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AN ACCURATE FOURTH ORDER EQUATION FOR CIRCULAR CYLINDRICAL SHELLS

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ABSTRACT

This paper presents a pair of complex conjugate fourth order equations for the homogeneous solutions of circular cylindrical shells as follows:

$$L\,w = 0\ ,\quad \bar{L}\,w = 0\ ,$$

where $L = (\nabla^2 + 1)\nabla^2 + i\,\{\frac{1}{k}\frac{\partial^2}{\partial\alpha^2} + k(1-\nu)[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2}]\}$,

$\bar{L}$ is the linear complex conjugate operator of L, w is the radial displacement of the midsurface,

$k = \frac{h}{2a\sqrt{3(1-\nu^2)}}$, h is the thickness, a is the radius of the cylinder and ν is Poisson's ratio. This pair of equations is as accurate as an equation can be within the scope of the Love-Kirchhoff assumptions. Because they are complex conjugates, we can simply replace them by a single equation, taking any one of the pair. Closed form solutions, which can be readily obtained from solving this single equation by means of the eigenfunctions, are given in the paper. This simplifies the calculations considerably in solving problems of cylindrical shells. The particular solutions can be found from the governing differential equation:

$$L\,\bar{L}\,w = \frac{a^4}{D}\nabla^4 Z$$

where D is the flexural rigidity and Z is surface load per unit area in radial direction.

Nomenclature

a = radius of midsurface of shell

$C^2 = \frac{h^2}{12a^2}$

e = base of natural logarithms

h, ℓ = wall thickness and length

$k = \frac{C}{\sqrt{1-\nu^2}}$

$k_1 = k(1-\nu)$

$i = \sqrt{-1}$ = imaginary unit

m = integer

n = real number

p = complex root of characteristic equation

u,v,w = axial, circumferential, and radial displacements of midsurface

x,y = axial and circumferential coordinates

X,Y,Z = surface loads per unit area in axial, tangential and radial directions

$D = \frac{Eh^3}{12(1-\nu^2)}$ = flexural rigidity

$K = \frac{Eh}{1-\nu^2}$ = extensional rigidity

E = Young's modulus

$L, \bar{L}$ = linear differential operator defined by eqn.(20) and its complex conjugate operator

M, N, S = stress couples and resultants per unit length

Q_1, Q_2 = transverse stress resultants per unit length

$Q_{1,eff.}$, $Q_{2,eff.}$ = effective transverse stress resultants per unit length

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α, β = dimensionless midsurface coordinates along the lines of curvatures, $\alpha = \frac{x}{a}$, $\beta = \frac{y}{a}$

$\xi = \frac{\alpha}{\sqrt{k}}$

$\zeta = \frac{\beta}{\sqrt{k}}$

ν = Poisson's ratio

$\epsilon_1, \epsilon_2, \omega$ = normal and shearing strains in midsurface

η_1, η_2, τ = bending and twisting strains

ϕ = displacement function

$$\nabla^2 = \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}$$

$$\nabla_1^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}$$

$$\delta \doteq \frac{9}{5} C^2$$

AN ACCURATE FOURTH ORDER EQUATION FOR CIRCULAR CYLINDRICAL SHELLS

Shun Cheng[†]

Introduction

In the present paper a pair of complex conjugate fourth order partial differential equations which govern the deformation of circular cylindrical shells is proposed as follows:

$$Lw = 0 \quad , \quad \bar{L}w = 0 \, ,$$

where

$$L = (\nabla^2+1)\nabla^2+i\{\frac{1}{k}\frac{\partial^2}{\partial\alpha^2} + k(1-\nu)[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2}]\}$$

and $\bar{L}$ is the linear complex conjugate operator of L. The preceding pair of equations, as will be shown, has at least the same accuracy as the Flügge equation [1, 19]. Because they are complex conjugates, only one of the equations need be considered. Closed form solutions of the characteristic equations which arise from solving the preceding equations by means of the eigenfunctions $e^{p\alpha}\cos n\beta$ and $e^{p\beta}\cos n\alpha$ can be easily obtained. The particular solutions can be found from the nonhomogeneous equation

$$L\bar{L}w = \frac{a^4}{D}\nabla^4 Z \, .$$

The formulation of the basic equations for thin elastic shells and, in particular, circular cylindrical shells, due to their importance in application and their exhibition of nearly every type of behavior found in general shell theory, has received repeated attention in the literature [1-20]. Many sets of basic equations for circular cylindrical shells based on linear thin

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shell theory have been derived since the inception of Love's first approximation. The basic assumptions known as Love-Kirchhoff assumptions of the linear shell theory [1, 2] are: (a) the thickness h of the shell is small compared with the least radius of curvature of the middle surface; (b) the strains and displacements are sufficiently small so that the quantities of the second and higher orders may be neglected in the components of strain; (c) the component of stress normal to the middle surface is small compared with other normal components of stress and may be neglected in the stress-strain relations; and (d) the normals to the undeformed middle surface remain normal to the deformed middle surface and undergo no extension.

The variety of resulting equations found in the literature, although all of them are based on the same basic Love-Kirchhoff hypotheses, are due to variations in rigor in their derivations and modified approximations in the subsequent analyses. However, it is inconvenient to assess and select between a variety of these equations with each new work on shells. A common difficulty with the unreduced equations for circular cylindrical shells, such as Flügge's [1], Sander's [2,3], Koiter's [4], Vlasov's [7], Goldenveizer's [12], Timoshenko's [13], Love's [14] and others, is that their general solutions remain unknown because of the algebraic complexities. For this reason many attempts have been made throughout the history of shell theory to simplify the basic equations.

A number of equations of a simplified nature have been suggested. Examples are the Donnell equations [6], Morley equations [8], Lukasiewicz equations [10] and Feinburg equations [11]. In order to evaluate the accuracy of these equations and other simplified equations, the more accurate Flügge equations are usually used as the standard for comparison [8,17]. Unfortunately, such comparisons are difficult because the general solutions of Flügge's equations can not be obtained explicitly. These difficulties can be removed by using the proposed equations.

Fundamental Equations

In accordance with the Love-Kirchhoff assumptions for the classical shell theory we can deduce the following basic equations for circular cylindrical shells. These relations are exact in the sense that in deriving these equations we have kept all terms without introducing further simplifications or approximations beyond these basic assumptions (even those terms with coefficients C^2 or higher are kept).

The strain displacement relations of the middle surface of the shell are: [1]

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{a}\frac{\partial u}{\partial \alpha} \\
\varepsilon_2 &= \frac{1}{a}\left(\frac{\partial v}{\partial \beta} + w\right) \\
\omega &= \frac{1}{a}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) \\
\eta_1 &= \frac{-1}{a^2}\frac{\partial^2 w}{\partial \alpha^2} \\
\eta_2 &= \frac{-1}{a^2}\left(\frac{\partial^2 w}{\partial \beta^2} + w\right) \\
\tau &= \frac{-1}{2a^2}\left(\frac{\partial u}{\partial \beta} - \frac{\partial v}{\partial \alpha} + 2\frac{\partial^2 w}{\partial \alpha \partial \beta}\right).
\end{aligned} \tag{1}$$

It may be shown that the stress resultants and couples (Fig. 1 and Fig. 2) are related to the midsurface displacements thru the stress-strain relations as:

$$N_1 = \frac{K}{a}\left[\frac{\partial u}{\partial\alpha} + \nu\left(\frac{\partial v}{\partial\beta} + w\right) - \frac{h^2}{12a^2}\frac{\partial^2 w}{\partial\alpha^2}\right]$$

$$N_2 = \frac{K}{a}\left[\frac{\partial v}{\partial\beta} + \nu\frac{\partial u}{\partial\alpha} + w + \frac{h^2}{12a^2}\left(\frac{\partial^2 w}{\partial\beta^2} + w\right)(1+\delta)\right]$$

$$S_1 = \frac{K(1-\nu)}{2a}\left[\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} - \frac{h^2}{12a^2}\left(\frac{\partial^2 w}{\partial\alpha\partial\beta} - \frac{\partial v}{\partial\alpha}\right)\right]$$

$$S_2 = \frac{K(1-\nu)}{2a}\left[\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} + \frac{h^2}{12a^2}\left(\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{\partial u}{\partial\beta}\right)(1+\delta)\right]$$

$$M_1 = -\frac{D}{a^2}\left[\frac{\partial u}{\partial\alpha} + \nu\frac{\partial v}{\partial\beta} - \left(\frac{\partial^2}{\partial\alpha^2} + \nu\frac{\partial^2}{\partial\beta^2}\right)w\right]$$

$$M_2 = \frac{D}{a^2}\left[\frac{\partial^2 w}{\partial\beta^2} + \nu\frac{\partial^2 w}{\partial\alpha^2} + w + \delta\left(\frac{\partial^2 w}{\partial\beta^2} + w\right)\right] \tag{2}$$

$$M_{12} = \frac{D(1-\nu)}{a^2}\left(\frac{\partial v}{\partial\alpha} - \frac{\partial^2 w}{\partial\alpha\partial\beta}\right)$$

$$M_{21} = -\frac{D(1-\nu)}{2a^2}\left[\frac{\partial u}{\partial\beta} - \frac{\partial v}{\partial\alpha} + 2\frac{\partial^2 w}{\partial\alpha\partial\beta} + \delta\left(\frac{\partial u}{\alpha\beta} + \frac{\partial^2 w}{\partial\alpha\partial\beta}\right)\right]$$

$$Q_1 = \frac{D}{a^3}\left[\frac{\partial^2 u}{\partial\alpha^2} - \frac{1-\nu}{2}(1+\delta)\frac{\partial^2 u}{\partial\beta^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial\alpha\partial\beta} - \frac{\partial}{\partial\alpha}\nabla^2 w - \frac{1-\nu}{2}\delta\frac{\partial^3 w}{\partial\alpha\partial\beta^2}\right]$$

$$Q_2 = \frac{D}{a^3}\left[(1-\nu)\frac{\partial^2 v}{\partial\alpha^2} - \frac{\partial w}{\partial\beta} - \frac{\partial}{\partial\beta}\nabla^2 w - \delta\frac{\partial}{\partial\beta}\left(\frac{\partial^2 w}{\partial\beta^2} + w\right)\right]$$

$$Q_{1,\,eff.} = Q_1 + \frac{1}{a}\frac{\partial M_{12}}{\partial\beta} = -\frac{D}{a^3}\left\{\frac{\partial}{\partial\alpha}\left[\nabla^2 w + (1-\nu)\frac{\partial^2 w}{\partial\beta^2}\right] - \frac{\partial^2 u}{\partial\alpha^2} + \frac{1-\nu}{2}(1+\delta)\frac{\partial^2 u}{\partial\beta^2} - \frac{3-\nu}{2}\frac{\partial^2 v}{\partial\alpha\partial\beta}\right.$$
$$\left. + \frac{1-\nu}{2}\delta\frac{\partial^3 w}{\partial\alpha\partial\beta^2}\right\}$$

$$Q_{2,eff.} = Q_2 + \frac{1}{a}\frac{\partial M_{21}}{\partial\alpha} = -\frac{D}{a^3}\left\{\frac{\partial}{\partial\beta}\left[\nabla^2 + (1-\nu)\frac{\partial^2}{\partial\alpha^2} + 1\right]w + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial\alpha\partial\beta} - \frac{3(1-\nu)}{2}\frac{\partial^2 v}{\partial\alpha^2}\right.$$
$$\left. + \delta\frac{\partial}{\partial\beta}\left(\nabla^2 w + w - \frac{1+\nu}{2}\frac{\partial^2 w}{\partial\alpha^2} + \frac{\partial u}{\partial\alpha}\right)\right\},$$

$$\text{where} \quad \delta = \frac{3(\tanh^{-1}\frac{h}{2a} - \frac{h}{2a})}{(\frac{h}{2a})^3} - 1 = 3[\frac{1}{5}(\frac{h}{2a})^2 + \frac{1}{7}(\frac{h}{2a})^4 + \frac{1}{9}(\frac{h}{2a})^6 + \ldots]. \qquad (3)$$

Equation (2) reduces to Flügge's equations [1] if δ is set equal to zero. The following equations of static equilibrium are well known and generally accepted:

$$\frac{\partial N_1}{\partial \alpha} + \frac{\partial S_2}{\partial \beta} + aX = 0$$

$$\frac{\partial N_2}{\partial \beta} + \frac{\partial S_1}{\partial \alpha} + Q_2 + aY = 0$$

$$-N_2 + \frac{\partial Q_1}{\partial \alpha} + \frac{\partial Q_2}{\partial \beta} + aZ = 0 \qquad (4)$$

$$\frac{\partial M_{12}}{\partial \alpha} - \frac{\partial M_2}{\partial \beta} - aQ_2 = 0$$

$$\frac{\partial M_{21}}{\partial \beta} - \frac{\partial M_1}{\partial \alpha} - aQ_1 = 0 .$$

Substituting Eqn. (2) into Eqn. (4), we obtain a system of three differential equations in the three basic functions $u=u(\alpha,\beta)$, $v=v(\alpha,\beta)$ and $w=w(\alpha,\beta)$. This system is presented in the form of the following table:

u	v	w	Load Terms	
$\frac{\partial^2}{\partial\alpha^2} + \frac{1-\nu}{2}\,[1+c^2(1+\delta)]\frac{\partial^2}{\partial\beta^2}$	$\frac{1+\nu}{2}\;\frac{\partial^2}{\partial\alpha\partial\beta}$	$\nu\frac{\partial}{\partial\alpha} - c^2[\frac{\partial^3}{\partial\alpha^3} - \frac{1-\nu}{2}(1+\delta)\frac{\partial^3}{\partial\alpha\partial\beta^2}]$	$+\frac{1-\nu^2}{Eh}\,a^2X$	$= 0$
$\frac{1+\nu}{2}\,\frac{\partial^2}{\partial\alpha\partial\beta}$	$\frac{\partial^2}{\partial\beta^2} + \frac{1-\nu}{2}(1+3c^2)\frac{\partial^2}{\partial\alpha^2}$	$\frac{\partial}{\partial\beta} - \frac{3-\nu}{2}c^2\,\frac{\partial^3}{\partial\alpha^2\partial\beta}$	$+\frac{1-\nu^2}{Eh}\,a^2Y$	$= 0$
$\nu\,\frac{\partial}{\partial\alpha} - c^2[\frac{\partial^3}{\partial\alpha^3} - \frac{1-\nu}{2}(1+\delta)\,\frac{\partial^3}{\partial\alpha\partial\beta^2}]$	$\frac{\partial}{\partial\beta} - \frac{3-\nu}{2}\,c^2\,\frac{\partial^3}{\partial\alpha^2\partial\beta}$	$c^2[\nabla^4+2\,\frac{\partial^2}{\partial\beta^2} + 1 + \delta(\frac{\partial^4}{\partial\beta^4} +$ $2\,\frac{\partial^2}{\partial\beta^2} + \frac{1-\nu}{2}\,\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + 1] + 1$	$-\frac{1-\nu^2}{Eh}\,a^2Z$	$= 0$

Table 1

where $C^2 = \frac{h^2}{12a^2}$.

The three differential equations presented in the table and possessing a symmetrical structure belong to the class of linear partial differential equations with constant coefficients. These equations can be reduced to a single differential equation of higher order which is more convenient to handle than a system of equations. We shall consider these three equations as algebraic equations in u, v, and w having coefficients of the symbols of differentiations $\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}$ together with other quantities and high order differentiations [12]. In order to make our discussion self-contained, we record here a method for reducing the three equations of equilibrium to a single governing differential equation. Let D_0 be the 3X3 determinant of the preceding table and calculate its cofactors $D_{11}, D_{12} \ldots D_{33}$. Let

$$u = D_{11}\phi_1 + D_{12}\phi_2 + D_{13}\phi_3$$

$$v = D_{21}\phi_1 + D_{22}\phi_2 + D_{23}\phi_3 \qquad (5)$$

$$w = D_{31}\phi_1 + D_{32}\phi_2 + D_{33}\phi_3 ,$$

and substitute these expressions in the three equations of equilibrium (Table 1). Then, in accordance with well-known theorems of the theory of linear algebra, we obtain

$$D_0\phi_1 + \frac{1-\nu^2}{Eh} a^2 X = 0$$

$$D_0\phi_2 + \frac{1-\nu^2}{Eh} a^2 Y = 0 \qquad (6)$$

$$D_0\phi_3 - \frac{1-\nu^2}{Eh} a^2 Z = 0 .$$

The homogeneous solutions of the three equations of equilibrium satisfy the following equations

$$D_0\phi_1 = 0 \quad ; \quad D_0\phi_2 = 0 \quad ; \quad D_0\phi_3 = 0. \tag{7}$$

Then the homogeneous solutions for u, v, w may be found from equation (5). In particular we can let

$$\phi_1 = \phi_2 = 0 \qquad \phi_3 = \frac{2}{1-\nu}\phi . \tag{8}$$

Then each solution of the equation

$$D_0\phi = 0 \tag{9}$$

corresponds to an integral of the homogeneous equations of equilibrium, given by the formulas:

$$u = \frac{2}{1-\nu} D_{13}\phi \quad , \quad v = \frac{2}{1-\nu} D_{23}\phi \quad , \quad w = \frac{2}{1-\nu} D_{33}\phi. \tag{10}$$

Calculating the cofactors D_{13}, D_{23}, D_{33} and D from Table 1, we obtain from Eqns. (9) and (10):

$$u = \frac{\partial}{\partial\alpha}\left[\frac{\partial^2}{\partial\beta^2} - \nu\frac{\partial^2}{\partial\alpha^2} + c^2\left(\frac{\partial^4}{\partial\alpha^4} - \frac{\partial^4}{\partial\beta^4} - 3\nu\frac{\partial^2}{\partial\alpha^2}\right)\right]\phi \tag{11}$$

$$v = \frac{\partial}{\partial\beta}\left[-\frac{\partial^2}{\partial\beta^2} - (2+\nu)\frac{\partial^2}{\partial\alpha^2} + 2c^2\left(\frac{\partial^4}{\partial\alpha^4} + \frac{\partial^4}{\partial\alpha^2\partial\beta^2}\right)\right]\phi \tag{12}$$

$$w = \left[\nabla^4 + c^2\left(\nabla^4 + 2\frac{\partial^4}{\partial\alpha^4} - 2\nu\frac{\partial^4}{\partial\alpha^2\partial\beta^2}\right)\right]\phi \tag{13}$$

$$
\begin{aligned}
D_0\phi = C^2 \frac{1-\nu}{2}\{(\nabla^2+1)^2\nabla^4 &+ \frac{1-\nu^2}{C^2}\frac{\partial^4}{\partial\alpha^4} + 2(1-\nu)\frac{\partial^2}{\partial\alpha^2}[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2} + \\
&\frac{3(1+\nu)}{2}\frac{\partial^2}{\partial\alpha^2}] + C^2[\underline{2\frac{\partial^8}{\partial\alpha^8} + \frac{4}{5}(8-3\nu)\frac{\partial^8}{\partial\alpha^6\partial\beta^2} + \frac{24}{5}(2-\nu)\frac{\partial^8}{\partial\alpha^4\partial\beta^4} +} \\
&\underline{\frac{4}{5}(10-3\nu)\frac{\partial^8}{\partial\alpha^2\partial\beta^6} + \frac{14}{5}\frac{\partial^8}{\partial\beta^8} + 6\nu\frac{\partial^6}{\partial\alpha^6} + \frac{28}{5}\frac{\partial^6}{\partial\beta^6} + \frac{24}{5}(2-\nu+\nu^2)\frac{\partial^6}{\partial\alpha^4\partial\beta^2}} \\
&\underline{+ \frac{2}{5}(40-17\nu)\frac{\partial^6}{\partial\alpha^2\partial\beta^4} + \frac{\partial^4}{\partial\beta^4} + \frac{9}{5}\nabla^4 + 3\frac{\partial^4}{\partial\alpha^4} + \frac{44}{10}(1-\nu)\frac{\partial^4}{\partial\alpha^2\partial\beta^2}}]\}\phi = 0.
\end{aligned}
\tag{14}
$$

In equations (11), (12), (13), and (14) terms with coefficients higher than those shown have been omitted. From equations (13) and (14) we obtain

$$D_0 w = 0, \tag{15}$$

where the linear operator D_0 is given by equation (14). The preceding equation is an eighth order partial differential equation for the normal deflection w.

Equation (15) is an accurate governing differential equation for the bending theory of circular cylinders because this equation is derived from a clear set of assumptions without introducing further approximations in its derivation aside from dropping several negligibly small terms with coefficients C^6 or higher. It is now appropriate to note that in the theory of thin shells $\frac{h}{a} \leq \frac{1}{20}$, thus $C^2 \leq 2\times10^{-4}$ and C^2 is always a small number. Therefore the underlined terms in eqn. (14), which have coefficients C^2 as compared with other terms in the same equation, may be wholly or in part dropped from equation (14), and similarly those terms in (15), without any significant loss in accuracy of the final solutions of the problem.

These negligibly small terms have been wholly dropped in all the known existing shell equations such as the Flügge equation [1], the Vlasov equation [7] and the Goldenveizer equation [12]. However there is the fundamental advantage in keeping some of these negligibly small terms so that equation (15) can be reduced to a pair of fourth order complex conjugate equations. This not only tremendously simplifies calculation of the roots of the characteristic equation which arise from solving the equation by separation of variables but also enables us to obtain solutions in simple explicit forms as will be shown in the following sections. Without having found this pair of fourth-order complex conjugate equations, the task of finding solutions in explicit forms is almost prohibitive due to the algebraic complexity.

When some of the underlined negligibly small terms in eqn. (14) are kept, the governing equation for the normal deflection w may be obtained from equations (6), (8) and (13) as

$$\{(\nabla^2+1)^2\nabla^4 + \frac{1}{k^2}\frac{\partial^4}{\partial\alpha^4} + 2(1-\nu)\frac{\partial^2}{\partial\alpha^2}[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2} + \frac{3(1+\nu)}{2}\frac{\partial^2}{\partial\alpha^2}] + k_1^2[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2} + \frac{3(1+\nu)}{2}\frac{\partial^2}{\partial\alpha^2}]^2\}\, w = \frac{a^4}{D}[\nabla^4 Z + c^2(\nabla^4 + 2\frac{\partial^4}{\partial\alpha^4} - 2\nu\frac{\partial^4}{\partial\alpha^2\partial\beta^2})Z]. \tag{16}$$

The term on the right side of the preceding equation represents the surface load which comes from equations (6), (8) and (13). Equation (16) can be put into the equivalent form

$$\{(\nabla^2+1)\nabla^2 + i(\frac{1}{k}\frac{\partial^2}{\partial\alpha^2} + k_1[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2} + \frac{3(1+\nu)}{2}\frac{\partial^2}{\partial\alpha^2}])\}\cdot\{(\nabla^2+1)\nabla^2 - i(\frac{1}{k}\frac{\partial^2}{\partial\alpha^2} + k_1[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2} + \frac{3(1+\nu)}{2}\frac{\partial^2}{\partial\alpha^2}])\}w = \frac{a^4}{D}[\nabla^4 Z + C^2(\nabla^4 + 2\frac{\partial^4}{\partial\alpha^4} - 2\nu\frac{\partial^4}{\partial\alpha^2\partial\beta^2})Z]. \tag{17}$$

It should be noted that the terms $k_1 \frac{3(1+\nu)}{2} \frac{\partial^2}{\partial\alpha^2}$ in the preceding equation have the order of magnitude k^2 as compared with the terms $\frac{1}{k} \frac{\partial^2}{\partial\alpha^2}$ in the same equation. Comparison of many numerical results of the characteristic roots of eigenfunction solutions for various values of the parameters of equation (17) and the same equation without the terms $\frac{3(1+\nu)}{2} k_1 \frac{\partial^2}{\partial\alpha^2}$ reveals that this term can be discarded for all values of the parameters. Thus equations (16) and (17) reduce to:

$$\{(\nabla^2+1)^2\nabla^4 + \frac{1}{k^2}\frac{\partial^4}{\partial\alpha^4} + 2(1-\nu)\frac{\partial^2}{\partial\alpha^2}[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2}] + k_1^2[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2}]^2\}\ w = \frac{a^4}{D}\nabla^4 Z \qquad (18)$$

or

$$L\bar{L}w = \frac{a^4}{D}\nabla^4 Z \qquad (19)$$

where

$$L = (\nabla^2+1)\nabla^2+i\{\frac{1}{k}\frac{\partial^2}{\partial\alpha^2} + k_1[(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2})\nabla^2 + \frac{\partial^2}{\partial\beta^2}]\}\ , \qquad (20)$$

$\bar{L}$ is the complex conjugate linear differential operator of L,

$$k = \frac{C}{\sqrt{1-\nu^2}} = \frac{h}{2a\sqrt{3(1-\nu^2)}} \qquad k_1 = k(1-\nu) \quad \text{where} \quad \frac{1}{5000} \le k \le \frac{1}{50}$$

and $D = \frac{Eh^3}{12(1-\nu^2)}$.

It is seen from eqns. (18) and (19) that the terms with coefficients c^2 on the right side of the eqns. (16) or (17) have been neglected. This simplification only affects the particular solutions slightly since c^2 is very small.

The small parameters k and k_1 appear constantly throughout the rest of this paper. When k is greater than $\frac{1}{50}$, the assumptions of thin shell theory become invalid. When k is smaller than $\frac{1}{5000}$ the shell is so thin-walled that it can hardly fulfill structural requirements [17].

The homogeneous solutions of equation (19) are obtained from the following equation

$$L\bar{L}\, w = 0. \tag{21}$$

The general integrals of the preceding equation are the sum of the general integrals of the following two equations.

$$L\, w=0 \quad \text{and} \quad \bar{L}\, w=0. \tag{22}$$

Since the solutions obtained from the preceding two equations are always complex conjugate to each other, we need now only consider a single equation, say:

$$L\, w=0 \tag{23}$$

along with

$$w = \sum_{i=1}^{4} (A_i w_i + B_i \bar{w}_i)\ , \tag{24}$$

where A_i and B_i are arbitrary constants.

Having obtained the governing differential equation for w, we obtain the other two displacements u and v in terms of w from equations (11), (12) and (13) as

$$[\nabla^4+c^2(\nabla^4+2\frac{\partial^4}{\partial\alpha^4} - 2\nu\frac{\partial^4}{\partial\alpha^2\partial\beta^2})]u = \frac{\partial}{\partial\alpha}[\frac{\partial^2}{\partial\beta^2} - \nu\frac{\partial^2}{\partial\alpha^2} + c^2(\frac{\partial^4}{\partial\alpha^4} - \frac{\partial^4}{\partial\beta^4} - 3\nu\frac{\partial^2}{\partial\alpha^2})]w \quad (25)$$

$$[\nabla^4+c^2(\nabla^4+2\frac{\partial^4}{\partial\alpha^4} -2\nu\frac{\partial^4}{\partial\alpha^2\partial\beta^2})]v = - \frac{\partial}{\partial\beta}[\frac{\partial^2}{\partial\beta^2} + (2+\nu)\frac{\partial^2}{\partial\alpha^2} - 2c^2(\frac{\partial^4}{\partial\alpha^4} + \frac{\partial^4}{\partial\alpha^2\partial\beta^2})]w. \quad (26)$$

The displacement functions given by equations (21), (25) and (26) constitute the complementary solution of the problem of a cylindrical shell. From these, the moments and forces can be obtained from equation (2) in which terms with coefficients δ can be discarded.

Equation (19) is the governing differential equation for the normal deflection of circular cylindrical shells and is the equation we set out to propose. This equation, or its homogeneous equation (23), as we have seen, is as accurate as an equation within the scope of the Love-Kirchhoff assumptions can be since it is derived without introducing additional approximations or simplifications along the way except some very small terms which could only affect the solutions slightly have been neglected as previously explained.

We have already gained confidence in the Flügge equations and are not aware of any problem of circular cylindrical shells in which such equations fail to yield the correct solution. It seems therefore that equation (23) can be applied with confidence to all problems of circular cylindrical shells. However, the advantage in using equation (23) lies in the fact that the solutions of this equation can be readily found in simple explicit forms. This is possible because the order of equation (23) is half that of those existing equations such as the Flügge equation [1,19] and other equations [2,3,4,7,12,13] for which solutions can only be found approximately by trial and error.

Since moments and forces depend on the derivatives of the displacement functions with resepct to the coordinate variables α and β, an inaccurate root p of the characteristic equation could possibly cause increased inaccuracies in the calculation of moments and forces in the shell. It is therefore advisable to calculate the roots of the characteristic equation as accurately as possible, even if the rest of the analysis is carried out in accordance with one of the simplified methods. Thus the following equations are appropriate for circular cylindrical shells.

The governing differential equation for the radial displacement w is given by eqn. (19):

$$L\bar{L}\, w = \frac{a^4}{D} \nabla^4 Z$$

where L is defined by equation (20) and $\bar{L}$ is the complex conjugate differential operator of L. Neglecting terms with coefficients C^2 in eqns. (25), (26) and (2) yields

$$\nabla^4 u = \frac{\partial}{\partial\alpha}\left(\frac{\partial^2}{\partial\beta^2} - \nu\frac{\partial^2}{\partial\alpha^2}\right)w$$

$$\nabla^4 v = -\frac{\partial}{\partial\beta}\left[\frac{\partial^2}{\partial\beta^2} + (2+\nu)\frac{\partial^2}{\partial\alpha^2}\right]w \qquad (27)$$

$$N_1 = \frac{K}{a}\left[\frac{\partial u}{\partial\alpha} + \nu\left(\frac{\partial v}{\partial\beta} + w\right)\right]$$

$$N_2 = \frac{K}{a}\left(\frac{\partial v}{\partial\beta} + \nu\frac{\partial u}{\partial\alpha} + w\right)$$

$$S_1 = S_2 = \frac{K(1-\nu)}{2a}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right)$$

$$M_1 = -\frac{D}{a^2}\left[\frac{\partial u}{\partial \alpha} + \nu\frac{\partial v}{\partial \beta} - \left(\frac{\partial^2}{\partial \alpha^2} + \nu\frac{\partial^2}{\partial \beta^2}\right)w\right]$$

$$M_2 = \frac{D}{a^2}\left(\frac{\partial^2 w}{\partial \beta^2} + \nu\frac{\partial^2 w}{\partial \alpha^2} + w\right) \quad (28)$$

$$M_{12} = \frac{D(1-\nu)}{a^2}\left(\frac{\partial v}{\partial \alpha} - \frac{\partial^2 w}{\partial \alpha^2 \beta}\right)$$

$$M_{21} = -\frac{D(1-\nu)}{2a^2}\left(\frac{\partial u}{\partial \beta} - \frac{\partial v}{\partial \alpha} + 2\frac{\partial^2 w}{\partial \alpha \partial \beta}\right)$$

$$Q_1 = \frac{D}{a^3}\left[\frac{\partial^2 u}{\partial \alpha^2} - \frac{1-\nu}{2}\frac{\partial^2 u}{\partial \beta^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial \alpha \partial \beta} - \frac{\partial}{\partial \alpha}\nabla^2 w\right]$$

$$Q_2 = \frac{D}{a^3}\left[(1-\nu)\frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial w}{\partial \beta} - \frac{\partial}{\partial \beta}\nabla^2 w\right].$$

The strain displacement relations can be obtained from eqn. (1).

Solution by eigenfunctions with trigonometric expressions along a generator

It may be shown that the homogeneous equation (23) and suitable boundary conditions are satisfied by taking the following solution when the eigenfunctions are trigonometric along a generator:

$$w = e^{p\beta}\cos n\alpha \quad (29)$$

where $n = \frac{m\pi a}{\ell}$

m is an arbitrary integer and ℓ is the length of the shell, whereupon p must be found from the differential equation (23). When the expression (29) is substituted into the governing equation (23), we obtain

$$(1+ik_1)p^4-(2n^2-1-ik_1)p^2+n^2[n^2-1-i(\frac{1}{k}+k_1n^2)] = 0, \tag{30}$$

which is quadratic in p^2. From this, we obtain the following expression for the roots p_1^2 and p_2^2.

$$p^2 = \frac{(1-ik_1)}{2(1+k_1^2)}\{2n^2-1-ik_1\pm[1-4(1-\nu)n^2-4k_1^2n^4-k_1^2 i\frac{2}{k}(2n^2+kk_1)]^{1/2}\}. \tag{31}$$

Four roots are obtained from the preceding equation and the other four roots are the complex conjugate numbers to these four roots. Hence the eight roots of the equation (21) are all complex, and come in two sets of four: $p = \pm\eta_1\pm i\mu_1$ and $p = \pm\eta_2\pm i\mu_2$; but, since k^{-1} appears in the last term of eqn. (30), these roots are not of different orders of magnitude. After separation of real and imaginary parts, the expression for w assumes the form

$$w = [e^{-\eta_1\beta}(A_1\cos\mu_1\beta+A_2\sin\mu_1\beta)+e^{-\eta_2\beta}(A_3\cos\mu_2\beta+A_4\sin\mu_2\beta)$$

$$+ e^{\eta_1\beta}(A_5\cos\mu_1\beta+A_6\sin\mu_1\beta)+e^{\eta_2\beta}(A_7\cos\mu_2\beta+A_8\sin\mu_2\beta)]\cos n\alpha .$$

Of course, the exponential functions can be combined into hyperbolic sines and cosines.

Solution by eigenfunctions with trigonometric expressions in the circumferential direction.

When the eigenfunctions are trigonometric in the circumferential direction, the homogeneous eqn. (23) and boundary conditions are satisfied by taking w to be

$$w = e^{p\alpha}\cos n\beta \tag{32}$$

where n is a real number. It is an integer number when the cylinder is closed and a noninteger value when the shell is open. Substitution of this into the homogeneous equation (23) yields

$$(1-ik_1)p^4 - (2n^2-1-\frac{i}{k})p^2+n^2[n^2-1+ik_1(n^2-1)] = 0\,, \tag{33}$$

which is again quadratic in p^2. The roots p_1^2, p_2^2 are found to be

$$p^2 = \frac{1}{1+k_1^2}\{n^2 - \frac{\nu}{2} - \frac{i}{2k}[1-2n^2kk_1 \pm$$

$$\pm(1+ik_1)\sqrt{1-k^2[1+k_1^2(n^2-1)]+ik[4n^2-2+kk_1(4n^2+1)(n^2-1)]}]\}. \tag{34}$$

Four roots are obtained from this equation and the other four roots are the complex conjugatesof these roots as seen from eqn. (21). Equations (31) and (34) represent the accurate characteristic roots for circular cylindrical shells, which are found for the first time in explicit forms.

Simplified equations for circular cylindrical shells.

We now proceed to show that the equation we have proposed may be used to obtain those simplified equations which may at times be inaccurate and not always applicable. Nevertheless, depending on the degree of simplifications of the original equation, a number of simplified equations can be obtained.

First let us rewrite equation (18) as

$$(\nabla^2+1)^2\nabla^4 w + \frac{1}{k^2}\frac{\partial^4 w}{\partial\alpha^4} + 2(1-\nu)\frac{\partial^2}{\partial\alpha^2}\left[\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2}\right)\nabla^2 + \frac{\partial^2}{\partial\beta^2}\right]w = 0 . \qquad (35)$$

For simplicity of presentation the surface load term and also k_1^2 terms which are small $(O[(\frac{h}{a})^2])$ as compared with the rest terms in the same equation have now been omitted. Equation (35) can be transformed into another form which is more convenient for the present purpose, if we introduce new dimensionless coordinates ξ and ζ by 'stretching' the variables α and β such that

$$\alpha = \sqrt{k}\,\xi, \qquad \beta = \sqrt{k}\,\zeta . \qquad (36)$$

Equation (35) becomes

$$\nabla_1^8 w + \frac{\partial^4 w}{\partial\zeta^4} + 2k\left[\nabla_1^6 w + (1-\nu)\frac{\partial^2}{\partial\xi^2}\left(\frac{\partial^2}{\partial\zeta^2} - \frac{\partial^2}{\partial\xi^2}\right)\nabla_1^2 w\right] +$$

$$k^2\left[\nabla_1^4 w + 2(1-\nu)\frac{\partial^4 w}{\partial\xi^2\partial\zeta^2}\right] = 0 . \qquad (37)$$

where $\nabla_1^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}$.

Once we transformed our original equation to the preceding form, the relative importance of each term in equation (37) becomes apparent. As noted previously k^2 is always a small number which is less than 4×10^{-4} in theory of thin shells, a number of relatively unimportant terms appearing in equation (37) may be discarded in order to obtain simplified equations. By dropping different terms consistant with the degree of approximation, we obtain various shell equations some of which are already well known in the literature. This way of deriving simplified equations seem more direct and systematic than those given before, which were derived individually by introducing various degrees of approximations in the strain displacement relations, the stress strain relations, etc.

(1) Donnell's equations:

When all terms with coefficients k and k^2 are omitted in equation (37), the resulting equation is the well known Donnell's equation

$$\nabla_1^8 w + \frac{\partial^4 w}{\partial \xi^4} = 0 ,$$

or

$$\nabla^8 w + \frac{1}{k^2} \frac{\partial^4 w}{\partial \alpha^4} = 0 , \qquad (38)$$

which can be factored into the form

$$(\nabla^4 + \frac{i}{k}\frac{\partial^2}{\partial\alpha^2})(\nabla^4 - \frac{i}{k}\frac{\partial^2}{\partial\alpha^2})\ w = 0\ . \qquad (39)$$

Thus the Donnell theory appears as the simplest possible governing equation for cylindrical shells under the Love-Kirchhoff hypotheses. The explanations given by Donnell [6] for neglecting a number of terms in the equations of equilibrium, in the relationships between the changes of curvature and twist and the displacements, and in the relations of stress resultants and moment resultants in terms of the displacements can not be easily followed. It is not easy to visualize the effect of so many reductions on the final solutions in spite of Donnell's seemingly reasonable arguments. It is known that as the characteristic circumferential wavelength increases so does the error in Donnell's equations [17].

(2) Morley's equation:

When only a part of the small terms are omitted in equation (35), the following equation is obtained

$$(\nabla_1^8 + \frac{\partial^4}{\partial\xi^4} + 2k\nabla_1^6 + k^2\nabla_1^4)\ w = 0\ ,$$

or

$$(\nabla^2 + 1)^2\nabla^4 w + \frac{1}{k^2}\frac{\partial^4 w}{\partial\alpha^4} = 0 \qquad (40)$$

which can be factored into the form

$$(\nabla^4 + \nabla^2 + \frac{i}{k}\frac{\partial^2}{\partial\alpha^2})(\nabla^4 + \nabla^2 - \frac{i}{k}\frac{\partial^2}{\partial\alpha^2})\ w = 0\ . \qquad (41)$$

Equation (40) was originally proposed by Morley [8]. This equation contains several improvements over Donnell's equations as has been discussed in the literature [8,18]. Lukasiewicz's equations [10] also reduce to Morley's equation.

(3) Extended Donnell's equations [9]:

$$(\nabla_1^8 + 2k\frac{\partial^6}{\partial\zeta^6} + k^2\frac{\partial^4}{\partial\zeta^4} + \frac{\partial^4}{\partial\xi^4})\, w = 0 \,,$$

or

$$(\nabla^8 + 2\frac{\partial^6}{\partial\beta^6} + \frac{\partial^4}{\partial\beta^4} + \frac{1}{k^2}\frac{\partial^4}{\partial\alpha^4})\, w = 0 \,. \tag{42}$$

Again, this equation is a special case of our original equation (37) or (35). However, this equation unlike Donnell's and Morley's equation can not be factored into two brackets.

(4) By observing the preceding equations (37), (38), (40) and (42) the following may also be obtained [11] :

$$(\nabla_1^8 + \frac{\partial^4}{\partial\xi^4} + 2k\,\nabla_1^4\frac{\partial^2}{\partial\zeta^2} + k^2\frac{\partial^4}{\partial\zeta^4})\, w = 0 \,,$$

or

$$\left(\nabla^8 + 2\frac{\partial^2}{\partial\beta^2}\nabla^4 + \frac{\partial^4}{\partial\beta^4} + \frac{1}{k^2}\frac{\partial^4}{\partial\alpha^4}\right) w = 0 \qquad (43)$$

which can be factored into the form

$$\left(\nabla^4 + \frac{\partial^2}{\partial\beta^2} \pm \frac{i}{k}\frac{\partial^2}{\partial\alpha^2}\right) w = 0 . \qquad (44)$$

In this way equation (43) comes naturally as a result of observing the preceding three simplified equations and our original equation. Equation (43) improves the accuracy of Donnell's equation yet retaining its essential simplicity [20].

(5) From equation (37), the following equation is another one of the possibilities

$$\left\{\nabla^4 + \frac{\partial^2}{\partial\beta^2} \pm i\left[\frac{1}{k}\frac{\partial^2}{\partial\alpha^2} + k_1\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2}\right)\nabla^2\right]\right\} w = 0 . \qquad (45)$$

(6) A slightly simplified equation of the proposed equation:

From eqn. (37) and also eqns. (31) and (34), it is seen that the term $ik_1 \frac{\partial^2}{\partial\beta^2}$ in the proposed equation has relatively small effect on the roots p of the characteristic equation. When this term is neglected in the proposed equation (19) one obtains the following equation:

$$L\bar{L}\, w = \frac{a^4}{D}\nabla^4 Z , \qquad (46)$$

where

$$L = (\nabla^2+1)\nabla^2 + i\left[\frac{1}{k}\frac{\partial^2}{\partial\alpha^2} + k_1\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2}\right)\nabla^2\right]. \qquad (47)$$

Thus we see that a number of equations of simplified nature can be deduced from our original equation. The method presented here is a more systematic and logical way of obtaining these simplified equations. Axial and circumferencial displacements or forces and moments can be found either from the accurate expressions (25), (26) and (2) or their simplified forms (27) and (28).

Comparison of Solutions of the Present Equation and Other Equations.

In order to prove the accuracy of the present equation based on its solutions, a comparison of the numerical results of characteristic roots has been made between the present equation and other known equations such as Flügge, Morley and Novozhilov. It is found that the difference between the characteristic roots from the present equation and the Flügge equation are always negligibly small for all values of n and k . Roots from the present equation are closer to the Flügge equation than those from any other equations such as Morley equation and Novozhilov equation. Especially when the values e^p from these equations are compared, results of the present equation are much closer to that of the Flügge equation than all other known equations. These numerical results will be shown in a second paper [20].

Conclusions.

A pair of complex conjugate fourth order equations (22) for circular cylindrical shells, which is as accurate as an equation can be under the Love-Kirchhoff assumptions has been proposed in the present paper. Because they are complex conjugates, only one of the equations, either one of the pair, need be considered. Closed form solutions of the characteristic roots which arise from solving the equations by means of the eigenfunctions are obtained.

A more direct and systematic way of obtaining these simplified equations for circular cylindrical shells, such as the Donnell equation, the Morley equation, the extended Donnell equation and other possible forms of simplified equations from the proposed equation is also presented in the paper. The essence of the present theory is contained in equations (19) to (26) and (1) to (4) or equations (19), (27), (28) and (1).

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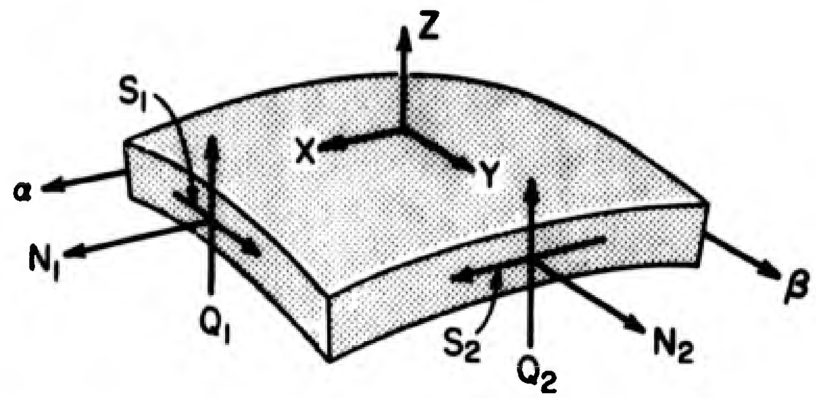

Figure 1.

Stress resultan's and surface loads acting on differential element.

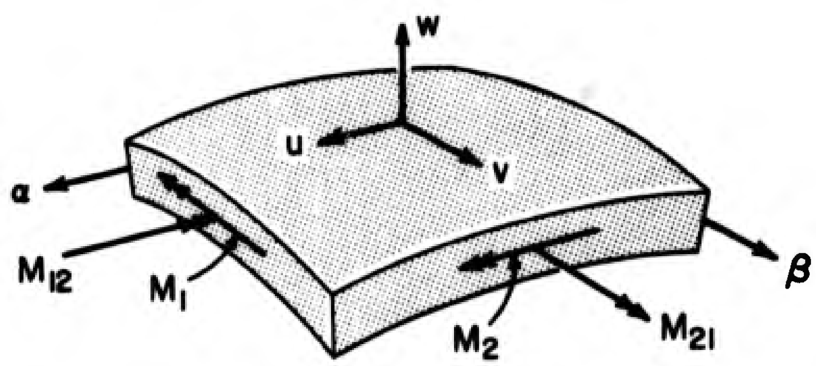

Figure 2.

Stress couples acting on differential element and midsurface displacements.