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POSED ERLANG RENEWAL PROCESSES**

P. A. W. Lewis, et al

**Naval Postgraduate School
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by

P. A. W. Lewis
R. D. Haskell
W. J. Hayne
R. D. Rantschler
J. Y. Schrader, Jr.
J. N. Swan

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P. A. W. Lewis R. D. Rantschler
R. D. Haskell J. Y. Schrader, Jr.
W. J. Hayne J. N. Swan

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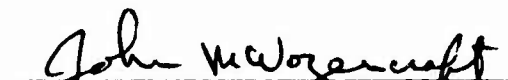
Prepared by:



P. A. W. Lewis, Professor
Department of Operations Research
and Administrative Sciences

Approved by:

Released by:


J. R. Borsting, Chairman
Department of Operations Research
and Administrative Sciences
J. M. Wozencraft
Dean of Research

I. Introduction.

Much work has been done recently in the area of the superposition of independent stationary point processes, extending earlier results summarized in Cox and Lewis (1966, Ch. 8) and Cinlar (1972). Various approaches have been utilized in deriving the probabilistic properties of the superposition process, including the development of expressions for the description of counts of events occurring in some interval and for the description of the properties of intervals between events starting at an "arbitrary" event. Following Lawrence (1971) we call these the synchronous intervals of the process.

The present paper considers the problem of describing the second-order joint moment structure of the synchronous intervals for the special case of the superposition of identical Erlang renewal processes.

Notationally, let p be the number of component processes to be superposed and let each component process be an Erlang renewal process, i.e. have independently distributed intervals between events, each interval having the same Erlang probability density function

$$g(x) = \left(\frac{k}{\mu}\right)^k \frac{x^{k-1} e^{-kx/\mu}}{\Gamma(k)} \quad (x \geq 0). \quad (1.1)$$

The superposition of p of these independent processes gives the superposed process to be studied. We use the description "Erlang process" rather than "Gamma process" to indicate that the shape parameter k in (1.1) is an integer.

Downton (1972) considered the problem of deriving the distributions of synchronous intervals in this superposition process and utilized

both an imbedded Markov property and an integral equation approach to develop limiting forms for the distributions, i.e. for large times of observation and large numbers of superpositions. Downton was concerned with applications to reliability studies; note that the superposed process is the overall failure pattern for a system consisting of p renewable components operating in series, i.e. the system fails whenever a component fails and is renewed.

Lawrance (1973), amongst other things, extended Downton's results and showed that the synchronous intervals between events in the superposed process are correlated. He calculated the first two serial correlation coefficients for specific cases of superposed Erlang renewal processes, and obtained joint and univariate distributions of synchronous intervals.

Barnett (1970) used simulation to investigate some of the correlational properties of superposed processes in the specific area of "studying the supply pattern of goods delivered to a supply depot by a fleet of vehicles." His processes may be considered to be Erlang processes with large values of k .

Here we develop somewhat more general expressions than those given previously for the serial correlations and the spectrum of synchronous intervals in superposed Erlang renewal processes. Specific forms for the spectra are given for a few cases. A general expression for the generating function of the distribution of counts is used to compute the spectrum of intervals and serial correlations in other cases. Some general conclusions about the shape of the spectrum of the intervals in the superposed process are given; these utilize results of Enns (1970) to the

effect that if intervals in the component processes have increasing failure rates (decreasing failure rates), then so do the marginal synchronous intervals in the superposed process. Enns (1970) also discussed serial correlations and tried to relate it to the properties of the failure rates of the intervals in the component processes.

2. Preliminaries.

Cox and Lewis (1966, Ch. 4) used the fundamental identity of point processes,

$$(N(t) < k) \Leftrightarrow (S_k > t) \quad (k = 1, 2, \dots) \quad (2.1)$$

and Palm-Khinchine formulae to relate the generating function of the asynchronous counts, $\phi(\xi, t) = E\{\xi^{N(t)}\}$, to the spectrum of synchronous intervals $\{X_i\}$ in a stationary point process. The development leans heavily on Laplace transforms; for (1.1) we have

$$\begin{aligned} g(s) &= \int_{0-}^{\infty} g(x) e^{-sx} dx = \left(\frac{k/\mu}{s + k/\mu} \right)^k \quad (\operatorname{Re}(s) \geq 0) \\ &= \left(\frac{\mu}{k} s + 1 \right)^{-k}, \end{aligned} \quad (2.2)$$

which is a rational function in s .

Note that the parametrization for (2.2) gives an expected value of μ ; the intervals X can be construed to be the sum of k independent exponentially distributed variables, each with mean $\mu/k = \beta$.

Denoting the Laplace transform of the generating function for counts in an interval $(0, t]$ following an arbitrary event (synchronous case) in the point process by $\phi_f^*(\xi; s)$, we have for a renewal process (Cox, 1962, p. 37)

$$\phi_f^*(\xi; s) = \frac{1 - g^*(s)}{s[1 - \{g^*(s)\}]} \quad (2.3)$$

This expression is simple looking but in general difficult to invert.

Using $\phi_f^*(\xi, s)$ the Palm-Khinchine formulae (Cox and Lewis, 1966, Ch. 4) give the Laplace transform $\phi^*(\xi; s)$ of the generating function for the

number of events in $(x, x+t)$, where x is an arbitrary time in a stationary process (i.e. the asynchronous case). The relationship is (Cox and Lewis, 1966, Ch. 4, p. 68)

$$\phi^*(\xi; s) = \frac{1}{s} + \frac{(\xi-1)}{\mu s} \phi^*(\xi; s). \quad (2.4)$$

If p renewal processes are superposed, we need to develop a general expression for the transform of the generating function of the asynchronous counts in the superposed process, denoted by $\phi^{*(p)}(\xi; s)$. This is because the spectrum can be obtained as (Cox and Lewis, 1966, p. 77)

$$\begin{aligned} f_+(\omega) &= \frac{1}{\pi} \left\{ 1 + 2 \sum_{k=1}^{\infty} \rho_k \cos(k\omega) \right\} \\ &= \frac{1}{\pi} \frac{\phi^*(e^{i\omega}; 0+) + \phi^*(e^{-i\omega}; 0+)}{2\phi^*(0, 0+) - \mu}, \end{aligned} \quad (2.5)$$

where the ρ_k 's are the serial correlation coefficients for synchronous intervals k lags apart, i.e. $\text{corr}\{X_1, X_{1+k}\}$, and the denominator is $\text{var}(X_1)/E(X_1) = \text{var}(X_1)/\mu$.

In principle it is easy, in the case of superposition of p identical processes, to obtain $\phi^{*(p)}(\xi; s)$ or $\phi^{*(p)}(\xi; t)$, since in the time domain

$$\begin{aligned} \phi^{(p)}(\xi; t) &= E\{\xi^{N^{(p)}}(t)\} = E\{\xi^{N_1(t) + \dots + N_p(t)}\} \\ &= \prod_{i=1}^p E\{\xi^{N_i(t)}\} \\ &= \{\phi(\xi; t)\}^p. \end{aligned} \quad (2.6)$$

However, if $\phi^*(\xi; s)$ is known, as for the renewal process, to obtain $\phi^{*(p)}(\xi; s)$ it is necessary to perform the operations (using 2.6)

$$\phi^*(\xi; s) \xrightarrow{L^{-1}} \phi(\xi; t) \rightarrow \phi^{(p)}(\xi; t) \xrightarrow{L} \phi^{*(p)}(\xi; s),$$

where L^{-1} denotes inverse Laplace transformation and L a Laplace transform. Obviously the procedure would be cumbersome for large p , even if $\phi^*(\xi; s)$ could be inverted to give $\phi(\xi; t)$.

Section 4 of this paper is concerned with the development of $\phi^{(p)}(\xi; t)$ or its transform in the special case of the superposition of independent, identical Erlang renewal processes, while Section 5 develops the spectral density, $f_+(\omega)$ for the superposed process.

Section 6 gives specific functional forms for $f_+(\omega)$ and the ρ_k 's in the cases of $k = 2$ and $p = 2, 3, 4$; the results are surprisingly simple. In Section 7 computational results for $f_+(\omega)$ are given, using the general results of Section 5, for $k = 2, 3$ and $p = 2, 3, \dots, 9$. These illustrate the general form of the spectrum of the superposed process. Some results for superpositions of non-identical Erlang processes are also given, corresponding to cases discussed by Lawrence (1973).

In Section 3 we review some results of Enns (1970) and show how they can be used to derive some properties of the spectrum of superposed renewal processes, not necessarily Erlang. All the results are used in Section 8 in the discussion of the general shape of the spectrum of intervals of superposed Erlang renewal processes.

3. Enn's results and the initial point on the spectrum.

It is clear that the marginal time between events in a stationary superposed process $X^{(p)}$ will have mean

$$E(X^{(p)}) = E(X)/p,$$

where $E(X)$ is the common mean time between events in the p component processes. Let the spectral density (2.5) of the synchronous intervals in the superposition process be denoted by $f_+^{(p)}(\omega)$ and the coefficient of variation squared by $C^2(X^{(p)}) = \text{var}(X^{(p)})/[E(X^{(p)})]^2$. Then we have the following relationship between the asymptotic slope of the variance time curve $V^{(p)}(t) = \text{var}\{N^{(p)}(t)\}$ and the initial point on the spectrum (Cox and Lewis, 1966, p. 78):

$$\lim_{t \rightarrow \infty} \frac{dV^{(p)}(t)}{dt} = \pi \frac{C^2(X^{(p)})}{E(X^{(p)})} f_+^{(p)}(0+). \quad (3.1)$$

Now $V^{(p)}(t) = pV(t)$, where $V(t)$ is the variance time curve for the component processes, and if the component processes are renewal processes it is well known (see Cox and Lewis, 1966, p. 31) that

$$\lim_{t \rightarrow \infty} V'(t) = \lim_{t \rightarrow \infty} \frac{dV(t)}{dt} = \frac{C^2(X)}{E(X)}.$$

Putting these results together we get, for superposed renewal processes,

$$\begin{aligned} f_+^{(p)}(0+) &= p \frac{V'(\infty)E(X^{(p)})}{\pi C(X^{(p)})} \\ &= \frac{p C^2(X)}{\pi E(X)} \times \frac{E(X)}{p C^2(X^{(p)})} = \frac{C^2(X)}{\pi C^2(X^{(p)})} \end{aligned} \quad (3.2)$$

There are two consequences of this result.

Enns (1970) showed that if X has an increasing hazard rate distribution (IHR), then so does $X^{(p)}$, and if X has a decreasing failure rate distribution (DFR), so does $X^{(p)}$. Thus, using results of Barlow and Proschan (1964) we have that

$$C^2(X^{(p)}) \leq 1 \quad \text{if } X \text{ is IHR,}$$

$$C^2(X^{(p)}) \geq 1 \quad \text{if } X \text{ is DFR.}$$

Consequently

$$\frac{1}{\pi C^2(X^{(p)})} \geq f_+^{(p)}(0+) \geq \frac{C^2(X)}{\pi} \quad \text{if } X \text{ is IHR,}$$

$$\frac{1}{\pi C^2(X^{(p)})} \leq f_+^{(p)}(0+) \leq \frac{C^2(X)}{\pi} \quad \text{if } X \text{ is DFR.}$$

Note that for the Erlang density function (1.1), X is IHR if $k > 1$ and DFR if $k < 1$, and $C^2(X) = \frac{1}{k}$. When $k = 1$ we have an exponential density and a Poisson process for the component processes and the superposition process; the spectrum then has the constant value $1/\pi$ and $f_+(0+) = f_+^{(p)}(0+) = C^2(X)/\pi = 1/\pi$.

The second consequence of (3.2) is that in the limit as $p \rightarrow \infty$, when the superposed process goes to a Poisson process and $C(X^{(p)}) \rightarrow 1$, the spectrum of intervals is tied down at $\omega = 0$:

$$\lim_{p \rightarrow \infty} f_+^{(p)}(0+) = \frac{C^2(X)}{\pi}.$$

For the Erlang process this value is $1/(\pi k)$. Thus the spectrum, which is converging to the flat spectrum of the limiting Poisson process, has a "notch" in it at ω near zero. This notch corresponds to the fact that if we could observe a superposed process well past the mean of

intervals in the component processes, the variance time curve would not be equal to $t/E(X)$, i.e. would depart from linearity. In other words low frequency effects and long time effects correspond.

It is not known if $f_+^{(p)}(0+)$ goes monotonically from $1/\pi$ at $p = 1$ to $1/(wk)$ as $p \rightarrow \infty$. Computational results are given in Section 7. It would also be interesting to know the derivative of $f_+^{(p)}(\omega)$ at $\omega = 0+$.

4. General results for the generating function for an Erlang renewal process.

For the Erlang renewal process, using (2.2) and (2.3) and letting $\lambda = k/\mu = 1/\beta$, we have

$$\begin{aligned}\phi_f^*(\xi; s) &= \frac{1 - (\frac{\lambda}{\lambda+s})^k}{s\{1 - \xi(\frac{\lambda}{\lambda+s})\}} \\ &= \frac{(\lambda+s)^k - \lambda^k}{s\{(\lambda+s)^k - \lambda^k\}},\end{aligned}\quad (4.1)$$

Hence, using (2.4),

$$\begin{aligned}\phi^*(\xi; s) &= \frac{1}{s} + \frac{\lambda(\xi-1)}{ks} \cdot \frac{(\lambda+s)^k - \lambda^k}{s\{(\lambda+s)^k - \xi\lambda^k\}} \\ &= \frac{(\frac{1}{s})\{(\lambda+s)^k - \xi\lambda^k\} + \{\frac{\lambda(\xi-1)}{ks^2}\}\{(\lambda+s)^k - \lambda^k\}}{(\lambda+s)^k - \xi\lambda^k}\end{aligned}\quad (4.2)$$

$$= \frac{P(s)}{Q(s)}, \quad (4.3)$$

where $P(s)$ and $Q(s)$ are polynomials in s .

By expanding and collecting terms in $P(s)$ one finds that

$$\begin{aligned}P(s) &= \frac{1}{s} Q(s) + \frac{\lambda(\xi-1)}{ks^2} \{Q(s) + \lambda^k(\xi-1)\} \\ &= a_0 + a_1s + a_2s^2 + \dots + a_{k-2}s^{k-2} + a_{k-1}s^{k-1},\end{aligned}\quad (4.4)$$

so that $P(s)$ is a proper polynomial in s of order $k-1$, with

$$\begin{aligned}a_0 &= \lambda^{k-1}\{k + (\xi-1)(\frac{k-1}{2})\}, \\ a_1 &= \lambda^{k-2}\{\frac{k-1}{2}\}\{k + (\xi-1)(\frac{k-2}{3})\},\end{aligned}$$

$$a_2 = \lambda^{k-3} \left\{ \frac{(k-1)(k-2)}{2 \cdot 3} \right\} \left\{ k + (\xi-1) \left(\frac{k-3}{4} \right) \right\},$$

$$\vdots$$

$$a_{k-2} = \lambda \left\{ k + \frac{\xi-1}{k} \right\},$$

$$a_{k-1} = 1.$$

The first derivative of $Q(s)$ with respect to s is

$$Q'(s) = k(\lambda+s)^{k-1}. \quad (4.5)$$

Now let $\{\tau_1, \dots, \tau_k\}$ be the k distinct, complex-valued roots of ξ , i.e.,

$$\tau_j^k = \xi \quad (j = 1, \dots, k).$$

The roots s_j of $Q(s)$, i.e. $Q(s_j) = 0$, are clearly

$$s_j = \lambda(\tau_j - 1) \quad (j = 1, 2, \dots, k), \quad (4.6)$$

and these too, like τ_j , are distinct and complex valued.

We are now in a position to use a standard inversion formula (Gardner and Barnes, 1942, p. 338), for the ratio of two polynomials when the denominator has distinct roots:

$$L^{-1} \left\{ \frac{P(s)}{Q(s)} \right\} = \sum_{j=1}^k \frac{P(s_j)}{Q'(s_j)} e^{s_j t}.$$

We note that from (4.4) and (4.6)

$$P(s_j) = \frac{\lambda^{k-1}}{k} \left(\frac{\xi-1}{\tau_j-1} \right)^2 \quad (j = 1, \dots, k),$$

and from (4.5)

$$Q'(s_j) = k(\lambda+s_j)^{k-1} = \frac{k\lambda^{k-1}\xi}{\tau_j} \quad (j = 1, \dots, k). \quad (4.7)$$

Consequently, the probability generating function for the asynchronous number of counts, $N(t)$, in an Erlang renewal process is

$$\begin{aligned}
 \phi(\xi; t) &= L^{-1}\{\phi^*(\xi; s)\} \\
 &= \sum_{j=1}^k \frac{P(s_j)}{Q'(s_j)} e^{s_j t} \\
 &= \frac{(\xi-1)^2}{k^2 \xi} \sum_{j=1}^k \frac{\tau_j}{(\tau_j-1)^2} e^{\lambda(\tau_j-1)t}. \quad (4.8)
 \end{aligned}$$

We note too that the expansion for the variance-time curve of an Erlang renewal process obtained by Serfling (1970) is obtainable from (4.8).

5. General Results for the Spectrum of Counts for Pooled Independent Erlang Renewal Processes.

We consider here the case in which each of the p component Erlang renewal process have the same scale parameter λ and shape parameter k . The probability generating function for counts in the pooled process starting at an arbitrary time is, from (2.6),

$$\phi^{(p)}(\xi, t) = \{\phi(\xi, t)\}^p.$$

Let $J = \{j_1, j_2, \dots, j_p\}$, be an index set. Using the result (4.8) from Section 4,

$$\begin{aligned} \phi^{(p)}(\xi, t) &= \left\{ \frac{(\xi-1)^2}{k^2\xi} \sum_{j=1}^k \frac{\tau_j}{(\tau_j-1)^2} e^{\lambda(\tau_j-1)t} \right\}^p \\ &= \left\{ \frac{(\xi-1)^2}{k^2\xi} \right\}^p \sum_{j_1=1}^k \dots \sum_{j_p=1}^k \left\{ \prod_{j \in J} \frac{\tau_j}{(\tau_j-1)^2} \right\} e^{\lambda \left\{ \sum_{j \in J} (\tau_j-1) \right\} t}. \end{aligned} \quad (5.1)$$

Taking the Laplace transform of $\phi^{(p)}(\xi, t)$ we have

$$\phi^{*(p)}(\xi, s) = \left\{ \frac{(\xi-1)^2}{k^2\xi} \right\}^p \sum_{j_1=1}^k \dots \sum_{j_p=1}^k \left\{ \frac{\prod_{j \in J} \frac{\tau_j}{(\tau_j-1)}}{s - \lambda \sum_{j \in J} (\tau_j-1)} \right\}. \quad (5.2)$$

We note the following in (2.5):

$$(i) \quad \int_0^\infty f_+(\omega) d\omega \triangleq 1,$$

- (ii) the denominator in the second expression of (2.5) for $f_+(\omega)$ is constant with respect to ω .

Consequently, if we can evaluate $\phi^{*(p)}(e^{i\omega}, 0+) + \phi^{*(p)}(e^{-i\omega}, 0+)$ for $0 \leq \omega \leq \pi$, and integrate it, then (i) will give us the value of $\pi\{2\phi^{*(p)}(0, 0+) - E(X)\}$. Thus we will have $f_+^{(p)}(\omega)$ for $0 \leq \omega \leq \pi$, and also $\text{var}(X^{(p)})$. Now when $s = 0+$,

$$\phi^{*(p)}(\xi, 0+) = \frac{1}{\lambda} \left\{ \frac{(\xi-1)^2}{k^2 \xi} \right\}^p \sum_{j_1=1}^k \dots \sum_{j_p=1}^k \left\{ \frac{\prod_{j \in J} \frac{\tau_j}{(\tau_j-1)^2}}{\sum_{j \in J} (1-\tau_j)} \right\}. \quad (5.3)$$

We are interested in the cases $\xi = e^{i\omega}$ and $\xi = e^{-i\omega}$. We note that

$$\frac{(\xi-1)^2}{\xi} = \xi - 2 + \xi^{-1}. \quad (5.4)$$

Thus,

$$\left. \frac{(\xi-1)^2}{\xi} \right|_{\xi=e^{i\omega}} = \left. \frac{(\xi-1)^2}{\xi} \right|_{\xi=e^{-i\omega}} = e^{i\omega} - 2 + e^{-i\omega} = -2(1-\cos \omega).$$

Recall that τ_j is a k^{th} root of ξ . We will now use the notation

$$\begin{aligned} \tau_j &= \text{a } k^{\text{th}} \text{ root of } e^{i\omega}, \\ \tau_j' &= \text{a } k^{\text{th}} \text{ root of } e^{-i\omega}. \end{aligned}$$

In particular we assign indices to these roots in the following manner:

$$\tau_j = \exp\left\{i\left(\frac{2\pi j + \omega}{k}\right)\right\}, \quad j = 1, \dots, k, \quad (5.5)$$

$$\tau_j' = \exp\left\{-i\left(\frac{2\pi j + \omega}{k}\right)\right\}, \quad j = 1, \dots, k. \quad (5.6)$$

Thus, $\tau_j' = \tau_j^{-1}$.

Taking real and imaginary parts,

$$\operatorname{Re}(\tau_j) = \operatorname{Re}(\tau_j') = \cos\left(\frac{2\pi j + w}{k}\right), \quad (5.8)$$

$$I(\tau_j) = -I(\tau_j') = \sin\left(\frac{2\pi j + w}{k}\right). \quad (5.9)$$

Because $\tau_j' = \tau_j^{-1}$, we see that

$$\frac{\tau_j}{(\tau_j - 1)^2} = \frac{1}{\tau_j - 2 + \tau_j^{-1}} = \frac{\tau_j'}{(\tau_j' - 1)^2}.$$

Thus, for any index set J ,

$$\begin{aligned} \prod_{j \in J} \frac{\tau_j}{(\tau_j - 1)^2} &= \prod_{j \in J} \frac{\tau_j'}{(\tau_j' - 1)^2} = \prod_{j \in J} \frac{1}{\tau_j - 2 + \tau_j^{-1}} \\ &= \prod_{j \in J} \frac{1}{-2\{1 - \cos(\frac{2\pi j + w}{k})\}}. \end{aligned}$$

We now consider the terms $1/\sum (1 - \tau_j)$ and $1/\sum (1 - \tau_j')$ in (5.3) when ξ is replaced by $e^{i\omega}$ and $e^{-i\omega}$ respectively.

$$\begin{aligned} (1) \quad \frac{1}{\sum_{j \in J} (1 - \tau_j)} &= \frac{1}{\sum_{j \in J} \{1 - \operatorname{Re}(\tau_j)\} - 1 \sum_{j \in J} I(\tau_j)} \\ &= \frac{\sum_{j \in J} \{1 - \operatorname{Re}(\tau_j)\} + 1 \sum_{j \in J} I(\tau_j)}{\left| \sum_{j \in J} \{1 - \operatorname{Re}(\tau_j)\} - 1 \sum_{j \in J} I(\tau_j) \right|^2}; \quad (5.11) \end{aligned}$$

$$\begin{aligned} (11) \quad \frac{1}{\sum_{j \in J} (1 - \tau_j')} &= \frac{1}{\sum_{j \in J} \{1 - \operatorname{Re}(\tau_j')\} - 1 \sum_{j \in J} I(\tau_j')} \\ &= \frac{\sum_{j \in J} \{1 - \operatorname{Re}(\tau_j')\} + 1 \sum_{j \in J} I(\tau_j')}{\left| \sum_{j \in J} \{1 - \operatorname{Re}(\tau_j')\} - 1 \sum_{j \in J} I(\tau_j') \right|^2} \end{aligned}$$

$$= \frac{\sum_{j \in J} \{1 - \operatorname{Re}(\tau_j)\} - 1 \sum_{j \in J} I(\tau_j)}{\left| \sum_{j \in J} \{1 - \operatorname{Re}(\tau_j)\} - 1 \sum_{j \in J} I(\tau_j) \right|^2}. \quad (5.12)$$

Thus, for any index set J ,

$$\frac{1}{\sum_{j \in J} (1 - \tau_j)} + \frac{1}{\sum_{j \in J} (1 - \tau_j')} = \frac{2a_J}{a_J^2 + b_J^2}. \quad (5.13)$$

where

$$\begin{aligned} a_J &= \sum_{j \in J} \{1 - \operatorname{Re}(\tau_j)\} \\ &= \sum_{j \in J} \left\{1 - \cos\left(\frac{2\pi j + \omega}{k}\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} b_J &= \sum_{j \in J} I(\tau_j) \\ &= \sum_{j \in J} \sin\left(\frac{2\pi j + \omega}{k}\right). \end{aligned}$$

Let

$$c_J = \prod_{j \in J} \left\{1 - \cos\left(\frac{2\pi j + \omega}{k}\right)\right\}. \quad (5.14)$$

Gathering results (5.3), (5.4), (5.10), (5.13) and (5.14) we have

$$\begin{aligned} f_+^{(p)}(\omega) &\sim \phi^{*(p)}(e^{i\omega}, 0+) + \phi^{*(p)}(e^{-i\omega}, 0+) \\ &= \frac{1}{\lambda k^{2p}} (1 - \cos \omega)^p \sum_{j_1=1}^k \dots \sum_{j_p=1}^k \frac{2a_J}{(a_J^2 + b_J^2)c_J}. \quad (5.15) \end{aligned}$$

Because a_J , b_J and c_J depend only on the indices in J and not on their order, the number of distinct terms to be computed for each

value of ω is equal to $\binom{k+p-1}{p}$, the figurate number for k types of objects taken in sets of size p . The computation time for a_j , b_j and c_j is proportional to p , so total computation time should be approximately proportional to p times the figurate number.

A FORTRAN program was written to compute $f_+^{(p)}(\omega)$ at intervals of 0.05π . Various values of k and p having figurate numbers no greater than 56 were used. Computing times were less than 2 seconds per run on the IBM 360/67 computer. Some results of the computations will be shown in Section 7. First we obtain some specific results for small p and k .

6. Specific results for $k = 2$ and $p = 2, 3, 4$.

The generating function $\phi^{(?) }(\xi; t)$ for the superposition of p identical Erlang renewal processes is analytically tractable for $k = 2$ and small p .

From (4.8), with $a = -\lambda(1+\xi^{1/2})$ and $b = -\lambda(1-\xi^{1/2})$, we obtain

$$\phi(\xi; t) = \frac{1}{4\xi^{1/2}} \{ -(1-\xi^{1/2})^2 e^{at} + (1+\xi^{1/2})^2 e^{bt} \}. \quad (6.1)$$

By taking $\{\phi(\xi; t)\}^p$ we obtain the generating functions for the superpositions of p such Erlang renewal processes.

(i) For the case $p = 2$, we have

$$\begin{aligned} \{\phi(\xi; t)\}^2 = \phi^{(2)}(\xi, t) = \frac{1}{16\xi} \{ & (1-\xi^{1/2})^4 e^{2at} \\ & + (1+\xi^{1/2})^2 e^{bt} - 2(1-\xi)^2 e^{(a+b)t} \}, \end{aligned} \quad (6.2)$$

whose Laplace transform is

$$\phi^{*(2)}(\xi; s) = \frac{1}{16\xi} \left\{ \frac{(1-\xi^{1/2})^4}{s-2a} + \frac{(1+\xi^{1/2})^4}{s-2b} - \frac{2(1-\xi)^2}{s-a-b} \right\}. \quad (6.3)$$

From (6.3) we can obtain the spectrum of intervals using (2.5). The result is

$$f_+^{(2)}(\omega) = \frac{1}{\pi} \left(1 - \frac{\cos \omega}{5} \right) \quad (0 \leq \omega \leq \pi). \quad (6.4)$$

From (2.5) we have immediately that $\rho_1 = -1/10$, $\rho_k = 0$ for $k > 1$, and the spectrum is that of a first order moving average process. This generalizes results of Lawrance (1973), who gave $\rho_1 = -1/10$, but no higher correlations.

(11) For $p = 3$ the generating function of the counting function of the superposition process is

$$\begin{aligned} \phi^{(3)}(\xi; t) = \frac{1}{(4\xi^{1/2})} \{ & -(1-\xi^{1/2})^6 e^{3at} + 3(1-\xi^{1/2})^2(1-\xi)^2 e^{(2a+b)t} \\ & - 3(1+\xi^{1/2})^2(1-\xi)^2 e^{(a+2b)t} + (1+\xi^{1/2})^6 e^{3bt} \}, \quad (6.5) \end{aligned}$$

and its transform is

$$\begin{aligned} \phi^{*(3)}(\xi; s) = \frac{1}{64\xi^{3/2}} \left\{ & \frac{-(1-\xi^{1/2})^6}{s-3a} + \frac{3(1-\xi^{1/2})^2(1-\xi)^2}{s-2a-b} \right. \\ & \left. - \frac{3(1+\xi^{1/2})^2(1-\xi)^2}{s-a-2b} + \frac{(1+\xi^{1/2})^6}{s-3a} \right\}. \quad (6.6) \end{aligned}$$

Again using (2.5) we obtain for the spectrum of intervals

$$f_+^{(3)}(\omega) = \frac{9}{25\pi} \left\{ \frac{121 - 48 \cos \omega - 9 \cos^2 \omega}{41 - 9 \cos \omega} \right\} \quad (0 \leq \omega \leq \pi). \quad (6.7)$$

This is the spectrum of a mixed second-order moving average and first-order autoregressive process for which the serial correlations are expressed by three constants, since for $k \geq 2$ $\rho_{k+1} = \rho\beta_k$. Equivalently, for $k \geq 2$, $\rho_k = C\beta^k$.

In the present case $\rho_1 = -0.1044447$, $\rho_2 = -0.0316033$, $\rho_3 = -0.0035118$, or $C = -2.55986$ and $\beta = 1/9$.

(iii) Skipping the details, we get for $p = 4$ the result for the spectrum

$$f_+^{(4)}(\omega) = \frac{8}{379\pi} \left\{ \frac{871 - 502 \cos \omega - 57 \cos^2 \omega - 24 \cos^3 \omega}{17 - 8 \cos \omega} \right\} \quad (0 \leq \omega \leq \pi) \quad (6.8)$$

This is the spectrum of mixed third-order moving average and first-order autoregressive process, so that the serial correlations may be expressed by four constants, since for $k > 3$, $\rho_{k+1} = \beta \rho_k$. Equivalently, for $k \geq 3$, $\rho_k = C\beta^k$.

For the case $p = 4$, $k = 2$ we get

$$\begin{aligned}\rho_1 &= -0.0979540 ; \\ \rho_2 &= -0.0442821 ; \\ \rho_3 &= -0.0150263 ; \quad C = -.961683 ; \\ \rho_4 &= -0.0037597 ; \quad \beta = 1/4 .\end{aligned}$$

In Figure 1 we show the spectra $f_+^{(p)}(\omega)$ for the superposition of independent renewal processes with Erlang ($k=2$) interval distributions. The calculations for $p = 2, 3, 4$, where p is the number of processes superimposed, are exact results using (6.4), (6.7) and (6.9) respectively. The result for $p = 5$ was obtained from the general results of Section 5.

Note that the initial points of the spectra seem to be decreasing monotonically with p to the value $\frac{1}{(k\pi)}$, while generally for the other ω the spectrum seems to be converging toward the flat spectrum of the Poisson limit, i.e. $\frac{1}{\pi}$.

7. Discussion of Results.

The results (6.4), (6.7) and (6.8) giving a simple structure to the spectra of the superposed processes for $p = 2$ and $k = 2, 3$ and 4 are at first quite surprising. However, some such structure could be anticipated from (5.15) which gives the spectrum $f_+^{(p)}(\omega)$ as a rational function in $\cos \omega$. It has not been possible, as yet, to relate the parameters p and k to the order of the moving average and autoregressive structure, or to give simple expressions for the spectrum or serial correlations. Knowing the order of the structure of the spectrum would be a help in knowing how many serial correlations one would have to compute from (5.15) for specific p and k . The correlations are computed by calculating $f_+(\omega)$ at a sufficiently fine grid and then inverting the series using a FFT algorithm. (See Cooley, Lewis, Welch, 1970.)

There are other theoretical questions raised by the results which will be discussed elsewhere. Briefly though, Downton (1971) discussed the possibility of looking at this particular superposition process as a Markovian progression through pk parallel channels, each channel having k stages. This may explain the structure. For $p = k = 2$ it is in fact possible to show that the intervals are not only uncorrelated but independent for lags greater than 2.

We discuss now the computational results.

In Figure 1 the spectrum is given for the superposition of Erlang ($k=2$) processes for several values of p , the number processes superposed. This was discussed in the previous section. A similar graph is given for $k = 3$ in Figure 2. Again as p increases the initial point of the spectrum decrease is converging slowly to the value $1/\pi$

but the remainder of the spectrum which would obtain for all ω for the limiting Poisson process.

Figure 3 by contrast has, for fixed number $p = 2$ of superposed processes, graphs for different values k . As k increases and the component processes become more regular with $C^2(X)$ decreasing, there is a larger peak at π , or equivalently at a period of $T = 2$. Thus the alternation of intervals induced by the superposition becomes pronounced in the spectrum of intervals.

In Figure 4 the case $p = 3$ is examined for different values of k . The tripling due to the superposition produces a more and more marked peak at $\omega = 2\pi/3$ (period $T = 3$) as the component processes become more regular.

Formulae such as (5.15) can be developed for non-identical component processes. In Figures 5, 6 and 7 several cases of the pooling of a Poisson and Erlangian process are given; the marginal distributions were given by Lawrance (1973). These are important as models for the case in which there is one dominant, fairly regular, process corrupting the superposition of many sparse component processes which sum (almost) to a Poisson process. Note carefully the vertical scale; departure from a flat spectrum is small and would be hard to detect in real data unless the Gamma process were very regular. The spectrum of counts would probably be much more informative in this case.

A short table of computed coefficients of variation of the marginal intervals X in the superposed process is given in Table 1. These came out of the computations of the spectrum.

POOLED ERLANG PROCESSES

Coefficient of Variation

p =	2	3	4	5	6	7	8	9
k=2	.791	.833	.860	.879	.893	.904	.913	.920
3	.714	.779	.819	.847	.867	.882	.894	
4	.675	.754	.802	.834				
5	.652	.740						
6	.637	.732						
7	.627							
8	.620							
9	.615							

TABLE 1

8. Conclusion.

The fact (Cox and Lewis, 1966, Ch. 8) that the variance-time curve and spectrum of counts in the superposition of identical processes is a scaled version of the corresponding components is very useful if p is known. The crux of most analysis though is that p is not known and must be determined. The spectra of intervals computed here should be a very useful tool in helping to determine p , and in verifying assumptions such as those made in Cox and Lewis (1966, Ch. 8) in analyzing nerve-pulse data.

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CAPTIONS

Figure 1. Computed spectra of intervals for superposed Erlang renewal processes ($k=2$) for several values of p , where p is the number of independent processes superposed. For $p = 2, 3$, and 4 these are exact results using (6.4), (6.7) and (6.9).

Figure 2. Computed spectra of intervals for superposed Erlang renewal processes ($k=3$) for several values of p , where p is the number of independent processes superposed. For all p the results are obtained from the computational formulae of Section 5.

Figure 3. Here the number p of superposed processes is held fixed at 2, while the parameter k in the pooled Erlang processes is increased. As k increases the intervals in the individual processes become more regular and the peak in the spectrum of the superposition process at $\omega = \pi$ becomes more pronounced.

Figure 4. Here the number p of superposed processes is held fixed at 3, while the parameter k in the pooled Erlang processes is increased. As k increases the intervals in the individual processes become more regular and the peak in the spectrum of the superposition process at $\omega = 2\pi/3$ becomes more pronounced.

Figure 5. The spectrum of intervals of superposed Poisson and Erlang renewal processes is plotted for increasing values of the parameter k in the Erlang process. The Poisson and Erlang process mean values are held constant.

Figure 6. The spectrum of intervals of superposed Poisson and Erlang renewal processes is plotted for increasing values of the parameter k in the Erlang process. The Poisson and Erlang process mean values are held constant.

Figure 7. The spectrum of intervals of superposed Poisson and Erlang renewal processes is plotted for increasing values of the parameter k in the Erlang process. The Poisson and Erlang process mean values are held constant.

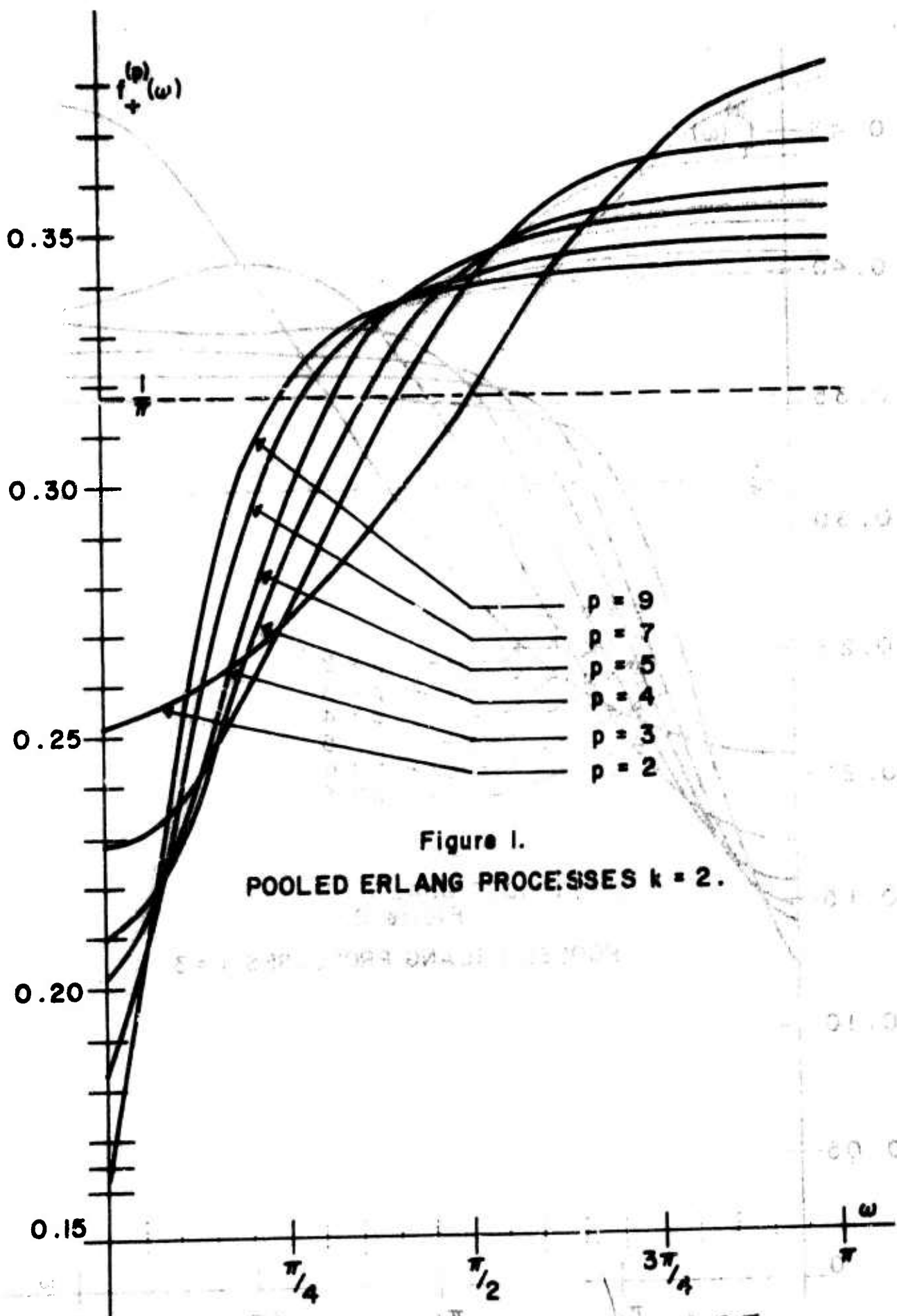
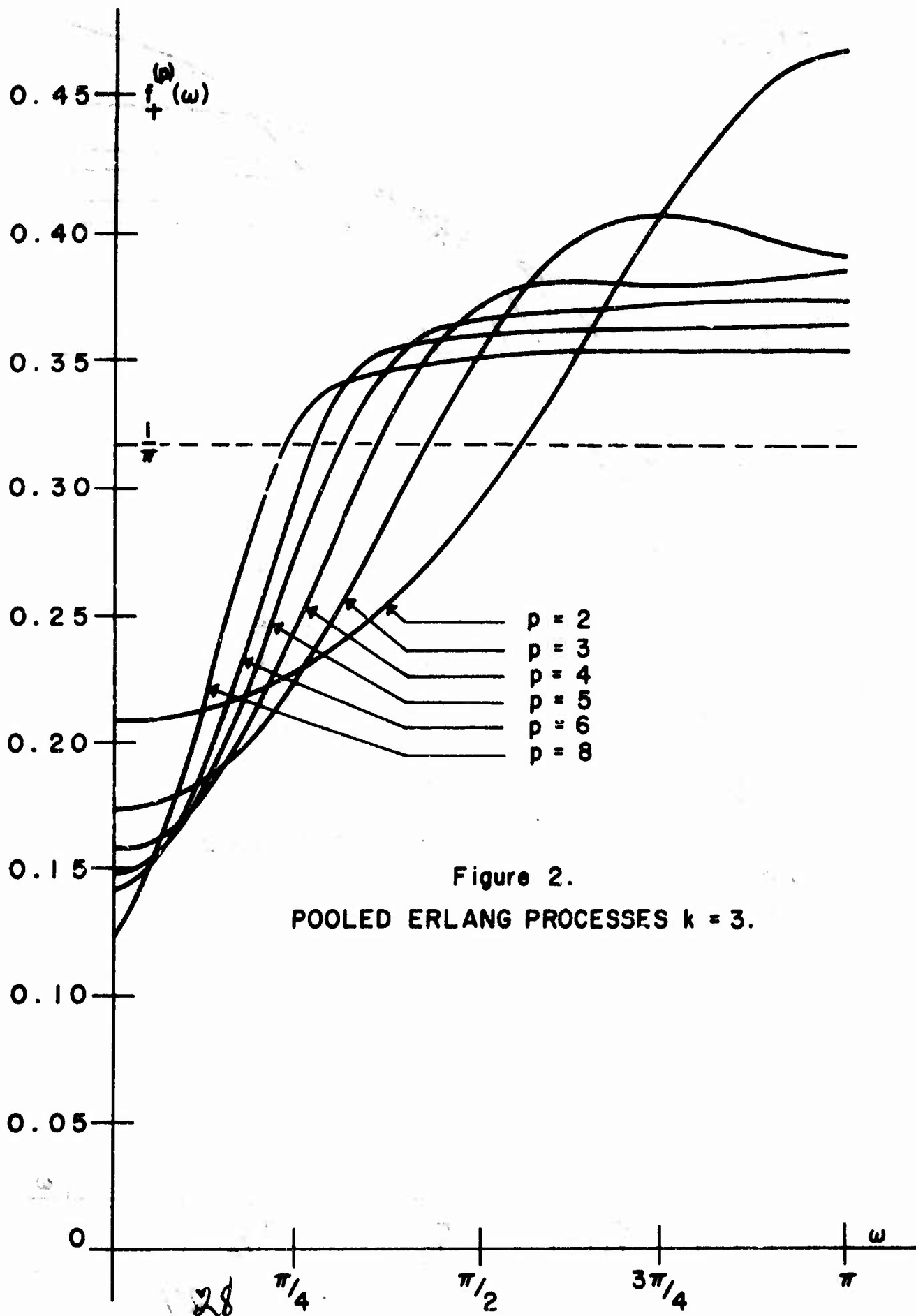
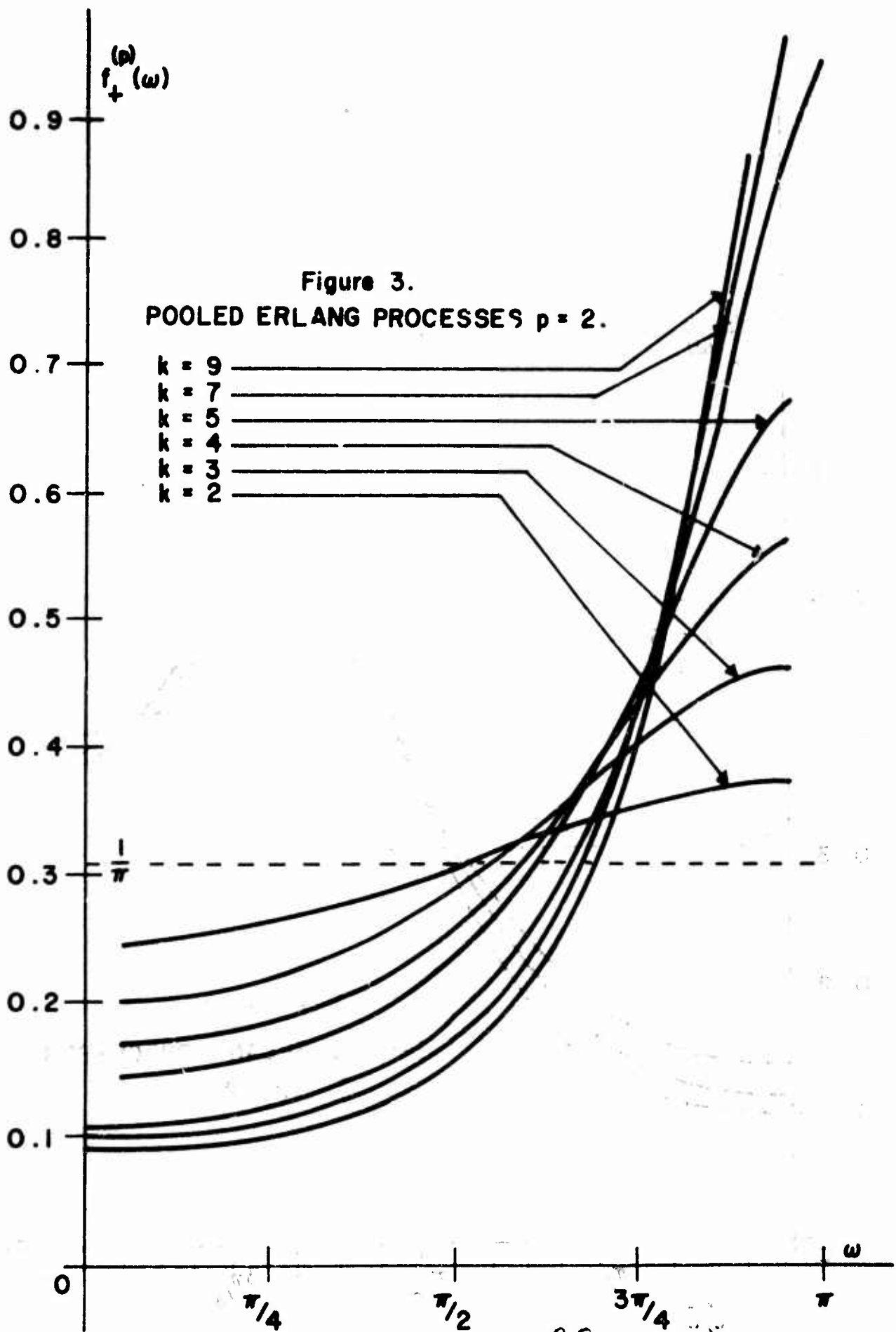


Figure 1.
 POOLED ERLANG PROCESSES $k = 2$.





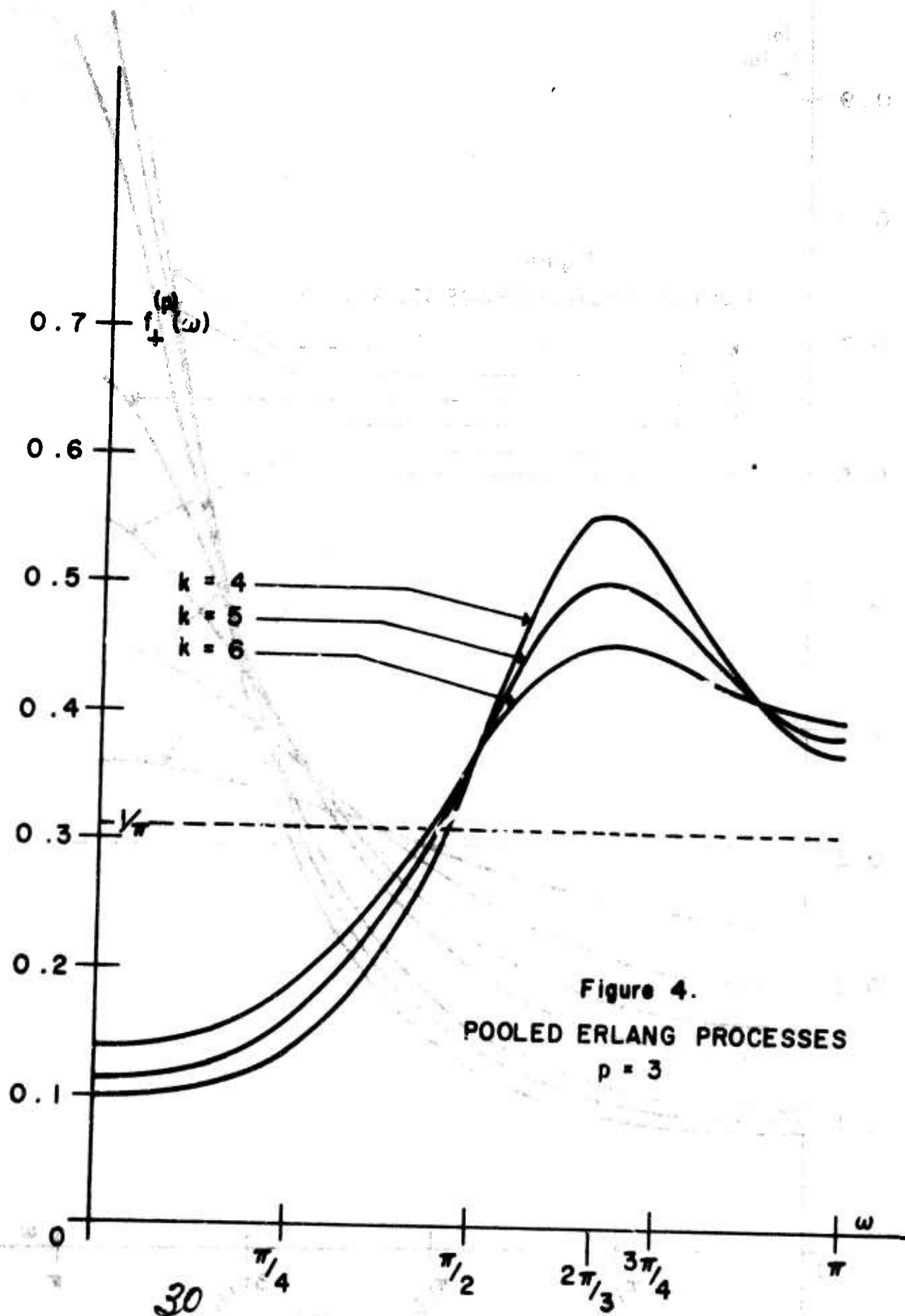


Figure 5.
SUPERPOSED POISSON AND ERLANG RENEWAL PROCESSES

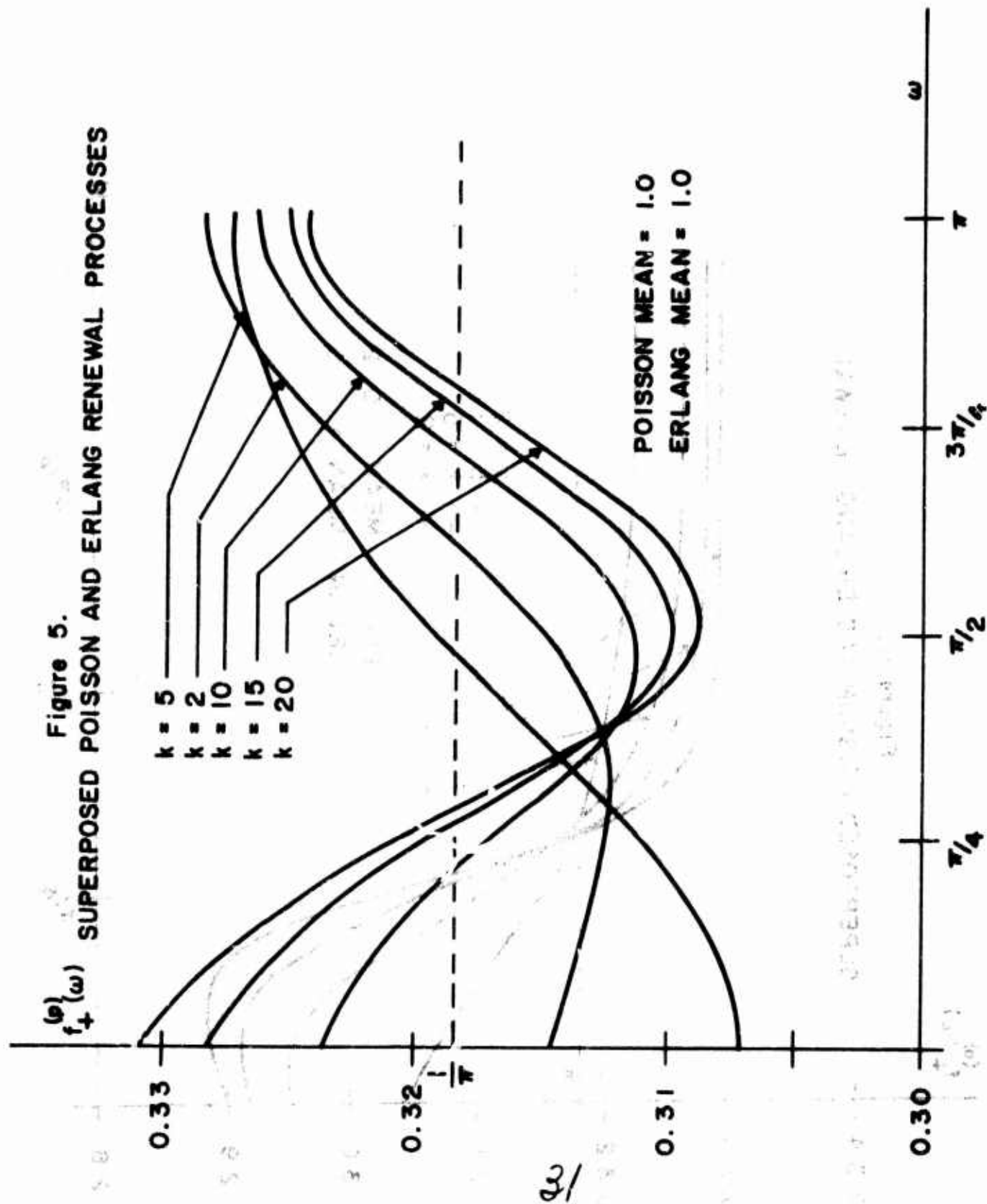


Figure 6.
SUPERPOSED POISSON AND ERLANG RENEWAL

