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ON THE APPROXIMATION OF ITO INTEGRALS USING BAND-LIMITED PROCESSES

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I

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ABSTRACT

Ito integrals involving observed data arise in many applications where we idealize the observation noise to be white. Since the Wiener process is not realizable, and any observed process must be smooth, the need arises for approximation in terms of a smooth process - such as a band-limited process which has no frequency components outside a finite band. We show that under a sufficient condition such an approximation is possible provided we also add suitable 'correction' terms.

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II

ON THE APPROXIMATION OF ITO INTEGRALS USING BAND-LIMITED PROCESSES

A. V. Balakrishnan[†]

Let $W(t;\omega)$, $-\infty < t < +\infty$, denote a standard Wiener process, say one-dimensional for simplicity, and let $\beta(t)$ denote the associated 'growing' sigma algebra of sets (that is the sigma algebra generated by $W(s;\omega)$, $s \leq t$). Let $f(t;\omega)$ be a function jointly measurable in t and ω, measurable $\beta(t)$ for each t, and further such that, say (for simplicity):

$$\int_0^1 E\, |f(t;\omega)|^2\, dt < \infty$$

In many problems of filtering and control, we need to evaluate the Ito-integral:

$$\int_0^1 f(t;\omega)\, dW(t;\omega) \qquad (1.1)$$

The standard approximation is to use partial sums of the form:

$$\Sigma\ f(t_i;\omega)(W(t_{i+1},\omega) - W(t_i;\omega)) \qquad (1.2)$$

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However, in practice, there is a serious difficulty with this procedure because, in dealing with Ito integrals with respect to observed data, one does not have a true Wiener process to work with (see section 2 under Application for more details on this). What one has rather is a 'smooth' approximation to the Wiener process. A most convenient approximation is the 'band-limited' version, by which is meant, precisely, the process:

$$y(t;\omega) = \int_{-\infty}^{\infty} M(t-s)\, dW(s;\omega)$$

where

$$\int_{-\infty}^{\infty} e^{2\pi i f t} M(t)dt = \Psi(f)$$

vanishes outside a finite interval, and is thus 'limited' to finite band:

$$\Psi(f) = 0 \qquad |f| > m > 0$$

and†

$$\Psi(f) = 1 \qquad |f| \leq m \tag{1.3}$$

For any function $g(\cdot)$ in $L_2(0,1)$ we note that

† We can generalize this condition, of course, so long as (1.4) holds.

$$\int_0^1 g(t)\, y(t;\omega)dt$$

converges as m goes to infinity with probability one and in the L_2-mean to:

$$\int_0^1 g(t)\, dW(t;\omega) \tag{1.4}$$

However, the situation is quite different in the case where $g(\cdot)$ is not deterministic. To take a classic example, consider the Ito integral:

$$\int_0^1 (W(t;\omega) - W(0,\omega))\, dW(t;\omega) \tag{1.5}$$

If we approximate by band-limited processes above:

$$\int_0^1 \int_0^t y(s;\omega)ds \quad y(t;\omega)dt \tag{1.6}$$

the limit as m goes to infinity is

$$(W(1;\omega) - W(0;\omega))^2/2 \tag{1.7}$$

while the Ito integral itself is

$$= (W(1;\omega) - W(0;\omega))^2/2 - 1/2 \qquad (1.8)$$

In other words (1.6) does not converge to (1.5), but it does if we add the correction term $(-\frac{1}{2})$. Wong and Zakai [1] showed that if we consider the Ito integral arising from a stochastic differential equation in the Ito sense, for example:

$$dx(t;\omega) = m(x(t;\omega))dt + \sigma(x(t;\omega))\, dW(t;\omega)$$

then if we take the sequence of solutions obtained by replacing $W(t;\omega)$ by a sequence of smooth processes (such as the band-limited process), we do not have convergence to the solution of the Ito equation but rather to another equation obtained by adding a correction term

$$\frac{1}{2}\sigma(x(t;\omega))\sigma'(x(t;\omega))dt$$

analagous to (1.8). They do not however deal with integrals of the form (1.1) directly. McShane in his recent work [2]† has examined many approximations to (1.1) but they are time-domain approximations (extension of the form (1.2)), and in fact he cites the need for examining band-limited approximations in view of the negative results of Wong-Zakai. In this paper we study the problem of approximating integrals of the form (1.1) by functionals on the band-limited process, one area of application being in the calculation of likelihood functionals. We

† See also his 'Stochastic Equations and Stochastic Models', Holt, Rhinehart and Winston, 1972.

show that under a sufficiency condition on the function $f(t;\omega)$, it is possible to approximate in the desired manner, and indicate what the precise 'correction' terms must be. We begin with a special case first along with a direct application of it. Our general result is given in section 3.

2. The Linear Case

We begin first with the notationally simple case where $f(t;\omega)$ is a linear functional but still exhibits the essential features of the more general situation. Let $W(t;\omega)$ denote an n-by-1 standard Wiener process on $(-\infty, \infty)$. See [3] for an explicit construction. As therein, we may set $W(0;\omega)$ to be zero. Let

$$f(t;\omega) = \int_0^t L(t;s)\, dW(s;\omega) \qquad 0 \leq t \leq 1 \tag{2.1}$$

where $L(t;s)$ is an n-by-n matrix function, Lebesgue measurable in s, t and such that†

$$\int_0^1 \int_0^t \|L(t;s)\|^2 ds \quad dt < \infty \tag{2.2}$$

In particular, of course, condition (2.2) implies that:

$$\int_0^1 E(\|f(t;\omega)\|^2)dt = \int_0^1 \int_0^t \|L(t;s)\|^2 ds \quad dt < \infty$$

Our first result is:

<u>Theorem 2.1</u> Let H denote the <u>real</u> Hilbert space of n-by-1 square integrable functions $L_2((0,1), E_n)$. With $L(t;s)$ as in (2.2),

---

† $\|B\|^2 = \text{Tr. } BB^*$; $[A, B] = \text{Tr. } AB^*$

define the linear transformation L by:

$$Lf = g; \quad g(t) = \int_0^t L(t;s)\, f(s)ds \qquad 0 < t < 1$$

mapping H into itself. Suppose $(L+L^*)$ is trace-class (or, 'nuclear', as it is referred to in the more recent literature). Then

$$\eta = \int_0^1 [\int_0^t L(t;s)\, dW(s;\omega),\ dW(t;\omega)]$$

$$= \underset{m}{\text{Lim}} \int_0^1 [\int_0^t L(t;s)\, y_m(s;\omega)ds,\ y_m(t;\omega)]dt - \frac{1}{2}\, \text{Tr.}(L+L^*) \qquad (2.3)$$

where

$$y_m(t;\omega) = \int_{-\infty}^{\infty} M(t-s)\, dW(s;\omega)$$

$$M(s) = I_n \int_{-m}^{m} e^{2\pi i f s}\, df = I_n(\sin 2\pi m s)/\pi s$$

$$I_n = \text{n-by-n identity matrix}$$

and the limit may be taken in the L_2-mean.

Proof: We begin with a Lemma (Cf. [4]):

<u>Lemma</u> Let $\phi_i(\cdot)$ denote any orthonormal basis of functions in H. Then

$$\eta = \sum_i \sum_j [L\phi_i, \phi_j] \zeta_i \zeta_j - \frac{1}{2} \text{Tr.} (L+L^*) \tag{2.4}$$

where

$$\zeta_i = \int_0^1 [\phi_i(t), dW(t;\omega)]$$

and the convergence is in the L_2-mean.

<u>Proof</u> See [4]. Here we shall indicate the main steps. Let

$$a_{ij} = [L\phi_i, \phi_j]$$

Then since we are in a real Hilbert space:

$$\frac{1}{2} \text{Tr.} (L+L^*) = \Sigma a_{ii}$$

and we note that

$$\Sigma |a_{ii}| < \infty \tag{2.5}$$

For any finite subdivision of [0, 1] with subdivision points $\{t_i\}$,

$$t_o = 0, \quad t_p = 1 \tag{2.5}$$

we note that

$$\eta_p = \sum_{i=0}^{p-1} [f(t_i;\omega), W(t_{i+1};\omega) - W(t_i;\omega)]$$

$$= \sum_i \sum_j [L_p \phi_i, \phi_j] \zeta_i \zeta_j$$

where L_p is the operator on H defined by:

$$L_p f = g; \quad g(t) = \int_0^{t_i} L(t_i;s) f(s) ds \qquad t_i < t < t_{i+1}$$

and we observe that L_p is trace-class with trace zero. Next we can calculate that, if we assume for the moment that $L(t;s)$ is continuous in $0 \leq s \leq t \leq 1$,

$$E((\sum_i \sum_j ([L_p \phi_i, \phi_j] - a_{ij}) \zeta_i \zeta_j + \sum_i a_{ii})^2)$$

$$\leq 2 \| L_p - L \|^2_{H-S} \longrightarrow 0 \text{ as } p \longrightarrow \infty$$

where H-S stands for the Hilbert-Schmidt norm. Since η_p converges in the L_2-mean to η, we obtain (2.4). Next given an arbitrary, that is non-continuous, kernel $L(t;s)$, we can approximate it by continuous kernels L_n such that

$$\|L_n - L\|^2_{H-S} \longrightarrow 0$$

$$\text{Tr.}\,(L_n + L_n^*) \longrightarrow \text{Tr.}(L+L^*)$$

Since (2.4) holds for each L_n, we can proceed to take limits on both sides to obtain the desires result for L.

Remark In the case that $L(t;s)$ is continuous in $s \leq t$, we know that

$$\frac{1}{2}\,\text{Tr.}(L+L^*) = \frac{1}{2}\int_0^1 \text{Tr.}\; L(t;t)dt$$

On the other hand, even if

$$\int_0^1 |\text{Tr.}\; L(t;t)|\, dt \;<\; \infty$$

it is not necessary that $(L+L^*)$ be trace-class, and (2.4) need not hold. For example, we know that we can by a classical construction due to Carleman, find a continuous function $h(t)$, $0 \leq t \leq 1$, such that $h(t)$ has the Fourier series expansion:

$$h(t) = \sum_0^\infty c_k \cos 2\pi kt$$

where

$$\sum_{0}^{\infty} |c_k| = +\infty$$

If we now take

$$L(t;s) = h(t-s)$$

we note that L is not trace-class, and (2.4) does not hold if we take the orthonormal basis of trigonometric functions.

Let us now return to the proof of the theorem. Let us first assume that $L(t;s)$ is continuous in $s \leq t$. Then because $(L+L^*)$ is trace-class, we know that

$$\text{Tr.}(L+L^*) = \int_0^1 \text{Tr. } L(t;t)dt$$

First we note that letting for fixed m:

$$R(t;s) = E(y_m(t;\omega)\ y(s;\omega)^*)$$

we have:

$$R(t;s) = I_n(\sin 2\pi m(t-s))/\pi(t-s) = M(t-s) \tag{2.6}$$

Define the transformation, mapping H into itself, by:

$$Rf = g; \quad g(t) = \int_0^1 R(t;s)\, f(s)ds \qquad 0 \leq t \leq 1$$

Let ϕ_i denote the orthonormalized eigen-functions of R, and it is an easy matter to see that R is trace-class, and that

$$\int_0^1 [\int_0^t L(t;s) y_m(s;\omega)ds; y_m(t;\omega)]dt = \sum_i \sum_j a_{ij} \zeta_i^m \zeta_j^m$$

where

$$\zeta_i^m = \int_0^1 [y_m(t;\omega), \phi_i(t)]dt$$

$$a_{ij} = [L \phi_i, \phi_j]$$

Let ζ_i be defined as before:

$$\zeta_i = \int_0^1 [\phi_i(t), dW(t;\omega)]$$

The $\phi_i(\cdot)$ being an orthonormal basis, we have, of course:

$$\Sigma[L\phi_k, \phi_k] = \frac{1}{2} \,\mathrm{Tr.}(L+L^*)$$

Next let us note that, because of the circumstance (2.6),

$$E(\zeta_i \zeta_j^m) = [R\phi_i, \phi_j]$$

as a ready calculation shows. Next let

$$\eta_m = \sum_i \sum_j a_{ij} \zeta_i^m \zeta_j^m$$

$$\eta_o = \sum_i \sum_j a_{ij} \zeta_i \zeta_j$$

Denoting by λ_i the eigenvalues corresponding to ϕ_i, we can readily calculate that

$$E((\eta_m - \eta_o)^2) = (\Sigma\, a_{ii}(1-\lambda_i))^2 + 2 \sum_i \sum_j (b_{ij})^2 (1-\lambda_i \lambda_j)$$

where

$$b_{ij} = (a_{ij} + a_{ji})/2$$

If we denote by k(t;s) the kernel corresponding to (L+L*) we note that

$$\Sigma\, a_{ii}(1-\lambda_i) = \frac{1}{2}\, \text{Tr.}(L+L^*) - \int_0^1 \int_0^1 \text{Tr. } k(t;s)\, R(s;t)\, ds\, dt$$

$$= \int_0^1 \text{Tr. } L(t;t)\, dt - \int_0^1 \int_0^1 \text{Tr. } L(t;s)\, \frac{\sin 2\pi m(t-s)}{\pi(t-s)}\, ds\, dt$$

Again

$$(b_{ij})^2 \lambda_i \lambda_j = \sum_i \sum_j [\tfrac{1}{2}(L+L^*) R \phi_i, \phi_j] \cdot [R \tfrac{1}{2}(L+L^*) \phi_i, \phi_j]$$

$$= [\tfrac{1}{2}(L+L^*) R, R \tfrac{1}{2}(L+L^*)]_{H-S}$$

(H-S denoting the inner product in the space of H-S operators)

But now it is standard analysis to show that

$$\lim_m \int_0^1 \int_0^1 Tr.\ L(t;s) \frac{\sin 2\pi m(t-s)}{(t-s)} \, ds \, dt$$

$$\longrightarrow \int_0^1 Tr.\ L(t;t) dt$$

$$\lim_m [\tfrac{1}{2}(L+L^*) R, R \tfrac{1}{2}(L+L^*)]_{H-S}$$

$$\longrightarrow [\tfrac{1}{2}(L+L^*), \tfrac{1}{2}(L+L^*)]_{H-S} = \sum_i \sum_j b_{ij}^2$$

Hence (2.3) has been proven for the case where the kernel $L(t;s)$ is continuous. If $L(t;s)$ is not continuous then we can use the approximation [as in [5] for example)

$$L_h(t;s) = \frac{1}{4h^2} \int_{t-h}^{t+h} \int_{s-h}^{s+h} L(u;v) \, du \, dv$$

and apply the theorem for each h, sufficiently small, and then as h goes to zero exploit the fact that

$$Tr.(L+L^*) = \lim_h Tr.(L_h+L_h^*)$$

$$\|L_h - L\|^2_{H-S} \longrightarrow 0$$

An Application:

We shall now indicate one application of Theorem 2.1, which was in fact the motivation for the present work. Consider the linear stochastic differential system:

$$x(t;\omega) = \int_0^t Ax(s;\omega)ds + BW(t;\omega) \qquad 0 \le t \le 1 \tag{2.7}$$

$$Y(t;\omega) = \int_0^t Cx(s;\omega)ds + DW(t;\omega) \qquad -\infty < t < \infty \tag{2.8}$$

$$x(t;\omega) = 0, \ t \le 0; \ DD^* = \text{Identity matrix}; \ BD^* = 0$$

$W(t;\omega)$ a standard Wiener process as before, with $W(0;\omega)$ zero. We know that the process $Y(t;\omega)$, $0 \le t \le 1$, induces a probability measure on the Banach space $C(0,1)$ which is absolutely continuous with respect to the Wiener measure induced by $W(t;\omega)$ $0 \le t \le 1$. Moreover the R-N derivative is given by:

$$\exp -\frac{1}{2}\left\{\int_0^1 [C\hat{x}(s;\omega),\ C\hat{x}(s;\omega)]ds - 2\int_0^1 [C\hat{x}(s;\omega),\ dY(s;\omega)]\right\} \tag{2.9}$$

where

$$\hat{x}(t;\omega) = \int_0^t \phi(t)\, \phi(s)^{-1}\, P(s)C^*\, dY(s;\omega)$$

where $\phi(t)$ is a fundamental matrix solution of

$$\dot{\phi}(t) = (A - P(t)C^*C)\, \phi(t)$$

and $P(t)$ is the non-negative definite solut

$$\dot{P}(t) = AP(t) + P(t)A^* + BB^* - P(t)C^*CP(t); \quad P(0) = 0$$

Unfortunately, what is observed in practice is not (2.8), but a band-limited version, albeit of large enough band-width to allow the use (in theory) of (2.8). The main question then is the approximation of the Ito integral in (2.9). Here we can use Theorem 2.1 to state:

<u>Theorem 2.2</u> Let $M(\cdot)$ be as in Theorem 2.1, and define:

$$y(t;\omega) = \int_{-\infty}^{\infty} M(t-s)\, dY(s;\omega) \qquad 0 \le t \le 1$$

Then the Ito integral in (2.9) can be approximated:

$$\int_0^1 [C\hat{x}(s;\omega),\ dY(s;\omega)] = \lim \int_0^1 [\int_0^t CL(t;s)y(s;\omega)ds, y(t;\omega)]dt$$

$$- \int_0^1 Tr.\ CP(t)C^*dt \qquad (2.10)$$

where

$$L(t;s) = \Phi(t)\ \Phi(s)^{-1}\ P(s)C^*$$

<u>Proof</u> We note first of all that we can write:

$$y(t;\omega) = Cx_m(t;\omega) + z(t;\omega)$$

$$Cx_m(t;\omega) = \int_{-\infty}^{\infty} M(t-s)\ Cx(s;\omega)ds = \int_0^1 M(t-s)Cx(s;\omega)ds$$

$$z(t;\omega) = \int_{-\infty}^{\infty} M(t-s)\ D\ dW(s;\omega).$$

we note that using Theorem 2.1, and the fact the operator:

$$Lf = g; \quad g(t) = \int_0^t CL(t;s)f(s)ds \qquad 0 \le t \le 1$$

is such that $(L+L^*)$ is trace-class (see [4] for a proof if necessary),

$$\int_0^1 [\int_0^t CL(t;s)\, z(s;\omega)ds,\ z(t;\omega)]dt \to \int_0^1 [\int_0^t CL(t;s)DdW(s;\omega),\ DdW(t;\omega)]$$

$$- \int_0^1 \text{Tr. } CP(t)C^*dt$$

The theorem is thus proved if we can show that

$$\int_0^1 [\int_0^t CL(t;s)Cx(s;\omega)ds,\ z(t;\omega)]dt \to \int_0^1 [\int_0^t CL(t;s)Cx(s;\omega),\ DdW(t;\omega)] \quad (2.11)$$

and that

$$\int_0^1 [\int_0^t CL(t;s)Cx_m(s;\omega),\ Cx_m(t;\omega)]dt \to \int_0^1 [\int_0^t CL(t;s)Cx(s;\omega)ds,\ Cx(t;\omega)]dt$$

(2.12)

Because random variables are involved, we shall proceed to prove this in some detail. Let

$$\Psi(f;\omega) = \int_0^1 e^{2\pi ifs}\, x(s;\omega)ds$$

Then

$$Cx_m(t;\omega) = \int_{-m}^m e^{2\pi ift}\, \Psi(f;\omega)df$$

Since $x(t;\omega)$ is continuous in t, omitting a set of measure zero, we note that

$$\int_0^1 \|Cx_m(t;\omega) - Cx(t;\omega)\|^2 dt \leq \int_{|f|>m} \|\Psi(f;\omega)\|^2 df$$

and since

$$\int_0^1 \|Cx_m(t;\omega) - Cx(t;\omega)\|^2 dt \leq \int_{|f|>m} \|\Psi(t;\omega)\|^2 df$$

it follows that:

$$\int_0^1 E(\|Cx_m(t;\omega) - Cx(t;\omega)\|^2) dt \to 0 \text{ as } m \to \infty$$

This is clearly enough to establish (2.12). To handle (2.11), it is convenient first to integrate by parts. Thus

$$\int_0^1 [\int_0^t CL(t;s)Cx(s;\omega), DdW(t;\omega)] = \int_0^1 [\int_0^t \phi(s)^{-1} P(s)C^*Cx(s;\omega)ds,$$

$$\phi(t)^* C^* DdW(t;\omega)]$$

$$= [\int_0^1 \phi(s)^{-1} P(s)C^*Cx(s;\omega)ds, \int_0^1 \phi(s)^* C^* DdW(s;\omega)]$$

$$- \int_0^1 [\phi(t)^{-1} P(t)C^*Cx(t;\omega), \int_0^t \phi(s)^* C^* DdW(s;\omega)]$$

We perform a similar integration by parts on the left-member of (2.11), and establish the necessary convergence term by term.

For $0 \leq t \leq 1$, let

$$\eta(t;\omega) = \int_0^t \Phi(d)^* C^* \, z(s;\omega)ds - \int_0^t \Phi(s)^* C^* \, DdW(s;\omega)$$

$$= \int_{-\infty}^{\infty} (h_m(s) - h(s))DdW(s;\omega)$$

where

$$h(s) = \Phi(s)^* C^* \qquad 0 \leq s \leq t$$

$$= 0 \quad \text{otherwise}$$

$$h_m(s) = \int_0^t M(s-\sigma)\, h(\sigma)\, d\sigma$$

Then

$$E(\|\eta(t;\omega)\|^2) = \int_{-\infty}^{\infty} \|h_m(s) - h(s)\|^2 ds$$

and the integral on the right goes to zero as m goes to infinity. With this additional estimate, we can see that (2.11) follows.

3. Generalization

Let us now go on to consider the general case. In order to avoid notational complication we shall restrict ourselves to the case where $W(t;\omega)$, the standard Wiener process, is one dimensional. The extension to the multidimensional case can be made either using polynomials as in [6], or using tensor-product Hilbert spaces as in [7].

Suppose then we are given an Ito integral of the form:

$$\int_0^1 f(t;\omega)\, dW(t;\omega)$$

where

$$\int_0^1 E(|f(t;\omega)|^2)dt < \infty$$

Note that

$$\eta(\omega) = \int_0^1 f(t;\omega)\, dW(t;\omega)$$

defines a measurable, square integrable functional on the Wiener process $W(t;\omega)$, $0 < t < 1$, and as Ito has shown in [8], it can be approximated by sums of the form:

$$\sum_{1}^{n} \int_0^1 \cdots \int_0^1 K_p(t_1, \ldots t_p) dW(t_1;\omega) \ldots dW(t_p;\omega)$$

where each term is an Ito multiple integral. For example the 'linear' case of section 2:

$$\int_0^1 \left(\int_0^t L(t;s) dW(s;\omega) \right) dW(t;\omega)$$

can be expressed using the convention that $L(t;s) = 0$ for $s > t$:

$$\int_0^1 \int_0^1 \frac{L(t;s) + L(s;t)}{2} \, dW(s;\omega) dW(t;\omega)$$

Hence our main result can be stated:

<u>Theorem 3.1</u> Let $K(t_1, \ldots t_p)$ be a continuous symmetric (real-valued) function on $0 \le t_i \le 1$, $i = 1..p$. For each integer ν, $2\nu \le p$, and each fixed function $h(t_1, \ldots t_{p-2\nu})$ continuous in $0 \le t_i \le 1$, $i = 1,..p-2\nu$, define the operator L_ν by:

$$L_\nu f = g; \; g(t_1, \ldots t_\nu)$$
$$= \int_0^1 \ldots \int_0^1 K(t_1, \ldots t_\nu, s_1 \ldots s_\nu, \tau_{2\nu+1} \ldots \tau_p) h(\tau_{2\nu+1} \ldots \tau_p) d\tau_{2\nu+1} \ldots d\tau_p$$
$$f(s_1, \ldots s_\nu) ds_1 \ldots ds_\nu$$

mapping $L_2((0,1)^\nu)$ into itself. Suppose L_ν is trace-class for each ν and each arbitrary chosen h(...). Then the Ito integral:

$$\int_0^1 \cdots \int_0^1 K(t_1, \ldots t_p) dW(t_1;\omega) \ldots dW(t_p;\omega)$$

$$= \lim \sum_{\nu=0}^{[p/2]} \frac{p!(-1)^\nu}{(p-2\nu)!2^\nu \nu!} \int_0^1 \cdots \int_0^1 K(\sigma_1, \sigma_1, \sigma_2, \ldots \sigma_\nu, \sigma_\nu, t_{2\nu+1}, \ldots t_p) . d\sigma_1 \ldots d\sigma_\nu$$

$$y(t_{2\nu+1};\omega) \ldots y(t_p;\omega) dt_{2\nu+1} \ldots dt_p \tag{3.1}$$

where

$$y(t;\omega) = \int_{-\infty}^{\infty} (\sin 2\pi (t-s))/\pi(t-s) \cdot dW(s;\omega)$$

the limit being taken in the L_2 norm, and $[c]$ denotes largest integer $\leq c$.

<u>Proof</u> To clarify the notation in (3.1), let us look at (3.1) for the case $p = 2$. We have:

$$\int_0^1 \int_0^1 K(t_1, t_2)\, dW(t_1;\omega)\, dW(t_2;\omega)$$

$$= \lim \int_0^1 \int_0^1 K(t_1, t_2)\ \ y(t_1;\omega)\ \ y(t_2;\omega)\ \ dt_1\, dt_2$$

$$- \int_0^1 K(s, s) ds$$

But this is a special case of Theorem 2.1. For, the Ito double integral is given by the sum of the integrated integrals:

$$\int_0^1\int_0^t K(t;s)dW(s;\omega)\, dW(t;\omega) + \int_0^1\int_0^t K(s;t)\, dW(s;\omega)\, dW(t;\omega)$$

which, since the kernel is symmetric

$$= 2\int_0^1\int_0^t K(t;s)dW(s;\omega)dW(t;\omega)$$

and this in turn, by Theorem 2.1 is

$$= \lim 2\int_0^1\int_0^t K(t;s)\, y(s;\omega)ds\, y(t;\omega)dt - \int_0^1 K(t;t)dt$$

$$= \lim \int_0^1\int_0^1 K(t;s)\, y(s;\omega)\, y(t;\omega)ds\, dt - \int_0^1 K(t;t)dt$$

Illustrating the case when p is odd, let us calculate (3.1) for $p = 3$:

$$\int_0^1\int_0^1\int_0^1 K(t_1, t_2, t_3)dW(t_1, \omega)\, dW(t_2;\omega)dW(t_3;\omega)$$

$$= \lim \int_0^1\int_0^1\int_0^1 K(t_1, t_2, t_3)\, y(t_1;\omega)\, y(t_2;\omega)y(t_3;\omega)dt_1\, dt_2\, dt_3$$

$$- 3\int_0^1\int_0^1 K(s;s;t)ds\, y(t;\omega)\, dt$$

The main tool we shall use in the proof is the decomposition formula for multiple Ito integrals. Let $\phi_i(\cdot)$ denote an orthonormal basis in $L_2(0,1)$. Let us use the notation:

$$I_n(K(t_1, \ldots t_n))$$

for the associated Ito multiple integral. Then we have using the Ito decomposition formula (Cf. [8]):

$$I_p(\phi_{i_1}(t_1)\,\phi_{i_2}(t_2)\ldots\phi_{i_p}(t_p))$$

$$= I_{p-1}(\phi_{i_1}(t_1)\ldots\phi_{i_{p-1}}(t_{p-1}))\,\zeta_{i_p}(\omega)$$

$$- \sum_{k=1}^{p-1} I_{p-2}(\phi_{i_1}(t_1)\ldots\phi_{i_{k-1}}(t_{k-1})\,\phi_{i_{k+1}}\ldots\phi_{i_{p-1}}(t_{p-1}))\,[\phi_{i_k}\cdot\phi_{i_p}] \tag{3.2}$$

where

$$\zeta_j(\omega) = \int_0^1 \phi_j(t)\, dW(t;\omega)$$

Next let us note that

$$\phi_{i_1}(t_1)\ldots\phi_{i_\nu}(t_\nu)$$

is an orthonormal basis for $L_2((0,1)^{\nu})$, $\nu \leq p$, and in particular

$$K(t_1, \ldots t_p) = \sum_{i_1} \ldots \sum_{i_p} a_{i_1 i_2 \ldots i_p} \phi_{i_1}(t_1) \ldots \phi_{i_p}(t_p) \tag{3.3}$$

the series converging in $L_2((0,1)^p)$, where the Fourier coefficients are also symmetric in the variables. Because of the trace-class condition on the operators L_ν it is readily seen that

$$\sum_{i_1} \ldots \sum_{i_\nu} |a_{i_1 i_1 i_2 i_2 \ldots i_\nu i_\nu i_{2\nu+1} \ldots i_p}| < \infty \tag{3.4}$$

for each fixed set of indices $i_{2\nu+1}, \ldots i_p$, and every ν, $2\nu \leq p$. Because of (3.3) we have that the Ito integral in (3.1) is the limit in the L_2 norm of the series:

$$\sum_{i_1} \ldots \sum_{i_p} a_{i_1 i_2 i_3 \ldots i_p} I_p(\phi_{i_1}(t_1) \ldots \phi_{i_p}(t_p)) \tag{3.5}$$

This series we shall now show is expressible as

$$\sum_{\nu=0}^{[p/2]} \frac{p!(-1)^{\nu}}{(p-2\nu)!2^{\nu}\nu!} \sum_{i_1} \ldots \sum_{i_\nu} \sum_{i_{2\nu+1}} \ldots \sum_{i_p} a_{i_1 i_1 i_2 i_2 \ldots i_{2\nu+1} \ldots i_p} \zeta_{i_{2\nu+1}}(\omega) \ldots \zeta_{i_p}(\omega) \quad \ldots \tag{3.6}$$

We can prove this by induction using (3.2). (The coefficients are obtained by the same combinatorial argument as in Wiener [9]).

Thus, substituting in (3.5) from (3.2), and assuming the result true for integers less than p, we have:

$$\sum_{i_p} \cdot\cdot \sum_{i_p} a_{i_1 \cdot\cdot i_p} I_p(\phi_{i_1}(t_1) \cdot\cdot \phi_{i_p}(t_p))$$

$$= \sum_{0}^{[\frac{p-1}{2}]} \frac{(p-1)!(-1)^{\nu}}{(p-1-2\nu)!2^{\nu}\nu!} \sum_{i_1} \cdot\cdot \sum_{i_\nu} \cdot\cdot \sum_{i_p} a_{i_1 i_1 \cdot\cdot i_\nu i_\nu i_{2\nu+1} \cdot\cdot i_p} \zeta_{2\nu+1}(\omega) \cdot\cdot \zeta_p(\omega)$$

$$-(p-1) \sum_{0}^{[\frac{p-2}{2}]} \frac{(p-2)!(-1)^{\nu}}{(p-2-2\nu)!2^{\nu}\nu!} \sum_{i_1} \cdot\cdot \sum_{j_1} \cdot\cdot \sum_{j_{p-2}} a_{i_1 i_1 j_1 j_1 \cdot\cdot j_\nu j_\nu j_{2\nu+1} \cdot\cdot j_{p-2}} \cdot$$

$$\cdot \zeta_{2\nu+1}(\omega) \cdot\cdot \zeta_{p-2}(\omega)$$

And combining the two sums, taking the ν sum in the first term with the $(\nu-1)$ sum in the second term for $\nu \geq 1$, and noting that

$$\frac{(p-1)!}{(p-1-2\nu)!\,2^{\nu}\nu!} + \frac{(p-1)\,(p-2)!}{(p-2-2(\nu-1))!\,2^{\nu-1}(\nu-1)!}$$

$$= \frac{p!}{(p-2\nu)!\,2^{\nu}\nu!}$$

we have (3.6). Since we have already proved the result for $p = 2$, the induction is complete. Next for each m, let ϕ_i be the ortho-normalized eigenfunctions of the operator R defined by:

$$Rf = g; \quad g(t) = \int_0^1 (\sin 2\pi(t-s))/\pi(t-s)ds$$

mapping $L_2(0,1)$ into itself, and letting

$$\zeta_i^m(\omega) = \int_0^1 \phi_i(t)\, y(t;\omega)dt$$

we note that:

$$\int_0^1 \cdot\cdot \int_0^1 K(t_1, t_1, \ldots t_\nu, t_\nu, t_{2\nu+1}, \ldots, t_p)\, \phi_{i_{2\nu+1}}(t_{2\nu+1}) \cdot\cdot \phi_{i_p}(t_p)$$

$$dt_1 \cdot\cdot dt_\nu\, dt_{2\nu+1} \cdot\cdot dt_p$$

$$= \sum_{i_1} \cdot\cdot \sum_{i_\nu} a_{i_1 i_1 \cdot\cdot i_\nu i_\nu i_{2\nu+1} \cdot\cdot i_p}$$

by virtue of the trace-class assumption of L_ν. Hence we can readily see that for fixed m, the right-side of (3.1) is given by:

$$\sum_0^{[\frac{p}{2}]} \frac{p!(-1)^\nu}{(p-2\nu)!\, 2^\nu \nu!} \sum_{i_1} \cdot\cdot \sum_{i_\nu} \sum_{i_{2\nu+1}} \cdot\cdot \sum_{i_p} a_{i_1 i_1 \cdot\cdot i_\nu i_\nu i_{2\nu+1} \cdot\cdot i_p}\, \zeta^m_{i_{2\nu+1}}(\omega) \cdot\cdot \zeta^m_{i_p}(\omega)$$

... (3.7)

which is the same as (3.6) except for replacing $\zeta_i(\omega)$ by $\zeta_i^m(\omega)$. It only remains to show that as m goes to infinity, (3.7) converges to (3.6), to conclude the proof of (3.1). Here we can again exploit the fact that

$$E(\zeta_i^m(\omega)\,\zeta_j(\omega)) = [R\phi_i,\ \phi_j]$$

The proof is similar to the one we used in Theorem 2.1, only more tedious. Let us first note that the difference of the $\nu = 0$ sums in (3.6) and (3.7) is:

$$\sum_{i_1}\cdots\sum_{i_p} a_{i_1 i_2 \cdots i_p}(\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_p} - \zeta_{i_1}^m\zeta_{i_2}^m\cdots\zeta_{i_p}^m)$$

and we can readily calculate that the expected value of the square of this is given by:

$$p!\sum_{i_1}\cdots\sum_{i_p}(a_{i_1\cdots i_p})^2(1-\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_p})$$

$$+\sum_{1}^{[p/2]}\frac{p!}{2^\nu \nu!}\sum_{i_{2\nu+1}}\cdots\sum_{i_p}\Big(\sum_{i_1}\cdots\sum_{i_\nu} a_{i_1 i_1\cdots i_\nu i_\nu i_{2\nu+1}\cdots i_p}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_\nu}\Big)^2\cdot$$

$$\cdot\lambda_{i_{2\nu+1}}\cdots\lambda_{i_p}$$

$$-\sum_{1}^{[p/2]}\frac{p!}{2^\nu \nu!}\sum_{i_{2\nu+1}}\cdots\sum_{i_p}\Big(\sum_{i_1}\cdots\sum_{i_\nu} a_{i_1 i_1\cdots i_\nu i_\nu i_{2\nu+1}\cdots i_p}\Big)^2 \qquad (3.8)$$

We can now proceed by induction. If we assume the result to be true integers less than p, then to prove it for p we only need to show that (3.8) goes to zero. But this is readily done analagous to the case $p = 2$. Thus the first term in (3.8) can be expressed:

$$p!([K_p, K_p] - [R_p K_p, K_p]) \tag{3.9}$$

where we denote by R_ν the operator:

$$R_\nu f = g; \; g(t_1, \ldots t_\nu) = \int_0^1 \ldots \int_0^1 \frac{\sin 2\pi m(t_1 - s_1)}{\pi(t_1 - s_1)} \ldots$$

$$\ldots \frac{\sin 2\pi m(t_\nu - s_\nu)}{\pi(t_\nu - s_\nu)} f(s_1 \ldots s_\nu)\, ds_1 \ldots ds_\nu$$

mapping $L_2((0,1)^\nu)$ into itself. By K_p we mean the function $K(t_1, \ldots t_p)$ as an element of $L_2((0,1)^p)$. Clearly (3.9) goes to zero as m goes to infinity. Next let us look at the second term. Note that we can write

$$\sum_{i_1} \ldots \sum_{i_\nu} a_{i_1 i_1 \ldots i_\nu i_\nu i_{2\nu+1} \ldots i_p} \lambda_{i_1} \ldots \lambda_{i_\nu}$$

$$= \text{Trace } R_\nu L_\nu$$

where L_ν is defined by:

$$L_\nu f = g; \quad g(t_1 \ldots t_\nu)$$

$$= \int_0^1 \ldots \int_0^1 \int_0^1 \ldots \int_0^1 K(t_1, \ldots t_\nu, s_1, \ldots s_\nu, \sigma_{2\nu+1}, \ldots \sigma_p)\, \phi_{i_{2\nu+1}}(\sigma_{2\nu+1}) \cdot$$

$$\ldots \phi_{i_p}(\sigma_p) d\sigma_{2\nu+1} \ldots d\sigma_p \; f(s_1, \ldots s_\nu) ds_1 \ldots ds_\nu$$

mapping $L_2((0,1)^\nu)$ into itself. And hence

$$\sum_{i_{2\nu+1}} \ldots \sum_{i_p} \Big(\sum_{i_1} \ldots \sum_{i_\nu} a_{i_1 i_1 \ldots i_\nu i_\nu i_{2\nu+1} \ldots i_p} \lambda_{i_1} \ldots \lambda_{i_\nu} \Big)^2 \lambda_{i_{2\nu+1}} \ldots \lambda_{i_p}$$

$$\int_0^1 \ldots \int_0^1 \Big(\int_0^1 \!\! \cdot \!\! \int_0^1 \frac{\sin 2\pi m(t_1 - s_1)}{\pi(t_1 - s_1)} \ldots \frac{\sin 2\pi m(t_\nu - s_\nu)}{\pi(t_\nu - s_\nu)} K(s_1, \ldots s_\nu, t_1 \ldots t_\nu\; \sigma_{2\nu+1} \ldots \sigma_p) \cdot$$

$$\cdot \quad ds_1 \ldots ds_\nu \ldots dt_1 \ldots dt_\nu \Big)^2 d\sigma_{2\nu+1} \ldots d\sigma_p$$

which as m goes to infinity clearly goes to

$$\int_0^1 \cdots \int_0^1 \left(\int_0^1 \cdots \int_0^1 K(t_1, \ldots t_\nu, t_1, \ldots t_\nu, \sigma_{2\nu+1}, \ldots \sigma_p) dt_1 \ldots dt_\nu \right)^2 d\sigma_{2\nu+1} \ldots d\sigma_p$$

$$= \sum_{i_{2\nu+1}} \cdots \sum_{i_p} \left(\sum_{i_1} \cdots \sum_{i_\nu} a_{i_1 i_1 \ldots i_\nu i_\nu i_{2\nu+1} \ldots i_p} \right)^2$$

Hence (3.8) goes to zero, concluding the proof of the Theorem.

Finally let us remove the condition of continuity on the kernel.

Corollary Suppose $K(t_1, \ldots t_p)$ is symmetric and $\epsilon\ L_2((0,1)^p)$. Suppose further that the operator: L_ν defined by:

$$L_\nu f = g; \quad g(t_1, \ldots t_\nu) = \int_0^1 \cdots \int_0^1 K(t_1, \ldots t_\nu, s_1, \ldots s_\nu, \sigma_{2\nu+1}, \ldots \sigma_p) \cdot f(s_1, \ldots s_\nu)\, ds_1 \ldots ds_\nu$$

mapping $L_2[(0,1)^\nu]$ into itself is trace-class a.e., in the variables σ_i, $2\nu+1 \leq i \leq p$, and

$$\int_0^1 \cdots \int_0^1 (\mathrm{Tr.}\ L_\nu(\sigma_{2\nu+1}, \ldots \sigma_p))^2\, d\sigma_{2\nu+1} \ldots d\sigma_p < \infty$$

for each ν, $2\nu \leq p$. Then the Ito integral:

$$\int_0^1 \cdots \int_0^1 K(t_1, \ldots t_p) dW(t_1, \omega) \ldots dW(t_p, \omega)$$

$$= \lim \sum_{\nu=0}^{[p/2]} \frac{p!(-1)^{\nu}}{(p-2\nu)!2^{\nu}\nu!} \int_0^1 \cdots \int_0^1 \mathrm{Tr}.L_{\nu}(t_{2\nu+1},\ldots t_p)y(t_{2\nu+1},\omega)\cdot$$

$$\cdots y(t_p,\omega)dt_{2\nu+1}\cdots dt_p] \tag{3.10}$$

Proof For each h, the kernel

$$K_h(t_1,\ldots t_p) = (1/(2h)^p) \int_{t_1-h}^{t_1+h} \cdots \int_{t_p-h}^{t_p+h} K(s_1,\ldots s_p)ds_1 \cdots ds_p$$

satisfies the conditions of the theorem, and hence we can obtain (3.1) for $K_h(\ldots)$. Because of the trace-class conditions imposed on $K(\ldots)$, we may proceed to take limits on both sides and obtain (3.10) as required.

REFERENCES

1. E. Wong and M. Zakai: On the Relation Between Ordinary Integrals and Stochastic Integrals, International Journal of Engineering Science, Vol. 3, pp. 213-229, 1965.

2. E. McShane: Stochastic Functional Equations: Continuity Properties and Relation to Ordinary Equations, in Calculus of Variations and Control Theory, Academic Press, 1969.

3. E. P. McKean: Stochastic Integrals, Academic Press, 1969.

4. A. V. Balakrishnan: Stochastic Differential Systems, Springer-Verlag, 1973.

5. I. C. Gohberg and M. Krein: Introduction to the Theory of Non-Selfadjoint Operators. Vol. 18, American Mathematical Society Translations, 1969.

6. A. V. Balakrishnan: Stochastic Control - A Function Space Approach, SIAM Journal on Control, Vol. 10, pp. 285-297, May 1972.

7. G. Kallianpur: "The Role of Reproducing Kernel Hilbert Spaces in the Study of Gaussian Processes", in "Advances in Probability", Marcel-Dekker, 1972.

8. K. Ito: Multiple Wiener Integral, Journal of the Mathematical Society of Japan, Vol. 3, No. 1, May 1951.

9. N. Wiener: Nonlinear Problems in Random Theory, Technology Press, Massachusetts Institute of Technology, 1958.