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NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Maryland 20034



THE CONVERGING FACTORS FOR THE SINE AND COSINE INTEGRALS

John W. Wrench, Jr.

and

Vicki Alley

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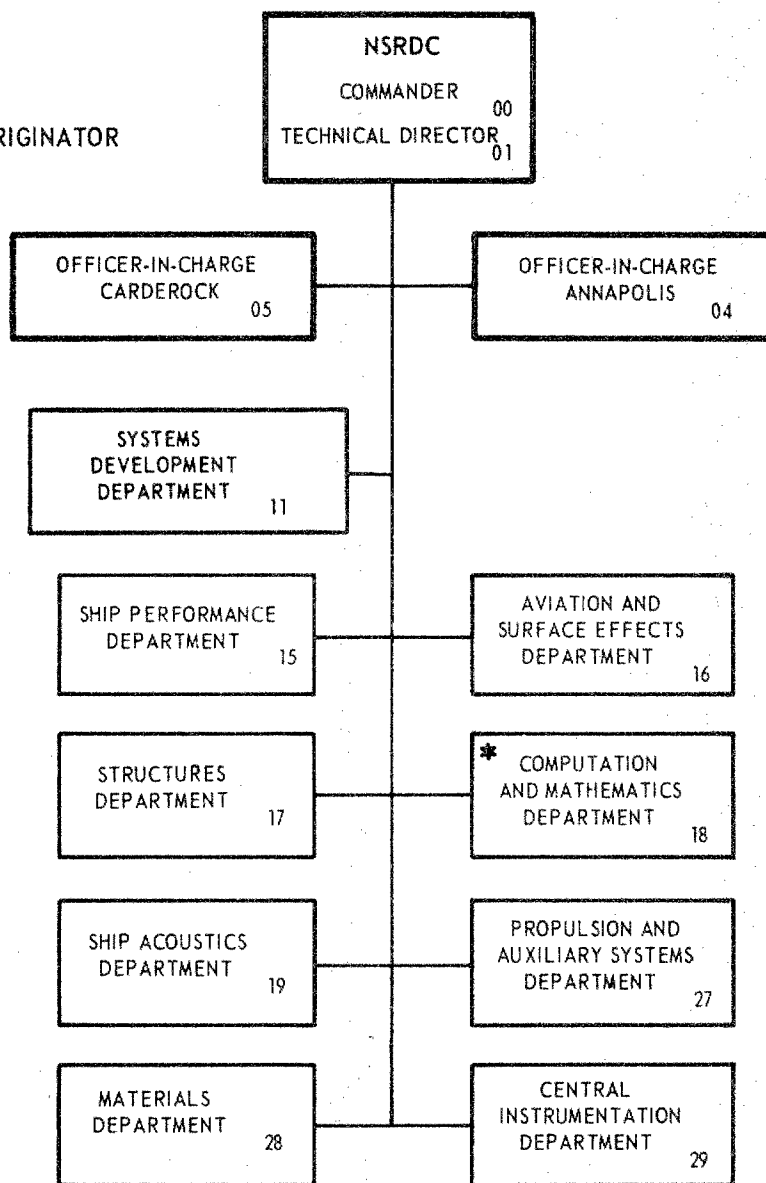
THE CONVERGING FACTORS FOR THE SINE AND COSINE INTEGRALS

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Naval Ship Research and Development Center
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DEPARTMENT OF THE NAVY
NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER
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ABSTRACT

The theory of the converging factors for the sine and cosine integrals is developed and then applied to the calculation on a CDC 6700 system of tables of these factors and their reduced derivatives to about 35 decimal places. The factors are then used in conjunction with appropriately truncated asymptotic series to produce tables of $\text{si}(x)$, $\text{Si}(x)$, and $\text{Ci}(x)$ to 28 decimals by a computer program, for successive integer values of x ranging from 1 to 70, inclusive. Interpolation in the tables by Taylor series is illustrated by numerical examples, which include a new, extended calculation of the Gibbs constant.

ADMINISTRATIVE INFORMATION

Work on this research was authorized by the Naval Ship Systems Command under the Mathematical Sciences Program. Necessary funds were allocated under Subproject SR 0140301, Task 15324.

INTRODUCTION

The sine and cosine integrals are known to be simply related to the exponential integral^{1,2}. This fact has motivated this study, which derives from a detailed examination by Murnaghan and Wrench³ of the converging factor for the exponential integral.

The first tables of numerical values of the sine and cosine integrals appear to be those of Bretschneider⁴, which first appeared in 1843. These consisted of 20-decimal approximations corresponding to the first 10 integer arguments. In 1870 Glaisher⁵ published more voluminous tables to 11 and 18 decimal places. The British Association for the Advancement of Science⁶ has published such tables to 10 and 11 places. The most extensive tabulation has been that of the National Bureau of Standards^{7,8} (1940 and 1942) to 9 and 10 decimals (15 decimals near the extrema), occupying three volumes.

In addition to these tables, a large number of others to less precision (ranging from 3 to 7 decimals) have been published. These are listed (along with those cited) by A. Fletcher⁹ and his associates.

In this report methods are developed for the evaluation of the converging factors for the sine and cosine integrals to high precision, and these procedures are exemplified by the calculation of appended tables of these factors and their reduced derivatives to 33 and 35 decimal places. Application of these tables is illustrated by specific examples.

These algorithms have been programmed for the CDC 6700 system and were used to calculate 28-place tables of $si(x)$, $Si(x)$, and $ci(x)$ for $x = 1(1)70$, which are also appended to this report (Tables 7-9).

¹ A complete list of references is given on pages 98-99.

THE SINE AND COSINE INTEGRALS

The sine and cosine integrals are generally defined by the relations

$$\text{Si}(z) = \int_0^z \frac{\sin t}{t} dt, \quad (1)$$

$$\text{Ci}(z) = \int_{-\infty}^z \frac{\cos t}{t} dt \quad (|\arg z| < \pi) \quad (2)$$

These integrals possess the power-series developments

$$\text{Si}(z) = z - \frac{z^3}{3 \cdot 3!} + \frac{z^5}{5 \cdot 5!} - \dots, \quad (3)$$

$$\text{Ci}(z) = \gamma + \ln z - \frac{z^2}{2 \cdot 2!} + \frac{z^4}{4 \cdot 4!} - \dots, \quad (4)$$

which are convergent for all values of z (excluding zero in the second expansion). However, for argument z of absolute value exceeding 10, say, it is preferable to evaluate these functions by asymptotic series, especially when modified by an appropriate converging factor, which is the subject of this report.

Instead of $\text{Si}(z)$ we shall here consider the complementary integral

$$\text{si}(z) = \int_{\infty}^z \frac{\sin t}{t} dt = \text{Si}(z) - \frac{\pi}{2} \quad (5)$$

Then, if we use the Euler relations

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad (6)$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad (7)$$

we can write

$$\text{si}(z) = \frac{1}{2i} \{ \text{Ei}(iz) - \text{Ei}(-iz) \} = -\text{Im Ei}(-iz) , \quad (8)$$

$$\text{Ci}(z) = \frac{1}{2} \{ \text{Ei}(iz) + \text{Ei}(-iz) \} = \text{Re Ei}(-iz) , \quad (9)$$

where $\text{Ei}(-z)$ is the exponential integral, defined by

$$\text{Ei}(-z) = \int_{\infty}^z \frac{e^{-t}}{t} dt . \quad (10)$$

THE ASYMPTOTIC SERIES FOR THE SINE AND COSINE INTEGRALS AND THEIR CONVERGING FACTORS

In a previous report³ F. D. Murnaghan and the present writer developed the series

$$-\text{Ei}(-z) = \frac{e^{-z}}{z} \left\{ 1 - \frac{1}{z} + \frac{2!}{z^2} - \dots + (-1)^{n-1} \frac{(n-1)!}{z^{n-1}} + (-1)^n \frac{n!}{z^n} \Gamma_n(z) \right\}, \quad (11)$$

where $\Gamma_n(z)$ is the appropriate converging factor, which is given by the integral

$$\Gamma_n(z) = \frac{1}{n!} \int_0^{\infty} \frac{e^{-u} u^n}{1 + \frac{u}{z}} du, \quad n = 1, 2, 3, \dots, \quad (12)$$

and which satisfies both the recurrence relation

$$\Gamma_n(z) = 1 - \frac{n+1}{z} \Gamma_{n+1}(z) \quad (13)$$

and the differential equation

$$\frac{d}{dz} \Gamma_n(z) = \left(\frac{n+1}{z} + 1 \right) \Gamma_n(z) - 1. \quad (14)$$

From the asymptotic series for $Ei(-z)$ we then deduce, upon replacing z by iz , the series

$$si(z) = -z^{-1} [C(z) \cos z + D(z) \sin z], \quad (15)$$

$$Ci(z) = z^{-1} [C(z) \sin z - D(z) \cos z], \quad (16)$$

where

$$C(z) = 1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \dots + \frac{(-1)^k (2k)!}{z^{2k}} \operatorname{Re} \Gamma_{2k}(-iz), \quad (17)$$

$$D(z) = \frac{1}{z} - \frac{3!}{z^3} + \frac{5!}{z^5} - \dots + \frac{(-1)^k (2k+1)!}{z^{2k+1}} \operatorname{Re} \Gamma_{2k+1}(-iz), \quad (18)$$

This is equivalent to the corresponding series given by Dingle¹⁰.

If we set

$$\Pi_n(z) = \operatorname{Re} \Gamma_n(-iz) \quad (19)$$

and

$$\Omega_n(z) = -\operatorname{Im} \Gamma_n(-iz), \quad (20)$$

so that $\Gamma_n(-iz) = \Pi_n(z) - i \Omega_n(z)$, then we deduce from

Equation (12) the representations

$$\Pi_n(z) = \frac{z^2}{n!} \int_0^\infty \frac{e^{-u} u^n}{u^2 + z^2} du \quad (21)$$

$$\Omega_n(z) = \frac{z}{n!} \int_0^\infty \frac{e^{-u} u^{n+1}}{u^2 + z^2} du = \frac{n+1}{z} \Pi_{n+1}(z) \quad (22)$$

for the components of $\Gamma_n(-iz)$ as real improper integrals.

In the previously cited report of Murnaghan and Wrench³ the following series was developed for $\Gamma_n(z)$:

$$\Gamma_n(z) = a_0(\beta) + \sum_{j=1}^{\infty} \frac{a_j(\beta)}{z^j} \quad (23)$$

where $\beta = 1 - \frac{n+1}{z}$, $a_0(\beta) = \frac{1}{2-\beta}$, $a_j(\beta) = \frac{(1-\beta)P_{j-1}(1-\beta)}{(2-\beta)^{2j+1}}$. Here

$P_j(v)$ is a polynomial in v of degree j which satisfies the recurrence

$$P_{j+1}(v) = \{(j+1)v - (j+2)\} P_j(v) - v(v+1) \frac{d}{dv} P_j(v), \quad (24)$$

and $P_0(v) = 1$.

If we replace $n+1$ by s and z by $-is$, then Equation (23) becomes

$$\Gamma_{s-1}(-is) = \frac{1}{1+i} + i \sum_{j=1}^{\infty} \frac{P_{j-1}(i)}{(1+i)^{2j+1} (-is)^j} \quad (25)$$

Since $(1+i)^2 = 2i$ and $\frac{i}{1+i} = \frac{1}{2}(1+i)$, we have

$$\Gamma_{s-1}(-is) = \frac{1}{2}(1+i) + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(1+i) P_{j-1}(i)}{(2s)^j} \quad (26)$$

Now, if we set $n = s-1$ and $z = -is$ in Equation (13) we obtain

$$\Gamma_s(-is) = -i + i \Gamma_{s-1}(-is) \quad (27)$$

and consequently

$$\Gamma_s(-is) = \frac{1-i}{2} - \frac{1-i}{2} \sum_{j=1}^{\infty} \frac{P_{j-1}(i)}{(2s)^j} \quad (28)$$

Then if we set $P_{j-1}(i) = A_{j-1} + i B_{j-1}$ and $\Gamma_s(-is) = \Pi_s(s) - i \Omega_s(s)$, we can deduce the expansions

$$\Pi_s(s) = \frac{1}{2} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{A_{j-1} + B_{j-1}}{(2s)^j}, \quad (29)$$

$$\Omega_s(s) = \frac{1}{2} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{A_{j-1} - B_{j-1}}{(2s)^j}. \quad (30)$$

When expanded to ten terms these series are explicitly:

$$\begin{aligned} 2\Pi_s(s) = 1 - \frac{1}{2s} + \frac{1}{(2s)^2} + \frac{3}{(2s)^3} - \frac{55}{(2s)^4} + \frac{599}{(2s)^5} \\ - \frac{5823}{(2s)^6} + \frac{49595}{(2s)^7} - \frac{266743}{(2s)^8} - \frac{2679473}{(2s)^9} + \dots, \end{aligned} \quad (31)$$

$$\begin{aligned} 2\Omega_s(s) = 1 - \frac{1}{2s} + \frac{3}{(2s)^2} - \frac{13}{(2s)^3} + \frac{59}{(2s)^4} + \frac{185}{(2s)^5} \\ - \frac{1309}{(2s)^6} + \frac{45387}{(2s)^7} - \frac{832613}{(2s)^8} + \frac{12609823}{(2s)^9} - \dots \end{aligned} \quad (32)$$

The coefficients of the first 60 terms of these series have been calculated on the CDC 6700 system in the Computation and Mathematics Department. These numbers are listed in Tables 1 and 2. Dingle¹¹ has given the first four terms of the series for $\Pi_s(s)$.

Table 1. Coefficients in the Asymptotic Series $2\Pi_s(s) \sim 1 + \frac{a_1}{2s} + \frac{a_2}{(2s)^2} + \dots + \frac{a_k}{(2s)^k} + \dots$

k	
1	-1
2	1
3	3
4	-55
5	599
6	-5823
7	49595
8	-2 66743
9	-26 79473
10	1414 94849
11	-36721 18989
12	7 58170 24777
13	-126 64772 40185
14	1299 12747 01953
15	17923 98707 48011
16	-17 17911 68578 49399
17	716 59803 98678 89919
18	-22863 39467 40671 88479
19	5 76116 40787 21862 80803
20	-90 11585 44798 12324 81975
21	-1373 12307 50976 19279 53481
22	2 08827 66361 59497 57877 31137
23	-121 69178 34705 56823 88291 25605
24	5295 89538 42235 03570 16979 45737
25	-1 80060 61784 78714 07644 38106 73233
26	38 70985 50168 53454 95003 48440 97729
27	575 64405 63347 88761 72446 90115 85171
28	-1 32841 41270 98610 05858 45187 08904 90743
29	100 59880 30170 79671 32956 30131 03499 51975
30	-5583 41244 86055 82565 91489 36677 75596 18047
31	2 40688 17733 73565 56430 16266 63551 20113 85291
32	-66 90276 55197 81316 73940 23591 71494 76892 11319
33	-873 49040 62898 01147 70021 68810 80337 18961 51201
34	3 08000 42883 70745 45808 33044 38889 13997 13900 48641
35	-290 10482 67598 45285 57593 91632 86665 98363 80589 03357
36	19689 33992 82099 13521 69504 34830 04316 43651 44815 14185
37	-10 34072 99385 11278 31517 20921 97977 24902 35095 70716 56681
38	356 86003 97030 06463 95679 13958 18354 95970 27925 04263 91617
39	3588 99401 42698 42841 94933 08667 97198 33484 27823 51377 51675
40	-20 64813 95753 43537 60156 72538 37790 22993 00971 16709 49585 96343
41	2349 60566 39679 01453 08948 61122 67508 16108 92946 38376 72570 19407
42	-1 89473 27778 42918 87985 91212 55888 86504 68617 79745 75943 85298 38271
43	117 91722 63715 57621 15102 32208 14097 06618 23380 34471 85365 60087 25491
44	-4907 28266 58743 83887 14849 19447 01286 51575 12453 28196 37606 73405 72663
45	-29797 28133 29637 10521 39806 95000 06566 15158 20186 13645 48786 56528 43705
46	340 64847 73622 23828 31451 06589 35986 33707 83513 35598 02634 32491 29789 79713
47	-45885 73677 84107 80684 30731 45132 94459 49454 86234 99213 90637 35756 08895 33909
48	43 05638 64711 71002 77183 42821 19221 44479 49301 90002 02124 26970 21302 56795 62441
49	-3110 91560 27407 29951 86287 82994 84793 07137 69780 55510 09573 06512 50598 50285 99361
50	1 52730 79571 90567 01688 74512 39868 91212 92923 03511 39709 64279 59443 81626 96170 00961
51	1 91322 22017 39457 89638 47636 98943 98465 88405 54367 41276 93922 13890 71451 39287 29123
52	-12280 53122 55014 72109 45158 53605 55350 76773 86800 63743 91838 04909 08616 66713 36514 06775
53	19 29474 74559 70016 33925 96926 65353 85197 52172 92567 25779 87742 11522 07835 74319 54520 39799
54	-2073 66136 07247 90292 32251 73859 65224 47290 80849 25190 33914 10336 57622 79532 48551 99786 54783
55	1 71230 94592 20203 94758 19900 48941 31813 56579 44574 68805 10624 16846 17523 37980 46532 88713 05435
56	-97 48060 42776 36270 36599 43865 77461 66159 36041 92477 73953 66847 87021 65290 62340 90075 29500 40183
57	417 49016 24812 41456 47834 59374 16700 43255 46624 92017 90962 54729 97237 17768 25801 10013 43198 48047
58	8 82988 57479 96115 00047 04993 57741 02488 46439 60048 96198 75897 80145 41578 54541 94556 33066 30883 07649
59	-1600 75508 90374 34137 17615 59182 55519 41104 42840 07901 60435 37033 65376 52020 88695 73923 34392 07823 04429
60	1 94624 84634 90560 13581 39582 74924 00188 06076 33146 81917 88932 32355 46478 98704 80577 75033 64438 87627 40617

Table 2—Coefficients in the asymptotic series $2\Omega_s(s) \sim 1 + \frac{b_1}{2s} + \frac{b_2}{(2s)^2} + \dots + \frac{b_k}{(2s)^k} + \dots$

k	
1	-1
2	3
3	-13
4	59
5	-185
6	-1309
7	45387
8	-8 32613
9	126 09823
10	-1585 44573
11	12292 07011
12	1 50570 55675
13	-109 97350 18057
14	3710 31774 22819
15	-87331 59462 67205
16	20 35465 18552 11419
17	-282 34769 05605 13521
18	-3874 95246 92383 40157
19	4 72305 89051 68228 81747
20	-235 62407 44709 18718 94981
21	8875 23874 61860 79837 00775
22	-2 62358 66661 97428 97245 67069
23	48 57059 91475 85708 16302 43627
24	740 93368 72921 48923 85310 98267
25	-1 39099 88832 33864 03020 66299 25377
26	93 03390 26282 16838 75960 16052 16067
27	-4601 75759 56346 23548 54556 08635 32349
28	1 77240 26703 25442 25816 19923 05477 53915
29	-43 59018 76303 41339 84392 44776 66009 27337
30	-616 76092 33188 27345 02887 51595 71341 63741
31	1 74959 86777 99747 31890 62842 65491 97959 54075
32	-148 41051 46152 42851 24862 70236 81614 35137 42181
33	9147 50507 70271 89509 45876 26680 58176 20147 76559
34	-4 37028 52738 33341 66805 33455 17658 22379 10509 78877
35	135 92842 50114 88023 52209 29284 72900 62988 94235 92627
36	1589 14129 61157 47470 18682 11318 39244 57202 58536 49979
37	-7 06297 67842 72401 48588 07380 35804 11391 49380 37207 85145
38	733 41235 57018 54905 83030 13823 21644 95347 20648 81845 36291
39	-54421 81228 22082 04076 45680 88650 36250 17577 00924 54428 56693
40	31 20490 47916 37097 64302 19353 83331 57005 71059 23618 93066 54107
41	-1186 26960 90930 46670 66941 44871 16369 24714 76743 69220 24778 05857
42	-9700 67408 74282 53374 26028 16667 71292 66894 11398 64509 94755 16093
43	75 55624 61596 27088 87348 76333 62307 80466 74007 07305 45567 01151 85571
44	-9374 41895 89447 16499 99625 07533 25982 05381 70423 37314 14820 66732 63685
45	8 17290 01061 70519 73995 37103 09357 24501 06069 02025 69904 76130 88140 87543
46	-549 26176 21184 89669 72576 38017 51793 15292 20389 96705 31055 08323 51763 25661
47	24887 20967 23615 13375 03777 10684 27537 26545 41957 76776 71543 38864 98278 54715
48	93540 88016 13077 37377 82349 27323 80133 74690 63134 11720 17070 31499 12983 16699
49	-1866 51679 92401 03534 41768 84369 86584 31956 28789 73889 96469 57774 96815 10170 75761
50	2 71983 02777 46247 19320 84231 18998 35109 57980 66996 64097 73474 91556 67510 12596 82243
51	-273 58437 57829 89528 33305 83746 68076 84141 11205 23363 30723 01135 46334 30226 22748 52973
52	21188 60204 99068 86703 34338 87272 51841 86990 05167 25652 78299 86749 50279 27210 46854 98939
53	-11 21274 92550 73261 96496 67414 17165 91459 04188 99590 75554 18029 08804 29939 57148 70249 33145
54	16 05532 41939 37968 34242 70004 31365 48344 48871 45469 16522 62362 64747 37709 15150 92563 71171
55	85991 70184 71188 72077 86041 24401 03812 23277 86490 22975 16342 60260 72789 04694 32585 41755 09227
56	-182 19241 31412 91972 60607 60673 12843 34147 08145 74273 37312 82273 80902 23331 26893 77107 95707 92293
57	18565 72464 89004 13535 89252 42957 24215 93218 78802 14093 81038 13998 47917 38411 16185 57876 33804 56383
58	-18 30997 62386 74849 55533 21189 05025 86471 11338 80864 13762 10740 25401 65085 92098 01757 16985 18956 91453
59	994 50021 05358 29807 32008 24397 02453 78492 47166 61160 28679 14998 42716 89544 12465 76381 25971 14467 94371
60	-7249 05537 14774 88648 38846 60138 89944 35077 12686 30741 77085 09661 43777 34047 12143 93197 12535 34266 30725

CALCULATION OF THE CONVERGING FACTORS AND THEIR DERIVATIVES

The recurrence formula for $\Gamma_n(z)$ implies that

$$\Gamma_n(-ix) = 1 - i \left(\frac{n+1}{x} \right) \Gamma_{n+1}(-ix) \quad (33)$$

from which we infer in particular that

$$\Pi_s(s) = 1 - \left(1 + \frac{1}{s} \right) \Omega_{s+1}(s) , \quad (34)$$

$$\Omega_s(s) = \left(1 + \frac{1}{s} \right) \Pi_{s+1}(s) . \quad (35)$$

Clearly, if we can develop a Taylor-series expansion of $\Pi_s(s+h)$ and of $\Omega_s(s+h)$ which is convergent for $|h| \leq 1$, then we can derive numerical values of $\Pi_{s+1}(s)$ and $\Omega_{s+1}(s)$ from those of $\Pi_{s+1}(s+1)$ and $\Omega_{s+1}(s+1)$, and thence deduce values of $\Pi_s(s)$ and $\Omega_s(s)$. For large s , the values of $\Pi_s(s)$ and $\Omega_s(s)$ are obtainable from the asymptotic series (29) and (30). This iterative process can then be used to derive a table of values of these converging factors for consecutive integral values of s in descending order.

Thus, there remains the derivation of the Taylor series for $\Pi_s(s+h)$ and $\Omega_s(s+h)$, which are required for interpolation in the tables.

First, we observe that

$$\begin{aligned} \frac{d}{dz} \Gamma_s(-iz) &= -i \frac{d}{d(-iz)} \Gamma_s(-iz) \\ &= \left(\frac{s+1}{z} - i \right) (\Pi_s(z) - i \Omega_s(z)) + i , \end{aligned} \quad (36)$$

whence

$$\begin{aligned}\frac{d}{dz} \Pi_s(z) &= \operatorname{Re} \frac{d}{dz} \Gamma_s(-iz) \\ &= \frac{s+1}{z} \Pi_s(z) - \Omega_s(z)\end{aligned}\quad (37)$$

and

$$\begin{aligned}\frac{d}{dz} \Omega_s(z) &= -\operatorname{Im} \frac{d}{dz} \Gamma_s(-iz) \\ &= \frac{s+1}{z} \Omega_s(z) + \Pi_s(z) - 1.\end{aligned}\quad (38)$$

If we set $d_j = \frac{d^j}{dz^j} \Pi_s(z)$ and $\delta_j = \frac{d^j}{dz^j} \Omega_s(z)$, evaluated at $z=s$,

then these relations can be written

$$d_1 = \left(1 + \frac{1}{s}\right) d_0 - \delta_0 \quad (39)$$

$$\delta_1 = \left(1 + \frac{1}{s}\right) \delta_0 + d_0 - 1 \quad (40)$$

Next we differentiate both sides of the equation

$$z \frac{d}{dz} \Gamma_s(-iz) = [(s+1) - iz] [\Pi_s(z) - i\Omega_s(z)] + iz \quad (41)$$

with respect to z and then set $z=s$. This yields the equation

$$d_1 - i\delta_2 + s(d_2 - i\delta_2) = -i(d_0 - i\delta_0) + (s+1-is)(d_1 - i\delta_1) + i, \quad (42)$$

which implies

$$d_2 = d_1 - \delta_1 - \frac{\delta_0}{s} \quad (43)$$

and

$$\delta_2 = d_1 + \delta_1 - \frac{1 - d_0}{s}. \quad (44)$$

In a similar manner we find

$$d_3 = (1 - \frac{1}{s})d_2 - \delta_2 - \frac{2}{s} \delta_1 , \quad (45)$$

$$\delta_3 = (1 - \frac{1}{s})\delta_2 + d_2 + \frac{2}{s} d_1 ; \quad (46)$$

and in general

$$d_k = (1 - \frac{k-2}{s})d_{k-1} - \delta_{k-1} - \frac{k-1}{s} \delta_{k-2} , \quad (47)$$

$$\delta_k = (1 - \frac{k-2}{s})\delta_{k-1} + d_{k-1} + \frac{k-1}{s} d_{k-2} , \quad (48)$$

valid for $k \geq 3$.

Then the Taylor series for $\Pi_s(s+h)$ and $\Omega_s(s+h)$ can be written

$$\Pi_s(s+h) = d_0 + d_1 h + \frac{d_2}{2!} h^2 + \frac{d_3}{3!} h^3 + \dots , \quad (49)$$

$$\Omega_s(s+h) = \delta_0 + \delta_1 h + \frac{\delta_2}{2!} h^2 + \frac{\delta_3}{3!} h^3 + \dots . \quad (50)$$

Thus we have computed appended Table 3 of the converging factor $\Pi_s(s)$, designated by d_0 , and its reduced derivatives $d_j/j!$ (designated as D_j) to 35 decimal places for $s = 2(1)70$, that is, for all integer values of s from 2 to 70, inclusive.

As useful by-products, there evolved tables of the converging factors $\Pi_{s+1}(s)$, $\Omega_s(s)$, and $\Omega_{s+1}(s)$, which have been rounded to 33 decimal places, and appear as Tables 4-6, respectively. (Appendixes B and C).

For small values of z and even integer values of s , the following series given by Dingle¹¹ is useful:

$$\begin{aligned} \Pi_s(z) = & \frac{z^2}{s(s-1)} - \frac{z^4}{s(s-1)(s-2)(s-3)} + \dots + (-1)^{(s-2)/2} \frac{z^s}{s!} \\ & + \frac{(-1)^{s/2} z^{s+1}}{s!} \left\{ \frac{\pi}{2} \cos z - \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1} (\psi(2k+1) - \ln z)}{(2k+1)!} \right\}, \end{aligned} \quad (51)$$

where $\psi(n) = -\gamma + 1 + \frac{1}{2} + \dots + \frac{1}{n}$, $\psi(0) = -\gamma$, and γ is Euler's constant.

For odd integer values of s , Dingle gives the companion series

$$\begin{aligned} \Pi_s(z) = & \frac{z^2}{s(s-1)} - \frac{z^4}{s(s-1)(s-2)(s-3)} + \dots + (-1)^{(s-3)/2} \frac{z^{s-1}}{s!} \\ & + \frac{(-1)^{(s-1)/2} z^{s+1}}{s!} \left\{ \frac{\pi}{2} \sin z + \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k} (\psi(2k) - \ln z)}{(2k)!} \right\}. \end{aligned} \quad (52)$$

When $z=1$ these series simplify appreciably; thus, we obtain

$$\Pi_1(1) = \frac{\pi}{2} \sin 1 + \psi(0) - \frac{\psi(2)}{2!} + \frac{\psi(4)}{4!} - \frac{\psi(6)}{6!} + \dots, \quad (53)$$

$$2\Pi_2(1) = 1 - \frac{\pi}{2} \cos 1 + \psi(1) - \frac{\psi(3)}{3!} + \frac{\psi(5)}{5!} - \dots; \quad (54)$$

and from Equation (35) we obtain the relation, $2\Pi_2(1) = \Omega_1(1)$.

The corresponding numerical values to 45D, obtained from evaluating 20 terms of each of these series, are:

$$\Pi_1(1) = 0.34337\ 79615\ 56427\ 03283\ 25330\ 03858\ 31243\ 40012\ 44019,$$

$$\Omega_1(1) = 0.37855\ 03757\ 64186\ 64236\ 07342\ 71784\ 66067\ 61068\ 35323.$$

The values of these converging factors when computed to about 35 decimal places on the CDC 6700 system, as the final step in the descending iterative procedure beginning with $\Pi_{70}(70)$ and $\Omega_{70}(70)$, differed from the above values by less than $1 \cdot 10^{-33}$.

APPLICATIONS

In the approximation of a function with a discontinuity by a finite number of terms of its expansion in a Fourier series, a paradoxical situation occurs that is known as the Gibbs phenomenon, named after the physicist J.W. Gibbs, who first described it in 1898.

To illustrate this phenomenon consider the Fourier series

$\sum_{n=1}^{\infty} \frac{\sin nt}{n}$ which has the values $(\pi-t)/2$ for $0 < t < \pi$, $(t-\pi)/2$ for $-\pi < t < 0$, and 0 for $t = 0$ or π . The approximating curves

$y_n = \sum_{\nu=1}^n \frac{\sin \nu t}{\nu} = f_n(t)$ behave in an extraordinary manner; namely,

near zero the n^{th} curve remains above the highest point, $\frac{\pi}{2}$, of the straight line by a definite ratio $K = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt$, which is called Gibbs' constant.

To evaluate this constant to high precision, we note that

$$K = \frac{2}{\pi} \text{Si}(\pi) = 1 + \frac{2}{\pi} \text{si}(\pi), \quad (55)$$

and then calculate $\text{si}(\pi)$ by means of the truncated asymptotic series

$$\text{si}(\pi) = \pi^{-1} - 2\pi^{-3} + 24\pi^{-5} \Pi_4(\pi). \quad (56)$$

The converging factor $\Pi_4(\pi)$ can be calculated from the Taylor series

$$\Pi_4(\pi) = \Pi_4(4) + d_1 h + \frac{d_2}{2!} h^2 + \dots, \quad (57)$$

where $h = \pi - 4$, and the coefficients are the reduced derivatives of $\Pi_s(s)$ evaluated at $s = 4$. If 43 terms of this series are summed, each carried to 35 decimal places, there results the value

$$\Pi_4(\pi) = 0.34852\ 97445\ 99239\ 66719\ 19407\ 36171\ 10389.$$

Multiplying this converging factor by $24\pi^{-5}$ and adding the product to $\pi^{-1} - 2\pi^{-3}$, we obtain

$$\text{si}(\pi) = 0.28114\ 07251\ 87569\ 55112\ 97316\ 78518\ 4\dots$$

As a check on the accuracy of this approximation, the value of $\text{si}(\pi)$ was recalculated by means of the series

$$\text{si}(\pi) = \frac{\pi}{2} - \frac{\pi^3}{3 \cdot 3!} + \frac{\pi^5}{5 \cdot 5!} - \frac{\pi^7}{7 \cdot 7!} + \dots, \quad (58)$$

of which a total of 27 terms sufficed to give the 45-place value

$$\text{si}(\pi) = 0.28114\ 07251\ 87569\ 55112\ 97316\ 78518\ 24000\ 15518\ 69618.$$

This shows that the value computed by means of the asymptotic series as modified by the introduction of an appropriate converging factor is in error by less than $2 \cdot 10^{-31}$.

The value of Gibbs' constant is then calculated to be

$$K = 1.17897\ 97444\ 72167\ 27023\ 20288\ 45824\ 90979\ 25874\ 21979\dots$$

to 45 decimal places. This confirms the accuracy of a value calculated to 24 decimal places by D. H. Lehmer¹².

As a by-product of this calculation we infer from the relation in Equation (5) that

$$\text{Si}(\pi) = 1.85193\ 70519\ 82466\ 17036\ 10533\ 70157\ 99144\ 36504\ 54318\dots$$

A companion calculation consists of evaluating $\text{Ci}(\pi)$ by the truncated asymptotic series

$$\text{Ci}(\pi) = \pi^{-2} - 6\pi^{-4} \Pi_3(\pi), \quad (59)$$

where the converging factor can be evaluated from the Taylor series

$$\Pi_3(\pi) = \Pi_3(3) + d_1 h + \frac{d_2}{2!} h^2 + \dots, \quad (60)$$

and $h = \pi - 3$. Here the coefficients $d_k/k!$ are the reduced derivatives of $\Pi_s(s)$ evaluated at $s = 3$. A total of 24 terms of this series yield the value

$$\Pi_3(\pi) = 0.44894\ 66750\ 45702\ 27893\ 45552\ 80548\ 12960.$$

If we multiply this converging factor by $6\pi^{-4}$ and subtract the product from π^{-2} , we obtain

$$Ci(\pi) = 0.07366\ 79120\ 46425\ 48599\ 01009\ 65230\ 0\dots$$

As a check, a total of 27 terms of the series

$$Ci(\pi) = \gamma + \ln \pi - \frac{\pi^2}{2 \cdot 2!} + \frac{\pi^4}{4 \cdot 4!} - \frac{\pi^6}{6 \cdot 6!} + \dots \quad (61)$$

yielded the value

$$Ci(\pi) = 0.07366\ 79120\ 46425\ 48599\ 01009\ 65230\ 14967\ 18698\ 77462,$$

which reveals an error in the previous result of about $1.5 \cdot 10^{-31}$.

For $x = \pi$ we conclude that the calculation of $Si(x)$ and $Ci(x)$ to a specified high accuracy by power series involves less computation than that by asymptotic series in conjunction with converging factors.

However, for an argument such as 2π this situation is reversed. Thus, by means of the series

$$si(2\pi) = -\frac{1}{2\pi} + \frac{2!}{(2\pi)^3} - \frac{4!}{(2\pi)^5} + \frac{6!}{(2\pi)^7} \Pi_6(2\pi) \quad (62)$$

we find the value

$$si(2\pi) = -0.15264\ 47506\ 62268\ 16898\ 55415\ 29340\ 00201,$$

after determining the converging factor to 35 decimal places from $\Pi_6(6)$, using 23 terms of the corresponding Taylor series. On the other hand, the calculation of this value by the power series requires the evaluation of 30 terms, each to at least 35 places. Again, by Equation (5) we deduce the value

$$\text{Si}(2\pi) = 1.41815\ 15761\ 32628\ 45024\ 57801\ 62299\ 74943\ ,$$

to 35 decimal places.

Similarly, $\text{Ci}(2\pi)$ can be calculated to 35 places from the series

$$\text{Ci}(2\pi) = -\frac{1}{(2\pi)^2} + \frac{3!}{(2\pi)^4} - \frac{5!}{(2\pi)^6} + \frac{7!}{(2\pi)^8} \Pi_7(2\pi) \quad (63)$$

when the converging factor is evaluated to that precision from $\Pi_7(7)$, using 30 terms of the Taylor series. This result is

$$\text{Ci}(2\pi) = -0.02256\ 06617\ 46346\ 06764\ 35387\ 78543\ 04643\ ;$$

it was checked by evaluating 32 terms of the corresponding power series.

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* Informally documented in CMD Technical Note 45-71, entitled "The Lehmer-Weinberger Multiprecision Integer Subroutine Package," October 1971.

APPENDIX A

VALUES OF $\Pi_s(s)$ AND OF ITS REDUCED DERIVATIVES

In this appendix are tabulated to 35 decimal places the values of the converging factor $\Pi_s(s)$ and its reduced derivatives D_i , defined as $\frac{1}{i!} \frac{d^i}{dx^i} \Pi_s(x)$ at $x = s$. This table (Table 3) has been photographically reproduced from computer sheets upon which the output was left-justified. Accordingly, the position of the decimal point for each tabular entry may be inferred from the right-hand indentation.

TABLE 3 - TABLE OF $\Pi_s(s)$ AND ITS REDUCED DERIVATIVES D_1
TO 35 D FOR $s = 2(1)70$

$s = 2$

C SUB 1

0	40391	60456	23264	61242	93335	56951	69424
1	10405	52805	84226	60047	88123	65348	64553
2	- 31749	17037	61893	24140	44536	58834	2010
3	14915	11505	90193	16353	66277	69133	920
4	14719	99540	24558	56275	17292	27421	389
5	- 78780	80540	88462	88072	80001	57228	82
6	29876	07055	61404	27253	62537	42981	66
7	- 99959	38707	51720	35272	27015	18805	0
8	31867	02613	50387	23845	27502	42895	2
9	- 10023	83255	27755	49637	61208	60196	7
10	31682	17075	27209	28651	17504	23640	
11	- 10160	69307	07708	28635	41586	53946	
12	23227	20561	59410	46378	69262	0504	
13	- 11102	14132	42406	15888	67225	6361	
14	37914	01406	50098	13763	23585	355	
15	- 13224	42632	06427	04766	68164	058	
16	47062	51380	19091	74046	65505	24	
17	- 17063	04525	18092	81018	66032	99	
18	62937	76720	57185	18185	22163	7	
19	- 23584	81220	69028	50965	65796	6	
20	89670	35740	15971	49404	20503		
21	- 34549	01157	88150	04468	25557		
22	13474	56836	51330	85498	61525		
23	- 53144	02111	28118	38802	0368		
24	21177	08327	00721	22615	4006		
25	- 85192	45078	62162	61556	517		
26	34573	69507	20721	20512	237		
27	- 14145	50731	23165	95890	247		
28	58313	00465	42516	21580	68		
29	- 24207	93082	18400	02411	76		
30	10115	52525	11252	32812	22		
31	- 42527	55059	87133	47645	6		
32	17981	80538	53847	85218	5		
33	- 76440	35526	16466	27581			
34	32658	57403	39527	48292			
35	- 14619	34580	81531	76074			
36	60450	02840	24184	6015			
37	- 26175	47615	38067	3016			
38	11379	47827	15075	4301			
39	- 49657	83428	72745	246			
40	21747	32747	25700	233			
41	- 95564	82937	83697	70			
42	42130	00228	26516	38			
43	- 18630	23801	22703	12			
44	82625	84481	25898	2			
45	- 36747	29504	61598	8			
46	16386	72156	87505	0			
47	- 73259	66282	75573				
48	32831	90245	37749				
49	- 14748	25523	29174				
50	66398	25319	4170				
51	- 29557	43051	1254				
52	13544	00851	6529				
53	- 61354	95068	036				
54	27847	02146	424				
55	- 12662	02441	715				
56	57675	50245	64				
57	- 26316	22565	68				
58	12027	21052	60				
59	- 55054	54851	5				
60	25239	73800	4				
61	- 11588	23554	3				
62	53280	62461					
63	- 24531	25408					
64	11309	70926					
65	- 52208	8155					
66	24131	2654					
67	- 11167	1783					
68	51738	987					
69	- 23598	728					
70	11143	965					
71	- 51803	43					
72	24106	33					
73	- 11229	10					
74	52358	5					
75	- 24436	9					
76	11415	9					
77	- 53379						
78	24981						
79	- 11701						
80	5485						
81	- 2573						
82	1208						
83	- 567						
84	267						
85	- 125						
86	55						
87	- 27						
88	13						
89	- 6						
90	2						
91	- 1						

S = 3

I	D SUE I						
0	43050	94642	81085	40741	01666	92627	25099
1	13258	35246	98383	77782	73629	17844	89729
2	- 16820	54841	05703	41502	11800	32173	7465
3	50276	76456	83257	90903	94528	45706	73
4	41108	56382	57077	15632	26495	46383	58
5	- 15180	34612	18376	18923	52516	15167	87
6	37422	85848	94707	39411	02755	63914	1
7	- 78049	55634	60656	55096	69517	50684	
8	14755	95156	69832	01356	74199	39462	
9	- 25948	31138	33191	14833	78945	5772	
10	42552	75261	16953	31899	77356	275	
11	- 63529	60391	84789	04772	81621	73	
12	78915	69777	91644	30914	60616	2	
13	- 49549	52895	99196	30006	40966		
14	- 15147	30631	62851	15999	13199		
15	90385	68225	37728	16657	5038		
16	- 32995	31426	64789	51594	3003		
17	10413	19009	22300	28987	4796		
18	- 30806	99488	62592	92105	419		
19	88197	26836	12069	76009	58		
20	- 24815	74932	76085	40357	60		
21	69207	00450	40835	96627	4		
22	- 19225	88789	50297	63025	6		
23	53365	60464	59458	50326			
24	- 14828	79647	85567	43245			
25	41299	52424	64170	4790			
26	- 11537	41913	48579	8071			
27	32344	35094	07024	016			
28	- 91018	68670	54844	48			
29	25713	64219	23436	81			
30	- 72931	80804	13879	7			
31	20767	47208	40061	7			
32	- 59366	11241	42752				
33	17035	11621	43905				
34	- 49063	51608	2677				
35	14181	76251	7406				
36	- 41134	67290	274				
37	11971	24783	632				
38	- 34951	95799	25				
39	10236	51571	46				
40	- 30069	78823	1				
41	88584	27535					
42	- 26168	76006					
43	77511	0996					
44	- 23017	3210					
45	68519	400					
46	- 20445	600					
47	61147	01					
48	- 18327	48					
49	55048	8					
50	- 16568	2					
51	49964						
52	- 15095						
53	4569						
54	- 1385						
55	420						
56	- 128						
57	39						
58	- 11						
59	3						
60	- 1						

S = 4

I	D SUB I							
0		44554	95690	46484	46885	07495	58466	67808
1		10400	23226	45417	56348	53247	00395	70667
2	-	10474	60219	75494	69616	29661	48012	1640
3		22026	34557	28394	87790	96976	11909	22
4		16447	75006	97940	64443	98764	62808	29
5	-	45389	81576	99321	64452	54073	69632	7
6		82006	39392	42102	58645	82558	53512	
7	-	12051	28584	81101	99437	32228	45679	
8		15057	81204	60041	71227	24701	8963	
9	-	15448	77106	05234	57159	56232	931	
10		10170	86934	04116	85845	46102	83	
11		61137	15450	69142	60775	82912		
12	-	42852	00264	41185	60028	88071		
13		11691	71474	51475	73343	00907		
14	-	25866	53966	75997	23841	9826		
15		52307	06426	22721	48660	729		
16	-	10103	44948	63209	28709	416		
17		19044	15953	48877	79326	91		
18	-	35452	67051	80750	55571	6		
19		65659	56086	70126	29249			
20	-	12154	04229	00850	58316			
21		22553	95738	99988	9367			
22	-	42039	14348	58745	739			
23		78806	04605	13740	31			
24	-	14868	85217	74562	10			
25		28248	89795	48086	4			
26	-	54053	74112	35597				
27		10417	77103	89773				
28	-	20222	06094	7063				
29		39529	54146	799				
30	-	77801	23408	64				
31		15414	46172	84				
32	-	30736	05074	4				
33		61665	47302					
34	-	12445	30377					
35		25260	0453					
36	-	51549	645					
37		10575	010					
38	-	21802	36					
39		45165	0					
40	-	93991						
41		19646						
42	-	4123						
43		869						
44	-	183						
45		39						
46	-	8						
47		1						

S = 5

I	D SUB I						
0	45523	70588	04917	28727	42872	20562	70980
1	85708	24288	59819	56121	90664	86225	4647
2	- 71677	67335	41081	76012	56897	20697	536
3	11266	21312	04918	53617	71493	29568	73
4	79164	99391	11338	93014	88195	69132	1
5	- 17398	53823	42393	97572	21273	38345	0
6	24629	03184	92338	90650	70386	48470	
7	- 27312	12827	58467	03616	38113	2960	
8	23748	43486	29823	97200	34694	236	
9	- 12790	75741	70633	83057	90890	64	
10	- 63406	42894	76011	25688	56902		
11	34294	62211	12072	32489	61099		
12	- 71822	04031	45119	84204	2115		
13	11997	96014	11506	23468	9661		
14	- 18029	87827	35033	82783	818		
15	25492	16996	91292	81095	98		
16	- 34670	26445	18613	70514	7		
17	45926	02069	55554	85577			
18	- 59693	75523	97773	1725			
19	76458	01393	97558	737			
20	- 96695	61294	72495	99			
21	12074	98314	05821	57			
22	- 14857	90859	15400	2			
23	17927	78487	94177				
24	- 21018	99451	7777				
25	23529	93466	108				
26	- 24234	89871	31				
27	20773	07639	6				
28	- 87482	6079					
29	- 19849	9765					
30	79406	300					
31	- 19549	458					
32	41326	75					
33	- 81206	6					
34	15308	6					
35	- 28125						
36	5081						
37	- 907						
38	160						
39	- 28						
40	4						

S = 6

I	D SUB I
0	46200 07695 48806 52637 10099 53089 58386
1	72955 07296 53203 90317 34121 26900 7795
2	- 52200 64852 82343 94371 70191 92048 843
3	63915 44126 27590 73346 10340 99471 2
4	42980 12634 48898 74713 95086 84541 6
5	- 78352 43311 02051 28244 47408 13618
6	90712 00899 36966 90958 03480 1000
7	- 79346 19403 94529 07610 55064 565
8	49142 60749 22494 66151 18328 65
9	- 74743 35756 28030 15639 25337
10	- 38211 99562 31010 40342 27183
11	81783 31208 50613 95216 0797
12	- 11891 95244 29071 92726 4912
13	14702 70419 77061 09397 350
14	- 16485 67428 88888 90101 95
15	17203 42147 57889 69273 4
16	- 16862 01123 66908 62911
17	15474 88403 87988 8932
18	- 13034 99132 32266 720
19	94933 55569 78807 4
20	- 47412 99741 54835
21	- 14066 06184 7616
22	92317 83312 470
23	- 19135 96043 014
24	31673 66968 38
25	- 47596 55227 1
26	67914 05263
27	- 93976 5566
28	12758 9196
29	- 17116 763
30	22795 15
31	- 30229 7
32	40008
33	- 5292
34	700
35	- 92
36	12
37	- 1

$$S = 7$$

I	D SUB I							
0		46699	13110	61784	03082	68894	97670	49347
1		63540	02606	71043	92071	35748	90286	4991
2	-	39741	39989	38954	91638	93780	07062	162
3		39073	02778	47304	92543	63874	40918	7
4		25410	65652	20332	28247	34116	64389	9
5	-	39529	09780	07664	16107	51532	87784	
6		38569	22190	21476	10082	03183	4234	
7	-	27453	97634	35901	71206	92485	248	
8		12157	54004	92525	33533	18335	98	
9		25378	67229	32832	50111	08197		
10	-	13811	45044	27553	55609	06323		
11		20637	23264	23960	94230	6010		
12	-	23226	62662	08487	23270	157		
13		22441	00635	02342	72034	15		
14	-	19342	32523	08275	45593	0		
15		14918	58986	52492	37439			
16	-	99555	85516	44518	344			
17		50065	20322	12900	16			
18	-	41533	31209	73431				
19	-	36380	73883	30011				
20		70897	13551	6627				
21	-	99498	70853	955				
22		12271	13676	201				
23	-	14125	96653	76				
24		15592	14613	8				
25	-	16744	17816					
26		17648	7635					
27	-	18362	920					
28		18933	67					
29	-	19398	5					
30		19786						
31	-	2011						
32		204						
33	-	20						
34		2						

S = 8

I	D SUB I
0	47082 51123 20588 52328 73241 13458 10303
1	56296 33874 21036 57669 13643 26977 1388
2	- 31280 82175 84145 22022 83337 21935 208
3	25278 89825 27979 70353 45054 59039 8
4	16011 76916 62335 10832 36043 37558 1
5	- 21703 65455 94010 86956 58116 00911
6	18248 29790 59734 98964 73965 1468
7	- 10819 58432 09176 99855 40754 449
8	33760 92043 63845 40556 65045 5
9	20494 06461 92995 38846 20769
10	- 49827 59046 50806 51275 0194
11	58326 94482 18892 45457 190
12	- 53180 99931 13928 65879 54
13	41221 75508 60952 14032 2
14	- 27464 96987 09568 58599
15	14912 74792 80389 3049
16	- 49469 60708 59533 32
17	- 21238 82523 27573 7
18	65830 39144 97683
19	- 89621 13466 9143
20	98324 76132 461
21	- 96991 25369 61
22	89609 71759 2
23	- 79084 44704
24	67385 6036
25	- 55755 614
26	44909 26
27	- 35201 0
28	26753
29	- 1954
30	134
31	- 8

S = 9

I		D SUB I						
0		47386	24328	69194	14993	42317	43801	00225
1		50546	47550	49676	30295	93601	21220	5967
2	-	25269	07945	19697	02726	52968	41988	079
3		17100	25259	51073	09134	17442	52435	7
4		10602	09133	49910	63786	81847	52560	8
5	-	12724	77200	07183	36981	53064	99599	
6		93789	49639	60278	45866	87292	640	
7	-	47168	13451	82771	31198	72247	21	
8		99754	22190	96038	21641	73644		
9		11320	94304	32542	25590	99819		
10	-	19063	76683	57341	19479	4514		
11		18388	90439	75480	29805	140		
12	-	13946	37909	82728	68920	32		
13		87933	22716	27114	99023			
14	-	44607	52584	11392	6957			
15		14325	19794	10779	695			
16		37025	55219	00375	7			
17	-	12454	52527	43804	3			
18		15161	31754	13004				
19	-	14459	82219	0262				
20		12185	39920	593				
21	-	94629	66424	4				
22		68941	64620					
23	-	47407	9236					
24		30675	492					
25	-	18391	37					
26		97971						
27	-	4056						
28		41						
29		17						
30	-	2						

S = 10

I		D SUB I						
0		47632	79825	24009	77020	41272	49830	83659
1		45869	30025	41280	95038	79163	59913	2747
2	-	20842	29448	31042	88941	16810	21331	021
3		11992	52804	78884	66874	14559	98006	8
4		73046	43931	89132	58452	27162	30327	
5	-	78622	69978	72942	45893	15455	0837	
6		51498	51182	21843	34508	79301	404	
7	-	22291	05827	56772	23247	80574	65	
8		29413	46596	18132	53556	11674		
9		59130	66521	24207	36118	4290		
10	-	77973	86152	31361	40678	571		
11		63832	36463	52048	08008	05		
12	-	40983	58894	94778	82226	5		
13		21144	03410	16966	26429			
14	-	78047	95419	91028	350			
15		44694	47215	09598	2			
16		27359	80684	09620	4			
17	-	35062	78171	18012				
18		31334	64721	2359				
19	-	23769	08849	426				
20		16183	24550	94				
21	-	10074	55245	0				
22		57251	4193					
23	-	28945	409					
24		11937	38					
25	-	25605						
26	-	2042						
27		386						
28	-	42						
29		3						

S = 11

I	D SUB I							
0		47836	92135	27719	26957	07035	84154	28736
1		41988	93742	72430	90743	48207	73466	1937
2	-	17487	30026	38868	47719	24067	73235	571
3		86650	14498	00369	38461	52613	56861	
4		51995	23923	83013	38330	65485	77792	
5	-	50709	58504	53445	93673	67454	3870	
6		29847	72499	56901	52396	34586	574	
7	-	11253	98936	63695	52453	69102	33	
8		76058	13812	81512	51206	4450		
9		31076	96985	27549	77242	9270		
10	-	33976	78718	95856	33081	695		
11		24079	67980	93250	27576	61		
12	-	13254	48455	44634	40830	3		
13		55916	35874	38296	4207			
14	-	13443	74752	53322	322			
15	-	50044	30435	76785	3			
16		10112	80642	28562	7			
17	-	93168	48297	6451				
18		67362	03477	259				
19	-	42155	21357	06				
20		23456	11166	7				
21	-	11522	98688					
22		47316	870					
23	-	12772	50					
24	-	24351						
25		7545						
26	-	799						
27		66						
28	-	4						

S = 12

I		O	SUB	I				
0		48008	69007	75239	68016	15505	64374	18352
1		38716	94961	03773	20174	28049	45791	3308
2	-	14883	38105	17504	11051	73505	82393	855
3		64198	49190	07864	26605	30933	67021	
4		38033	25325	55455	18927	92634	60675	
5	-	33898	76624	73589	95444	80148	3831	
6		18096	50994	48868	95278	57196	585	
7	-	60038	73102	81540	10679	03529	2	
8		87067	31080	27876	07389	301		
9		16730	78600	41657	41572	5482		
10	-	15677	36674	75068	13754	827		
11		97603	24775	53582	74735	3		
12	-	46494	58212	16890	31099			
13		15920	22389	75141	6505			
14	-	17647	00619	69677	84			
15	-	29568	03265	78207	5			
16		34408	74663	10762				
17	-	25558	11219	2265				
18		15495	70941	186				
19	-	81056	87561	0				
20		36547	38091					
21	-	13315	7072					
22		27926	72					
23		11224	0					
24	-	20377						
25		1819						
26	-	131						
27		8						

S = 13

I	D SUB I							
0		48155	22582	58320	87716	19357	50801	31532
1		35920	15362	86211	41060	00577	30174	8016
2	-	12821	59642	26484	34867	79728	32939	564
3		48594	11702	76379	15823	94279	44656	
4		28471	60003	64422	94883	86047	02158	
5	-	23359	01799	10684	35956	20937	2009	
6		11398	43453	49813	03934	32901	551	
7	-	33563	16023	57390	59313	20499	7	
8	-	10056	70110	87183	97659	7292		
9		92732	88009	65971	56681	410		
10	-	76124	05284	27985	69524	55		
11		42108	29755	36266	51355	2		
12	-	17483	60896	01022	16317			
13		47826	50863	28296	565			
14		15341	27274	61941	3			
15	-	13450	00000	21938	9			
16		11808	30672	12914				
17	-	73957	96018	793				
18		38251	83106	75				
19	-	16747	13921	2				
20		59271	5075					
21	-	13039	306					
22	-	28594	7					
23		61192						
24	-	5167						
25		342						
26	-	19						
27		1						

S = 14

I		D SUB I						
0		48281	70369	79771	77667	62017	84125	92488
1		33501	70448	14687	97979	71517	64591	1962
2	-	11161	02655	51938	64268	96948	38283	930
3		37469	92485	42458	72574	66726	12714	
4		21742	16462	37823	06734	58090	41786	
5	-	16521	03460	09972	12145	43968	9152	
6		74183	56022	46279	85975	94418	58	
7	-	19531	69867	86452	29656	08826	2	
8	-	13225	00110	66631	94952	9684		
9		52943	66242	53750	82656	588		
10	-	38678	24032	71318	21032	46		
11		19184	42929	64448	45256	0		
12	-	69806	77732	12123	5025			
13		14831	93464	90783	968			
14		30872	82023	93413	3			
15	-	58127	78267	59666				
16		41928	51833	3360				
17	-	22665	21404	487				
18		10086	32072	91				
19	-	36642	13531					
20		94089	660					
21	-	75957						
22	-	20281	4					
23		18113						
24	-	1158						
25		63						
26	-	3						

S = 15

I	D SUB I
0	48391 97479 27790 70479 43700 52906 22885
1	31389 46648 28422 10283 23422 93148 7375
2	- 98037 95741 06042 99299 88582 73902 96
3	29363 19221 63880 05674 14322 13642
4	16893 09089 75428 79177 95726 92120
5	- 11951 97784 11382 92720 99502 6894
6	49670 56758 51940 89360 51284 25
7	- 11769 93269 34952 74010 07768 1
8	- 11784 39530 17482 93084 1511
9	31098 35959 95872 89839 381
10	- 20462 01692 13736 77427 06
11	91700 55285 03943 93162
12	- 29361 57862 81743 0255
13	46124 38845 04785 67
14	21122 79096 42375 8
15	- 25155 18596 22031
16	15520 24193 2765
17	- 73432 63575 36
18	28209 47360 0
19	- 83255 4693
20	12340 475
21	61533 7
22	- 74152
23	4886
24	- 258
25	11

S = 16

I		D SUB I						
0		48488	96408	92146	53510	79695	22529	00881
1		29528	58924	14359	49099	68607	32923	5568
2	-	86801	88164	94524	25869	07183	40354	41
3		23340	52006	99017	64397	35026	16210	
4		13326	40381	91828	77883	75933	13959	
5	-	88196	05165	44967	43306	62820	437	
6		34094	89264	22855	92750	97932	67	
7	-	73131	29482	33004	88802	86733		
8	-	93778	41754	80287	78683	688		
9		18759	34036	06840	74821	188		
10	-	11222	76054	76489	94782	78		
11		45736	80771	89080	46605			
12	-	12924	61253	78265	7183			
13		13674	22783	56086	85			
14		12187	61165	85997	6			
15	-	11101	40340	63800				
16		59960	57316	745				
17	-	25058	80274	30				
18		83025	07781					
19	-	19084	9609					
20		13727	5					
21		31595	2					
22	-	24251						
23		1312						
24	-	59						
25		2						

S = 17

I

D SUB I

0		48574	93298	64189	70348	41325	62883	93744
1		27876	61300	52230	76893	24025	77514	1872
2	-	77394	46109	12075	84744	74211	69370	70
3		18789	36733	25239	35480	51111	91433	
4		10655	07332	34375	39890	77671	34630	
5	-	66231	62770	27259	73914	97000	848	
6		23923	10174	10304	59453	62946	05	
7	-	46686	67022	86058	34209	79821		
8	-	71349	16028	06758	78388	443		
9		11598	43845	57268	63194	793		
10	-	63578	63132	07646	34255	7		
11		23693	74580	89866	09850			
12	-	59209	39678	51242	631			
13		33921	90512	14584	1			
14		66881	15194	31797				
15	-	50302	98663	9828				
16		24146	28578	267				
17	-	89684	37505	7				
18		25499	18110					
19	-	41843	036					
20	-	93139	8					
21		12945	4					
22	-	7797						
23		361						
24	-	14						

$$S = 18$$

I	D SUB I						
0	48651	65733	28839	68722	26367	91275	18990
1	26400	15066	30670	26536	20133	60363	5197
2	- 69438	97684	64959	34255	33753	22328	60
3	15297	72957	07010	23280	92230	52719	
4	86219	65233	67399	98897	09104	2466	
5	- 50518	96464	56453	11301	51484	696	
6	17117	17930	86211	15687	62495	58	
7	- 30532	45551	10876	18466	95259		
8	- 53281	48916	84318	45281	830		
9	73357	74666	79675	54752	32		
10	- 37084	68200	62660	23432	7		
11	12699	33792	24730	51543			
12	- 28094	31110	78751	849			
13	29347	98645	04553				
14	36274	72035	62491				
15	- 23454	61508	6326				
16	10112	89418	240				
17	- 33519	20823	6				
18	80977	5969					
19	- 75896	64					
20	- 59922	2					
21	50164						
22	- 2540						
23	102						
24	- 3						

S = 19

I		D SUB I						
0		48720	55106	65459	04487	07447	01341	71691
1		25072	58331	83210	80065	32490	14877	2194
2	-	62651	03986	14256	50194	55493	31318	87
3		12582	36490	14292	59910	32369	43136	
4		70522	50096	80405	73597	15244	9565	
5	-	39076	24444	46670	64496	14930	907	
6		12463	70983	31861	56369	39093	95	
7	-	20404	64211	87028	56973	54334		
8	-	39535	21719	44808	72475	013		
9		47377	49944	17296	68701	33		
10	-	22209	98836	67535	85995	1		
11		70187	41602	94601	5860			
12	-	13749	46607	98458	244			
13	-	49159	94682	17380				
14		19756	00405	30342				
15	-	11254	89939	8799				
16		43936	96712	03				
17	-	13028	88836	7				
18		26306	4149					
19	-	33666	3					
20	-	30108	3					
21		19297						
22	-	848						
23		30						

S = 20

I	D	SUB	I
0	48782	75391	14022 43657 35804 59681 21257
1	23872	42526	57610 19839 03981 36959 0845
2	- 56812	68663	10104 57704 11409 25802 66
3	10444	71424	54122 74076 28708 67596
4	58245	85052	85903 84298 13271 6452
5	- 30608	48939	28780 53258 55703 344
6	92195	33884	01832 20934 01405 7
7	- 13905	10431	54936 69837 22196
8	- 29332	75121	88738 38986 910
9	31193	06370	31373 43716 77
10	- 13624	74043	26546 95653 5
11	39886	04491	87406 4572
12	- 69148	46119	85242 89
13	- 56908	81708	42077
14	10882	74450	45609
15	- 55536	84208	229
16	19751	24939	25
17	- 52466	28991	
18	86201	081	
19	76774	6	
20	- 14086	6	
21	7508		
22	- 292		
23	9		

S = 21

I		C SUB I						
0		48839	19475	71404	47552	31501	35649	35086
1		22782	14172	22652	05847	69514	71132	8907
2	-	51754	58035	68013	73826	85516	18204	26
3		87431	19053	49551	49404	81500	0563	
4		48531	80209	74598	88281	89657	7486	
5	-	24250	89838	43187	05563	64756	802	
6		69178	36150	79134	03179	82848	9	
7	-	96451	07774	40996	24325	7352		
8	-	21835	89794	08012	51622	817		
9		20905	00406	75194	49575	21		
10	-	85432	37952	62951	23239			
11		23247	97883	62431	9315			
12	-	35616	52406	32347	58			
13	-	46380	11420	84688				
14		60840	98249	6494				
15	-	28145	44726	811				
16		91642	50365	6				
17	-	21809	14502					
18		27889	148					
19		65104	0					
20	-	64583						
21		2979						
22	-	103						
23		2						

S = 22

I		D SUB I						
0		48890	63824	01013	70811	10240	30677	98415
1		21787	27933	39477	96928	20912	35012	8404
2	-	47343	54147	67594	12362	94037	18070	38
3		73748	99301	01811	58449	88931	5194	
4		40763	62537	30972	74680	12025	4429	
5	-	19414	50081	12261	45884	51718	591	
6		52586	23435	69467	69131	76022	0	
7	-	67989	75184	46012	76072	0995		
8	-	16340	40555	90873	38306	904		
9		14241	60045	14929	13342	18		
10	-	54655	17108	25570	29398			
11		13867	85083	46536	7377			
12	-	18730	28959	01328	46			
13	-	33839	38040	06395				
14		34571	24673	9561				
15	-	14628	84120	464				
16		43786	06802	9				
17	-	93259	5606					
18		85622	75					
19		40796	9					
20	-	29619						
21		1210						
22	-	37						

S = 23

I

D SUB I

0		48937	71949	97679	96182	63063	66974	33126
1		20875	81689	61957	83087	32694	13415	6155
2	-	43473	65774	86325	49479	29537	55138	03
3		62645	47441	76273	34832	80980	0902	
4		34491	21107	40401	56787	73745	1750	
5	-	15690	98179	60855	91376	75223	029	
6		40450	85002	68727	94714	38292	5	
7	-	48639	30600	36247	88188	5443		
8	-	12304	86643	14317	23410	503		
9		98503	56809	39946	97888	6		
10	-	35616	36011	51517	09886			
11		84501	79144	99761	707			
12	-	10027	17907	41881	77			
13	-	23563	92066	38956				
14		19975	55793	7851				
15	-	77867	77303	71				
16		21497	51668	0				
17	-	40889	7148					
18		22582	35					
19		23262	2					
20	-	13719						
21		503						
22	-	14						

$$S = 24$$

I	D SUB I							
0		48980	97047	00699	63996	57190	50851	42793
1		20037	67378	83556	88537	45840	21972	4033
2	-	40059	84522	83795	44533	11403	70800	55
3		53558	02078	30687	37021	32988	5773	
4		29381	50003	88282	57233	07910	5337	
5	-	12792	64774	01219	88768	15905	845	
6		31456	52369	60087	85243	16699	9	
7	-	35270	87980	18535	04876	9956		
8	-	93291	32755	99694	90863	92		
9		69095	64905	31960	06533	4		
10	-	23607	67426	43340	87349			
11		52508	09341	09859	196			
12	-	54487	00437	84766	8			
13	-	16070	22389	72097				
14		11735	80391	8724				
15	-	42387	36058	61				
16		10824	50647	8				
17	-	18322	8948					
18		31417	0					
19		12793	0					
20	-	6448						
21		214						
22	-	5						

S = 25

I	D SUB I							
0	49020	84001	67093	74833	06617	04568	88525	
1	19264	33312	40610	46505	02201	01239	0429	
2	- 37033	11683	63234	86260	75750	41468	11	
3	46062	33491	70968	81197	71359	7532		
4	25184	91909	10949	44475	13206	0606		
5	- 10513	76877	46532	97638	74178	444		
6	24708	46786	03171	35903	43905	4		
7	- 25898	19484	58388	39738	7375			
8	- 71227	93459	10234	52065	30			
9	49104	69413	55010	76682	0			
10	- 15895	98111	05806	40144				
11	33223	38846	84439	164				
12	- 29963	68087	68841	5				
13	- 10869	23986	38096					
14	70079	41995	356					
15	- 23564	52757	43					
16	55798	52263						
17	- 83638	396						
18	- 19762	6						
19	69489							
20	- 3082							
21	93							
22	- 2							

S = 26

I		D SUB I						
0		49057	70953	76837	24045	56530	93919	28473
1		18548	54952	50145	41388	47605	04088	5929
2	-	34337	05965	31443	95197	84435	78681	42
3		39834	85979	78554	44682	69763	3987	
4		21712	27632	17165	92820	92798	1719	
5	-	87052	13196	74549	15774	76426	00	
6		19588	46789	61603	61287	65367	8	
7	-	19237	05632	76854	29116	5972		
8	-	54766	68167	76209	75558	20		
9		35324	71907	84476	13089	9		
10	-	10860	66732	66206	66865			
11		21376	84454	64190	706			
12	-	16622	52046	29926	4			
13	-	73404	07876	7505				
14		42510	37764	080				
15	-	13362	03745	92				
16		29398	61577					
17	-	38756	065					
18	-	26612	9					
19		37712						
20	-	1500						
21		41						

S = 27

I		D SUB I						
0		49091	90517	61210	76504	23300	48626	79272
1		17884	12012	88557	99914	06348	22229	7667
2	-	31925	18032	89313	75868	76679	09977	26
3		34626	31806	39436	44185	33716	3674	
4		18818	61344	80719	54428	22464	3849	
5	-	72575	13795	52498	34788	12332	19	
6		15663	14318	65844	69747	49159	6	
7	-	14443	08585	35872	07248	3532		
8	-	42403	44093	01332	39164	02		
9		25701	87163	03607	40121	4		
10	-	75217	12243	62147	1020			
11		13970	46459	45063	765			
12	-	92686	84792	18714				
13	-	49690	01150	3836				
14		26179	05764	895				
15	-	77190	38177	6				
16		15807	99370					
17	-	18160	067					
18	-	21824	1					
19		20574						
20	-	743						
21		19						

S = 28

I	D SUE I							
0	49123	70747	35576	15773	27964	69603	77968	
1	17265	70340	98582	41106	00760	39556	0662	
2	-	29758	88233	62998	29967	41147	32618	90
3		30242	82754	29356	71709	99123	1873	
4		16391	76178	49413	67960	02975	5454	
5	-	60893	82172	40912	85634	89520	21	
6		12624	57795	93163	56399	94178	6	
7	-	10952	37540	78649	97192	5940		
8	-	33054	83324	29865	84751	94		
9		18899	96215	34293	22217	1		
10	-	52755	60012	99419	0030			
11		92636	87113	34749	38			
12	-	51717	77090	12968				
13	-	33794	14631	6818				
14		16356	14610	386				
15	-	45378	71261	8				
16		86633	0584					
17	-	85655	23					
18	-	15543	2					
19		11321						
20	-	375						
21		8						

S = 29

I		D SUB I						
0		49153	35906	82622	69545	38436	74775	95667
1		16688	67454	77942	75853	96402	24763	3520
2	-	27805	90878	49437	37387	61884	18190	24
3		26532	25537	79300	79768	65967	4851	
4		14344	12646	09793	65040	74094	5539	
5	-	51398	03174	59596	41437	91700	36	
6		10251	25842	59720	78428	63098	0	
7	-	83828	44902	66099	30445	684		
8	-	25937	62686	08427	85941	28		
9		14037	06378	21906	41882	7		
10	-	37441	35810	89187	3202			
11		62265	37196	94196	25			
12	-	28711	22376	12574				
13	-	23122	29180	0266				
14		10360	65577	661				
15	-	27120	45897	1				
16		48329	0652					
17	-	40431	98					
18	-	10425	5					
19		6295						
20	-	192						
21		4						

S = 30

I		D SUB I							
0		49181	07088	53420	89338	89345	05311	50074	
1		16149	00901	67674	00628	15653	55212	7172	
2	-	26039	13219	36091	73527	38488	77663	33	
3		23374	24334	20925	95400	25403	3235		
4		12606	71731	36543	87372	95497	6029		
5	-	43625	19063	38593	96397	43195	62		
6		83819	73345	53115	57616	37667			
7	-	64721	10298	29812	05337	674			
8	-	20482	77782	40761	15293	60			
9		10523	14819	11431	19543	7			
10	-	26868	24952	54884	0165				
11		42386	03016	48580	41				
12	-	15728	25658	40129					
13	-	15928	87125	5066					
14		66495	08848	14					
15	-	16462	22612	3					
16		27412	9696						
17	-	18946	92						
18	-	67951							
19		3541							
20	-	100							
21		2							

S = 31

I

C SUB I

0		49207	02715	23966	78460	78455	01975	89199
1		15643	18816	77487	97606	29483	24020	4179
2	-	24435	60556	85426	69257	82674	29616	65
3		20672	83811	78880	15069	56636	9418	
4		11124	76364	66162	91261	36317	9987	
5	-	37221	32139	76215	16570	64104	32	
6		68981	25649	30212	55215	07713		
7	-	50377	13169	18409	64723	089		
8	-	16274	50737	53380	36228	70		
9		79584	25438	80797	29133			
10	-	19482	08444	35835	9588			
11		29199	20644	42526	85			
12	-	83927	35758	0950				
13	-	11053	40752	7400				
14		43212	99715	97				
15	-	10140	24720	4				
16		15793	2418					
17	-	87066	6					
18	-	43691						
19		2016						
20	-	53						
21		1						

S = 32

I

D SUB I

0		49231	38949	26733	08677	70146	52381	51439
1		15168	12210	15711	43861	50350	24523	4412
2	-	22975	81221	92845	89256	74056	79384	32
3		18350	99152	35422	91466	20253	8234	
4		98544	59057	39146	06992	13992	374	
5	-	31913	18358	95307	27900	14743	87	
6		57115	84308	91941	69745	64505		
7	-	39512	78839	54157	51922	282		
8	-	13007	17047	97896	89086	28		
9		60687	49392	70853	18136			
10	-	14264	93063	65002	5144			
11		20341	40510	31500	28			
12	-	42624	46445	7908				
13	-	77278	05819	211				
14		28418	43447	35				
15	-	63332	60403					
16		92328	072					
17	-	38404	5					
18	-	27941						
19		1162						
20	-	28						

S = 33

I		D	SUB	I
0		49254	30028	77475 04929 96365 49918 96206
1		14721	08624	52238 53438 31337 48402 5023
2	-	21643	06811	21813 69986 06446 84937 09
3		16346	41766	27841 67740 24618 8693
4		87605	21634	18285 00213 15873 356
5	-	27488	13232	56029 99284 14037 90
6		47562	37122	36742 55598 89017
7	-	31214	87924	74965 68302 468
8	-	10454	66792	12518 18623 40
9		46640	02146	30456 30257
10	-	10541	33005	41959 6191
11		14320	85828	61581 35
12	-	19599	18401	7255
13	-	54436	57372	066
14		18901	59360	11
15	-	40077	58688	
16		54721	078	
17	-	15554	3	
18	-	17858		
19		677		
20	-	15		

$$S = 34$$

I	D SUB I							
0		49275	88545	73978	09362	47214	88152	28354
1		14299	66887	08450	87467	94994	21642	6541
2	-	20423	04230	62182	53616	18741	45746	25
3		14608	44474	31139	92154	96595	0798	
4		78143	48647	01708	39338	52771	737	
5	-	23779	40921	19331	87633	13228	07	
6		39820	48412	80414	88812	52417		
7	-	24827	07214	17577	71529	194		
8	-	84486	79419	26381	84208	5		
9		36109	30329	92887	17764			
10	-	78575	42722	40787	801			
11		10183	00668	02152	47			
12	-	70137	48023	245				
13	-	38635	08997	983				
14		12707	91411	40				
15	-	25678	62613					
16		32852	841					
17	-	51023						
18	-	11441						
19		400						
20	-	8						

S = 35

I	D SUB I
0	49296 25677 11893 21354 25690 27815 65169
1	13901 72741 49796 87810 72585 28002 1448
2	- 19303 36953 31483 16135 50428 84930 96
3	13095 60297 52836 96962 09126 8158
4	69926 08962 55598 87220 62311 728
5	- 20655 29610 84087 85981 45931 22
6	33508 33119 63164 74342 94522
7	- 19873 04981 87769 23284 459
8	- 68631 03762 69089 33275 5
9	28151 94523 35621 64750
10	- 59052 22851 43684 783
11	73091 01749 24590 5
12	- 37804 32187 91
13	- 27623 75515 096
14	86318 95187 7
15	- 16647 94006
16	19964 224
17	- 5657
18	- 7360
19	239
20	- 4

S = 36

I		D SUB I							
0		49315	51378	12711	32670	85595	20989	86310	
1		13525	35192	22763	95924	48243	61415	0385	
2	-	18273	33522	38304	79309	66882	07424	31	
3		11773	76207	61823	56361	11936	1755		
4		62761	60734	66180	57146	39652	437		
5	-	18011	04444	22300	09676	23370	89		
6		28332	20632	61775	42215	27327			
7	-	16003	96210	45470	65545	336			
8	-	56028	78032	27840	59036	3			
9		22093	67797	03197	37169				
10	-	44725	29396	48613	534				
11		52931	50338	11400	9				
12		28920	47523	875					
13	-	19894	23747	601					
14		59208	70272	4					
15	-	10914	66042						
16		12270	972						
17		11955							
18	-	4761							
19		144							
20	-	2							

S = 37

I		D SUB I						
0		49333	74544	68640	54336	30974	48250	68827
1		13168	83429	30490	69100	09148	67922	4751
2	-	17323	61789	77431	27053	81899	57236	03
3		10614	68260	31093	36617	78396	8075	
4		56492	12913	38387	59208	47932	469	
5	-	15762	81830	83747	24085	87460	68	
6		24064	52314	83217	36705	64824		
7	-	12962	09981	18329	18715	063		
8	-	45958	98353	27451	07275	0		
9		17448	22644	24004	56903			
10	-	34124	12386	94639	562			
11		38656	72615	33109	8			
12		42866	41606	639				
13	-	14429	05018	706				
14		40993	62679	4				
15	-	72323	8816					
16		76235	58					
17		16980						
18	-	3098						
19		88						
20	-	1						

S = 38

I		O	SUB	I				
0		49351	03150	63191	75028	28147	22519	81083
1		12830	64228	68022	62674	79077	96634	9194
2	-	16446	07727	65273	98943	75192	26773	40
3		95948	81368	69608	81480	69474	535	
4		50986	71012	98683	24456	77286	650	
5	-	13843	11211	05138	30219	80775	30	
6		20527	69221	87816	00219	95114		
7	-	10555	52283	27739	25748	496		
8	-	37871	47703	28689	34882	5		
9		13861	88781	95019	13347			
10	-	26217	94147	57893	511			
11		28458	38817	53211	9			
12		46639	07540	912				
13	-	10537	28635	244				
14		28636	12584	7				
15	-	48411	9357					
16		47842	20					
17		16665						
18	-	2030						
19		54						

S = 39

I

D SUB I

0		49367	44364	13464	39962	11770	88932	41816
1		12509	39744	47207	53202	70784	93643	7727
2	-	15633	57907	40373	73375	89691	31662	30
3		86947	37022	81511	85426	22995	743	
4		46136	23206	92464	16100	55883	243	
5	-	12197	25727	21959	11171	98587	06	
6		17582	21693	74283	77719	93608		
7	-	86401	65925	63150	38477	11		
8	-	31344	02483	33889	04398	5		
9		11075	29364	60879	81044			
10	-	20277	47470	33589	008			
11		21110	45244	05935	5			
12		45153	38296	380				
13	-	77465	99457	88				
14		20174	65581	0				
15	-	32720	1348					
16		30309	35					
17		14396						
18	-	1339						
19		34						

S = 40

I		D SUB I						
0		49383	04646	92299	41629	21854	62791	97367
1		12203	85625	84210	40689	65145	01811	3279
2	-	14879	84938	11764	83143	22971	25256	55
3		78977	80589	37438	52891	95709	686	
4		41849	34074	78978	29418	02900	806	
5	-	10780	73992	18547	26236	70599	84	
6		15117	83266	12386	07083	76003		
7	-	71071	01564	82342	75665	26		
8	-	26050	74675	31057	57474	7		
9		88967	77786	63156	0599			
10	-	15782	20424	17235	449			
11		15773	45909	84875	6			
12		41191	99652	846				
13	-	57318	34742	31				
14		14329	43991	7				
15	-	22319	0792					
16		19373	33					
17		11691						
18	-	889						
19		21						

S = 41

I	O SUB I
0	49397 89839 19627 91929 83778 95087 59562
1	11912 89404 11322 34620 25858 86782 4946
2	- 14179 35306 61254 77729 71864 25507 06
3	71901 29399 55465 24564 12905 261
4	38049 21507 39706 05100 38579 628
5	- 95571 27163 19963 93686 82016 6
6	13046 86009 09359 55066 44829
7	- 58733 94076 66828 65272 95
8	- 21738 75475 12460 29192 0
9	71836 60312 28852 3004
10	- 12357 48651 10575 015
11	11867 24717 17668 3
12	36298 28696 090
13	- 42676 15723 58
14	10257 23991 0
15	- 15358 7343
16	12486 93
17	9172
18	- 595
19	13

S = 42

I		D SUB I						
0		49412	05232	58039	58297	88036	16906	96863
1		11635	49105	92075	14942	34720	24711	0135
2	-	13527	19176	68699	37606	23378	92930	61
3		65600	32912	92411	28602	34178	612	
4		34670	98271	81833	97489	63839	809	
5	-	84964	53418	45784	93799	46381	6	
6		11299	18305	30527	34690	68356		
7	-	48754	77800	94297	88972	57		
8	-	18210	76608	43644	87816	1		
9		58289	81477	74571	8152			
10	-	97315	15327	36799	43			
11		89872	83500	92754				
12		31298	26804	924				
13	-	31966	55931	97				
14		73971	79021					
15	-	10658	1950					
16		81115	1					
17		7050						
18	-	401						
19		8						

S = 43

I		D SUB I						
0		49425	55633	04492	13999	42738	65654	14962
1		11370	72056	26715	72363	01413	89684	5987
2	-	12919	01794	89230	45333	39993	86989	37
3		59975	06372	59164	95249	16230	046	
4		31659	64167	22541	40291	95354	210	
5	-	75739	57008	52590	21490	63684	2	
6		98184	27251	53033	29287	1793		
7	-	40643	24761	77096	74261	98		
8	-	15312	09015	31375	36118	2		
9		47520	21755	60337	4809			
10	-	77055	90812	62702	63			
11		68490	98810	05070				
12		26606	38665	730				
13	-	24084	46797	64				
14		53727	99590					
15	-	74559	476					
16		53078	6					
17		5351						
18	-	272						
19		5						

$$S = 44$$

I	O SUB I							
0		49438	45415	35673	77365	82545	78154	77461
1		11117	73841	80632	01833	00742	03856	7361
2	-	12350	96219	98672	00833	16264	87795	47
3		54940	33811	06669	67664	27088	178	
4		28968	37998	93769	08896	48012	450	
5	-	67690	85131	00017	34758	33920	0	
6		85590	33530	78976	38333	6713		
7	-	34018	91676	37384	81842	75		
8	-	12920	83424	48024	23847	3		
9		38914	74487	25953	9219			
10	-	61334	00322	61662	12			
11		52510	31425	19071				
12		22404	69057	405				
13	-	18248	31327	95				
14		39292	21567					
15	-	52561	135					
16		34969	9					
17		4031						
18	-	185						
19		3						

$$S = 45$$

I	D SUB I							
0		49450	78570	36907	99318	19648	88641	14051
1		10875	77409	86887	10961	18768	46619	8629
2	-	11819	57148	05556	56137	53022	81440	01
3		50423	26115	23522	97912	99191	764	
4		26557	21069	34976	92116	68612	876	
5	-	60647	05823	96052	20665	10469	2	
6		74840	02103	20673	84099	3683		
7	-	28584	88846	72519	89385	88		
8	-	10940	49166	49417	92154	7		
9		32004	92227	28914	9063			
10	-	49064	58529	45803	97			
11		40490	15624	43161				
12		18747	50141	321				
13	-	13901	74994	00				
14		28924	34796					
15	-	37327	608					
16		23185	2					
17		3024						
18	-	127						
19		2						

S = 46

I	D	SUE	I
0	49462	58746	22200 39962 45665 46520 16470
1	10644	12282	92776 08814 29672 70105 8162
2	- 11321	75648	54387 20358 83491 94802 72
3	46361	23034	17955 10924 06931 840
4	24391	85749	72244 77262 17859 503
5	- 54464	80519	59398 89140 81570 4
6	65631	43451	62888 53886 2101
7	- 24108	18389	50500 50543 68
8	- 92943	02919	62116 09430
9	26430	66926	58102 7935
10	- 39437	93746	06156 45
11	31393	70170	76867
12	15622	14851	032
13	- 10646	25080	15
14	21426	85678	
15	- 26697	363	
16	15461	8	
17	2263		
18	- 88		
19	1		

S = 47

I

D SUB I

0		49473	89284	34949	04933	10736	35143	77827
1		10422	13871	60175	90553	42087	89872	7192
2	-	10854	74660	77692	49026	52430	73834	71
3		42700	30434	25907	55331	65971	874	
4		22442	84113	94070	04485	57060	720	
5	-	49023	61909	84060	53998	06996	2	
6		57717	14304	55176	56135	6448		
7	-	20405	01958	68361	34151	21		
8	-	79209	43622	07010	84240			
9		21913	66742	41792	2489			
10	-	31845	86168	71180	02			
11		24469	49331	52061				
12		12983	77278	830				
13	-	81945	80754	7				
14		15969	23825					
15	-	19224	556					
16		10366	5					
17		1692						
18	-	61						
19		1						

S = 48

I		C SUB I							
0		49484	73251	04108	63314	25400	28085	51368	
1		10209	22872	06596	73529	20881	95116	9855	
2	-	10416	05127	92825	01697	85170	43294	71	
3		39393	85962	49934	78434	64889	454		
4		20884	72698	67150	31812	15363	324		
5	-	44221	90134	62640	01802	17782	3		
6		50893	48063	38352	88362	3188			
7	-	17329	68912	54809	47943	93			
8	-	67712	10412	69644	02498				
9		18237	63850	52222	4707				
10	-	25828	74651	65789	48				
11		19169	19824	82605					
12		10774	85872	578					
13	-	63384	77411	5					
14		11971	25994						
15	-	13934	074						
16		69841							
17		1266							
18	-	43							

S = 49

I

O SUB I

0		49495	13465	18539	13594	73493	15978	46159
1		10004	84736	00870	26836	51230	33430	7069
2	-	10003	42667	34866	84223	57719	75909	39
3		36401	47711	62252	12149	76122	733	
4		19095	50287	44448	63769	49147	362	
5	-	39973	66522	60827	57807	11543	0	
6		44992	11709	36838	06078	2956		
7	-	14766	11653	65067	04752	23		
8	-	58054	61501	49899	87550			
9		15233	64813	32785	9573			
10	-	21037	24080	74980	17			
11		15090	10038	37998				
12		89358	18959	70				
13	-	49260	59205	7				
14		90246	3137					
15	-	10163	060					
16		47258						
17		948						
18	-	30						

S = 50

I		D SUB I							
0		49505	12522	72354	23803	22919	86150	73791	
1		98084	92031	41983	72152	86925	78264	128	
2	-	96148	46938	55729	08880	99823	14262	2	
3		33688	01591	12780	50789	59122	169		
4		17656	06260	47908	91552	12240	346		
5	-	36205	88810	41646	23634	89820	2		
6		39873	37887	74582	70125	8507			
7	-	12621	40259	45488	24719	32			
8	-	49916	30668	67038	16711				
9		12769	08855	01918	1305				
10	-	17204	31643	64487	75				
11		11934	58453	88439					
12		74103	87893	66					
13	-	38459	35741	7					
14		68401	0970						
15	-	74574	84						
16		32099							
17		712							
18	-	21							

S = 51

I		D SUB I							
0		49514	72818	25898	13068	09560	07561	79637	
1		96196	98878	16085	34708	28531	63049	866	
2	-	92484	79268	95654	07267	74331	80273	3	
3		31222	83975	32289	54671	86293	290		
4		16349	77553	09057	77814	91414	070		
5	-	32856	35333	50116	67609	62969	0		
6		35420	93217	33794	87774	7291			
7	-	10820	86670	50478	51175	92			
8	-	43036	79178	00286	31994				
9		10739	37171	22575	7170				
10	-	14124	77242	87447	09				
11		94813	89334	7438					
12		61480	52572	55					
13	-	30159	64620	2					
14		52114	0594						
15	-	55040	73						
16		21874							
17		535							
18	-	15							

S = 52

I		D SUB I						
0		49523	96564	20193	11734	91231	66810	53519
1		94380	39125	68731	47936	85836	49898	094
2	-	89026	62241	24282	58584	17449	79763	0
3		28979	16878	40839	07796	59745	788	
4		15162	12657	12598	68874	75059	191	
5	-	29871	88406	83200	44194	15577	1	
6		31537	53081	88141	92824	5711		
7	-	93042	17669	91199	95596	2		
8	-	37203	83699	58339	03970			
9		90616	31195	96007	301			
10	-	11640	10521	68073	95			
11		75650	71156	9127				
12		51048	37328	66				
13	-	23752	49128	2				
14		39904	6823					
15	-	40851	48					
16		14945						
17		403						
18	-	11						

S = 53

I	D	SUB	I
0	49532	85807	77053 21910 52304 52043 43741
1	92631	15825	05414 83106 42514 43262 668
2	- 85758	86934	50708 93380 03377 47720 9
3	26933	53441	71797 30260 66468 675
4	14080	41407	52659 79771 60691 719
5	- 27206	89207	06667 36794 58202 8
6	28141	60066	12995 28396 5362
7	- 80225	81312	92315 28671 7
8	- 32243	83213	20477 46605
9	76699	21642	23718 567
10	- 96272	73576	23056 6
11	60612	37727	9703
12	42432	14043	05
13	- 18784	09874	2
14	30703	8610	
15	- 30484	28	
16	10232		
17	305		
18	- 8		

S = 54

I		D SUB I						
0		49541	42446	12342	44439	43329	24995	41189
1		90945	60952	98166	20246	66522	20727	762
2	-	82667	80433	71440	60682	62626	22188	9
3		25065	31941	61525	16160	36042	171	
4		13093	49538	95350	52966	47601	969	
5	-	24822	18084	36301	65921	69563	3	
6		25164	48410	29568	27290	1099		
7	-	69361	81980	64248	84049	5		
8	-	28014	26449	63916	05055			
9		65115	40859	09649	325			
10	-	79903	07889	78660	6			
11		48758	63405	8935				
12		35315	84710	69				
13	-	14914	61237	5				
14		23735	0020					
15	-	22866	83					
16		7014						
17		231						
18	-	5						

S = 55

I

D SUB I

0		49549	68239	85903	10595	08390	42362	44933
1		89320	32823	37236	05500	55595	70933	244
2	-	79740	91378	27882	16202	29037	98115	0
3		23356	36863	13923	56505	78175	960	
4		12191	57189	98347	51204	11260	432	
5	-	22683	95486	69705	33013	69457	1	
6		22548	20809	22080	25758	9832		
7	-	60125	25595	63648	22195	1		
8	-	24397	75265	94709	65249			
9		55441	98093	49360	295			
10	-	66540	09108	62064	9			
11		39375	11049	0744				
12		29435	89465	85				
13	-	11888	26854	0				
14		18430	7638					
15	-	17239	29					
16		4809						
17		176						
18	-	4						

S = 56

I		D SUB I							
0		49557	64825	08354	76174	83595	82536	77354	
1		87752	13771	98904	16915	42453	96045	908	
2	-	76966	77271	09905	96958	76986	89930	4	
3		21790	65853	23560	72080	61529	802		
4		11366	00685	76759	06320	38824	032		
5	-	20762	99658	12587	41361	02253	3		
6		20243	66897	41747	81048	1294			
7	-	52249	69334	12433	24683	1			
8	-	21297	29795	35501	18504				
9		47338	23397	06618	124				
10	-	55591	91851	73980	4				
11		31916	36697	1806					
12		24573	78732	93					
13	-	95116	80423						
14		14374	3834						
15	-	13059	96						
16		3296							
17		134							
18	-	3							

S = 57

I		O SUB I
0		49565 33724 22158 94662 21015 33591 55724
1		86238 08081 23487 07676 03165 01001 659
2	-	74334 93307 13092 33709 94206 57200 0
3		20354 01582 88397 53117 44351 115
4		10609 17052 74058 46634 34335 423
5	-	19033 98039 24381 17206 42994 1
6		18209 15072 24878 58770 1907
7	-	45515 90760 89303 18043 3
8	-	18632 49081 82569 68224
9		40528 65149 36938 588
10	-	46590 74858 45530 8
11		25963 95856 5351
12		20549 21132 37
13	-	76379 55786
14		11258 0769
15	-	99403 7
16		2255
17		102
18	-	2

S = 58

I		O	SUB	I
0		49572	76355	71970 23810 77461 77165 80088
1		84775	40116	47220 79913 12246 78228 758
2	-	71835	82517	04417 95977 79283 21511 6
3		19033	87720	88854 42893 20425 770
4		99143	08175	18494 14670 27997 24
5	-	17474	89901	66574 05494 75310 0
6		16409	11088	04607 05095 7674
7	-	39742	89976	04237 44327 7
8	-	16336	46883	28869 28735
9		34789	61085	55989 494
10	-	39165	28537	16686 3
11		21195	38715	6310
12		17213	80167	04
13	-	61550	22839	
14		88533	689	
15	-	76003	6	
16		1537		
17		79		
18	-	1		

S = 59

I	D SUB I
0	49579 94042 77278 10361 54153 43247 20610
1	83361 52649 13646 68925 68840 17692 007
2	- 69460 67051 94990 40260 49664 97708 4
3	17819 08362 27656 83616 16696 661
4	92754 27215 91743 26265 68739 89
5	- 16066 58225 54556 58069 45608 7
6	14813 18232 24276 78880 3461
7	- 34780 73877 53673 72142 4
8	- 14353 46878 53964 26019
9	29938 96439 46484 962
10	- 33019 66093 79518 9
11	17360 99496 5044
12	14445 71061 55
13	- 49770 07349
14	69898 100
15	- 58367 7
16	1043
17	60
18	- 1

S = 60

I		D SUB I						
0		49586	88021	18627	25795	93574	40548	26755
1		81994	05345	15549	94396	92574	53815	790
2	-	67201	40460	66892	16309	30285	64659	9
3		16699	70367	95554	56252	54222	922	
4		86872	00477	49008	37212	96480	29	
5	-	14792	29207	95943	96639	84621	7	
6		13395	34966	43296	04527	4897		
7	-	30504	83371	27723	26383	9		
8	-	12636	84995	13847	84592			
9		25827	83790	44081	084			
10	-	27917	20326	04189	6			
11		14266	68117	3494				
12		12144	97498	07				
13	-	40378	33244					
14		55396	254					
15	-	45015	0					
16		702						
17		47						

S = 61

I	D SUB I
0	49593 59446 47238 33101 33833 53720 53011
1	80670 73399 96541 13778 07152 62429 586
2	- 65050 60832 41573 47130 01026 28741 2
3	15666 88164 84858 01177 63833 131
4	81448 83064 31085 40403 72528 55
5	- 13637 38091 94199 41905 84068 1
6	12133 26752 47579 97820 5663
7	- 26811 33249 32113 91787 3
8	- 11147 49251 31795 25803
9	22334 14851 55627 127
10	- 23667 88683 83776 5
11	11760 91095 6421
12	10229 62212 68
13	- 32864 51443
14	44065 822
15	- 34860 2
16	467
17	36

S = 62

I	D SUB I
0	49600 09400 26595 45093 92446 28423 97040
1	79389 46303 79575 74066 89844 80419 894
2	- 63001 44695 80274 11086 05955 47331 8
3	14712 70631 35062 79401 54046 602
4	76442 40728 27246 06678 05595 42
5	- 12589 00247 59639 99965 82356 2
6	11007 69440 52334 09789 6081
7	- 23613 41226 17780 81399 4
8	- 98524 95691 50528 8987
9	19357 46168 58798 055
10	- 20118 58876 61472 8
11	97249 04716 830
12	86324 29796 4
13	- 26832 66646
14	35178 749
15	- 27104 1
16	306
17	28

S = 63

I		D SUB I						
0		49606	38896	13489	76077	25049	28941	86984
1		78148	26722	94318	70541	78134	44536	071
2	-	61047	61580	30240	50743	27175	62447	8
3		13830	09754	93698	07043	87561	313	
4		71814	88003	90738	22582	40535	81	
5	-	11635	86630	78778	97172	62031	0	
6		10002	02113	17872	40814	3028		
7	-	20838	27801	92242	23566	4		
8	-	87241	16150	30996	6826			
9		16814	89536	69853	017			
10	-	17145	49165	56050	4			
11		80651	95123	940				
12		72982	49232	5				
13	-	21974	49619					
14		28181	802					
15	-	21155	3					
16		196						
17		22						

$$S = 64$$

I	D SUE I							
0	49612	48884	85080	41646	72305	04740	31839	
1	76945	29484	51464	46575	21807	08151	909	
2	- 59183	29159	21055	83937	20432	54481	3	
3	13012	70799	73353	05472	83773	093		
4	67532	34650	69151	45176	96867	96		
5	- 10768	02901	39838	75858	80318	6		
6	91018	76891	24416	75609	054			
7	- 18424	72600	48495	63372	4			
8	- 77388	99666	15360	7939				
9	14637	84629	43984	891				
10	- 14648	13657	25749	9				
11	67079	55495	778					
12	61818	04753	4					
13	- 18049	06934						
14	22652	798						
15	- 16574	2						
16	121							
17	17							

S = 65

I		D SUB I							
0		49618	40259	17734	38239	92045	13128	68322	
1		75778	80653	64234	02175	66211	20041	822	
2	-	57403	08904	11746	20024	06306	79450	2	
3		12254	83763	91422	92720	27903	113		
4		63564	39181	13549	99064	10653	90		
5	-	99767	16097	08703	96394	59533			
6		82947	99143	77082	89309	329			
7	-	16321	15902	71294	76965	9			
8	-	68769	68754	76497	5227				
9		12769	36418	44796	304				
10	-	12544	74948	49264	0				
11		55946	59439	001					
12		52458	91301	1					
13	-	14867	41019						
14		18268	231						
15	-	13032	5						
16		70							
17		13							

S = 66

I	C	SUE	I
0	49624	13858	22714 24361 98150 36162 55147
1	74647	16693	48896 21806 55466 93197 679
2	- 55702	02190	23137 75720 23103 01028 3
3	11551	35941	59240 98615 74185 564
4	59883	68447	99394 91343 35981 97
5	- 92541	69622	29325 09746 86751
6	75699	56281	68163 52057 037
7	- 14483	96473	97012 60734 1
8	- 61214	36545	89251 4492
9	11162	03843	35896 354
10	- 10768	55464	98251 7
11	46787	45473	664
12	44599	01465	1
13	- 12280	77390	
14	14779	217	
15	- 10283	8	
16	37		
17	10		

S = 67

I		O	SUB	I
0		49629	70471	43183 29836 25948 97318 18918
1		73548	83699	50320 78674 00036 25247 377
2	-	54075	46799	87338 13681 73556 98099 5
3		10897	65432	75599 72886 01634 476
4		56465	62425	23728 97519 43511 65
5	-	85935	17620	02524 06505 85255
6		69179	13974	65174 57101 796
7	-	12876	18637	09184 05778 9
8	-	54579	23170	25780 6584
9		97762	91670	08992 74
10	-	92648	55884	23197
11		39230	43624	763
12		37986	23558	1
13	-	10171	68032	
14		11993	531	
15	-	81428		
16		15		
17		8		

S = 68

I		D SUB I						
0		49635	10842	16475	94918	65438	31833	18002
1		72482	36700	37804	17382	57903	27935	119
2	-	52519	13778	28806	16108	69612	64546	2
3		10289	55468	83723	13760	39379	134	
4		53288	03451	48096	62839	87307	32	
5	-	79886	61867	77820	53324	74087		
6		63304	37802	96144	93127	282		
7	-	11466	42990	81551	26840	8		
8	-	48741	55633	33722	5678			
9		85789	97415	01786	91			
10	-	79887	17578	37994				
11		32977	82155	853				
12		32412	61682	5				
13	-	84470	3292					
14		97621	51					
15	-	64690						
16		0						
17		6						

S = 69

I	D SUB I
0	49640 35671 05127 56868 19801 53226 69037
1	71446 39019 02853 59145 01890 36155 664
2	- 51029 04601 77183 27059 50537 24308 7
3	97232 94415 73007 69407 01689 19
4	50330 89316 27553 62370 98236 04
5	- 74341 81245 14752 67315 95717
6	58003 26118 40254 16821 919
7	- 10227 96302 34221 42705 7
8	- 43596 35878 89017 8100
9	75423 56809 07330 13
10	- 69031 15874 41670
11	27790 37809 272
12	27706 37038 9
13	- 70328 1827
14	79691 21
15	- 51560
16	- 7
17	5

S = 70

I		D SUB I							
0		49645	45618	99762	02944	92641	67416	34942	
1		70439	61687	66075	98029	25281	36674	498	
2	-	49601	48623	15802	45403	62728	43923	7	
3		91954	65396	04803	80746	49586	65		
4		47576	09663	00779	01682	52778	88		
5	-	69252	48317	43018	16555	45476			
6		53212	68153	54428	88249	820			
7	-	91379	70021	74880	73075				
8	-	39053	64705	27594	1398				
9		66429	83096	49653	15				
10	-	59774	59761	24495					
11		23475	26439	757					
12		23725	36896	1					
13	-	58700	0837						
14		65238	96						
15	-	41225							
16	-	12							
17		4							

APPENDIX B

VALUES OF $\Pi_{s+1}(s)$

TABLE 4-II_{s+1}(s) TO 33 DECIMAL PLACES FOR s = 1(1)70

s	$\Pi_{s+1}(s)$							
1	0.	18927	51878	82093	32118	03671	35892	330
2	0.	28121	25252	33780	21211	01257	13387	264
3	0.	33107	18207	57297	57403	96445	04243	578
4	0.	36234	77109	30150	41806	24897	98150	113
5	0.	38381	35230	66598	99050	60316	80043	921
6	0.	39946	78498	64246	19181	32889	13640	946
7	0.	41139	37882	53067	68776	44516	94770	425
8	0.	42078	39223	27607	49424	80917	28837	913
9	0.	42837	06049	14723	28266	78893	32891	149
10	0.	43462	86186	56620	59289	61348	53475	084
11	0.	43987	93542	19413	10305	58450	13253	220
12	0.	44434	81780	57968	30769	29839	53993	445
13	0.	44819	78298	88886	96046	33589	61570	798
14	0.	45154	87794	63734	23189	51342	86097	413
15	0.	45449	21231	00126	13265	38379	62923	535
16	0.	45709	80274	88442	11242	59120	41783	262
17	0.	45942	14175	81479	01975	16278	75007	598
18	0.	46150	59042	79618	29366	20249	99240	751
19	0.	46338	65565	13054	01880	86860	44928	381
20	0.	46509	18960	03773	84625	06853	99018	443
21	0.	46664	53577	45605	86994	12593	13404	848
22	0.	46806	63760	81585	38061	44761	73415	886
23	0.	46937	12038	05492	33636	76180	48188	668
24	0.	47057	35378	63878	17896	97589	84742	077
25	0.	47168	50029	32419	66515	27559	25603	593
26	0.	47271	55291	67563	97837	78613	41673	717
27	0.	47367	36502	08385	52941	09116	90768	922
28	0.	47456	67404	08816	47666	49270	58888	022
29	0.	47540	12052	86421	56212	83451	19715	499
30	0.	47618	26356	11387	92503	91055	99968	335
31	0.	47691	59329	86460	13692	67473	83086	414
32	0.	47760	54128	88845	91697	19203	46973	181
33	0.	47825	48897	57110	72096	24500	39044	602
34	0.	47886	77476	70871	43722	72844	01478	426
35	0.	47944	69993	91774	07261	54744	48704	332
36	0.	47999	53359	42388	34689	01441	77717	049
37	0.	48051	51684	46461	18660	77452	11110	868
38	0.	48100	86636	04256	21229	19929	37206	665
39	0.	48147	77739	04861	66515	10377	15802	150
40	0.	48192	42634	64571	57171	69157	55298	186
41	0.	48234	97302	12808	35693	09826	06139	781
42	0.	48275	56250	14162	47536	06970	47051	474
43	0.	48314	32682	09154	01245	76918	66207	700
44	0.	48351	38639	71345	30964	37588	33777	672
45	0.	48386	85128	09929	90637	21073	71037	023
46	0.	48420	82224	91418	05908	29144	30722	574
47	0.	48453	39176	08848	49191	41823	57802	075
48	0.	48484	64479	89993	03458	33150	62033	238
49	0.	48514	65961	05653	84964	75672	58702	252
50	0.	48543	50836	14099	49566	43636	27540	305
51	0.	48571	25771	56932	67823	84126	40001	115
52	0.	48597	96935	04424	43113	64502	70622	856
53	0.	48623	70041	33944	51798	55257	61952	156
54	0.	48648	50393	13051	35804	28421	57737	357
55	0.	48672	42917	48662	39469	63201	53719	355
56	0.	48695	52198	55172	54534	96512	06715	972
57	0.	48717	82506	87152	26328	15260	09357	575
58	0.	48739	37825	76116	20313	32358	32871	349
59	0.	48760	21875	05627	24450	43893	17073	235
60	0.	48780	38132	54539	88146	12991	70518	604
61	0.	48799	89853	34369	13594	81214	77751	465
62	0.	48818	80087	43494	86428	50419	24013	489
63	0.	48837	11695	58092	56101	60483	27847	218
64	0.	48854	87363	77250	61249	67474	94690	823
65	0.	48872	09616	37632	07763	94802	14219	180
66	0.	48888	80828	11217	65400	90200	98587	768
67	0.	48905	03234	98084	54943	44213	32187	075
68	0.	48920	78944	24799	03848	79708	25702	805
69	0.	48936	09943	57799	44042	33997	18441	727
70	0.	48950	98109	40096	49161	53941	71702	657

APPENDIX C

VALUES OF $\Omega_s(s)$ AND OF $\Omega_{s+1}(s)$

TABLE 5— $\Omega_s(s)$ TO 33 DECIMAL PLACES FOR $s = 1(1)70$

s		$\Omega_s(s)$						
1	0.	37855	03757	64186	64236	07342	71784	661
2	0.	42181	87878	50670	31816	51885	70080	896
3	0.	44142	90943	43063	43205	28593	38991	437
4	0.	45293	46386	62688	02257	81122	47687	641
5	0.	46057	62276	79918	78860	72380	16052	705
6	0.	46604	58248	41620	55711	55037	32581	103
7	0.	47016	43294	32077	35744	50876	51166	200
8	0.	47338	19126	18558	43102	91031	94942	652
9	0.	47596	73387	94136	98074	20992	58767	943
10	0.	47809	14805	22282	65218	57483	38822	593
11	0.	47986	83864	21177	93060	63763	78094	421
12	0.	48137	71928	96132	33333	40659	50159	566
13	0.	48267	45860	34185	95742	20788	81691	629
14	0.	48380	22637	11143	81988	76438	77961	514
15	0.	48479	15979	73467	87483	07604	93785	104
16	0.	48566	66542	06469	74445	25315	44394	716
17	0.	48644	62068	50977	78561	93706	91184	515
18	0.	48714	51211	84041	53219	88041	65865	237
19	0.	48777	53226	45320	01979	86168	89398	296
20	0.	48834	64908	03962	53856	32196	68969	365
21	0.	48886	65652	57301	38755	75097	56900	317
22	0.	48934	21204	48930	17064	24069	08571	154
23	0.	48977	86474	49209	39447	05579	63327	306
24	0.	49018	07686	08206	43642	68322	75772	997
25	0.	49055	24030	49716	45175	88661	62627	736
26	0.	49089	68956	74008	74677	70098	54815	013
27	0.	49121	71187	34622	03050	02047	16352	957
28	0.	49151	55525	66274	20797	43887	39562	594
29	0.	49179	43502	96298	16771	89777	10050	517
30	0.	49205	53901	31767	52254	04091	19967	279
31	0.	49230	03179	21507	23811	79327	82540	814
32	0.	49253	05820	41622	35187	72928	57816	093
33	0.	49274	74621	73992	86402	19182	22045	953
34	0.	49295	20931	90602	95008	69104	13286	615
35	0.	49314	54850	88681	90326	16308	61524	456
36	0.	49332	85397	18565	80097	04259	60431	411
37	0.	49350	20648	90960	13759	71437	30330	081
38	0.	49366	67863	30684	00735	23085	40817	367
39	0.	49382	33578	51140	16938	56797	08515	026
40	0.	49397	23700	51185	86100	98386	49180	640
41	0.	49411	43577	78974	41441	71041	33118	800
42	0.	49424	98065	62118	72477	40469	76743	176
43	0.	49437	91581	67506	43135	20567	93328	810
44	0.	49450	28154	25239	52122	65715	34545	346
45	0.	49462	11464	27928	34873	59319	79282	291
46	0.	49473	44881	97753	23428	03690	92260	021
47	0.	49484	31498	98398	45982	72500	67542	544
48	0.	49494	74156	56451	22280	38007	92492	263
49	0.	49504	75470	46585	56086	48645	49696	176
50	0.	49514	37852	86381	48557	76509	00091	111
51	0.	49523	63531	79617	63271	36756	32942	313
52	0.	49532	54568	41047	97788	90743	14288	680
53	0.	49541	12872	30811	39568	33658	70668	234
54	0.	49549	40215	22552	30911	77096	05102	864
55	0.	49557	38243	25910	80187	26168	83786	980
56	0.	49565	08487	81157	76937	37521	21121	614
57	0.	49572	52375	41312	82930	40089	21802	445
58	0.	49579	71236	55014	75835	96709	33438	096
59	0.	49586	66313	61654	82491	97179	49566	001
60	0.	49593	38768	08782	21281	89874	90027	247
61	0.	49599	89687	00506	33489	80906	82304	768
62	0.	49606	20088	84518	97499	93167	93755	642
63	0.	49612	30928	84411	49055	59856	02892	412
64	0.	49618	23103	83145	15331	70091	74295	367
65	0.	49623	97456	62826	41729	54722	17514	860
66	0.	49629	54780	05327	01240	30961	60687	583
67	0.	49634	95820	57757	45315	73231	43115	241
68	0.	49640	21281	66340	20081	86762	79021	964
69	0.	49645	31826	81825	51927	01156	56390	158
70	0.	49650	28082	39240	72720	98998	02726	981

TABLE 6— $\Omega_{s+1}(s)$ TO 33 DECIMAL PLACES FOR $s = 1(1)70$

s		$\Omega_{s+1}(s)$						
1	0.	32831	10192	21786	48358	37319	54362	913
2	0.	39738	93029	17823	59171	37773	62032	204
3	0.	42711	79017	89185	94444	23749	80529	562
4	0.	44356	03447	62812	42491	94003	53226	658
5	0.	45396	91176	62568	92727	14273	16197	742
6	0.	46114	21975	29594	40596	77057	54494	642
7	0.	46638	26028	20938	97302	64716	89538	318
8	0.	47037	76779	37254	64596	68230	10259	464
9	0.	47352	38104	17725	26505	91914	30579	098
10	0.	47606	54704	32718	39072	35206	81971	967
11	0.	47816	15542	66257	33622	68550	47858	570
12	0.	47991	97838	99778	75677	39533	25193	061
13	0.	48141	57601	88702	04263	53453	74255	921
14	0.	48270	40988	18879	67510	22116	68149	137
15	0.	48382	52363	17696	21425	52780	75400	410
16	0.	48480	97497	48567	96695	72051	55266	815
17	0.	48568	11884	61598	61337	60970	23942	948
18	0.	48645	79831	62151	87526	27651	45107	715
19	0.	48715	47648	67813	90737	27925	33725	369
20	0.	48778	32960	81883	39373	94471	81255	988
21	0.	48835	31409	54568	45518	24475	97789	256
22	0.	48887	21559	64247	75745	90204	92394	972
23	0.	48934	68547	93890	03658	31230	64982	933
24	0.	48978	26834	87328	34563	29097	11182	629
25	0.	49018	42306	08563	70352	82098	99452	995
26	0.	49055	53896	37119	69437	60377	61411	059
27	0.	49089	94858	01689	61942	34674	53109	878
28	0.	49121	93761	17374	74425	79896	15554	971
29	0.	49151	75290	06798	06106	12844	47716	575
30	0.	49179	60882	06366	87736	55472	52924	354
31	0.	49205	69244	61157	17741	11496	69960	855
32	0.	49230	16776	46804	27948	89554	88599	744
33	0.	49253	17913	24803	62862	09409	95666	890
34	0.	49274	85412	70992	70905	02705	54366	353
35	0.	49295	30591	68992	70905	58356	67401	450
36	0.	49314	63523	98443	03347	27528	98496	349
37	0.	49332	93206	48955	26040	96156	42492	751
38	0.	49350	27699	38428	55100	64882	19083	261
39	0.	49366	74244	96872	21036	93523	38290	892
40	0.	49382	39368	85561	54508	07946	70446	855
41	0.	49397	28966	49887	03116	11072	92890	680
42	0.	49411	48377	47961	33755	55871	64881	566
43	0.	49425	02449	52428	13591	46869	04019	808
44	0.	49437	95593	87341	19908	97062	43581	998
45	0.	49450	31833	33459	57188	72082	61111	928
46	0.	49462	14844	12314	50249	51050	82129	201
47	0.	49473	47992	40779	05586	33237	32255	050
48	0.	49484	34366	32709	91039	09811	96977	456
49	0.	49494	76804	11831	64677	15976	70341	108
50	0.	49504	77918	89848	78624	28509	93969	865
51	0.	49514	40120	55369	14106	29085	31045	161
52	0.	49523	65635	12263	35656	31244	40110	418
53	0.	49532	56522	00299	61828	56071	48920	330
54	0.	49541	14689	26063	78186	73822	19095	414
55	0.	49549	41907	28130	87808	39973	69108	309
56	0.	49557	39820	97054	97091	38923	39963	872
57	0.	49565	09960	67878	27659	55209	06642	780
58	0.	49572	53752	00436	03711	44190	12277	687
59	0.	49579	72524	60676	53144	48415	79140	247
60	0.	49586	67520	14464	99217	11238	28968	917
61	0.	49593	39899	43846	15819	65099	26178	188
62	0.	49599	90748	94461	61971	05846	51392	283
63	0.	49606	21086	61721	01673	95654	60572	847
64	0.	49612	31867	22382	35917	07268	87640	302
65	0.	49618	23987	17382	80521	29046	46161	145
66	0.	49623	98288	91057	61076	25702	62884	651
67	0.	49629	55564	91275	27955	45020	86465	902
68	0.	49634	96561	34487	47036	68843	39642	663
69	0.	49640	21981	39231	39658	49052	77533	691
70	0.	49645	32488	31220	53434	57958	91279	656

APPENDIX D
VALUES OF THE SINE AND COSINE INTEGRALS

TABLE 7— $si(x)$ TO 33 DECIMAL PLACES FOR $x = 1(1)70$

x		$si(x)$						
1	−0.	62471	32564	27713	60428	99683	778	
2	0.	03461	66500	07798	22934	53984	566	
3	0.	27785	62012	04571	63716	64085	595	
4	0.	18740	68121	54156	43887	42376	117	
5	−0.	02086	50818	50222	48195	69132	909	
6	−0.	14610	87755	14390	08346	22906	637	
7	−0.	11619	97125	46803	02861	65531	978	
8	0.	00339	04949	12045	43285	16497	596	
9	0.	09424	37490	34705	87587	53317	363	
10	0.	08755	12674	23977	43009	96501	877	
11	0.	00751	04801	50830	79648	79540	535	
12	−0.	06582	50852	68523	24870	41731	595	
13	−0.	07143	46039	32072	05500	37149	094	
14	−0.	01458	52767	17231	56552	76897	988	
15	0.	04739	81169	13472	11989	26669	398	
16	0.	06050	59414	75136	26691	52816	252	
17	0.	01934	00890	75804	50320	25257	183	
18	−0.	03418	82299	33711	15687	01477	978	
19	−0.	05216	62950	25532	68765	96871	414	
20	−0.	02255	46257	51456	77906	76783	495	
21	0.	02409	46412	74762	70221	22667	843	
22	0.	04528	74097	99469	92388	31093	355	
23	0.	02466	31055	18260	21956	78927	114	
24	−0.	01605	76350	71977	51917	32659	077	
25	−0.	03931	37757	94935	29660	01384	986	
26	−0.	02592	74638	08558	06134	35844	313	
27	0.	00948	93571	73776	11125	41737	632	
28	0.	03394	94114	95475	33348	92680	551	
29	0.	02651	81882	49224	38162	86729	478	
30	−0.	00403	97867	64545	50824	75903	827	
31	−0.	02902	92895	01107	14690	72349	174	
32	−0.	02655	45497	35755	10385	70639	013	
33	−0.	00051	16286	26210	71451	80734	751	
34	0.	02445	98583	87572	00573	57929	860	
35	0.	02612	58777	13409	00612	20695	532	
36	0.	00431	08827	61880	38152	63169	366	
37	−0.	02018	88557	69319	97574	66044	853	
38	−0.	02530	33895	59197	87866	94305	609	
39	−0.	00745	92690	98525	96467	75278	911	
40	0.	01618	87925	59887	88754	43442	704	
41	0.	02414	70246	47298	39350	70721	373	
42	0.	01003	05569	20119	62773	59211	237	
43	−0.	01244	73301	65561	89987	20906	100	
44	−0.	02270	94954	70314	79949	57945	025	
45	−0.	01208	13258	98483	77281	29753	503	
46	0.	00896	13635	21708	92742	99872	906	
47	0.	02103	87170	20935	61391	13018	596	
48	0.	01365	74061	74167	95711	79077	921	
49	−0.	00573	32100	58625	46653	53640	049	
50	−0.	01917	92543	08960	72450	33360	968	
51	−0.	01479	79959	62744	98483	90623	664	
52	0.	00276	80565	70212	01420	85714	223	
53	0.	01717	31564	90128	80314	66984	921	
54	0.	01553	83360	67068	46608	93399	618	
55	−0.	00007	21933	96744	84196	44242	888	
56	−0.	01506	03985	88975	40875	74393	911	
57	−0.	01591	14541	69725	34659	16171	650	
58	−0.	00234	82357	00524	98962	89698	116	
59	0.	01287	90685	01941	72474	33178	752	
60	0.	01594	92894	65050	79309	51184	407	
61	0.	00448	81302	69302	71332	67505	525	
62	−0.	01066	53882	57229	90227	73770	974	
63	−0.	01568	33923	47207	34735	87181	263	
64	−0.	00634	40765	82866	08706	82213	305	
65	0.	00845	36290	83705	75085	62118	791	
66	0.	01514	53253	85134	29094	97632	313	
67	0.	00791	48013	70666	04146	39882	175	
68	−0.	00627	59615	01426	39967	09696	243	
69	−0.	01436	68501	65808	57961	57054	532	
70	−0.	00920	14776	16890	51370	83394	869	

TABLE 8-- $Si(x)$ TO 33 DECIMAL PLACES FOR $x = 1(1)70$

x		$Si(x)$					
1	0.	94608	30703	67183	01494	13533	139
2	1.	60541	29768	02694	84857	67201	482
3	1.	84865	25279	99468	25639	77302	511
4	1.	75820	31389	49053	05810	55593	033
5	1.	54993	12449	44674	13727	44084	007
6	1.	42468	75512	80506	53576	90310	279
7	1.	45459	66142	48093	59061	47684	938
8	1.	57418	68217	06942	05208	29714	512
9	1.	66504	00758	29602	49510	66534	279
10	1.	65834	75942	18874	04933	09718	794
11	1.	57830	68069	45727	41571	92757	452
12	1.	50497	12415	26373	37052	71485	321
13	1.	49936	17228	62824	56422	76067	823
14	1.	55621	10500	77665	05370	36318	928
15	1.	61819	44437	08368	73912	39886	314
16	1.	63130	22682	70032	88614	66033	169
17	1.	59013	64158	70701	12243	38474	100
18	1.	53660	80968	61185	46236	11738	939
19	1.	51863	00317	69363	93157	16345	502
20	1.	54824	17010	43439	84016	36433	422
21	1.	59489	09680	69659	32144	35884	759
22	1.	61608	37365	94366	54311	44310	272
23	1.	59545	94323	13156	83879	92144	030
24	1.	55473	86917	22919	10005	80557	839
25	1.	53148	25509	99961	32263	11831	930
26	1.	54486	88629	86338	55788	77372	603
27	1.	58028	56839	68672	73048	54954	548
28	1.	60474	57382	90371	95272	05897	467
29	1.	59731	45150	44121	00085	99946	394
30	1.	56675	65400	30351	11098	37313	090
31	1.	54176	70372	93789	47232	40867	743
32	1.	54424	17770	59141	51537	42577	904
33	1.	57028	46981	68685	90471	32482	165
34	1.	59525	61851	82468	62496	71146	776
35	1.	59692	22045	08305	62535	33912	448
36	1.	57510	72095	56777	00075	76386	282
37	1.	55060	74710	25576	64348	47172	063
38	1.	54549	29372	35698	74056	18911	308
39	1.	56333	70576	96370	65455	37938	006
40	1.	58698	51193	54784	50677	56659	620
41	1.	59494	33514	42195	01273	83938	289
42	1.	58082	68837	15016	24696	72428	153
43	1.	55834	89966	29334	71935	92310	816
44	1.	54808	68313	24581	81973	55271	892
45	1.	55871	50008	96412	84641	83463	414
46	1.	57975	76903	16605	54666	13089	822
47	1.	59183	50438	15832	23314	26235	512
48	1.	58445	37329	69064	57634	92294	838
49	1.	56506	31167	36271	15269	59576	867
50	1.	55161	70724	85935	89472	79855	949
51	1.	55599	83308	32151	63439	22593	253
52	1.	57356	43833	65108	63343	98931	139
53	1.	58796	94832	85025	42237	80201	837
54	1.	58633	46628	61965	08532	06616	534
55	1.	57072	41333	98151	77726	68974	029
56	1.	55573	59282	05921	21047	38823	005
57	1.	55488	48726	25171	27263	97045	266
58	1.	56844	80910	94371	62960	23518	800
59	1.	58367	53952	96838	34397	46395	668
60	1.	58674	56162	59947	41232	64401	323
61	1.	57528	44570	64199	33255	80722	442
62	1.	56013	09385	37666	71695	39445	942
63	1.	55511	29344	47689	27187	26035	653
64	1.	56445	22502	12030	53216	31003	611
65	1.	57924	99558	78602	37008	75335	708
66	1.	58594	16521	80030	91018	10849	229
67	1.	57871	11281	65562	66069	53099	092
68	1.	56452	03652	93470	21956	03520	674
69	1.	55642	94766	29088	03961	56162	384
70	1.	56159	48491	78006	10552	29822	047

TABLE 9— $C(x)$ TO 33 DECIMAL PLACES FOR $x = 1(1)70$

x		$C(x)$					
1	0.	33740	39229	00968	13466	26462	039
2	0.	42298	08287	74864	99569	85651	532
3	0.	11962	97860	08000	32762	64722	812
4	-0.	14098	16978	86930	41163	91448	987
5	-0.	19002	97496	56643	87861	84589	001
6	-0.	06805	72438	93247	12620	41683	048
7	0.	07669	52784	82184	51838	29157	630
8	0.	12243	38825	32009	55729	22959	583
9	0.	05534	75313	33133	60708	56416	484
10	-0.	04545	64330	04455	37263	45328	300
11	-0.	08956	31354	95479	97948	24676	305
12	-0.	04978	00068	84113	67559	59212	087
13	0.	02676	41255	64034	55503	67498	668
14	0.	06939	63559	27584	54727	43832	682
15	0.	04627	86776	74360	43960	43108	309
16	-0.	01420	01901	20190	02239	87727	863
17	-0.	05524	26822	60813	85052	73400	995
18	-0.	04347	51029	99501	00478	34411	492
19	0.	00515	03710	08426	12856	98550	925
20	0.	04441	98208	45353	31653	97687	170
21	0.	04089	05001	54692	36971	03028	992
22	0.	00164	06919	15737	74972	66809	806
23	-0.	03565	98603	82261	68292	95312	957
24	-0.	03833	30155	51247	14767	84692	866
25	-0.	00684	85971	79702	59091	89387	780
26	0.	02829	51510	31757	13190	84211	300
27	0.	03572	32177	17395	86892	52412	198
28	0.	01086	95343	08760	86639	85990	747
29	-0.	02194	69729	74023	04438	17170	542
30	-0.	03303	24172	82071	14377	92264	410
31	-0.	01395	27170	88765	38111	27878	536
32	0.	01638	88233	76158	67976	52215	894
33	0.	03025	74341	71747	31064	64190	945
34	0.	01626	49164	37355	76698	16563	519
35	-0.	01147	98563	55303	12949	95524	260
36	-0.	02740	89958	44977	44812	19314	772
37	-0.	01792	44197	10374	87873	38841	573
38	0.	00712	97618	01971	37971	35513	765
39	0.	02450	58334	17995	32807	86490	871
40	0.	01902	00078	96208	76696	19812	034
41	-0.	00327	89946	01272	07220	24609	597
42	-0.	02157	13793	86110	01039	32740	436
43	-0.	01962	20686	19766	02916	38534	024
44	-0.	00011	29602	15923	15149	71254	859
45	0.	01863	17437	03556	53210	28213	528
46	0.	01978	93378	69101	64954	65696	877
47	0.	00307	48622	02395	11907	40950	517
48	-0.	01571	43603	39703	32720	89430	021
49	-0.	01957	30908	02871	69756	81183	920
50	-0.	00562	83863	24116	30544	01858	955
51	0.	01284	70085	40631	13274	41575	407
52	0.	01901	97802	86359	47487	24315	938
53	0.	00779	12035	06269	68417	72858	007
54	-0.	01005	70583	27335	24933	98636	613
55	-0.	01817	26961	21802	92912	23331	524
56	-0.	00957	90490	58468	44996	70886	149
57	0.	00737	08611	60588	57036	19831	016
58	0.	01707	29871	71210	07125	73603	436
59	0.	01100	71216	77743	47251	35276	728
60	-0.	00481	32433	77443	21528	88723	439
61	-0.	01576	02584	39320	85240	81906	190
62	-0.	01209	10233	69152	33626	71637	466
63	0.	00240	70795	07390	65877	50053	950
64	0.	01427	28792	13325	48725	35777	047
65	0.	01284	73736	99511	79921	86448	815
66	-0.	00017	29331	94092	48378	73381	330
67	-0.	01264	80926	03150	42962	90900	197
68	-0.	01329	41857	91036	14117	61348	849
69	-0.	00187	12407	20841	47510	74718	745
70	0.	01092	19884	73464	97703	78113	544

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