear Law: $\alpha = 0.2$, $\beta = 0.5$

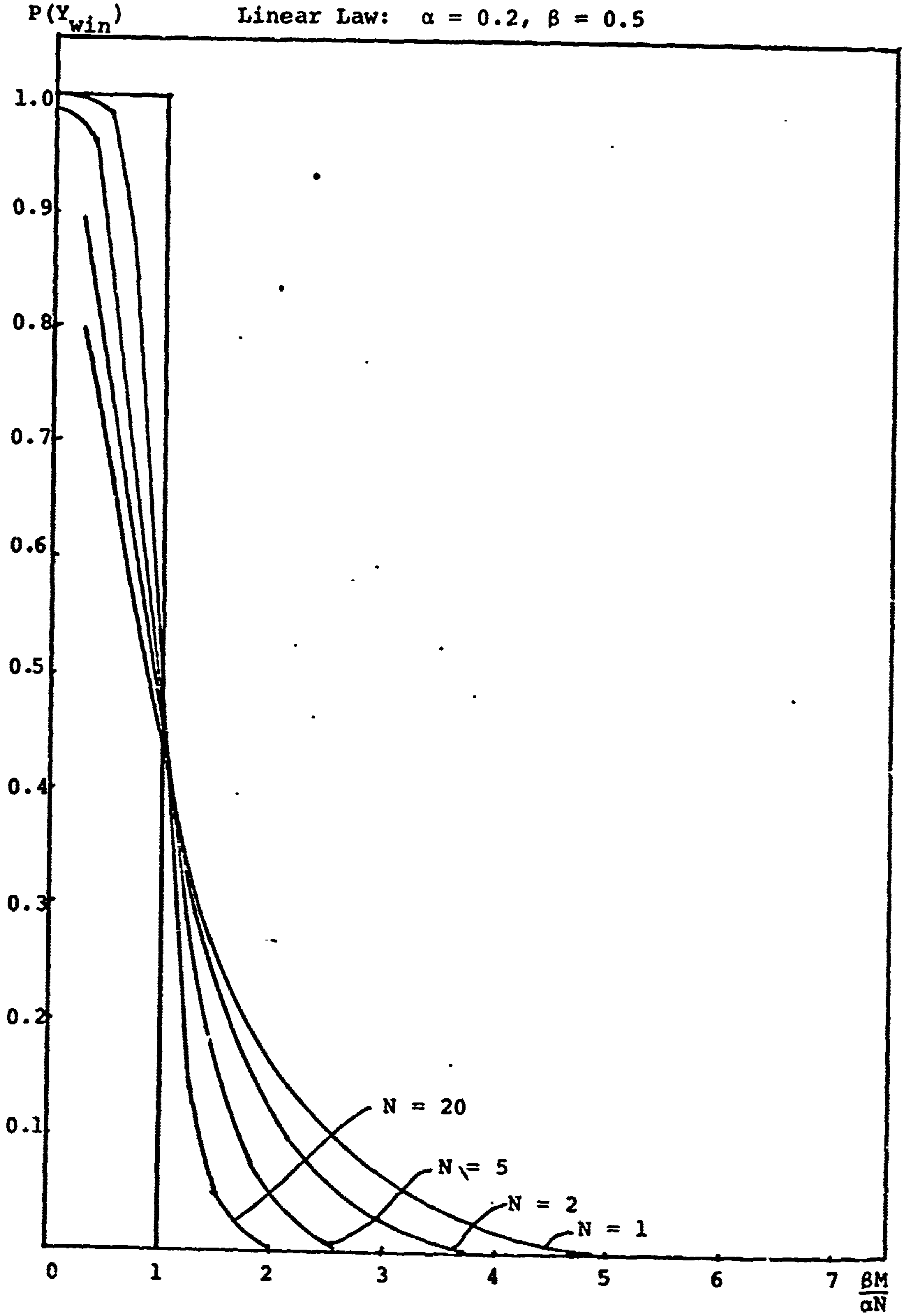


Figure 3

Linear Law: $\alpha = 0.05$, $\beta = 0.20$

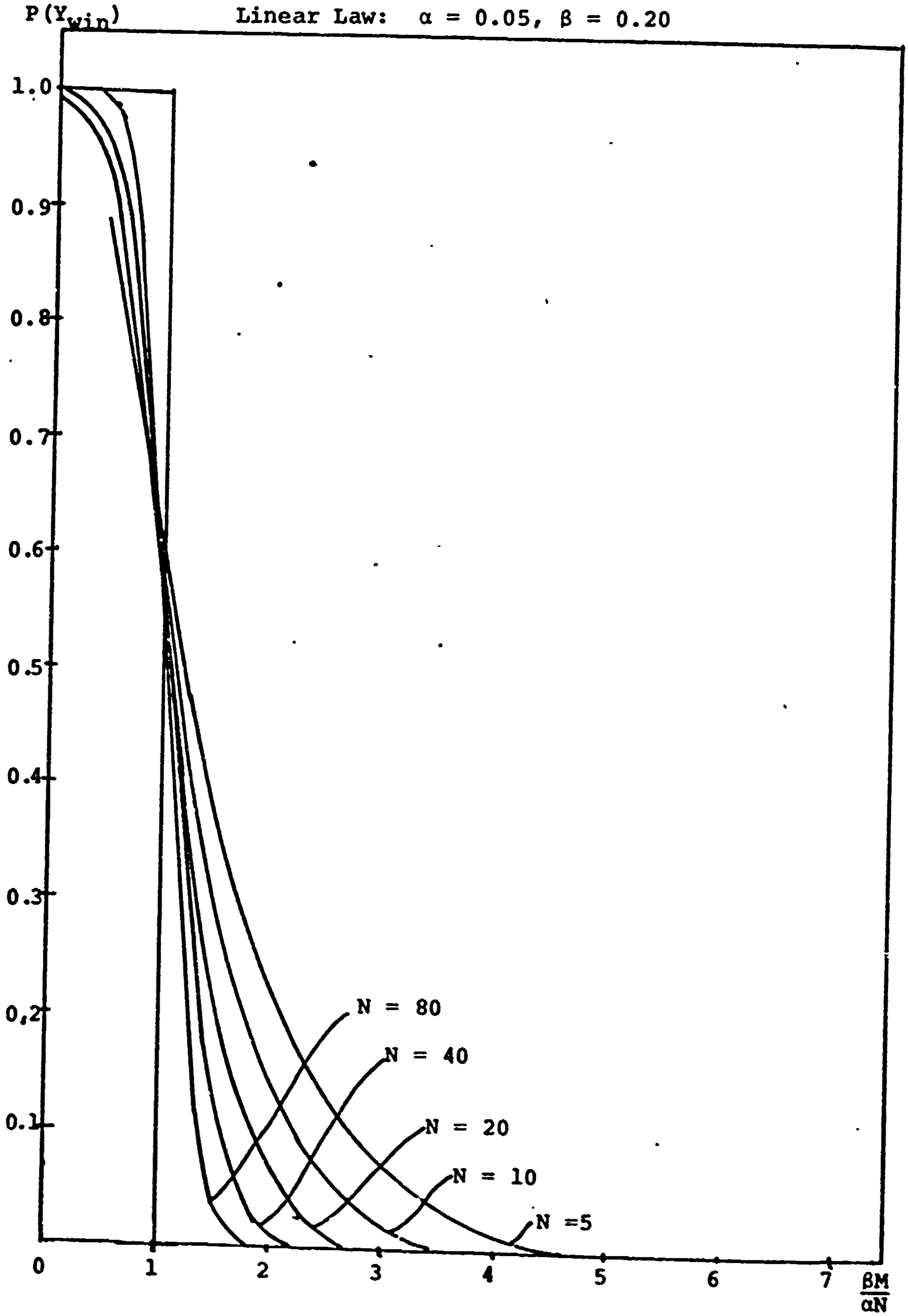


Figure 4

Linear Law: $\alpha = 0.3$, $\beta = 0.1$

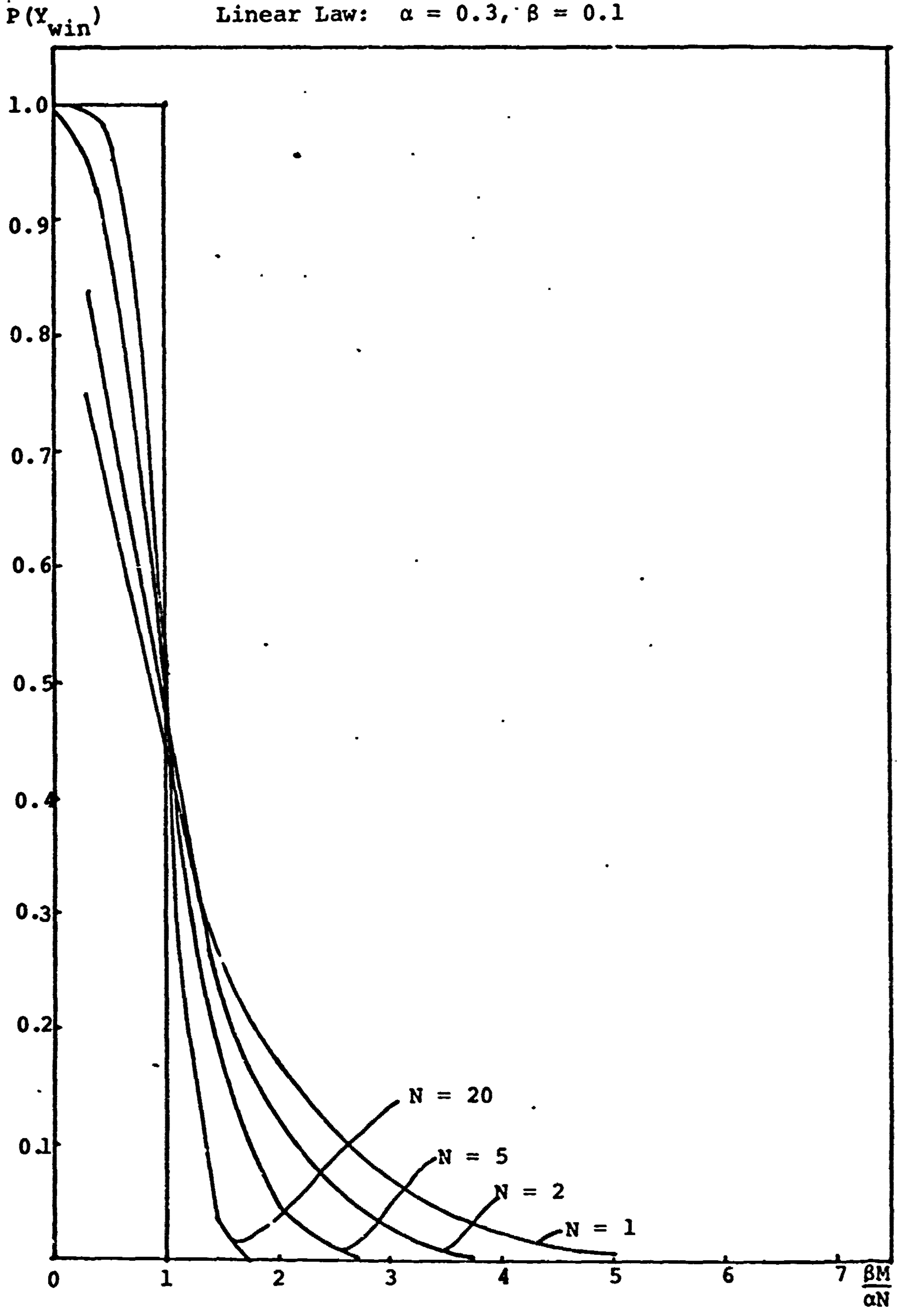


Figure 5

Linear Law: $\alpha = 0.1, \beta = 0.3$

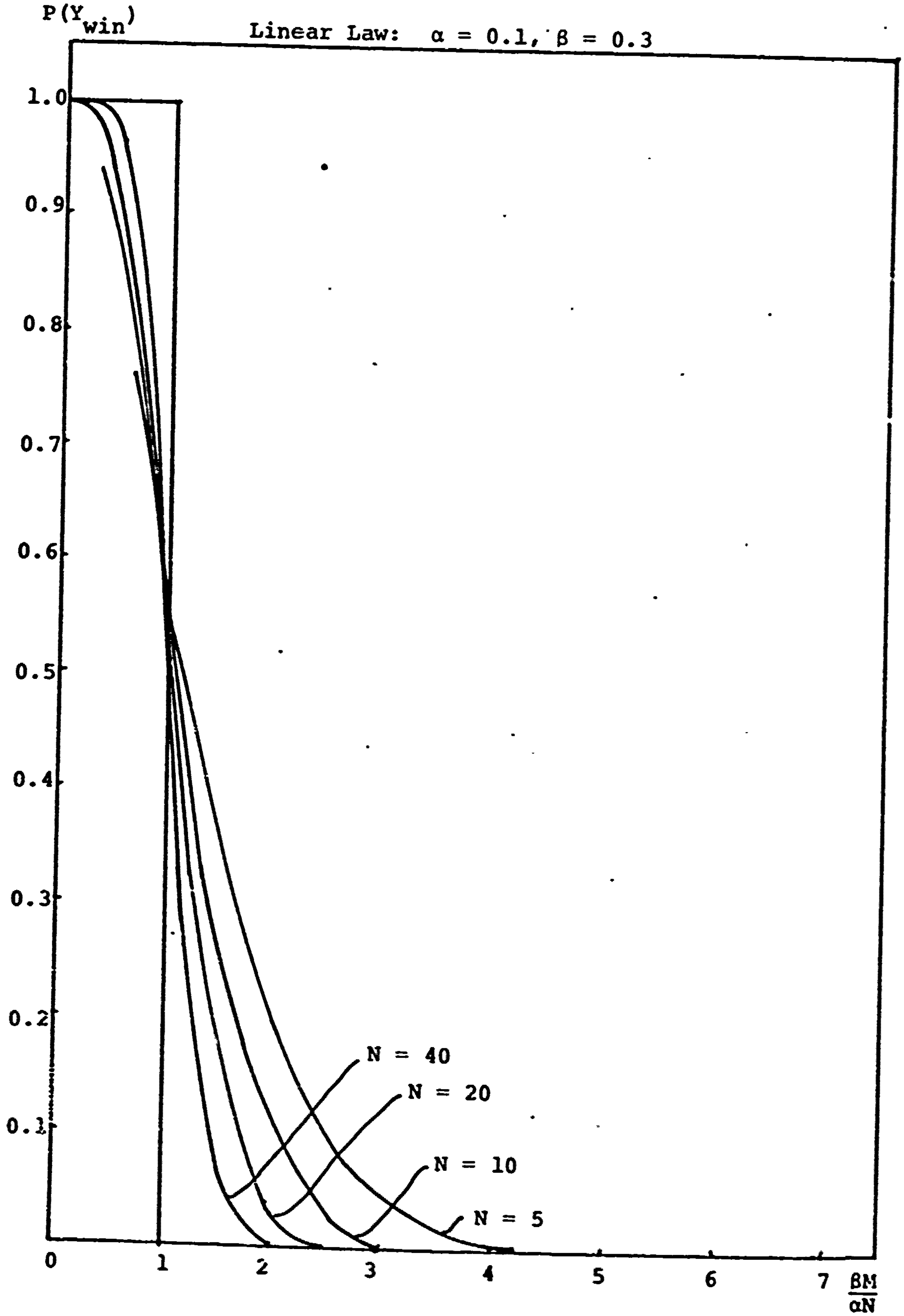


Figure 6

Linear Law: $\alpha = 0.5$, $\beta = 0.5$

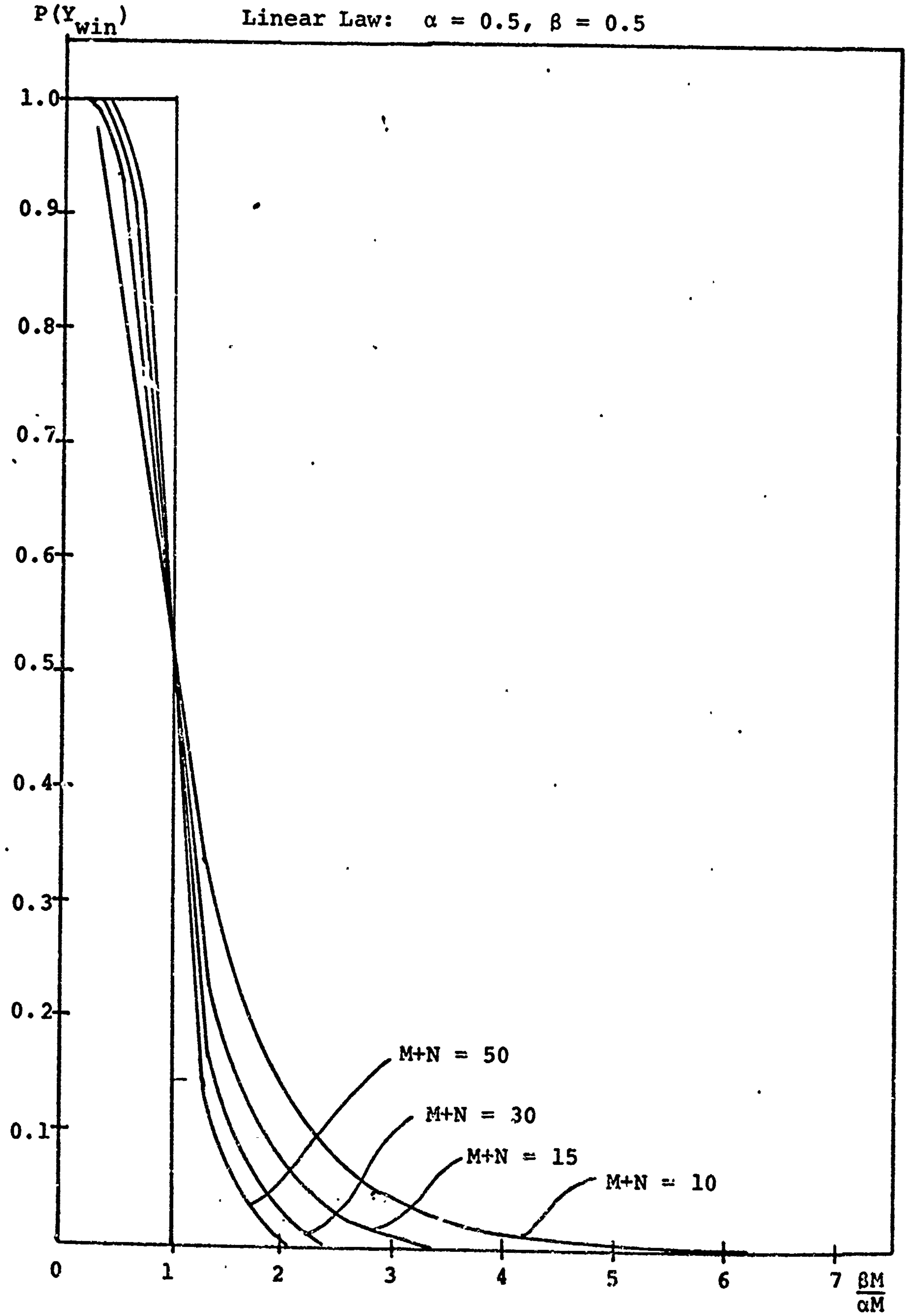


TABLE I

Initial Forces		P(B,N)						
		$\alpha = .05,\ \beta = .05$			$\alpha = .1,\ \beta = .3$		$\alpha = .05,\ \beta = .2$	
M	N	linear	square	mixed	linear	square	linear	square
1	3	0.8750	0.9583		0.57812		0.51200	
2	6	0.9375	0.9938	0.1600	0.55505	0.87179	0.57680	0.64338
3	9	0.9672	0.9990	0.1348	0.54480	0.92326	0.61740	0.84266
4	12	0.9824	0.9998	0.3362	0.53871	0.95268	0.64816	0.88234
5	15	0.9904	0.9999	0.4536	0.53458	0.97028	0.67329	0.91067
6	18	0.9947	1.0	0.5742	0.53158	0.98110	0.69469	0.93148
7	21	0.9970	1.0	0.6874	0.52917	0.98787	0.71339	0.94703
8	24	0.9983	1.0	0.7833	0.52727	0.99219	0.73003	0.95881
9	27	0.9991	1.0	0.8582	0.52570	0.99554	0.74501	0.96804
10	30	0.9995	1.0	0.9123	0.52437	.	0.75864	0.98188
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
20	60	1.0	1.0	1.0	0.51720	.	0.85079	.

$$\alpha N^2 \geq \beta M^2$$

if N = 3M the winning condition will be

$$\alpha (3M)^2 \geq \beta M^2$$

This can be written

$$\frac{\beta}{\alpha} \leq 9$$

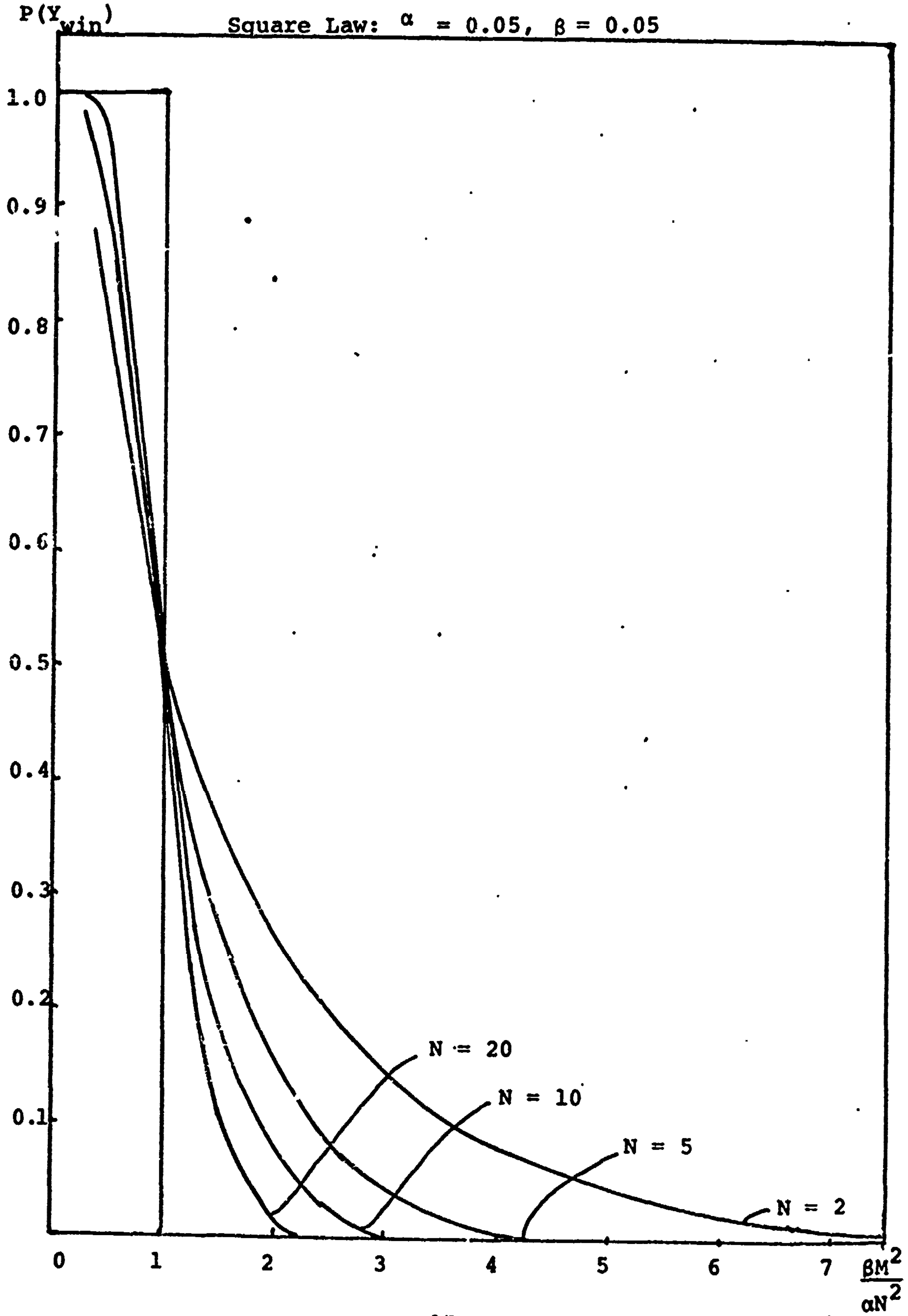
By investigating the column 4, 7, 9 of Table I, we can expect the same result as linear law case.

C. MIXED LAW

All results are similar to those for cases A and B above. From column 5 of Table I the probability of winning is small compared to case A and B. This is caused from the effect of idealized assumption for mixed law. The actual situation of combat is quite different from the idealized situation, attacker uses area fire while the defender use aimed fire.

Below the line where the initial force ratio 8 to 24, the probability of winning become much greater. This shows the difference between the deterministic and stochastic model more clearly.

Figure 7



$P(Y_{win})$
Square Law: α = 0.05, β = 0.05
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
$\frac{\beta M^2}{\alpha N^2}$
N = 20
N = 10
N = 5
N = 2

Figure 8

Square Law: $\alpha = 0.05$, $\beta = 0.2$

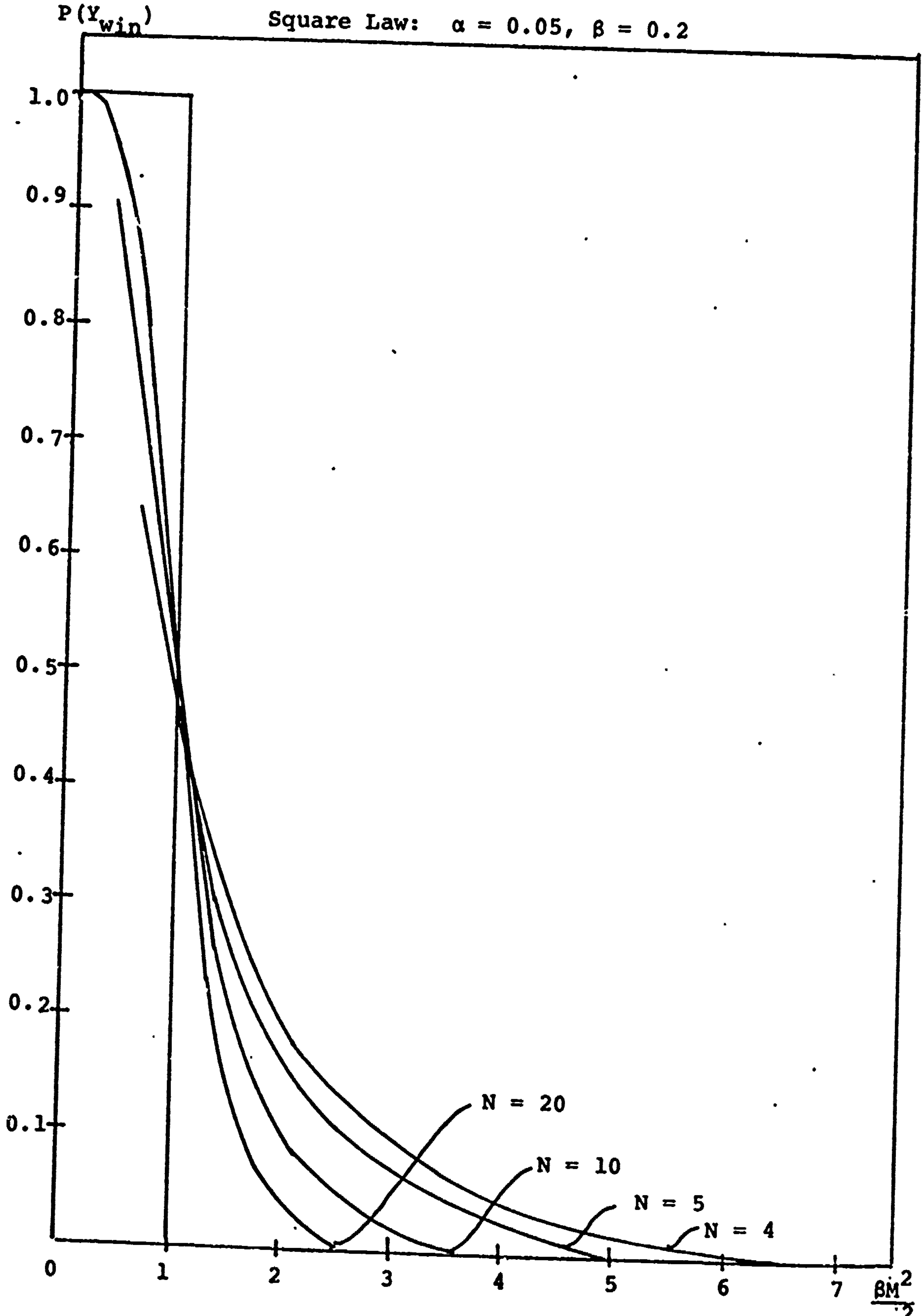


Figure 9

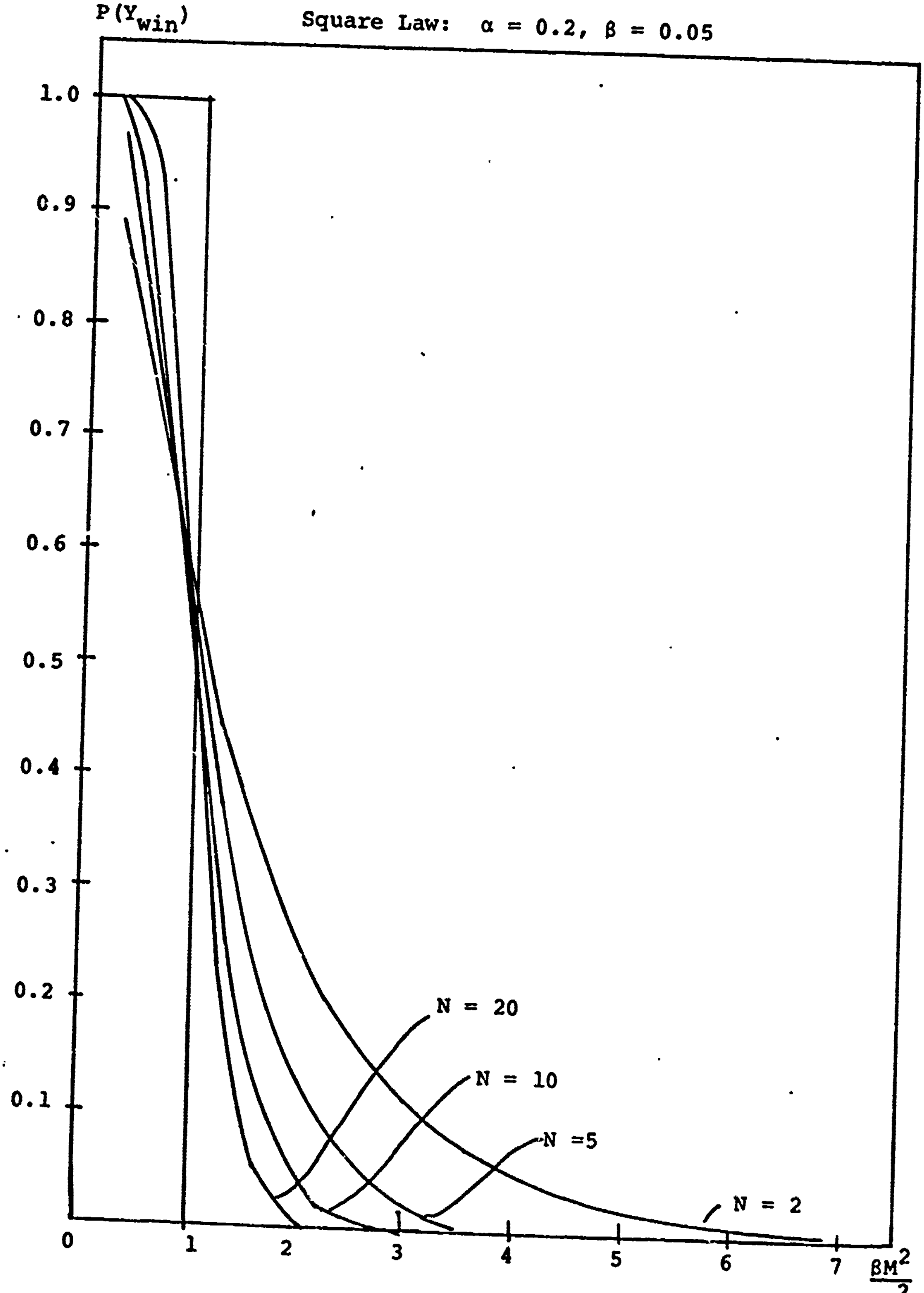

P(Ywin)
Square Law: α = 0.2, β = 0.05
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
βM²/αN²
N = 20
N = 10
N =5
N = 2

Figure 10

Square Law: $\alpha = 0.3,\ \beta = 0.1$

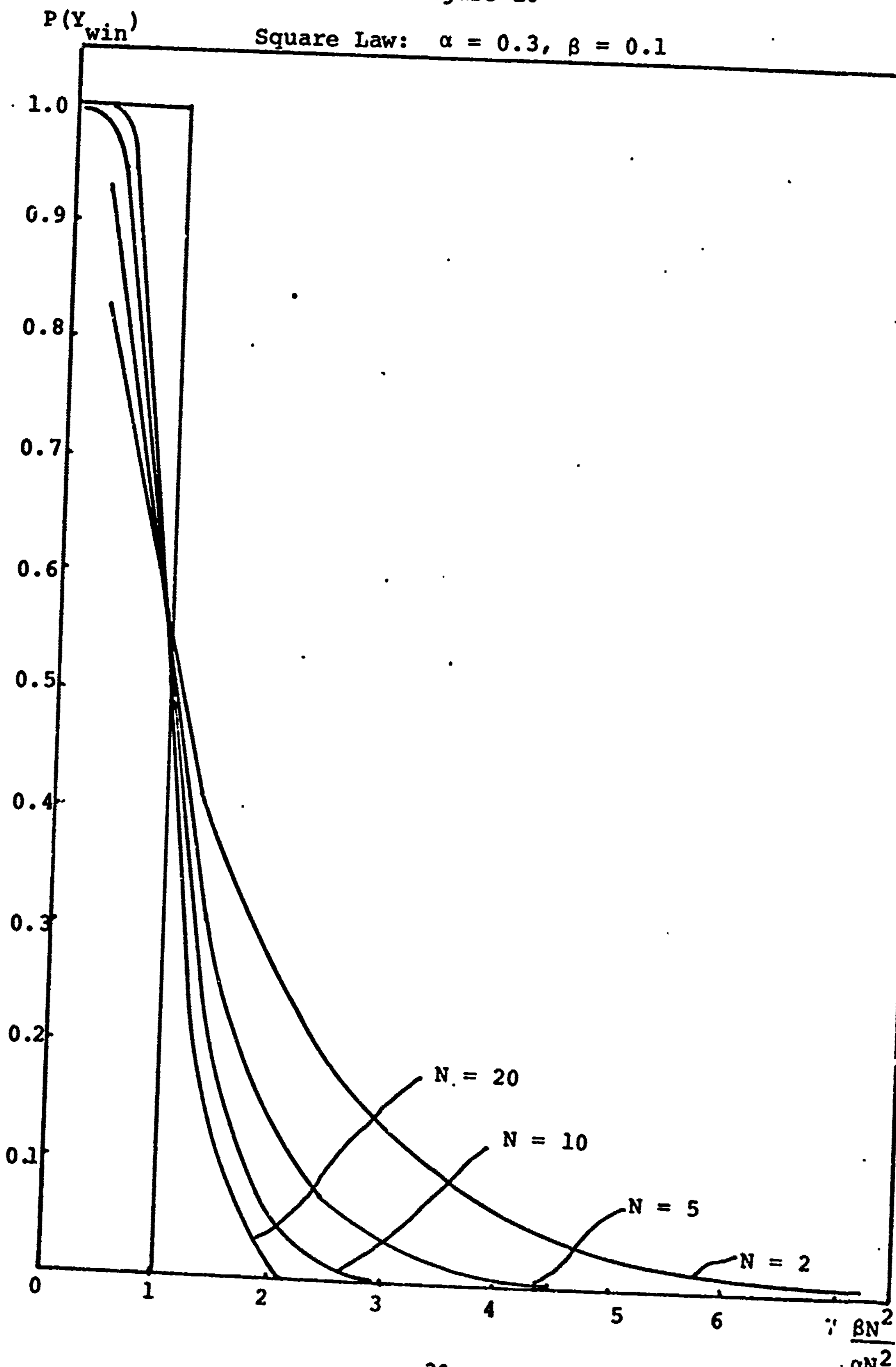


Figure 11

Square Law: $\alpha = 0.1, \beta = 0.3$

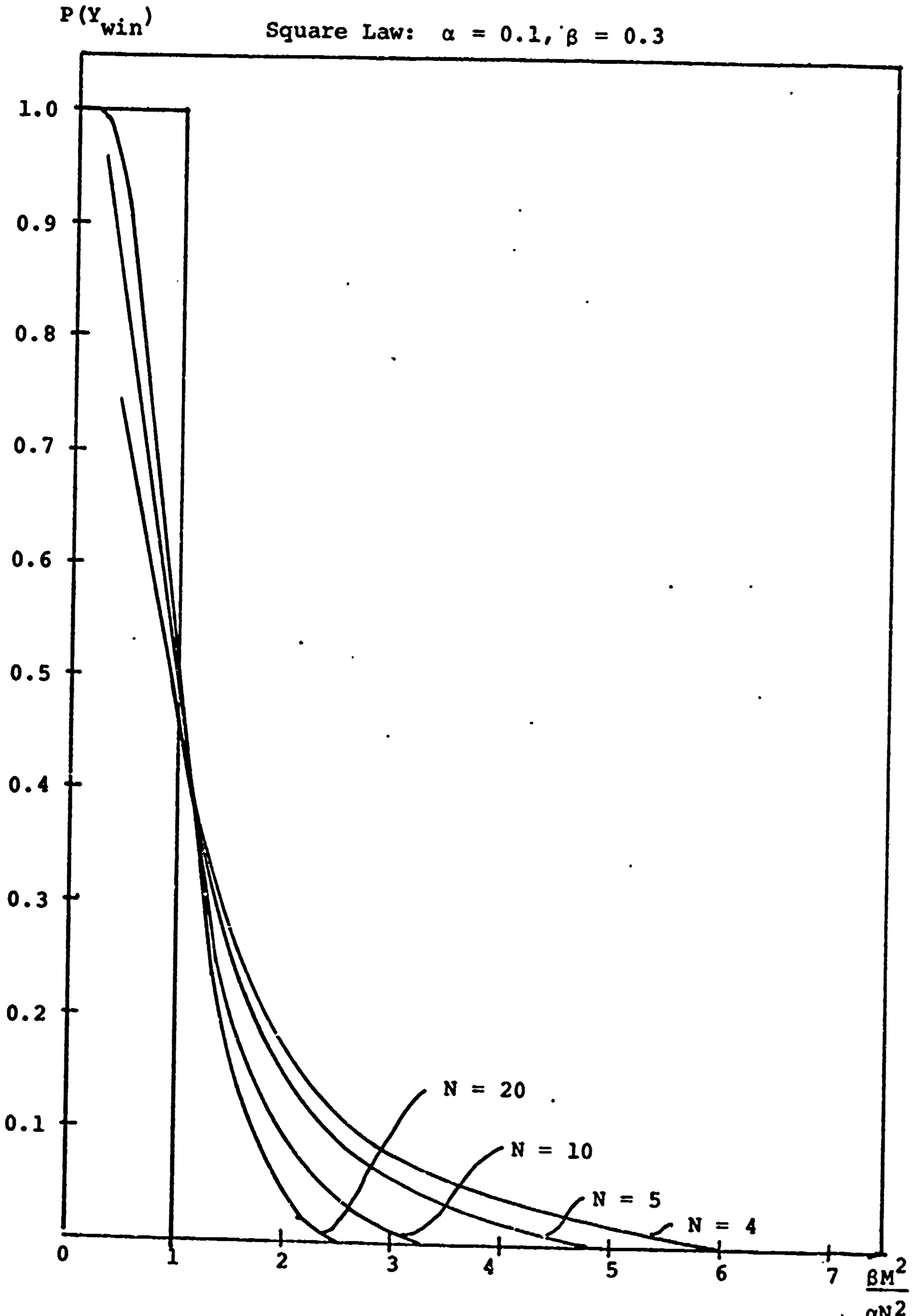
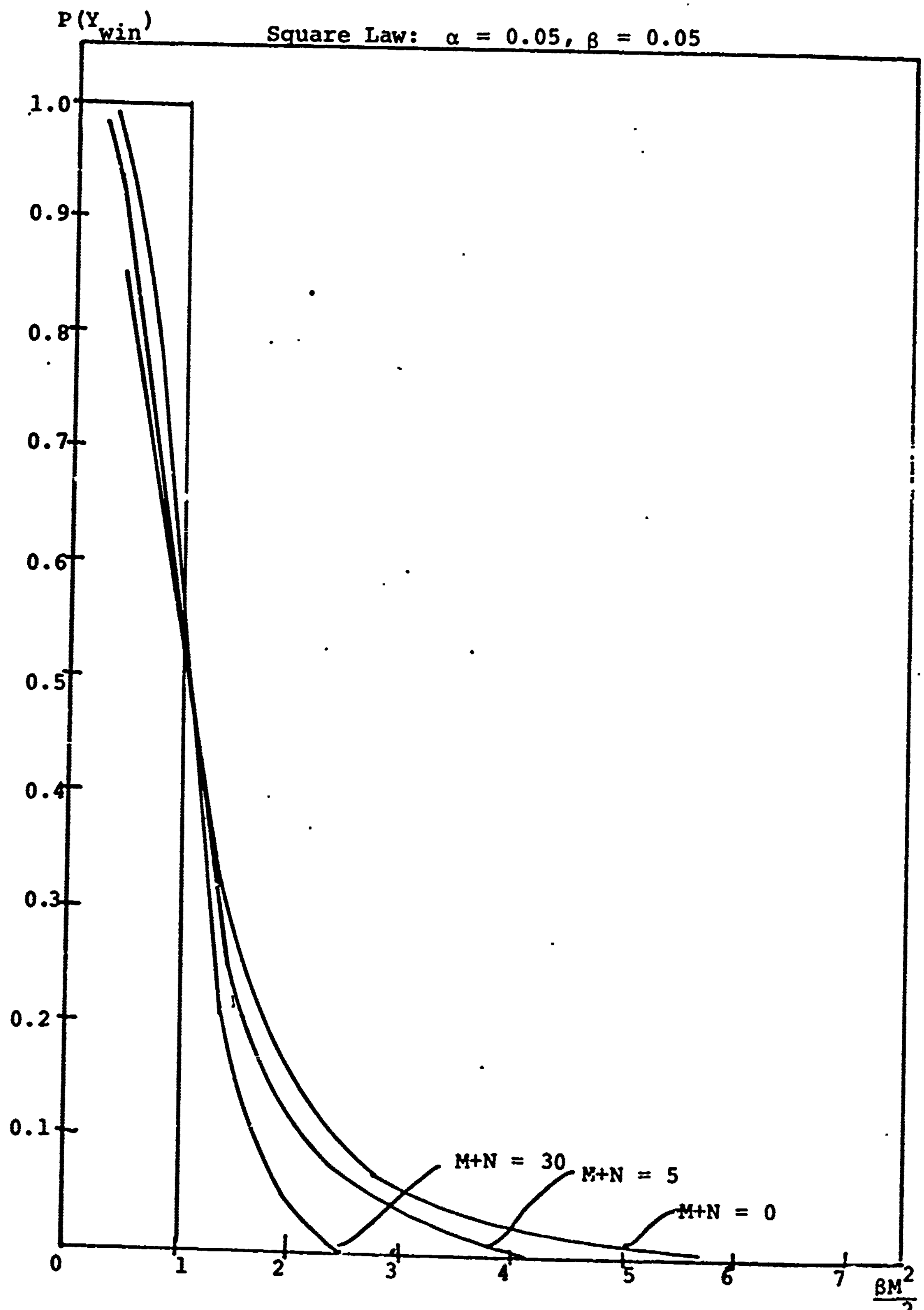


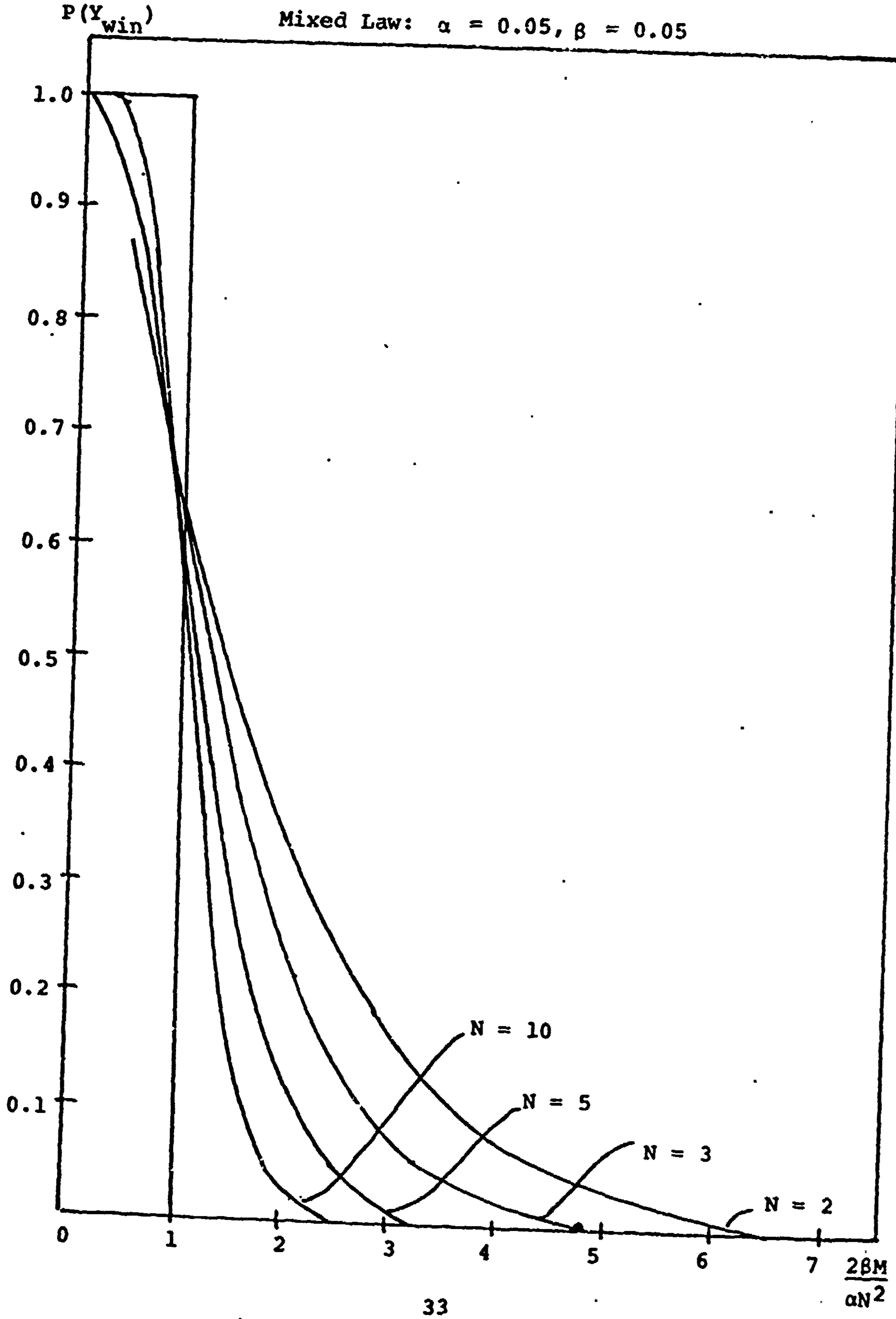
Figure 12



P(Y_win)
Square Law: α = 0.05, β = 0.05
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
M+N = 30
M+N = 5
M+N = 0
βM²/αN²

Figure 13

Mixed Law: $\alpha = 0.05, \beta = 0.05$



IV. SUGGESTED FUTURE WORK

The stochastic model solution includes many factorial and gamma functions. This apparently gives computational difficulties due to the large numbers generated causing "overflows" in the computer. To overcome such difficulties recursive formulas were developed for the case of linear-law model. These are shown in the following table.

TABLE II

N	P(B,N)
1	$D^M \cdot 1$
2	$D^M\left\{1+MC\right\}$
3	$D^M\left\{1+MC + \frac{M(M+1)}{2} C^2\right\}$
4	$D^M\left\{1+MC+\frac{M(M+1)}{2} C^2 + \frac{M(M+1)(M+2)\ C^2}{1\cdot 2\cdot 3}\right\}$
.	. .
.	. .

$$C = \frac{\beta}{\alpha+\beta}, \quad D = \frac{\alpha}{\alpha+\beta}$$

Because the recursive formulas for the case of square and mixed model were not developed at this time, another approach had to be taken to eliminate overflow problems for the "Square-Law" and "mixed-law" models. This was accomplished by using logarithms in the appropriate formulas for the probability of winning. Although this took much computing time and was very inefficient, it did allow results to be computed for force levels up to approximately 20 to 50. Development of such recursive formulas for square law and mixed law cases was left for future work.

Secondly the current work examined only the simplest cases of combat between two forces in the Lanchester theory of combat. If we introduce the time to the model then the situation becomes more complicated. We can expect more interesting results by introducing reality into the model and do the same work.

V. CONCLUSION

Idealized combat situations in the Lanchester theory of combat were examined using both deterministic and stochastic models. In the simplest case of combat between two homogeneous forces, the probability of winning was studied and results contrasted for these two types of models. Based on the results of numerical computation of the probability of winning, it has been concluded that the deterministic model (even though win probabilities are either zero or one depending upon whether the "deterministic win criterion" is satisfied) yields satisfactory forecasts of combat outcomes (probability of winning) when there are large numbers of combatants (at least on one side). The curve of the probability of winning (in a fight-to-the finish) versus the "normalized force ratio" (or appropriate equivalent) is characterized by a steep slope in the probability of winning for normalized forces ratios between 0.5 and 1.5. This slope becomes steeper as the number of combatants increases. Thus, in such circumstances the addition of a relatively few additional forces on one side may significantly increase that side's chances of winning.

The results of tedious computation shows the attrition-rate coefficients influence more the probability of winning than increasing the initial number of combatants. The three to one force superiority is needed to win the combat with much confidence. The results of three cases, linear law, square law and mixed law, are very similar. Still there is much work to be done in the future.

COMPUTER PROGRAM

```
C
C
C
C     COMPUTER PROGRAM FOR LINEAR-LAW
C
C
      IMPLICIT REAL*8(A-H,O-Z,$)
 9000 FORMAT (' ',5X,F9.5,5X,F9.5,5X,F9.5,5X,F9.5,5X,F9.5,
     *5X,I5,5X,I5)
 9100 FORMAT (' ',8X,'A',11X,'B',10X,'A/B',9X,'BM/AN',
     *5X,'PROB WIN',7X,'M',7X,'N')
      WRITE (6,9100)
      A=0.05
      B=0.05
      AOB=A/B
      C=B/(A+B)
      D=A/(A+B)
      DO 1000 N=1,100
      DO 1000 M=1,100
      XM=DFLOAT(M)
      XN=DFLOAT(N)
      XMON=(B*XM)/(A*XN)
      TERM1=D**M
      SUM=1.0
      TERM=1.0
      IF(N.LT.2) GO TO 200
      DO 100 I=2,N
      XI=DFLOAT(I)
      M2I=M-2+I
      XM2I=DFLOAT(M2I)
      TERM=TERM*XM2I/(XI-1.0)*C
      SUM=SUM+TERM
  100 CONTINUE
  200 PROBY=TERM1*SUM
      WRITE(6,9000)A,B,AOB,XMON,PROBY,M,N
 1000 CONTINUE
      STOP
      END
C
C
C     COMPUTER PROGRAM FOR SQUARE-LAW
C
C
      IMPLICIT REAL*8(A-H,O-Z,$)
 9000 FORMAT(' ',5X,F9.5,5X,F9.5,5X,F9.5,5X,F9.5,
     *5X,F9.5,5X,I5,5X,I5)
      A=0.3
      B=0.1
      AOB=A/B
      DO 1000 N=2,100
      XN=DFLOAT(N)
      DO 1000 M=2,100
      XM=DFLOAT(M)
      MN=M+N
      XMN=DFLOAT(MN)
      XMON=(B*XM**2)/(A*XN**2)
      IF(XMON.LT.0.2) GO TO 1000
      IF(XMON.GT.10.0) GO TO 1000
      TERM1=XM*DLOG(AOB)
      SUM=0.0
      DO 100 K=1,N
      XK=DFLOAT(K)
      TERM2=0.0
      TERM3=0.0
      TERM4=0.0
      NMINK=N-K
      DO 111 K1=1,NMINK
      XK1=DFLOAT(K1)
      TERM2=TERM2+DLOG(XK1)
  111 CONTINUE
      DO 112 K2=1,K
      XK2=DFLOAT(K2)
```

```
      TERM3=TERM3+DLOG(XK2)
  112 CONTINUE
      XKAB=XK*AOB
      DO 113 K3=1,M
      XK3=DFLOAT(K3)
      XK3XK=XK3+XKAB
      TERM4=TERM4+DLOG(XK3XK)
  113 CONTINUE
      XJCOE=(-1.0)**NMINK
      XKMNS=XMN*DLOG(XK)
      STUB=TERM1+XKMNS-TERM2-TERM3-TERM4
      SUM=SUM+XJCOE*DEXP(STUB)
  100 CONTINUE
      PROBY=SUM
      WRITE(6,9000) A,B,AOB,XMON,PROBY,M,N
 1000 CONTINUE
      STOP
      END
C
C
C     COMPUTER PROGRAM FOR MIXED-LAW
C
C
      IMPLICIT REAL*8(A-H,O-Z,$)
 9000 FORMAT(' ',5X,F9.5,5X,F9.5,5X,F9.5,5X,F9.5,
     *5X,F9.5,5X,I5,5X,I5)
      A=0.05
      B=0.05
      AOB=A/B
      BOA=B/A
      DO 1000 N=2,100
      XN=DFLOAT(N)
      DO 1000 M=2,100
      XM=DFLOAT(M)
      MN=M+N-1
      XMON=(2.0*B*XM)/(A*XN**2)
      XMN=DFLOAT(MN)
      SUM=0.0
      DO 100 L=1,N
      XL=DFLOAT(L)
      TERM1=0.0
      TERM2=0.0
      NML=N-L
      XNML=DFLOAT(NML)
      LM1=L-1
      XLM1=DFLOAT(LM1)
      XJCOE=(-1.0)**NML
      IF(NML.LE.1) GO TO 200
      DO 111 L1=1,NML
      XL1=DFLOAT(L1)
      TERM1=TERM1+DLOG(XL1)
  111 CONTINUE
      GO TO 300
  200 TERM1=0.0
  300 IF(LM1.LE.1) GO TO 400
      DO 112 L2=1,LM1
      XL2=DFLOAT(L2)
      TERM2=TERM2+DLOG(XL2)
  112 CONTINUE
      GO TO 500
  400 TERM2=0.0
  500 XLBOA=XL+BOA
      STUB=XMN*DLOG(XL)-XM*DLOG(XLBOA)-TERM1-TERM2
      SUM=SUM+XJCOE*DEXP(STUB)
  100 CONTINUE
      PROBY=SUM
      WRITE(6,9000) A,B,AOB,XMON,PROBY,M,N
 1000 CONTINUE
      STOP
      END
```

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