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A STUDY OF THE SENSITIVITY OF COUPLED  
REACTION SYSTEMS TO UNCERTAINTIES IN  
RATE COEFFICIENTS. II. APPLICATIONS

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A STUDY OF THE SENSITIVITY OF COUPLED  
REACTION SYSTEMS TO UNCERTAINTIES  
IN RATE COEFFICIENTS  
II. APPLICATIONS†

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## ABSTRACT

The Fourier amplitude method developed in Part I as a diagnostic tool for determining the sensitivity of the results of complex calculations to the parameters which enter these calculations has been applied to two chemical reaction systems involving sets of coupled, non linear rate equations. These were: a) a 5 reaction set describing the high temperature (6000°K) dissociation of air and b) a 9 reaction set describing the constant temperature (2000°K) combustion of H<sub>2</sub> and O<sub>2</sub>. We have evaluated the Fourier amplitudes for all the species at a number of different times for both reaction systems. The analysis of these results verifies the claims made in part I. The relative magnitudes of the Fourier amplitudes showed a several order of magnitude distribution which permitted a clear distinction of the relative sensitivity of the species concentration to uncertainties in the rate coefficients. The conclusions based on the Fourier amplitude method for these two reaction systems are in excellent agreement with sensitivity predictions which could be made on the basis of previous kinetic studies of these systems.

## I. INTRODUCTION

In the preceding paper, hereafter referred to as I,<sup>(1)</sup> we developed a diagnostic technique to investigate the sensitivity of the solution of large sets of coupled, non-linear equations to uncertainties or changes in the coupling parameters. For brevity, we will refer to this method as FAST, the Fourier Amplitude Sensitivity Test. In this paper we wish to demonstrate the utility of FAST by applying it to a study of the sensitivity of two sets of coupled non-linear rate equations describing, respectively, the high temperature dissociation of air and the high temperature  $H_2 - O_2$  reaction. The specific objective of the calculations described here was to determine the sensitivity of the various species concentrations at a particular time to assumed uncertainties in the rate coefficients which enter into these calculations.

In Section II we summarize the steps which must be carried out in the application of FAST to any system of coupled non-linear equations and discuss the choice of frequencies and the number of sampling points to be used. In Section III, we present the sensitivity analysis for the two chemical reaction systems. Section IV contains some concluding remarks.

## II. GENERAL REMARKS ON APPLICATION OF FAST

We wish to determine the sensitivity of an "output function", in our case the calculated concentration of a particular species at a particular time, to assumed uncertainties in the rate coefficients entering the rate equations. We write this output function as

$$c(k_1, k_2, \dots, k_n) \equiv c(\vec{k}) \quad (2.1)$$

where  $\vec{k}$  is the  $n$ -component vector of the rate coefficients  $k_1, k_2, \dots, k_n$ .

The application of FAST is carried out via the following steps:

Step 1) One must choose a set of  $n$  integer frequencies,  $\{\omega_1, \omega_2, \dots, \omega_n\}$ , where  $n$  is the number of independent rate coefficients which determine the value of the output function.

Step 2) One frequency of this set is now assigned arbitrarily to each rate coefficient by (see I.2.2):

$$k_i = k_i^{(0)} e^{u_i} \quad (2.2)$$

$$u_i = u_i^{(0)} \sin \omega_i s \quad (2.3)$$

where  $k_i^{(0)}$  is the best estimate of the rate coefficient,  $s$  is a parameter and the  $u_i^{(0)}$  are the endpoints of the estimated ranges of uncertainties of the rate coefficients  $k_i$ . If a rate coefficient is known precisely, i.e. with zero uncertainty, then  $k_i = k_i^{(0)}$  which implies  $u_i^{(0)} = 0$ . The endpoints  $u_i^{(0)}$  are specified as part of the input data and their values should be based on the best judgement of the investigator.

Step 3) The output function  $c(\vec{k}) = c[\vec{k}(s)]$  is now Fourier analyzed in  $s$  to obtain the sine amplitudes  $A_{\omega_i}$  corresponding to the input frequencies  $\omega_i$ . This analysis requires the evaluation of  $c[\vec{k}(s)]$  for  $N$  values of  $s$ , where  $N$  is an integer which depends on the frequency spectrum of  $c[\vec{k}(s)]$ .

These steps will now be discussed in more detail.

a) Choice of Frequencies  $\omega_i$

In section III of part I we have presented a discussion of the choice of the set of frequencies  $\{\omega_i\}$  and have also given there a set of references (ref. 4) which contain both discussions of optimal choices of sets of frequencies

as well as tables of frequencies which could be used for FAST. For the calculations presented here we have developed our own set of frequencies. We have done so to obtain a better insight in how to handle the calculations involved in the Fourier analysis method. In a follow-on study we plan to investigate whether our method of choosing frequencies yields better, equivalent or worse sets for use in FAST than those presented in reference 4 of part I.

Since we will carry out computer calculations, we are limited to rational (or, equivalently, integer) frequencies. For the coupled non-linear equations under study here, this means that in addition to the input frequencies  $\omega_i$  and their multiples, various linear combinations of integer multiples of  $\omega_i$  ("interference frequencies") will appear in the spectrum of  $c[\vec{k}(s)]$ . This presents a problem in our analysis since the Fourier sine amplitude of frequency  $\omega_\ell$  is to reflect only the effect of the uncertainty in the value of the rate coefficient  $k_\ell$ . Clearly, if an interference frequency coincides with one of the input frequencies  $\omega_i$ , say  $\omega_\ell$ , the corresponding Fourier amplitude  $A_{\omega_\ell}$  will not only reflect changes in  $k_\ell$  but also in other rate coefficients. Thus, for example, if  $\omega_1 + \omega_3 - \omega_4 = \omega_2$ , the amplitude  $A_{\omega_2}$  is identical with  $A_{(\omega_1 + \omega_3 - \omega_4)}$  and will therefore reflect not only changes in  $k_2$ , but also in  $k_1$ ,  $k_3$  and  $k_4$ .

One therefore needs to choose a linearly independent set of input frequencies  $\omega_i$  such that

$$\sum_{i=1}^n a_i \omega_i \neq 0 \quad ; \quad a_i \text{ integer} \quad (2.4)$$

for

$$\sum_{i=1}^n |a_i| \leq M + 1 \quad (2.5)$$

where  $M$  is a parameter at the disposition of the investigator. We refer to such a set as being free of interferences to order  $M$ . The larger the chosen value of  $M$ , the greater the likelihood that the Fourier amplitude of each input frequency reflects solely the uncertainty of the corresponding rate coefficient. On the other hand, as discussed in section III of part I and below, the larger the chosen value of  $M$ , the larger the maximum value,  $\omega_{\max}$ , of the input frequencies  $\omega_i$  which will still satisfy (2.4) and (2.5) and, correspondingly, the larger the set  $N$  of points  $s$  which will be required for the evaluation of the Fourier amplitudes. For the calculations presented in this paper we have chosen  $M = 4$ . In Table I we present sets of frequencies which are free of interferences to 4th order for systems with dimensionality  $n$  from 5 to 19, i.e. for systems with from 5 to 19 independent rate coefficients. These frequencies were determined via computer by a trial and error procedure. The sets of frequencies listed here have the smallest value of  $\omega_{\max}$  satisfying conditions (2.4) and (2.5) and are referred to as "minimal sets".

b) Assignment of Frequencies to Rate Coefficients

The frequencies  $\{\omega_1, \omega_2, \dots, \omega_n\}$  are assigned to the rate coefficients  $\{k_1, k_2, \dots, k_n\}$  arbitrarily since the results of these calculations, in order to be useful, must be independent of both the frequency set and the assignment of the frequencies. To check that the spectrum of the output function does not contain interfering frequencies which coincide with the input frequencies, and thus affect the results, one should reassign the frequencies to the various rate coefficients and repeat the Fourier analysis. If the calculated Fourier amplitudes are invariant in magnitude and sign to these permutations, then the set of frequencies is free of interferences to the order of the dimensionality of the set of rate equations and the results of these calculations are reliable. If,

on the other hand, some of the Fourier amplitudes vary greatly with the permutations of the frequency assignments, a better set of input frequencies must be generated. By "better" we refer to a set  $\{\omega\}$  with a larger parameter  $M$  and a higher  $\omega_{\max}$ . We have carried out such a permutation analysis for the  $N_2 - O_2$  system discussed below and verified that our sets of input frequencies yielded Fourier amplitudes  $A_{\omega_\ell}^{(i)}$  which were invariant to the frequency permutations.

c) Calculation of Fourier Amplitudes

As discussed in section II of part I, we are only interested in the Fourier sine amplitudes. For an input frequency  $\omega_\ell$ , the amplitude is

$$A_{\omega_\ell} = \frac{1}{2\pi} \int_0^{2\pi} c[\vec{k}(s)] \sin \omega_\ell s \, ds \quad (2.6)$$

To compute this integral, one must evaluate the function  $c[\vec{k}(s)]$  at a set of points in the interval  $0 \leq s \leq 2\pi$ . In the absence of any information on the form of the output function  $c$  we take the points to be equally spaced in that interval,

$$s = \frac{2\pi q}{N}, \quad q = 1, 2, \dots, N \quad (2.7)$$

where  $N$  is some integer. With these equally spaced points we can now conveniently relate the Fourier coefficient  $A_{\omega_\ell}$  as defined by the integral (2.6) to the sum

$$A_{\omega_\ell}^* = \frac{2}{N} \sum_{q=1}^N \sin \omega_\ell s_q c[\vec{k}(s_q)] \quad (2.8)$$

where

$$A_{\omega_\ell}^* = A_{\omega_\ell} + \epsilon_{\omega_\ell} \quad (2.9)$$

The error term  $\epsilon_{\omega_\ell}$  is given by

$$\epsilon_{\omega_\ell} = \sum_{m=1}^{\infty} \sum_j A(mN - \omega_j) \quad (2.10)$$

where  $\omega_j$  is any frequency in the spectrum of  $f[k(s)]$  which satisfies

$$mN - \omega_j = \omega_\ell \quad (2.11)$$

This error term is due to "aliasing" which inevitably occurs whenever a finite number of points are chosen on an interval to evaluate Fourier amplitudes. For example, if  $N$  equally spaced points are used, the amplitudes  $A_{\omega_\ell}^{(i)}$  for  $\omega_\ell$  will unavoidably include the amplitudes of a component of frequency  $\omega_j$  present in  $c(s)$  which satisfies the relation  $\omega_\ell = mN - \omega_j$  where  $m$  is an arbitrary integer. The value of  $N$  is chosen to be the smallest integer such that

$$\sum_{i=1}^n b_i \omega_i \neq mN, \quad b_i, m \text{ integers} \quad (2.12)$$

for

$$\sum_{i=1}^n |b_i| \leq M + 1 \quad (2.13)$$

The Fourier amplitudes  $A_{\omega_\ell}$  have been evaluated using Eqs. (2.8) through (2.10).

If the output function  $c(k)$  could be expressed as a polynomial of degree less than or equal to  $M$  in the variables  $u_i$ ,  $i = 1, \dots, n$ , where (see Eq. 2.2)

$$u_i = \ln(k_i / i_i^{(0)}) \quad (2.14)$$

then the conditions (2.12) and (2.13) imply that  $\epsilon_{\omega_\ell} = 0$  so that

$$A_{\omega_\ell}^* = A_{\omega_\ell} \quad (2.15)$$

The conditions (2.11) through (2.13) also determine N the minimum number of s points required for the evaluation of  $c[\tilde{K}(s)]$ . These are listed in Table I.

The Fourier amplitudes are related to a multiple integral of an output function over the uncertainty space (see I. 2.12). One way of obtaining the Fourier amplitudes without the troublesome interferences and aliasing involved in the method discussed above would be to evaluate the n-dimensional integral of Eq. (I. 2.12) directly. We have done this via simple Monte Carlo integrations. Our results show that the number of integration points required by the Monte Carlo method to give a comparable accuracy is much greater than the number required in the Fourier analysis. Stated another way, for a given number of integration points (we used 200), the degree of accuracy of the Fourier method detailed above far surpasses that of the Monte Carlo integration.

d) Choice of the Weight Function

As emphasized in I, our final result, i.e. the relation

$$A_{\omega_\ell}^{(i)} \propto \langle \partial c_i / \partial u_\ell \rangle \quad (2.16)$$

involves a weight function  $p(u_i)$  which can be interpreted as a distribution function in u space which weights the uncertainty in the rate coefficients. In our treatment in part I, we used the weight function (see Eq. 2.20)

$$p(u_i) = \frac{a_i}{\cosh a_i u_i} \quad (2.17)$$

which had several desirable properties. It is, for instance, bell shaped about  $u_i=0$  corresponding to  $k_i=k_i^{(0)}$ , the "best" value of the rate coefficient  $k_i$ . For computational convenience, we have used two different weight functions for the calculations presented in section III below corresponding to two different

transformations  $u_i = u(\omega_i s)$ . As we had hoped and expected, our results did not depend significantly on the choice of the weight function. This indicates that our output functions  $c(k)$  were "well behaved", i.e. did not have any large local excursions in  $\vec{u}$  space. It is also in agreement with the discussion in the last paragraph of section III of part I.

As is shown in part I, the form of the weight function  $p(u_i)$  depends upon the choice of the transformation function  $f$  of Eq. (I.2.3). For the Fourier amplitude method to be at all useful, our results on the relative magnitudes of the Fourier amplitudes must clearly be independent of the explicit form of the transformation function. The insensitivity of our results to our choice of transformation function (and thus of the weight functions) verifies the validity of FAST in this respect.

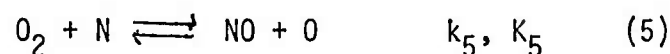
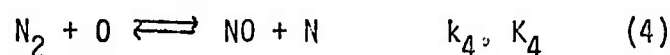
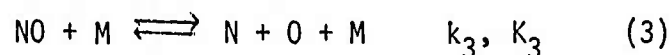
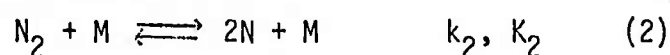
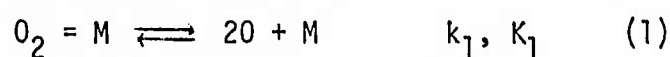
### III. APPLICATION OF FAST TO CHEMICAL SYSTEMS

We have applied FAST to two relatively simple reaction systems as a test of its utility as a diagnostic tool. Both systems were assumed, for simplicity, to react at constant temperatures and volume. These restrictions are in no way necessary for the application of FAST. The concentrations were

integrated in time with a code using Gear's<sup>(2)</sup> algorithm. These integrations were carried out once for each of the  $N$  values of  $s$  with the corresponding values of the  $n$  rate coefficients,  $\tilde{k}(s)$ , up to some time  $t$ . This yielded the output function  $c_i[\tilde{k}(s),t]$ , i.e. the concentration of species  $i$ ,  $i=1, 2, \dots, m$ , as a function of time. The Fourier analysis of  $c_i[\tilde{k}(s),t]$  was then carried out at several selected times. The uncertainties in the rate coefficients were arbitrarily assumed to be much larger than warranted by the available experimental data in order to check out FAST. It should therefore be pointed out again that the primary purpose of the calculations presented here is to demonstrate the validity and utility of FAST as a diagnostic tool and not to provide new information about the selected reaction systems.

a) High Temperature Air Reaction

We consider the following simplified reaction system<sup>(3)</sup>



The rate coefficients  $k_i$ ,  $i=1, \dots, 5$ , refer to the forward reactions, the rate coefficients  $k_{-i}$  for the reverse reactions are obtained from the equilibrium constant  $K_i$  through the relation  $K_i = k_i/k_{-i}$ . In the system and the one discussed below in IIIb, the equilibrium constants are assumed to be known exactly, i.e. with zero uncertainty. The forward and reverse rate coefficients  $k_i$ ,  $k_{-i}$  for each reaction therefore have identical uncertainties and the number of

independent uncertainties is equal to the number of equilibrium constants. It is, of course, not necessary to proceed this way and one can instead assign different uncertainties to the forward and reverse rate coefficients which reflect uncertainties in the equilibrium constants.

The system was assumed to react at a constant temperature of 6000°K and constant volume. The rate coefficients, equilibrium constants and initial conditions used in this calculation are shown in Table II.

The equilibrium constants represent the latest and "best" literature values we could find and the rate coefficients listed in this table are the  $k_i^{(0)}$  of Eq. (2.2). In Fig. 1 we show a plot of the time evolution of this reaction system obtained from the integration of the rate equations using the rate coefficients  $k_{\pm i}^{(0)}$  of Table II.

While the real uncertainty of these rate coefficients is not too large, we assumed arbitrarily for the purpose of testing FAST that the experimental uncertainties were  $\pm$  two orders of magnitude for each of the five independent sets of rate coefficients. To take account of this uncertainty we then write (see Eq. (2.2))

$$k_i^{(0)} e^{-4.606} \leq k_i \leq k_i^{(0)} e^{4.606} \quad ; \quad i = 1, \dots, 5 \quad (3.1)$$

For our sensitivity analysis we used the frequency set (see Table I)

$$\{\omega\} = (2, 42, 62, 74, 90) \quad (3.2)$$

for which the smallest number  $N$  of evenly spaced points satisfying the conditions of Eqs. (2.4, 2.5, 2.12, 2.13) in the  $s$  interval,  $0 \leq s \leq 2\pi$ , is 191.

We carried out the Fourier analysis of the output function  $c_i[\vec{K}(s)]$  at  $t = 10^{-6}$  sec and  $10^{-4}$  sec. From Eqs. (2.2), (2.3) and (3.1) we have

$$u_i = 4.606 \sin \omega_i s \quad (3.3)$$

The concentrations of the various species,  $N_2$ ,  $O_2$ ,  $NO$ ,  $O$ ,  $N$ , were calculated at  $10^{-6}$  sec and  $10^{-4}$  sec for each of the 191  $s$  values by integrating the rate equations. The 191 values of the concentration of the above species at a given time define the output function  $c_i[\tilde{k}(s)]$  which is now Fourier analysed by computing the Fourier coefficients from Eqs. (2.8) through (2.13) using the set of input frequencies of Eq. (3.2). An example of the function  $c_i[\tilde{k}(s)]$  for  $NO$  at  $t = 10^{-4}$  sec is shown in Fig. 2. This curve has been drawn by connecting the 191 calculated concentrations of  $NO$  (one for each of the 191 evenly spaced values of  $s$ ) by straight lines.

The results of the Fourier analysis at  $t = 10^{-6}$  sec are shown in Table III. The high values of the Fourier amplitudes associated with the rate coefficients  $k_1$ ,  $k_{-1}$  for the concentration of  $O$  and  $O_2$  (see blocks 1 and 2 of Table III) indicate that a variation in  $k_1$  and  $k_{-1}$  has a greater effect on the concentration of  $O$  and  $O_2$  than the variation of any of the other rate coefficients. This follows immediately from Eq. (2.16), where now  $i$  refers to the  $O$  atoms and  $O_2$  molecules and  $\omega_\ell = 2$  refers to the rate coefficients  $k_1$  and  $k_{-1}$ . We can rephrase this by saying that, at this early time in the reaction, only reaction (1) is important in the production of  $O$  atoms and the disappearance of  $O_2$  molecules. The sign of the amplitudes in the last column can be understood from Eq. (2.16) according to which the amplitude  $A_{\omega_\ell}^{(i)}$  is proportional to  $\langle \partial c_i / \partial u_\ell \rangle$ . Thus the negative sign of the Fourier amplitude for  $k_1$  for the concentration of  $O_2$  (block 1, Table III) simply indicates that  $[O_2]$  decreases with increasing  $k_1$ . The positive sign for the Fourier amplitude for  $k_1$  for the concentration of  $O$  atoms (block 2, Table III) indicates that  $[O]$  increases with increasing  $k_1$ . From the relative magnitudes of the Fourier amplitudes in blocks 4 and 5 of Table III, it will be noted that the formation of  $N$  and  $NO$  at these early times

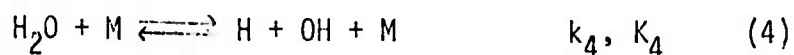
in the reaction is most sensitive to reactions (1) and (4). This is quite reasonable since at  $t = 10^{-6}$  sec the formation of N and NO via reactions (2) and (5) cannot play any important role owing to the small value of  $k_2^{(0)}$  as compared to  $k_1^{(0)}$  (see Table II). A number of other conclusions as to the sensitivity of the concentrations of the various species with respect to variation of the rate coefficients can be drawn from a study of the results in Table III. All these conclusions are in accord with one's knowledge and intuition of this rather simple reaction system.

In Table IV we present our results for  $t = 10^{-4}$  sec. At this time, as can be seen from Fig. 1, the concentrations of reactants and products are, except for  $N_2$ , significantly different from their initial values. The coupling between the various reactions has come into play much more strongly at this later time and one would expect that the influence of the uncertainties in the rate coefficients on the concentrations of the various species is more complex. This can readily be verified from the data in Table IV, where there is now no longer such a pronounced order of magnitude effect in the Fourier amplitudes. Let us arbitrarily use a factor of about 10 to distinguish between the "relatively important" amplitudes (i.e. the concentration of the species is very sensitive to the change of a specific rate coefficient) and the "relatively unimportant" amplitudes (i.e. the concentration of the species is not very sensitive to the change of a specific rate coefficient). Then the data of Table IV indicates that  $[O_2]$  and  $[O]$  are affected primarily by uncertainties in  $k_1$ ,  $[N_2]$  is affected about equally by uncertainties in all of the rate coefficients,  $[N]$  is least affected by uncertainty in  $k_1$ , and  $[NO]$  is least affected by uncertainty in  $k_2$ . The least "sensitive" rate coefficient is  $k_2$ , which determines the rate

of dissociation of  $N_2$ . This is not surprising since  $k_2^{(0)}$  is smaller than the other rate coefficients by 2 to 4 orders of magnitude. The important point to note is that owing to the strong coupling of all the reactions at this stage of the kinetic development, a number of the elementary rate processes contribute significantly to the production and removal of most of the species, so that the concentration of these species are quite sensitive to uncertainties in several rather than just one rate coefficient. This result is certainly not unexpected.

b) The  $H_2 - O_2$  Reaction

The  $H_2 - O_2$  combustion system was modeled by the following set of chemical reactions<sup>(4)</sup>:



The system was assumed to react at a constant temperature of 2000°K and constant volume, with only  $H_2$  and  $O_2$  present at  $t = 0$ .

The initial conditions, rate coefficients and equilibrium constants are shown in Table V. The rate coefficients listed in this table are the  $k_i^{(0)}$  of Eq. (2.2); the equilibrium constants are assumed to be known with zero uncertainty.

We assumed arbitrarily that the experimental uncertainties for each of the nine independent rate coefficients were  $\pm$  one order of magnitude so that

$$k_i^{(0)} e^{-2.303} \leq k_i \leq k_i^{(0)} e^{2.303} \quad ; \quad i = 1, 2, \dots, 9 \quad . \quad (3.4)$$

For our analysis we used the frequency set

$$\{\omega\} = (19, 59, 91, 113, 133, 143, 149, 157, 161) \quad (3.5)$$

for which the smallest number  $N$  of evenly spaced points in the  $s$  interval,  $0 \leq s \leq 2\pi$ , is 630.

A plot of the concentrations of the various species for the time interval  $10^{-4} \leq t \leq 10$  seconds, calculated with the  $k_i^{(0)}$  of Table V is shown in Fig. 3. We have employed FAST to study the sensitivity of the species concentrations to uncertainties in the rate coefficients at  $10^{-8}$  seconds and  $10^{-3}$  seconds. These results are shown in Tables VI and VII.

At the very early time of  $10^{-8}$  seconds (Table VI), the  $H_2$  and  $O_2$  concentrations have not changed significantly from their original value (to within 8 significant places) and the Fourier amplitudes are merely round-off error. For H atoms, reaction (1) seems to be the major contributor to its formation at this early time and its rate coefficient is the most sensitive in determining the accuracy of  $[H]$ . Reactions (7) and (8), which together also produce H atoms, are also important but the concentration of H atoms is not as sensitive to  $k_7$  and  $k_8$  as it is to  $k_1$ . The sensitivity of  $[H]$  to these two rate coefficients has identical values which is not surprising since  $[H]$  is proportional to  $k_7 \cdot k_8$  at this early time. The same argument also holds for the production of  $H_2O$  since its concentration at  $10^{-8}$  seconds is proportional to  $k_7 \cdot k_8$ . The  $H_2O$  concentration

is also sensitive to  $k_5$ , since reaction (5) appears to be partly responsible for the production of the OH which enters into reaction (7). The concentration of O atoms is most sensitive to  $k_2$  which is again as expected. Finally, the OH concentration is most sensitive to  $k_8$  which is also most reasonable, since at this early time, only the direct reaction between  $O_2$  and  $H_2$  would be expected to effect the OH concentration. Again, the results of FAST are in good agreement with what could be predicted on the basis of chemical knowledge.

At  $t = 10^{-3}$  seconds (Table VII) the story is quite different. As can be seen from Fig. 3, significant changes from the initial concentrations have occurred by this time, and the reactions are now strongly coupled. This is reflected in the array of Fourier amplitude in Table VII as compared to those of Table VI. Now, for instance, rate coefficient  $k_5$  corresponding to the chain branching step (5) which produces H and O atoms is the most influential one in controlling the concentration of all species. The H atoms concentration, in addition, is equally sensitive to  $k_7$ . All other rate coefficients, except for  $k_3$ , which is very small for all species, have about equal sensitivities to within a factor of 10 as measured by their Fourier amplitudes. According to our theory and calculations then, the rate coefficient which needs to be determined with the highest accuracy for the accurate calculation of all species concentration at  $10^{-3}$  seconds is  $k_5$  with  $k_7$  next in line. Uncertainties in the other rate coefficients will have a smaller effect on the species concentrations.

#### IV. CONCLUDING REMARKS

It should be realized that the applications presented here are only a first cut at the subject. It appears that FAST is a useful diagnostic tool in determining the sensitivity of the results of complex calculations to the parameters which enter the calculations. The Fourier amplitude method clearly needs further testing on more complicated systems than the ones dealt with in this paper. Such tests should also involve comparison with various types of "brute force" calculations.

In addition to applications to larger systems, there are also a number of theoretical-computational questions which need further investigation. Some examples are: the influence of the range of uncertainties, with different uncertainties for different coupling parameters, on the relative magnitudes of the Fourier amplitudes; the influence of the choice of the input frequencies  $\{\omega\}$  and the spacing of the  $N$  values of the parameter  $s$  on the Fourier amplitudes; and the question whether the absolute magnitude of the Fourier amplitudes can be used as a predictor for the actual change of the output function for a given change of the coupling parameter. Additional work is being carried out to answer these questions.

#### V. ACKNOWLEDGEMENT

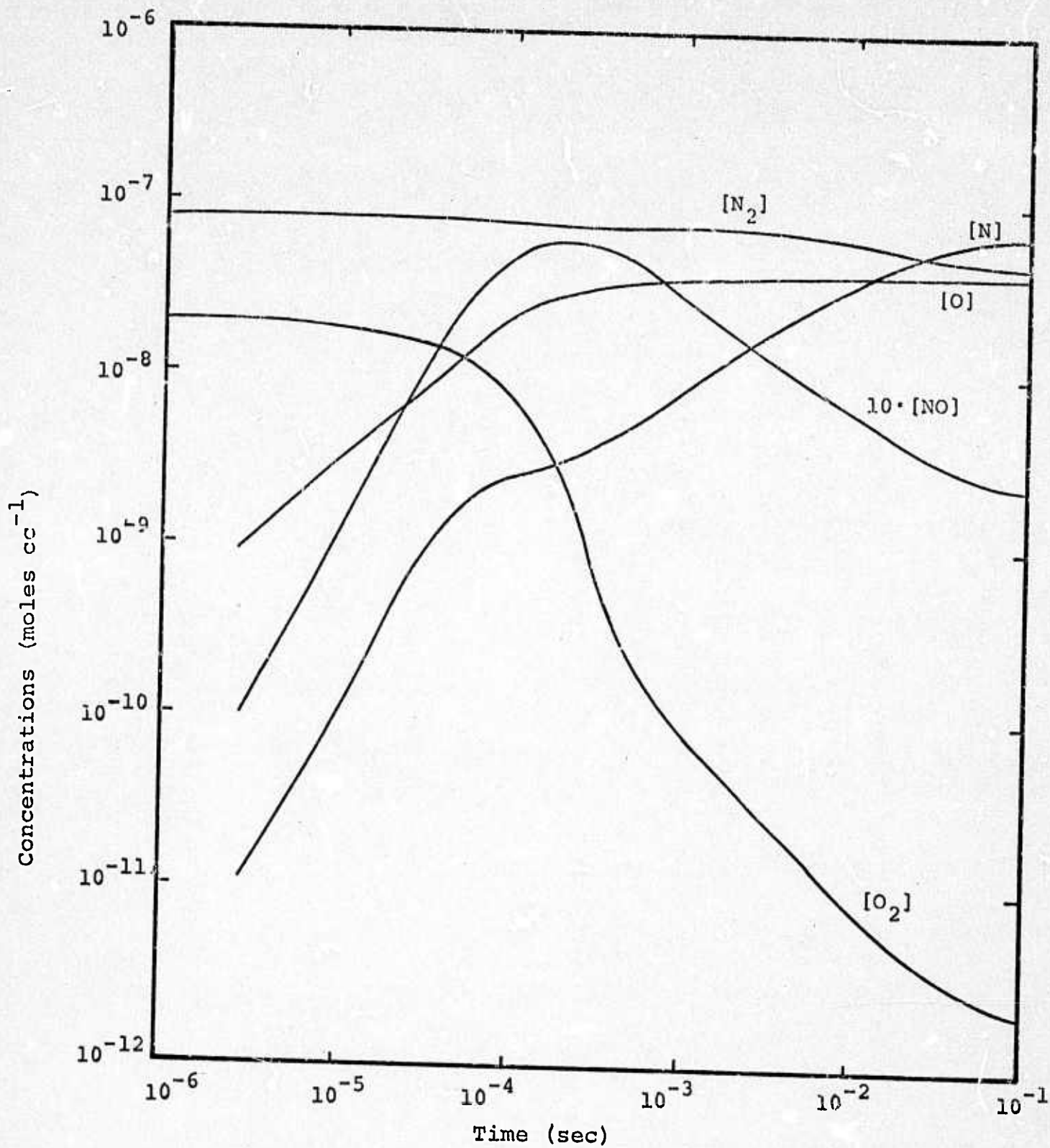
We wish to acknowledge with thanks many helpful conversations with our colleague R. I. Cukier.

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- 3) The equilibrium constants were calculated from data in B. J. McBride, Thermodynamic Properties to 6000°K for 210 Substances Involving the First 18 Elements, NASA, SP-3001 (1963). The rate coefficients were taken from: J. C. Schexnayder and J. S. Evans, NASA Technical Report R-108, 1961; H. S. Glick, J. J. Klein, and W. Squire, J. Chem. Phys. 27, 850 (1957); G. B. Kistiakowski and G. G. Volpi, J. Chem. Phys. 27, 1141 (1957). We are aware that more recent "nominal" values of the various rate coefficients exist in the literature but feel that the values used in our calculations are adequate for the purpose of this paper.
- 4) The equilibrium constants were calculated from the data in Mc Bride (see ref. 2); the rate coefficients were obtained from the compilation of G. S. Bahn, Reaction Rate Compilation for the H - O - N System, Gordon and Breach Science Publishers, New York (1968).

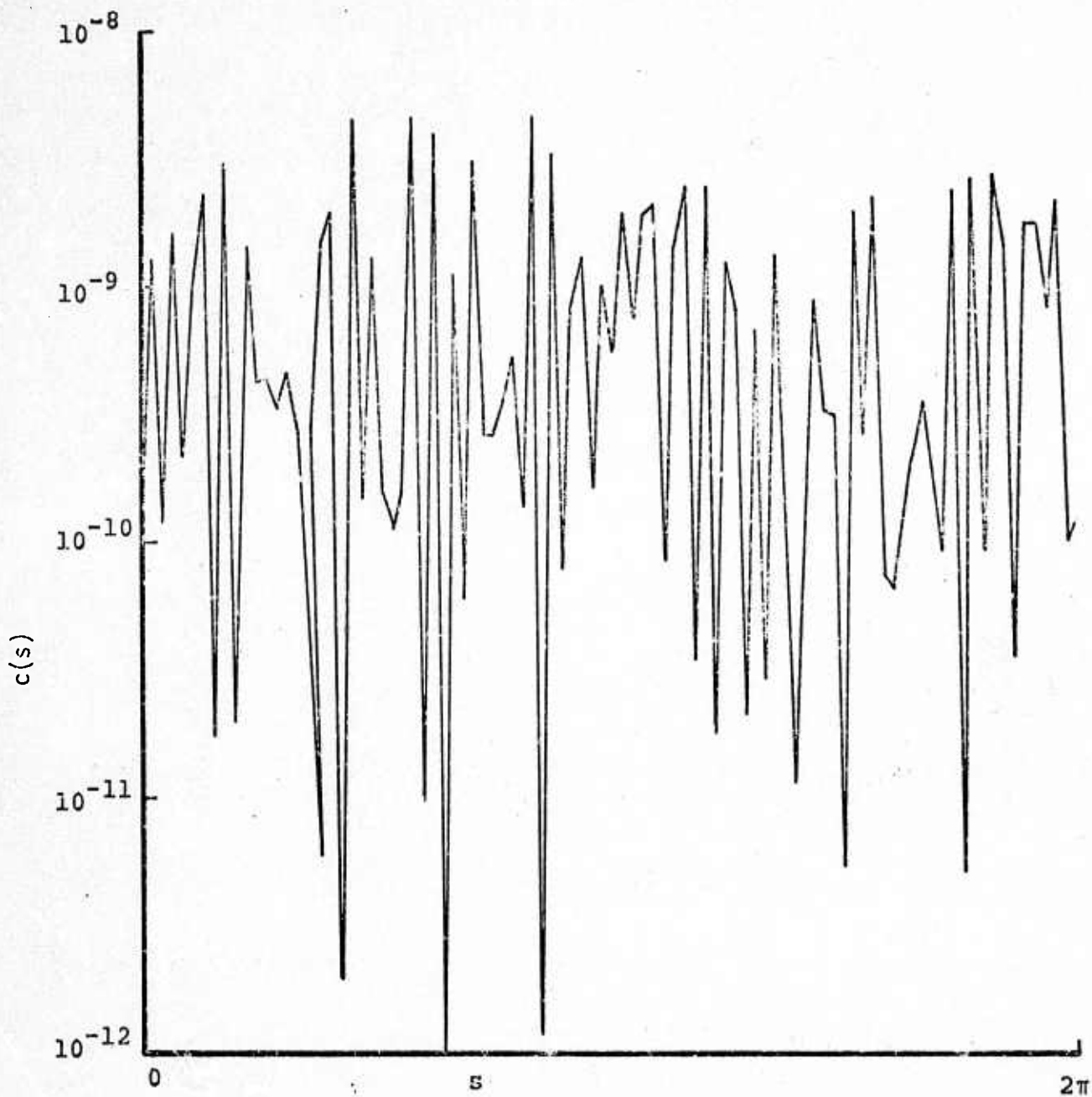
### FIGURE CAPTIONS

- Fig. 1. Time evolution of the  $N_2 - O_2$  system based on the kinetic data in Table II.
- Fig. 2. The function  $c_i[\vec{k}(s)]$  for NO at  $10^{-4}$  sec between 0 and  $2\pi$  as a function of  $s$ . This curve has been drawn by connecting the 191 calculated values of the concentrations of NO (corresponding to the 191 values of  $s$ ) by straight lines.
- Fig. 3. Time evolution of the  $H_2 - O_2$  systems between  $t = 10^{-4}$  sec to  $t = 10$  sec based on the kinetic data in Table V.



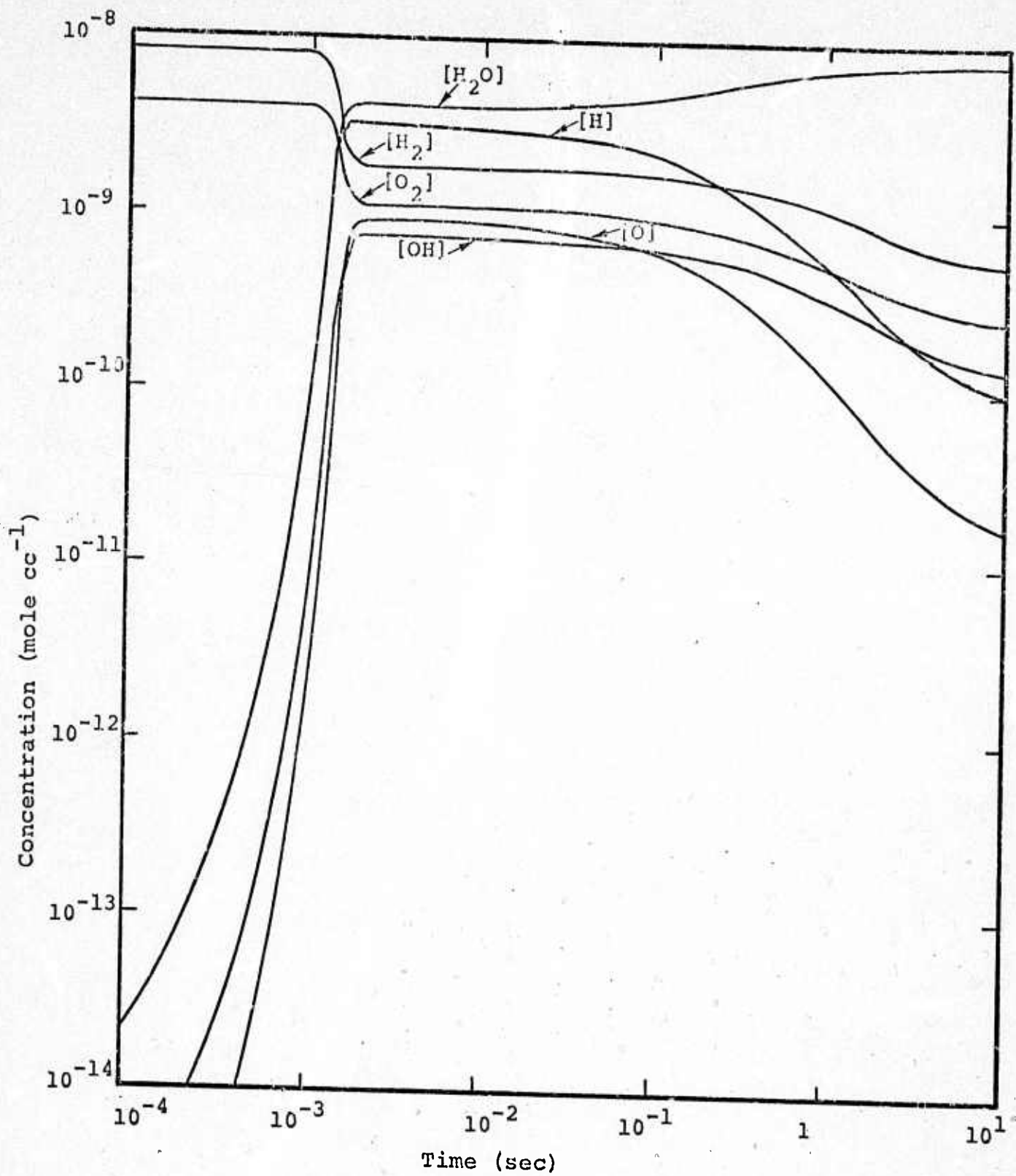
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Fig. 1, Schachtly + Shuler  
20



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21  
Fig. 2, Solari & Skuler



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Fig. 3, Schairly + Shuler 22

Table I

FREQUENCY SETS AND CORRESPONDING NUMBER OF  
POINTS AVOIDING INTERFERENCES  
THROUGH 4th ORDER

Dimension (n)	Frequency Set { $\omega$ }	Minimum Number of Points (N)
5	11, 21, 27, 35, 39	142
5*	2, 42, 62, 74, 90	191
6	1, 21, 31, 37, 45, 49	182
6*	2, 42, 62, 74, 90, 98	231
7	17, 39, 59, 69, 75, 83, 87	334
8	23, 55, 77, 97, 107, 113, 121, 125	486
9	19, 59, 91, 113, 133, 143 149, 157, 161	630
10	25, 63, 103, 135, 157, 177, 187, 193, 201, 205	806
11	41, 67, 105, 145, 177, 199, 219, 229, 235, 243, 247	974
12	31, 87, 113, 151, 191, 223, 245, 265, 275, 281, 289, 293	1158
13	23, 85, 141, 167, 205, 245 277, 299, 319, 329, 335, 343, 347	1374
14	87, 133, 195, 251, 277, 315, 355, 387, 409, 429, 439, 445, 453, 457.	1814
15	67, 143, 189, 251, 307, 333, 371, 411, 443, 465, 485, 495, 501, 509, 513	2038
16	73, 169, 245, 291, 353, 409, 435, 473, 513, 545, 567, 587, 597, 603, 611, 615	2446
17	85, 145, 241, 317, 363, 425, 481, 507, 545, 585, 617, 639, 659, 669, 675, 683, 687	2734
18	143, 229, 289, 385, 461, 507, 569, 625, 651, 689, 729, 761, 783, 803, 813, 819, 827, 831	3310
19	149, 275, 361, 421, 517, 593, 639, 701, 757, 783, 821, 861, 893, 915, 935, 945, 951, 959, 963	3848

\*These sets of 5 and 6 frequencies are not minimal, but have been used and are only included for completeness. They are also free of interferences to order less than 5.

Table II

INITIAL CONDITIONS, RATE COEFFICIENTS AND EQUILIBRIUM  
CONSTANTS FOR THE HIGH TEMPERATURE AIR REACTION

Initial Conditions

$$[N_2] = 8 \times 10^{-8} \text{ moles/cc}$$

$$[O_2] = 2 \times 10^{-8} \quad "$$

Rate Coefficients

$$k_1^{(0)} = 8.5 \times 10^{10} \text{ (moles/cc)}^{-1} \text{ sec}^{-1}$$

$$k_2^{(0)} = 3.0 \times 10^7 \quad "$$

$$k_3^{(0)} = 8.0 \times 10^9 \quad "$$

$$k_4^{(0)} = 9.0 \times 10^{10} \quad "$$

$$k_5^{(0)} = 8.0 \times 10^{11} \quad "$$

Equilibrium Constants

$$K_1 = 7.8 \times 10^{-4} \text{ (moles/cc)}$$

$$K_2 = 1.0 \times 10^{-7} \quad "$$

$$K_3 = 1.3 \times 10^{-5} \quad "$$

$$K_4 = 9.0 \times 10^{-3} \quad -$$

$$K_5 = 5.9 \times 10^1 \quad -$$

A

Table III

FOURIER AMPLITUDES OF THE CONCENTRATIONS AT  $10^{-6}$  SEC  
FOR THE HIGH TEMPERATURE AIR SYSTEM

Input Frequency	Rate Coefficient	Fourier Amplitude
$[O_2] \times 10^9$ moles/cc		
2	$k_1$	-4.47
90	$k_5$	-0.0244
74	$k_4$	-0.0227
42	$k_2$	$-4.7 \times 10^{-4}$
62	$k_3$	$-3.9 \times 10^{-4}$
$[O] \times 10^9$ moles/cc		
2	$k_1$	8.53
74	$k_4$	-0.36
90	$k_5$	0.018
62	$k_3$	-0.029
42	$k_2$	0.0027
$[N_2] \times 10^9$ moles/cc		
2	$k_1$	-0.389
74	$k_4$	-0.386

62	$k_3$	$-3.9 \times 10^{-4}$
$[O] \times 10^9$ moles/cc		
2	$k_1$	8.53
74	$k_4$	-0.36
90	$k_5$	0.018
62	$k_3$	-0.029
42	$k_2$	0.0027
$[N_2] \times 10^9$ moles/cc		
2	$k_1$	-0.389
74	$k_4$	-0.386
42	$k_2$	-0.0087
62	$k_3$	-0.0039
90	$k_5$	-0.00103
$[N] \times 10^9$ moles/cc		
2	$k_1$	0.367
74	$k_4$	0.362
90	$k_5$	-0.028
62	$k_3$	-0.013
42	$k_2$	-0.011
$[NO] \times 10^{10}$ moles/cc		
2	$k_1$	0.412
74	$k_4$	0.410
90	$k_5$	0.031
42	$k_2$	-0.0066
62	$k_3$	-0.0048

Table IV

A

FOURIER AMPLITUDES OF THE CONCENTRATIONS AT  $10^{-4}$  SEC  
FOR THE HIGH TEMPERATURE AIR SYSTEM

Input Frequency	Rate Coefficient	Fourier Amplitude
$[O_2] \times 10^8$ moles/cc		
2	$k_1$	-1.1
90	$k_5$	-0.147
74	$k_4$	-0.087
62	$k_3$	-0.069
42	$k_2$	-0.044
$[O] \times 10^8$ moles/cc		
2	$k_1$	2.17
62	$k_3$	0.228
90	$k_5$	0.164
42	$k_2$	0.078
74	$k_4$	-0.00146
$[N_2] \times 10^9$ moles/cc		
74	$k_4$	-2.6
2	$k_1$	-2.3
62	$k_3$	-1.0
90	$k_5$	-0.70
42	$k_2$	-0.55
$[N] \times 10^9$ moles/cc		
2	$k_1$	4.3
74	$k_4$	3.3
62	$k_3$	3.0

[O] x 10<sup>9</sup> moles/cc

2	k <sub>1</sub>	2.17
62	k <sub>3</sub>	0.228
90	k <sub>5</sub>	0.164
42	k <sub>2</sub>	0.078
74	k <sub>4</sub>	-0.00146

[N<sub>2</sub>] x 10<sup>9</sup> moles/cc

74	k <sub>4</sub>	-2.6
2	k <sub>1</sub>	-2.3
62	k <sub>3</sub>	-1.0
90	k <sub>5</sub>	-0.70
42	k <sub>2</sub>	-0.55

[N] x 10<sup>9</sup> moles/cc

2	k <sub>1</sub>	4.3
74	k <sub>4</sub>	3.3
62	k <sub>3</sub>	3.0
42	k <sub>2</sub>	0.98
90	k <sub>5</sub>	0.125

[NO] x 10<sup>9</sup> moles/cc

74	k <sub>4</sub>	1.77
90	k <sub>5</sub>	1.29
62	k <sub>3</sub>	-0.91
2	k <sub>1</sub>	0.23
42	k <sub>2</sub>	0.124

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B

Table V

Initial Conditions, the Rate Coefficients and Equilibrium Constants for the  $H_2 - O_2$  Reaction

Initial Conditions

$$[H_2] = 8.0 \times 10^{-9} \text{ moles/cc}$$

$$[O_2] = 4.0 \times 10^{-9} \text{ moles/cc}$$

Rate Coefficients

$$k_1^0 = 5.78 \times 10^4 \text{ (moles/cc)}^{-1} \text{ sec}^{-1}$$

$$k_6^0 = 4.0 \times 10^{13} \text{ (moles/cc)}^{-1} \text{ sec}^{-1}$$

$$k_2^0 = 4.47 \times 10^3 \quad "$$

$$k_7^0 = 1.51 \times 10^{13} \quad "$$

$$k_3^0 = 1.03 \times 10^5 \quad "$$

$$k_8^0 = 2.20 \times 10^6 \quad "$$

$$k_4^0 = 6.79 \times 10^5 \quad "$$

$$k_9^0 = 1.6 \times 10^{14} \quad "$$

$$k_5^0 = 1.23 \times 10^{12} \quad "$$

Equilibrium Constants

$$K_1 = 1.59 \times 10^{-11} \text{ moles/cc}$$

$$K_6 = 1.4 \quad -$$

$$K_2 = 2.69 \times 10^{-12} \quad "$$

$$K_7 = 9.76 \quad -$$

$$K_3 = 1.14 \times 10^{-11} \quad "$$

$$K_8 = 2.02 \quad -$$

$$K_4 = 1.63 \times 10^{-12} \quad "$$

$$K_9 = 6.76 \quad -$$

$$K_5 = 2.37 \times 10^{-1} \quad -$$

Table VI

FOURIER AMPLITUDES OF THE CONCENTRATIONS AT  $10^{-8}$  SEC  
FOR THE  $H_2 - O_2$  REACTION

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Rate Coefficient	Fourier Amplitude	Rate Coefficient	Fourier Amplitude
	[H <sub>2</sub> ]		[O <sub>2</sub> ]
	Inensitive to uncertainties in any rate constant.		Inensitive to uncertainties in any rate constant.
	[H] x 10 <sup>19</sup>		[O] x 10 <sup>20</sup>
k <sub>1</sub>	4.67	k <sub>2</sub>	1.79
k <sub>8</sub>	0.202	k <sub>6</sub>	-0.016
k <sub>7</sub>	0.202	k <sub>1</sub>	0.0065
k <sub>5</sub>	0.010	k <sub>5</sub>	0.0055
k <sub>6</sub>	0.0023	k <sub>3</sub>	2.25x10 <sup>-5</sup>
k <sub>2</sub>	0.00165	k <sub>4</sub>	-1.95x10 <sup>-5</sup>
k <sub>4</sub>	8.0x10 <sup>-5</sup>	k <sub>9</sub>	7.87x10 <sup>-6</sup>
k <sub>9</sub>	4.1x10 <sup>-5</sup>	k <sub>8</sub>	1.20x10 <sup>-6</sup>
k <sub>3</sub>	-3.0x10 <sup>-5</sup>	k <sub>7</sub>	-8.46x10 <sup>-7</sup>
	[OH] x 10 <sup>18</sup>		[H <sub>2</sub> O] x 10 <sup>20</sup>
k <sub>8</sub>	5.9	k <sub>8</sub>	2.03
k <sub>7</sub>	-0.0203	k <sub>7</sub>	2.03
k <sub>9</sub>	0.00176	k <sub>5</sub>	0.121
k <sub>5</sub>	0.0014	k <sub>6</sub>	0.0066
k <sub>2</sub>	1.61x10 <sup>-4</sup>	k <sub>4</sub>	8.35x10 <sup>-4</sup>
k <sub>6</sub>	9.73x10 <sup>-5</sup>	k <sub>1</sub>	-6.35x10 <sup>-4</sup>
k <sub>1</sub>	7.08x10 <sup>-5</sup>	k <sub>9</sub>	4.22x10 <sup>-4</sup>
k <sub>4</sub>	-7.43x10 <sup>-6</sup>	k <sub>3</sub>	-2.81x10 <sup>-4</sup>
k <sub>3</sub>	3.13x10 <sup>-6</sup>	k <sub>2</sub>	1.33x10 <sup>-4</sup>

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Table VII

FOURIER AMPLITUDES OF THE CONCENTRATIONS AT  $10^{-3}$  SEC  
FOR THE  $H_2 - O_2$  REACTION

Rate Coefficient	Fourier Amplitude	Rate Coefficient	Fourier Amplitude
	$[H_2] \times 10^9$		$[H] \times 10^9$
$k_5$	-3.71	$k_5$	1.92
$k_7$	-0.346	$k_7$	1.92
$k_6$	-0.313	$k_6$	0.122
$k_8$	-0.172	$k_8$	0.102
$k_2$	-0.108	$k_2$	0.061
$k_4$	-0.101	$k_9$	0.0466
$k_9$	-0.0905	$k_4$	0.0439
$k_1$	-0.0491	$k_1$	0.0176
$k_3$	0.0119	$k_3$	-0.00318
	$[O_2] \times 10^9$		$[O] \times 10^{10}$
$k_5$	-1.77	$k_5$	5.77
$k_7$	-0.153	$k_7$	0.420
$k_6$	-0.089	$k_1$	0.164
$k_8$	-0.070	$k_6$	0.162
$k_2$	-0.047	$k_9$	0.145
$k_4$	-0.047	$k_2$	0.116
$k_9$	-0.044	$k_8$	0.115
$k_1$	-0.031	$k_4$	0.10
$k_3$	0.0091	$k_3$	-0.061
	$[OH] \times 10^{10}$		$[H_2O] \times 10^9$
$k_5$	4.63	$k_5$	2.52
$k_7$	0.281	$k_7$	0.237
$k_6$	0.196	$k_6$	0.142

$k_1$	-0.0491	$k_1$	0.0176
$k_3$	0.0119	$k_3$	-0.00318

$[O_2] \times 10^9$		$[O] \times 10^{10}$	
$k_5$	-1.77	$k_5$	5.77
$k_7$	-0.153	$k_7$	0.420
$k_6$	-0.089	$k_1$	0.164
$k_8$	-0.070	$k_6$	0.162
$k_2$	-0.047	$k_9$	0.145
$k_4$	-0.047	$k_2$	0.116
$k_9$	-0.044	$k_8$	0.115
$k_1$	-0.031	$k_4$	0.10
$k_3$	0.0091	$k_3$	-0.061

$[OH] \times 10^{10}$		$[H_2O] \times 10^9$	
$k_5$	4.63	$k_5$	2.52
$k_7$	0.281	$k_7$	0.237
$k_6$	0.196	$k_6$	0.142
$k_8$	0.135	$k_8$	0.114
$k_9$	0.116	$k_4$	0.114
$k_2$	0.115	$k_2$	0.0718
$k_4$	0.0961	$k_9$	0.0614
$k_1$	0.0876	$k_1$	0.0358
$k_3$	-0.0359	$k_3$	-0.00851

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