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SYMMETRIC SIMPLE GAMES

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A symmetric simple game is an n-person game in which the winning coalitions are the coalitions of at least k players. All von Neumann-Morgenstern stable sets, in which (n-k) players are discriminated, are characterized for these games.

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SYMMETRIC SIMPLE GAMES

by

Robert James Weber

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III

A question of interest in the study of n-person games concerns conditions under which a set of players of a game may face discrimination which, in a sense, excludes them from the bargaining process. In [2, 4, 5], discriminatory solutions are given for several classes of games. In this paper, a characterization is given of all discriminatory solutions for n-person games in which all coalitions with at least k players win, and all coalitions with less than k players lose. The symmetric solutions of these games were given in [1] for the case $k > \frac{n}{2}$.

An n-person <u>game</u> is a function v from the coalitions (subsets) of a set of players $N = \{1,2,\ldots,n\}$ to the reals satisfying

$$v(\emptyset) = v(\{i\}) = 0 \quad \text{for all} \quad i \in N$$

$$0 \leq v(S) \leq 1 \qquad \text{for all} \quad S \subseteq N$$

$$v(N) = 1 .$$

It is assumed throughout that $n \geq 3$. For any real vector $x \in R^k$, define $x(S) = \sum_{i \in S} x_i$ and define x^S as the restriction of x to the coordinates in S. Let

$$X = \{x \in R^n\colon x(N) = 1 , x_i \geq 0 \ \text{ for all } \ i \in N\} .$$

If $x,y \in X$, then y <u>dominates</u> x with respect to a non-empty coalition S, written $y \operatorname{dom}_S x$, if $y^S > x^S$ and $y(S) \leq v(S)$. For $A \subseteq X$, define

$$\operatorname{dom} A = \{x \in X\colon y \operatorname{dom}_S x \ \text{ for some } \ S \subseteq N , \ y \in A\} .$$

A <u>solution</u>, or von Neumann-Morgenstein stable set, is any set $K \subseteq X$ satisfying

$$K \cup \operatorname{dom} K = X , \tag{1}$$

$$K \cap \operatorname{dom} K = \emptyset . \tag{2}$$

Any set K satisfying (1) is said to be externally stable; any set satisfying (2) is internally stable. Motivation for these definitions is given in [3].

A symmetric simple game, or (n,k)-game, is an n-person game satisfying

$$v(S) = 0 \quad \text{if} \quad |S| < k$$

$$v(S) = 1 \quad \text{if} \quad |S| \geq k ,$$

where $|S|$ denotes the number of players in the coalition S . Clearly domination in this game can occur only with respect to coalitions of at least k players. A p-*discriminatory* solution is a solution

$$D(\alpha_{i_1},\ldots,\alpha_{i_p};i_1,\ldots,i_p) = \{x \in X: x_{i_k} = \alpha_{i_k} \text{ for all } 1 \leq k \leq p\} .$$

The main result of this paper is a characterization of all m-discriminatory solutions for the $(n,n-m)$-game.

For $m < \frac{n}{2}$, let $M \subseteq N$ be a set of m players, and let $P = N-M$. Also let α be a non-negative m-vector, and write $K(\alpha)$ for $D(\alpha;M)$.

Theorem. $K(\alpha)$ is an m-discriminatory solution of the $(n,n-m)$-game if and only if

$$\alpha(M) + (n-m-1)\alpha_i < 1 \tag{3}$$

for all $i \in M$.

The proof follows a sequence of lemmas.

Lemma 1. For any α , $K(\alpha)$ is internally stable.

Proof. Assume on the contrary that $x,y \in K(\alpha)$ and $y \text{ dom}_S x$. Since $y^M = \alpha = x^M$ and $|S| \geq n-m$, it follows that $S = P$. However, $y(M) = x(M)$ and $y(N) = x(N)$ imply $y(P) = x(P)$ and therefore $y_i \leq x_i$ for some $i \in P$, a contradiction.

Lemma 2. Suppose $x \notin K(\alpha)$. Then $x \notin \text{dom } K(\alpha)$ if and only if

$$\alpha(M) + x(S \cap P) \geq 1 \quad \text{or} \quad x^{S\cap M} \not< \alpha^{S\cap M} \tag{4}$$

for all $S \subseteq N$ with $|S| = n-m$.

Proof. If $y \in K(\alpha)$ and $y \text{ dom}_T x$, then $|T| \geq n-m$. Take any $S \subseteq T$ with $|S| = n-m$. Then $y \text{ dom}_S x$, and $x^{S\cap M} < y^{S\cap M} = \alpha^{S\cap M}$. Since $2m < n$, $S \cap P \neq \emptyset$ and therefore

$$\alpha(M) + x(S \cap P) < y(M) + y(S \cap P) \leq 1 .$$

This establishes the sufficiency of (4). To establish necessity, assume that (4) fails for some S . Let

$$y_i = \alpha_i \qquad i \in M$$

$$y_i = x_i + (1-\alpha(M)-x(S \cap P))/|S \cap P| \qquad i \in S \cap P$$

$$y_i = 0 \qquad i \in P - S .$$

Then $y \in K(\alpha)$, $y \text{ dom}_S x$ and therefore $x \in \text{dom } K(\alpha)$.

Lemma 3. Let $\tau(x) = \{i \in M | x_i \geq \alpha_i\}$. If $\tau(x) = M$, then $x \in K(\alpha) \cup \text{dom } K(\alpha)$. If $\tau(x) \subsetneq M$, then $x \notin K(\alpha) \cup \text{dom } K(\alpha)$ if and only if

$$\alpha(M) + x(S \cap P) \geq 1 \tag{5}$$

for all $S \subseteq N$ with $|S| = n-m$ and $\tau(x) \cap S = \emptyset$.

Proof. Assume $\tau(x) = M$, and let $\epsilon = x(M) - \alpha(M)$. If $\epsilon = 0$ then $x \in K(\alpha)$. If $\epsilon > 0$, let

$$y_i = \alpha_i \qquad i \in M$$

$$y_i = x_i + \epsilon/(n-m) \qquad i \in P\ .$$

Then $y \in K(\alpha)$ and $y \text{ dom}_P x$. The remainder of the lemma is simply a restatement of Lemma 2.

Lemma 4. There exists $x \notin K(\alpha) \cup \text{dom } K(\alpha)$ with $\tau(x) = T$ if and only if

$$\alpha(M) + |S \cap P| \cdot (1-\alpha(T))/(n-m) \geq 1 \tag{6}$$

for all $S \subseteq N$ such that $|S| = n-m$ and $S \cap T = \emptyset$.

Proof. For any $x \in X$, let

$$y_i = \alpha_i \qquad i \in T$$

$$y_i = 0 \qquad i \in M-T$$

$$y_i = x_i + (x(M) - \alpha(T))/|P| \qquad i \in P\ .$$

If x satisfies (5), then y clearly also satisfies (5). Therefore by Lemma 3, there exists $x \notin K(\alpha) \cup \text{dom } K(\alpha)$ with $\tau(x) = T$ if and only if there exists some y such that

$$y(P) = 1-\alpha(T)$$

$$\alpha(M) + y(S \cap P) \geq 1 \tag{7}$$

for all $S \subseteq N$ with $|S| = n-m$ and $S \cap T = \emptyset$. By the symmetry of (7), such a y exists if and only if (7) is satisfied when

$$y_i = (1-\alpha(T))/|P| \quad i \in P .$$

This establishes the lemma.

Proof of theorem. Observe that (6) is satisfied if and only if it is satisfied when $|S \cap P|$ is minimized, that is $|S \cap P| = n - 2m + |T|$. Therefore, in view of the preceeding lemmas, $K(\alpha)$ is externally stable if and only if

$$\alpha(M) + (n-2m+t)(1-\alpha(T))/(n-m) < 1$$

for all $T \subsetneq M$, where $|T| = t$. Replacing T with $M-T$, this condition is equivalent to

$$t \cdot \alpha(M) + (n-m-t)\alpha(T) < t \tag{8}$$

for all $T \subseteq M$ with $|T| = t > 0$. For $t = 1$, this is exactly the condition (3) of the theorem. For $t > 1$, (3) implies

$$t \cdot \alpha(M) + (n-m-t)\alpha(T) < t \cdot \alpha(M) + t \cdot (n-m-1)\bar{\alpha} < t ,$$

where $\bar{\alpha} = \max_{i \in M} (\alpha_i)$. Thus (8) is equivalent to (3), completing the proof of the theorem.

Comments.

1. With slight modifications to the proof, the theorem may be shown to hold for all $0 \leq m \leq n-2$.

2. The theorem characterizes all m-discriminatory solutions to the $(n,n-m)$-game. It is easily verified that the game has no k-discriminatory solutions for $k \neq m$.

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