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SOME GENERALIZATIONS OF SOCIAL DECISIONS UNDER MAJORITY RULE

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SOME GENERALIZATIONS OF SOCIAL DECISIONS UNDER MAJORITY RULE*

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13. ABSTRACT

The existence of equilibrium points under majority rule is investigated for an n-dimensional issue space and an expanded class of indifference contours. This paper generalizes previous problem formulations and results.

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In recent years considerable theoretical effort has been centered on determining the existence and location of an optimal group decision or position on the basis of a variety of assumptions about the preferences of the individuals in the group. Such results as have been previously obtained suggest possible applications in phenomena ranging from small group and committee decision making to electoral strategy at the national level. Unfortunately, the applicability of previous results has been limited by the rather restrictive assumptions required to derive them. In this paper we propose a generalized theory of optimal decisions under majority rule which is based upon assumptions which are considerably less restrictive than those made in previous work.

Prior results then become important special cases of the general theory presented here. For example, it has already been shown that when only one issue exists, then the median position of the citizen's preferences on this issue has the property that no other issue position will be strictly preferred to it by a majority of the citizens (or members of the group). In the case of more than one issue, various symmetry and indifference contour assumptions have led to the result that the mean is such an "unbeatable point." For a further discussion on the history and the details of the above results see (for example) [6] and [19]. We will, however, review in more detail the formulation and results alluded to above in the next section. Furthermore, we will study a more general class of indifference contours. In particular, this class contains the "Euclidian contours" of [6] and [19] as a special case. With respect to this class of indifferent contours we generalize the median result in the one issue case to the case where the

position of all the citizens lies along some line in a multi-issue space. Then we consider special cases of our general indifferent contour class which are just as important (if not more so) than the "Euclidian contours." With respect to these special cases, we also generalize the median result from the one issue case to the two dimensional issue case. Furthermore, this generalization is made without symmetry assumptions. Finally, as in [6], we compare our results to those that result from a benevolent dictator analysis [23]. In conclusion, this paper is a generalization of not only the problem formulation but of well established results in the literature on optimal decisions under majority rule.

1. A Review of Spatial Theory and Democracy Analysis

Following [6] we consider a situation in which there are n (finite) issues and m (finite) citizens. Letting each of the n coordinate axes* of R^n correspond to an issue we can represent the location of each citizen $x_i (i=1,\ldots,m)$ in the issue space R^n. In particular,** $x_i=(x_i^1,x_i^2,\ldots,x_i^n)' \epsilon R^n$ is the (unique) most preferred position of citizen i in the issue space. Now consider an arbitrary position $\theta = (\theta^1,\theta^2,\ldots,\theta^n)' \epsilon R^n$. With respect to position θ and each citizen i we define:

$L_i(\theta) \equiv$ the loss citizen i sustains from the position θ.

In particular, $L_i(\theta)$ is a function from R^n into R^1.

To characterize the phenomena that citizen i becomes "less happy" as the "distance" between θ and x_i increases, we make the following assumption of "single peakedness" for each citizen's loss function:

(1.1) For every non-zero vector $v \epsilon R$ and every pair of scalars $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 > \lambda_2$, we require $L_i(x_i+\lambda_1 v) > L_i(x_i+\lambda_2 v)$.

In the above assumption note that $x_i + \lambda v$ defines a ray with an origin at x_i and direction v. Thus, the assumption has an interpretation of an increasing loss to each citizen i as we move farther from x_i along any ray. This is equivalent to the "single peakedness" assumption in [6].

We can now more meaningfully discuss the central point of this paper. In particular, we can now define what we mean by "best position" under majority rule. In our discussion such a position will be called an <u>equilibrium point</u>. Further, we distinguish different types of equilibrium points in the following definition.

*The set R is the set of all real numbers. The set R^n is the Cartesian product of "n" sets equal to R; i.e., RxRx ... xR. Any point on R^n is then an n-tuple of real numbers.
**We use the notation "'" to denote the transpose of a vector.

<u>Definition 1.1</u>

Consider a point $\theta^* \epsilon R^n$. For each $\theta \epsilon R^n$ define:

$p(\theta) \equiv$ the number of citizens who have $L_i(\theta^*) < L_i(\theta)$;

$q(\theta) \equiv$ the number of citizens who have $L_i(\theta^*) > L_i(\theta)$;

$r(\theta) \equiv$ the number of citizens who have $L_i(\theta^*) = L_i(\theta)$;

so that $p(\theta) + q(\theta) + r(\theta) = m$.

(i) If for every $\theta \epsilon R^n$ we have $p(\theta) \geq m/2$, then we say that θ^* is a <u>majority equilibrium point</u>.*

(ii) If for every $\theta \epsilon R^n$ we have $p(\theta) \geq q(\theta)$, then we say that θ^* is a <u>plurality equilibrium point</u>.*

(iii) If for every $\theta \epsilon R^n$ we have $p(\theta) + r(\theta) \geq m/2$, then we say that θ^* is a <u>non-minority equilibrium point</u>.

Note that any θ^* satisfying (i) will also satisfy (ii) and, similarly, any θ^* satisfying (ii) will satisfy (iii). Also note that with respect to any of the three types of equilibrium points we can consider the case where it is unique. In (i), for example, θ^* is unique if for every $\theta \neq \theta^*$ we have $p(\theta) > m/2$. Similarly in (ii) θ^* is unique if for every $\theta \neq \theta^*$ we have $p(\theta) > q(\theta)$. For a discussion of (iii) and its uniqueness properties we refer the reader to [6]. For our purposes, (iii) has some undesirable properties (e.g., see footnote 2 on page 150 of [6]) and we will not further consider it in this paper.

*Note that our use of the word "plurality" is somewhat different than usual. In particular, our use has an interpretation as a majority of those citizens who are not indifferent.

In the spatial theory literature the concept of an equilibrium point is often referred to as a dominant point. To be consistent with other literature [5], [23], we will later use dominant to have a somewhat different meaning.

We will now interpret the above definitions in a voting situation between positions θ_1 and θ_2 in R^n. To do this we make the following assumption.

(1.2) Each citizen i votes for θ_1 iff (i.e., if and only if) $L_i(\theta_1) < L_i(\theta_2)$ and, similarly, votes for θ_2 iff $L_i(\theta_2) < L_i(\theta_1)$. Thus, if $L_i(\theta_1) = L_i(\theta_2)$, citizen i is indifferent between θ_1 and θ_2 and he will not vote. Hence, if θ_1 is a majority equilibrium point, then at least one half of the citizens will always vote for θ_1 in any election regardless of the position of θ_2. If θ_1 is at a <u>unique majority equilibrium point</u> and if $\theta_2 \neq \theta_1$, then more than one half of the citizens will vote for θ_1 over θ_2. Similarly, if θ_1 is at a plurality equilibrium point, then θ_1 will always obtain at least as many votes as θ_2 (but because of indifferences not necessarily a majority of all citizens). If further θ_1 is a unique plurality equilibrium point and if $\theta_1 \neq \theta_2$, then θ_1 will obtain more votes than θ_2. It is perhaps appropriate to point out that positions θ_1 and θ_2 are often interpreted to represent the locations of two candidates in the issue space.

Again following [6] we let f(x) denote a multivariate density of preference "which characterizes the population in the sense that it represents a summary statement of the preferred positions of all citizens."

With respect to the above notation, we have the following important result in the one issue (R^1) case.

<u>Theorem 1.1</u>. Consider the case of just one issue. If θ^*

is a median of f(x) and if (1.1) holds, then θ^* is a majority equilibrium point. If further the median is unique (e.g., as in the case where m is odd), then θ^* is the unique majority equilibrium point.

The simple proof of this theorem is given on page 427 of [6].

The above result tends to strongly imply that candidates should locate at median positions on issues. This unfortunately is not in general true. Consider, for example, a situation of three citizens in a two-dimensional issue space whose loss functions have the form

$$L_i(\theta) = [(x_i^1 - \theta^1)^2 + (x_i^2 - \theta^2)^2]^{\frac{1}{2}}$$

(the usual Euclidian norm). Then for the situation illustrated in Figure 1.1, (insert Figure 1.1)where θ_1 corresponds to the median position on each issue and the circles are isoloss* contours, we can readily see that θ_2 is strictly preferred to θ_1 by citizens 1 and 2. Furthermore, θ_3 is strictly preferred to θ_2 by citizens 2 and 3 while θ_1 is strictly preferred to θ_3 by citizens 1 and 3. From this we can conclude that the "multidimensional median" θ_1 is not in general an equilibrium point. Furthermore, this same construction can be used to show that in general no equilibrium point θ^* need exist in a multidimensional issue space [6].

This does not, however, imply that equilibrium points cannot exist in multidimensional issue spaces. We will now review some conditions which guarantee the existence of an equilibrium point.

Again following [6] suppose that

(1.3) $$L_i(\theta) = \phi((x_i-\theta)' A(x_i-\theta))$$

where $\phi(.)$ is a strictly increasing function and A is a positive definite

*See Section 2 for a precise definition of this concept.

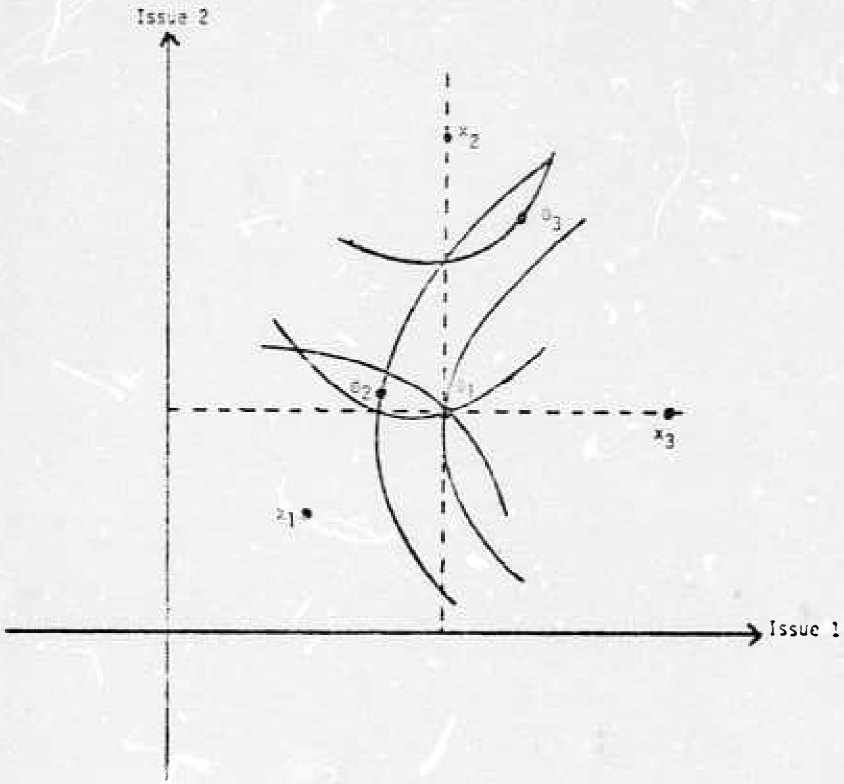


Figure 1.1

matrix (one can easily show that (1.1) is satisfied in this case). This yields isoloss contours that have the shape of a rotated ellipse about each citizen i. This is illustrated in Figure 1.2. (insert Figure 1.2). Since each citizen's loss function is defined by the same matrix A, the shape of the isoloss contours are implicitly assumed to be the same for all citizens.

Besides the assumptions implicit in the expression (1.3), following [6] we also assume that $f(x)$ is symmetric about some point θ^*. In that case, they prove that θ^* is an equilibrium point. We now state this result more precisely.

Theorem 1.2. If $f(x)$ is symmetric about θ^* and if each citizen's loss function has the form (1.3), then θ^* is a unique plurality equilibrium point.

Note that θ^* is the mean of $f(x)$. Thus, in contrast to the median results in Theorem 1.1, Theorem 1.2 gives conditions when the mean is an equilibrium point.

Before studying other multidimensional equilibrium existence questions, we first take a more detailed look at the nature of isoloss contours.

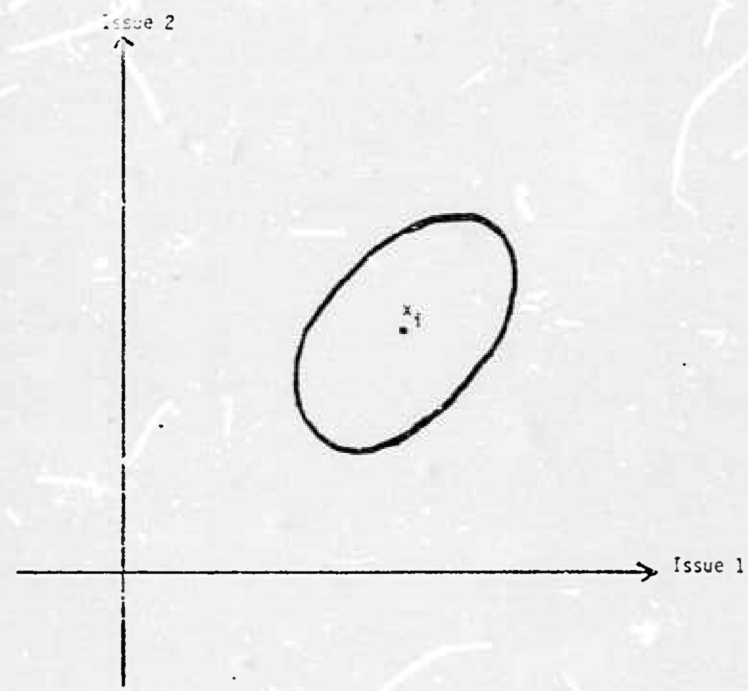


Figure 1.2

2. Indifference Contours and Norms

By an indifference contour for a citizen i we mean the locus of points about his position x_i over which he is indifferent. In Section 1 we called such a set of points an isoloss contour for citizen i. Figure 2.1a illustrates a general indifference contour for citizen i.

Consider a set of points S_I enclosed by an indifference contour I (see Figure 2.1).(insert Figure 2.1). Such a set S_I will either be convex* or nonconvex. Figure 2.1a illustrates a nonconvex set S_I while Figure 2.1b illustrates a convex set S_I. In this paper we restrict our analysis to the cases where S_I is convex. This restriction is analagous to the usual behavioral assumption of diminishing marginal rate of substitution in economics [3].

Another property that indifference curves may have is symmetry about the citizen's position x_i. More specifically, an indifference curve I for citizen i is said to be symmetric about x_i if for every $c \in I$, $2x_i - c$ is also on I. An example of a symmetric indifference curve is given in Figure 2.1c. Although symmetric indifference contours constitute a special class of contours, we will limit our discussion in this paper to them.

In summary, we consider the following class of indifference contours in this paper.

(2.1) For each citizen i we assume that each of his indifference contours

*A set S is convex if the points on a line segment connecting any two points of the set are also in S.

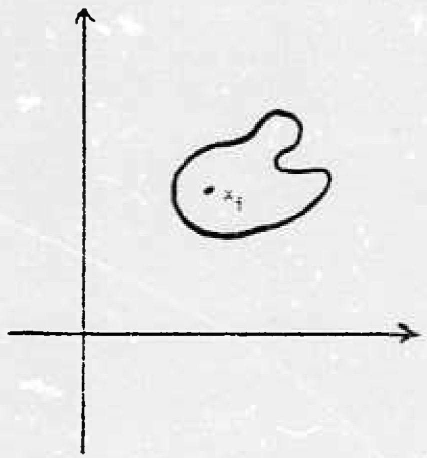


Figure 2.1a

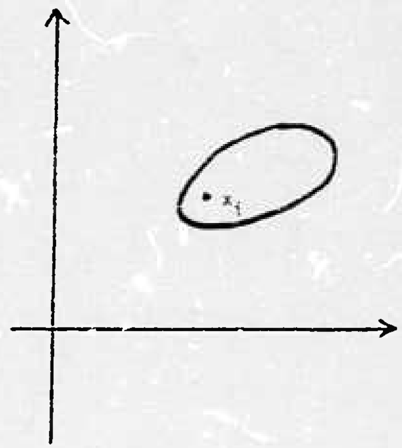


Figure 2.1b

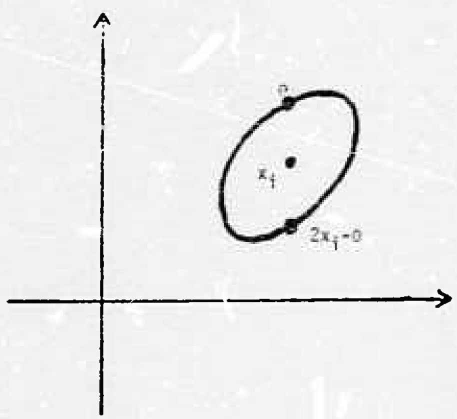


Figure 2.1c

I is symmetric about the position x_i and that each corresponding set S_I is convex.

Note that the elliptical indifference contours considered in [6] (e.g. as discussed in Section 1) satisfy assumption (2.1).

In the rest of this section we will show how indifference contours satisfying assumption (2.1) can be characterized by a function called a norm. Also, we will discuss some important special norms that have not yet been considered in democracy equilibrium problems. First, however, we define a norm.

Definition 2.1.

A function $||\cdot||$ from R^n into R^1 is a norm if it satisfies the following properties:

(i) $||\xi|| \geq 0$ for every $\xi \epsilon R^n$;

(ii) $||\xi|| = 0$ iff $\xi = 0$;

(iii) $||\alpha\xi|| = \alpha||\xi||$ where α is any positive scalar;

(iv) $||\xi_1 + \xi_2|| \leq ||\xi_1|| + ||\xi_2||$ where ξ_1 and ξ_2 are arbitrary vectors in R^n;

(v) $||\xi|| = ||-\xi||$.

Property (i) says that the norm is a function that is always greater than or equal to zero, while property (ii) says that the value of the norm is zero iff $\xi = 0 \epsilon R^n$. Property (iii) is a linear homogeneity property and property (iv) is often called the triangle inequality. Finally, property (v) is just one of symmetry.

Before giving some examples to illustrate functions which are norms, we first try to motivate our discussion. Consider the locus of those points θ that yield the same value for the function $||\theta - x_i||$. This locus of points will be called an <u>indifference concour about x_i under the norm</u>

[1.1]. Now we can state the following important result.

Theorem 2.1. To each indifference contour I for citizen i that satisfies assumption (2.1) there exists some norm $||.||$ such that $||\theta - x_i||$ generates the same indifference contour about x_i. Conversely, every indifference contour about x_i generated by a norm $||.||$ satisfies the assumptions in (2.1).

(See Theorem 15.2 of [19] for a proof of this equivalence between norms and sets S_I having the properties resulting from assumption (2.1)).

The above theorem essentially says that the class of indifference curves satisfying (1.2) is equivalent to the class of functions that are norms. More specifically, any indifference curve satisfying the assumptions in (2.1) corresponds to some norm $||.||$ and every norm generates indifference curves that satisfy (2.1). We now consider some particular norms that may have important interpretations in political theory.

Example 2.1.

Consider a norm defined as

$$||\xi|| \equiv [\xi' A \xi]^{\frac{1}{2}} \qquad (2.2)$$

where A is a positive definite matrix. For reasons to be explained below, we call this the "generalized" Euclidian norm. See Householder [12] to see how (2.2) satisfies the properties in Definition 2.1. As we will see, this norm is precisely the one considered in previous spatial theory articles [6], [18]. In fact, the shape of its indifference contours (e.g., in R^2) is that of a rotated ellipse.

For our purposes, it is important to point out that since A is positive definite one can find the eigenvectors of A and then use these vectors as a new coordinate system for the issue space. Furthermore, by making appropriate scaling charges along each of the new axes we get the

following *equivalent norm* to (2.2) (i.e., details of this equivalence are given in [6] and [18]):

$$||\xi|| = [\xi'\xi]^{\frac{1}{2}} \qquad (2.3)$$

The norm in (2.3) is just the well known Euclidian norm which we will refer to as the ℓ_2 norm (for reasons to become clearer later) and we distinguish it from other norms by writing $||.||^{(2)}$(i.e., $||\xi||^{(2)} = [\xi'\xi]^{\frac{1}{2}}$). Note that the indifference contours for the Euclidian norm (2.3) are now "circles" instead of the "ellipses" as in (2.2). From this difference one can observe that the above transition from (2.2) to (2.3) involved essentially a rotation and scaling change of the axes. Furthermore, we can conclude that the norm in (2.2) is mathematically no more general than the well known Euclidian norm (2.3).

By letting $\xi = x_i - \theta$ in (2.2) we get

$$||x_i-\theta|| = [(x_i-\theta)' A (x_i-\theta)]^{\frac{1}{2}}. \qquad (2.4)$$

Except for the square root, (2.4) is identical to the argument of ϕ in (1.3). Thus, one can see that the indifference contours to (1.3) have the same shape as that in (2.4). In particular, this is illustrated in Figure 1.2. Of course, under an appropriate basis change (as discussed above)(2.4) is equivalent to the Euclidian norm

$$||x_i-\theta||^{(2)} = [(x_i-\theta)'(x_i-\theta)]^{\frac{1}{2}} \qquad (2.5)$$

whose indifference contours are "circles" about x_i.

Example 2.2

Consider a norm defined as

$$||\xi||^{(1)} \equiv \sum_{j=1}^{n} |\xi^j| \qquad (2.6)$$

where $\xi \epsilon R^n$ (the superscript (1) is used to denote this type of norm).

The fact that (2.6) indeed satisfies the properties in Definition 2.1

is easy to orove. Such a norm is often called the city block norm, the Manhattan norm, or the ℓ_1 norm. The raason for the former names is that in various urban araas (a.g., Manhattan) the city straets ara in tha north-south and east-west directions. Given a north-south and aast-west coordinate system, a distance vector ε has a length given by the sum of absolute values of the coordinatas. This is illustrated in Figure 2.2. (insart Figure 2.2). The reason for it being called the ℓ_1 norm will become clear in Example 2.4.

Besides the above interpretation, the ℓ_1 norm has an interesting interpretation in distance percaption in an issue space. Consider a citizan's position at a point x_i and a candidate's position at the point θ. Then

$$||\theta - x_i||^{(1)} = \sum_{j=1}^{n} |\theta^j - x_i^j|$$

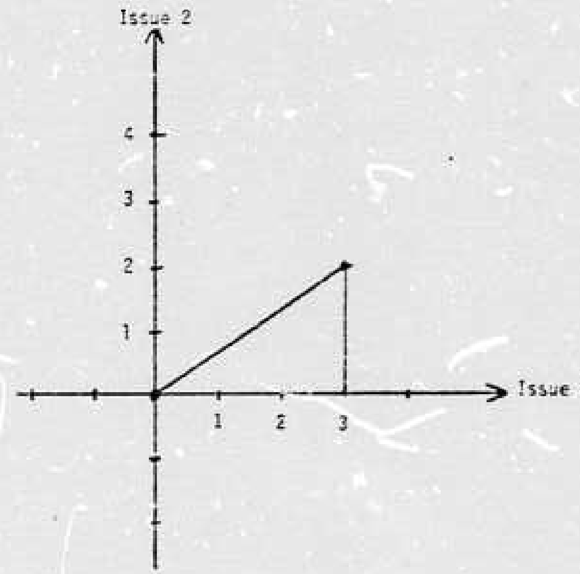
and the citizen views the distance from him to the candidate as the sum of the distance that their views vary on each issue.

Since tha empirical work of Attneava [2], the ℓ_1 norm has been of some interest to psychologists in analyzing perceptual data. Indaed the argument has baen made [2] that the ℓ_1 norm rather than the Euclidian norm should be considered as fundamental for ordaring perceptual data, sinca subjects saemed to judge dissimilarities in geometric stimuli by independently judging differences in components (dimensions). Further, tha ℓ_1 norm has nice additive properties not possessed by the Euclidian norm. Indifference contours of this type of individual are given in Figura 2.3. Note the diamond shape of these contours as opposad to the elliptical shape in Example 2.1. (insert Figure 2.3).

<u>Example 2.3</u>

Consider a norm defined as

(2.7) $$||\varepsilon||^{(\infty)} \equiv \max_{j=1,\dots,n} (|\varepsilon^j|)$$



$$||\varepsilon||^{(1)} = |3| + |2| = 3 + 2 = 5$$

$$||\varepsilon||^{(2)} = [(3)^2 + (2)^2]^{1/2} = [13]^{1/2}$$

$$||\varepsilon||^{(\infty)} = \mathrm{Max}(|3|,|2|) = 3$$

Figure 2.2

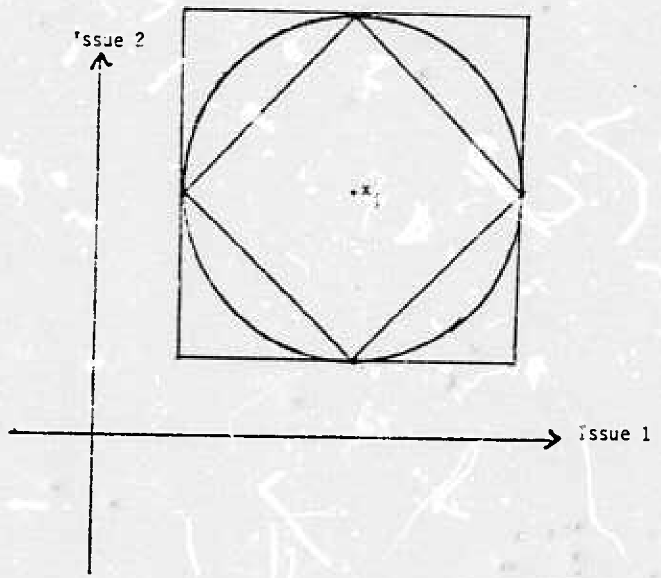


Figure 2.3

where $\xi \in R^n$ (the superscript (∞) is used to denote this type of norm).

Such a norm is often called the sup norm, the Tchebycheff norm, or the ℓ_∞ norm. Unlike the previous norm, the ℓ_∞ norm has an interpretation that the distance from a candidate to a citizen equals the maximum of the differences in positions over all issues. Mathematically, we write this as

$$||G-x_i||^{(\infty)} = \max_{j=1,\ldots,n} \{|\theta^j - x_i^j|\}.$$

This is illustrated numerically in Figure 2.2.

This norm might well be of special interest in political science. In particular, a citizen who measures distance under such a norm would ignore a candidate's position on all issues but the one which achieved maximum disparity (e.g., Vietnam).

As shown in Figure 2.3, the indifference contours for this type of norm are box-like. Since a rotation of the box is a diamond, one might suspect that the mathematical structure of ℓ_1 persons and ℓ_∞ persons is related. We will now investigate this relationship between these two norms.

<u>Theorem 2.2</u>. In R^2 the ℓ_1 norm and the ℓ_∞ norm are equivalent under a change of variables (i.e., a 45 degree rotation).

For a proof, see [29].

Just as significant as the above theorem is the fact that the norms are not equivalent in R^n for $n \geq 3$. In R^3, for example, the diamond will have 6 extreme points (e.g., corners) while a box has 8 extreme points. Thus, no matter how much one rotates the two figures, they will never become "equivalent."

Example 2.4

In a more general sense we define an ℓ_p norm for $p \geq 1$ as

$$||\varepsilon||^{(p)} \equiv \left[\sum_{j=1}^{n} |\varepsilon^j|^p\right]^{1/p}, \quad (2.8)$$

where superscript (p) is used to denote the particular ℓ_p norm. It is not especially easy to see that (2.8) defines a norm and, in particular, we must apply the Minkowski inequality to prove the triangle inequality in Definition 2.1. Note that Examples 2.1 and 2.2 are special cases when p=2 and 1, respectively. It can also be shown that Example 2.3 is a special case when $p=\infty$ [12].

To observe how the norm given in (2.2) is such a special case, note that its equivalent form (2.3) is just a special case of the ℓ_p norm when p=2. Thus, the norm previously considered in spatial theory analysis is just one of an (uncountably) infinite number. Furthermore, even ℓ_p norms do not exhaust all possible types. This is easily seen from the fact that

$$||\varepsilon|| \equiv ||\varepsilon||^{(1)} + ||\varepsilon||^{(2)}$$

is a norm that is not equivalent to any ℓ_p norm. Finally, we end this section with the following observation: althouth the ℓ_p norms can be "generalized" to allow for basis changes which correspond to rotations and scale changes (as in Example 2.1), we will, for simplicity, stick to the mathematically equivalent forms that are given in the Examples.

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3. Equilibrium Points in a Generalized Spatial Approach

Now that we know what a norm is, we will let $||.||_i$ denote a norm whose type is allowed to depend on the citizen i. For example, citizen 1 may measure distance via the ℓ_1 norm, citizen 2 may use the ℓ_2 norm, and citizen 3 may use the ℓ_∞ norm. In general, we assume that each citizen's loss function has the form

$$L_i(\theta) = C_i(||x_i-\theta||_i) \quad (3.1)$$

where $C_i(.)$ is a strictly increasing function. Note that such a formulation satisfies the "single peakedness" condition in (1.1).

If we just have one issue (i.e., n=1), then we are in the case considered by Theorem 1.1. If, on the other hand, we have more than one issue, then (3.1) results in a generalization of (1.3). In particular, if all of the norms $||.||_i$ are assumed to be of the form (2.2) and if $C_i(z) = \phi(z^2)$ for all i, then (3.1) degenerates into (1.3). Thus, previous spatial analysis becomes a special case of our formulation.

Before proceeding with our analysis of majority decision making, we will first review the definition of a multidimensional median.

Definition 3.1.

A point θ^* is said to be a multidimensional median of the points $\{x_1,\ldots,x_m\}$ if for each component θ^{j*} we have $\theta^{j*} \geq x_i^j$ for at least one half of the m citizens and $\theta^{j*} \leq x_i^j$ for at least one half of the m citizens.

When m is an even number, we have (just as in the R^1 case) the possibility that more than one point θ^* will satisfy the definition. Thus, we let M^* denote the set of median points. This is illustrated for six points in Figure 3.1. Of course, M^* consists of just one point θ^* when m is an odd number. (Insert Figure 3.1)

Now consider a generalization of the one dimensional issue space case to a multidimensional issue space where the points x_i are collinear (i.e., lie along some straight line). Such a situation is illustrated in Figure 3.2 where x_1, x_2, and x_3 lie along the line L. (Insert Figure 3.2). Of course, in a one dimensional issue space the positions are always collinear.

Given that the points $\{x_1, \ldots, x_m\}$ are collinear in R^n along some line L we now ask the following question:

(3.2) Under what conditions can we reduce the search for an equilibrium point to points on the line L?

Such a question is important since if such a reduction can be made then we will essentially be in the case considered by Theorem 1.1. In particular, then, an equilibrium position would exist at the median position(s) along line L (i.e., the point(s) $M^* \subset L$). This would be a direct multi-dimensional generalization of Theorem 1.1 to location at medians.

Unfortuantely, such a reduction can not always be made. If, for example, citizen 1 measures distance via the ℓ_1 norm, citizen 3 uses the ℓ_∞ norm, and citizen 2 uses a "generalized" Euclidian norm. (e.g., where the indifference contours are given in Figure 3.2), then no position θ_2 on L will ever beat θ_1 in a majority election. In particular, only citizen 1 could prefer positions θ_2 on L to the left of point A over the position θ_1. Similarly, only citizen 3 could prerer position θ_2 on L to the right of point B over the position θ_1. Finally, only citizen 2 could prefer position θ_2 between points A and B to the position θ_1. In short, no θ_2 along L could ever hope to beat the position θ_1 in an election.

There is, however, one important case when such a reduction can be made. Before stating this case, we first give a result upon which it depends.

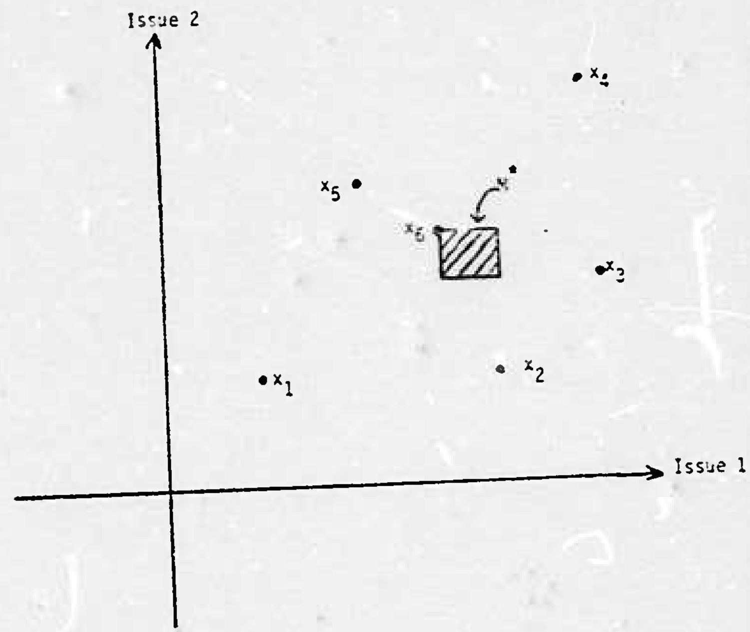

Figure 3.1

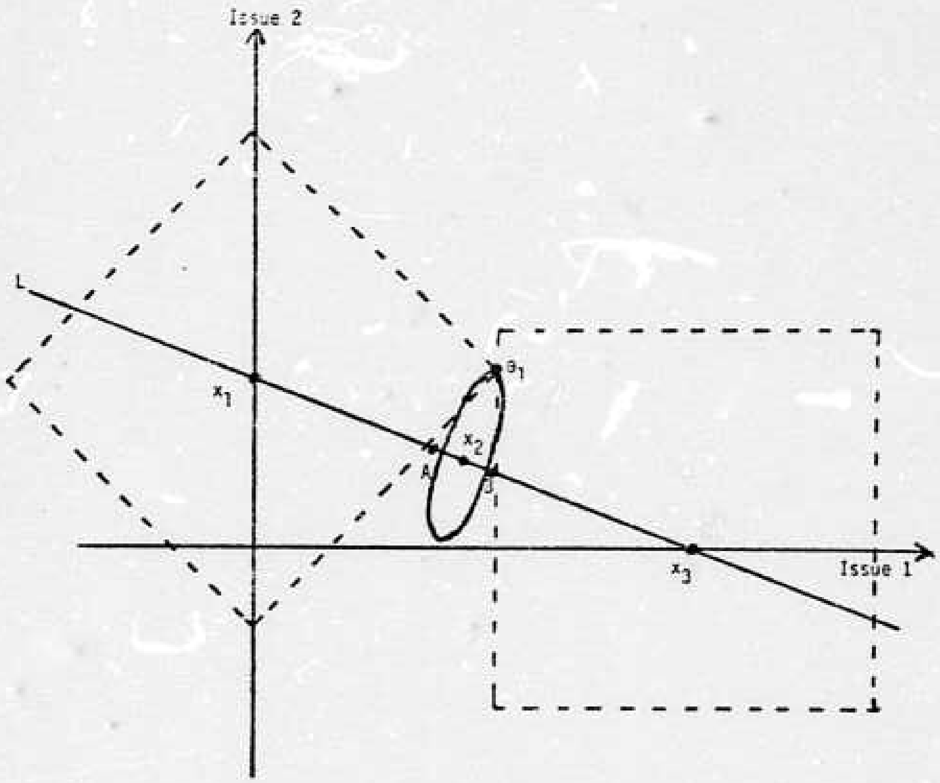

Figure 3.2

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<u>Lemma</u>. Let $||\cdot||$ be any norm. If the points $\{x_1,\ldots,x_m\}$ are collinear in R^n along a line L then for every $\theta_1 \notin L$ there exists some $\theta_2 \varepsilon L$ such that

(3.3) $\quad ||\theta_2 - x_i|| \leq ||\theta_1 - x_i||$ for $i=1,\ldots,m$.

The above Lemma follows from Theorems 2 and 3 in [25] (e.g., condition (3.3) is what is called dominance in [25] as distinct from the use of this word in spatial theory).

Note that the above Lemma says that <u>for every position θ_1 not on L there exists at least one position θ_2 on L such that each citizen i is either indifferent between θ_1 and θ_2 or the citizen strictly prefers θ_2 to θ_1</u>. Thus, positions not on L can be disregarded so that we get the following result.

<u>Theorem 3.1</u>. Suppose that the locations of citizens positions are collinear along some line L in R^n. Also suppose that each citizen measures distance in R^n via the same norm. Then a median θ^* of the positions $\{x_1,\ldots,x_m\}$ along L (i.e., $\theta^* \varepsilon M^* \subset L$) is a plurality equilibrium point. Furthermore, if m is an odd number then the median θ^* is a unique plurality equilibrium point.

The proof of the above theorem is given in the Appendix. Unlike Theorem 1.1 note that Theorem 3.1 only guarantees that the "median" is a plurality equilibrium point in contrast to a majority equilibrium point. Figure 3.3 illustrates a case where θ^* is not a majority equilibrium point (e.g., citizens 1 and 2 are indifferent between θ^* and θ). Under certain norms (e.g., the Euclidian norm) the result in Theorem 3.1 may be strengthened to assert the existence of a majority equilibrium point. We will not pursue this here. (insert Figure 3.3).

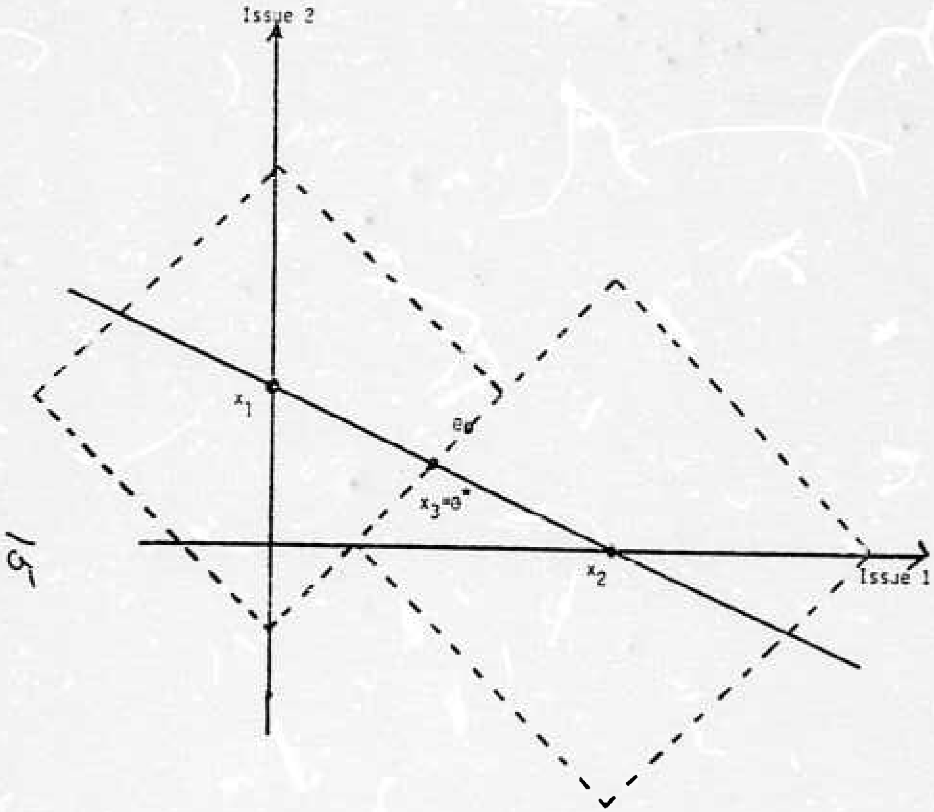

Figure 3.3

In the above discussion we have just shown that the "median" is an equilibrium point in a multidimensional issue space when the positions of the citizens are collinear and when they all measure distance via the same norm. We now proceed to again generalize the median equilibrium result from a one dimensional issue space to a multidimensional issue space situation. More specifically, we will generalize the result to a two dimensional issue space where all citizens measure distance via the ℓ_1 norm. No collinearity assumption or even symmetry assumptions about the distribution $f(x)$ of the citizens' positions will be made. In particular, we state this result as follows.

Theorem 3.2. If each of the m citizens measures distance via the ℓ_1 norm in a two dimensional issue space R^2, then the multidimensional median θ^* of the citizens' positions is a plurality equilibrium point.

Furthermore, if m is odd, then θ^* is a unique plurality equilibrium point.

Again, the proof of this theorem is given in the Appendix. Note that Figure 3.3 illustrates a case where θ^* is not a majority equilibrium point.

The above Theorem is interesting and important since it is a direct multidimensional generalization of the well known one dimensional result of location at a median. It is, however, also interesting that the result in Theorem 3.2 cannot be directly extended to R^n for $n \geq 3$. We illustrate this with an example.

Example 3.1

In the three dimensional issue space R^3 suppose that $x_1=(1,1,0)'$, $x_2=(1,0,1)'$, $x_3=(0,1,1)'$, $x_4=(-1,-1,-1)'$, and $x_5=(-1,-2,-3)'$. Note that $\theta^*=(0,0,0)'$ is the multidimensional median. Consider another position

$\bar{\theta} = (1,1,1)'$. Then, using the ℓ_1 norm $||\cdot||^{(1)}$, we have

$$||x_1-\theta^*||^{(1)} = |1-0| + |1-0| + |0-0| = 2$$

$$||x_2-\theta^*||^{(1)} = |1-0| + |0-0| + |1-0| = 2$$

$$||x_3-\theta^*||^{(1)} = |0-0| + |1-0| + |1-0| = 2$$

$$||x_1-\bar{\theta}||^{(1)} = |1-1| + |1-1| + |0-1| = 1$$

$$||x_2-\bar{\theta}||^{(1)} = |1-1| + |0-1| + |1-1| = 1$$

$$||x_3-\bar{\theta}||^{(1)} = |0-1| + |1-1| + |1-1| = 1.$$

From this we note that

$$||x_i-\theta^*||^{(1)} > ||x_i-\bar{\theta}||^{(1)} \text{ for } i=1,2,3.$$

Thus $q(\bar{\theta}) > p(\bar{\theta})$ and we conclude that θ^* is not an equilibrium point. In other words, $\bar{\theta}$ would beat θ^* in a majority election.

We now conclude this section with an examination of the equilibrium point existence question when the norm in (3.1) for all persons is the ℓ_∞ norm. Before doing this, however, we first make the following definition.

Definition 3.2.

Let $\{x_1,\ldots,x_m\}$ and θ^* be points in a two dimensional issue space R^2. Further, let $\{\bar{x}_1,\ldots,\bar{x}_m\}$ and $\bar{\theta}^*$ be the new coordinates of the points $\{x_1,\ldots,x_m\}$ and θ^* under a 45 degree rotation of the coordinate axes. Then θ^* is said to be a rotated two dimensional median of $\{x_1,\ldots,x_m\}$ if $\bar{\theta}^*$ is the multidimensional median of $\{\bar{x}_1,\ldots,\bar{x}_m\}$.

Although we will illustrate this definition in Example 3.2, we first state the following result.

Theorem 3.3. If each of the m citizens measures distance via the ℓ_∞ norm in a two dimensional issue space R^2, the rotated two dimensional

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median of $\{x_1,\ldots,x_m\}$ is a plurality equilibrium point. Furthermore, if m is odd then this is a unique plurality equilibrium point.

The proof of this theorem is a direct result of Theorem 2.2 and Theorem 3.2. We now illustrate the theorem and the definition via an example.

Example 3.2

Let $x_1 = (1,1)'$, $x_2 = (0,3)'$, $x_3 = (2,2)'$. Then, as illustrated in Figure 3.4, the coordinates under the rotations are (insert Figure 3.4)

$$\bar{x}_1 = (\sqrt{2},\ 0)',\ \bar{x}_2 = \left(\frac{3\sqrt{2}}{2},\ \frac{3\sqrt{2}}{2}\right)',\ \bar{x}_3 = (2\sqrt{2},\ 0)'.$$

The median with respect to $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ is $\bar{\theta}^* = \left(\frac{3\sqrt{2}}{2}, 0\right)'$. Note that $\bar{\theta}^*$ is the point resulting from a rotation of $\theta^* = (3/2, 3/2)'$. From the above theorem we know that θ^* is a plurality equilibrium point (and in this case a unique plurality equilibrium point since m=3). The equivalence between the results in Theorems 3.2 and 3.3 can be observed by viewing Figure 3.4 at a 45 degree angle. In particular, note that the box-like indifference contours of the ℓ_∞ norm become the diamond contours of the ℓ_1 norm.

We now make some observations about the results in Theorem 3.3. First, we believe that it is an important result in that it characterizes the equilibrium point for an important special case. Unfortunately, we are not able to give much "direct intuitive interpretation" about the point θ^*. This appears to be at least one case where the mathematics leads the intuition (at least to us). Also it should be pointed out that the extension of Theorem 3.3 to R^n for $n \geq 3$ is not clear. This is partially true because our definition of a rotated median only considers two dimensions, and since the relationship between the ℓ_1 and the ℓ_∞ norms breaks down for R^3. This is further discussed in Section 2.

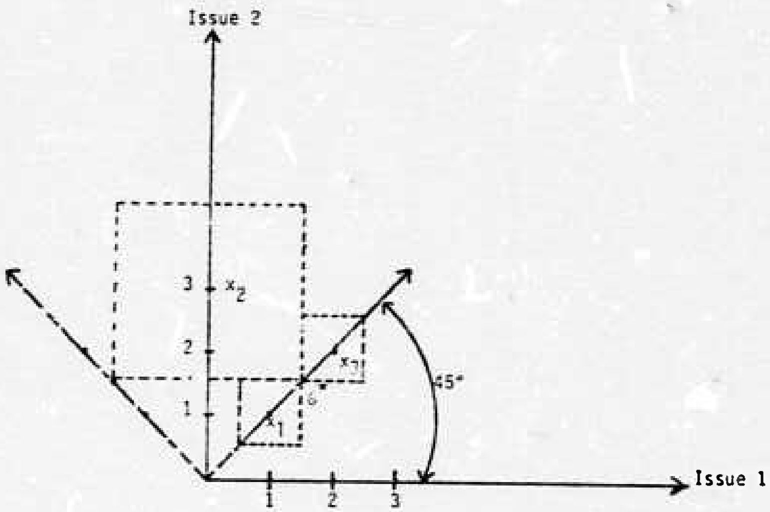

Figure 3.4

4. Conclusions

In this section, we will summarize and analyze our results. In particular, we will compare them to the optimal position that a benevolent dictator would select under a "similar" situation (the benevolent dictator problem has been extensively analyzed in [28]).

First we summarize our results of the paper. In Section 1 we define various kinds of equilibrium points (previously called dominant points) and review related results in the literature. In Section 2 we characterize a general class of indifference contours (e.g., see (2.1)) and point out that this class is equivalent to the class of functions called norms (see Theorem 2.1). Various examples are given including the "generalized" Euclidian norm used in previous multidimensional spatial theory (see example 2.1). Other examples include the ℓ_1 norm and the ℓ_∞ norm which have very important substantive interpretations in political science (see examples 2.2 and 2.3). We also point out the fact that the ℓ_1 and ℓ_∞ norms are equivalent in R^2 under a 45 degree coordinate rotation, but are not equivalent in R^n for $n \geq 3$ (see Theorem 2.2). In Section 3 we generalize the formulation of a loss function to allow for arbitrary norms and for the type of norm to depend upon the particular citizen (e.g., see (3.1)). In particular, the formulation in [6] becomes a special case. Then we consider the situation when the positions of the citizens are collinear (i.e., lie along some straight line L) in R^n. With respect to this situation we show that if all citizens use the <u>same</u> arbitrary norm to measure distance, then the median θ^* on line L will be an equilibrium point (see Theorem 3.1). If the citizens do not use the same norm then this result is not true (e.g., see Figure 3.2). Then we consider a noncollinear case in R^2 where the citizens all measure distance via the ℓ_1 norm. In that case we show that the multidimensional median is an equilibrium point (see Theorem 3.2). Example 3.1 shows that this result is not true in R^n for $n \geq 3$. Similar to that case, we also consider a situation when again we are in a two dimensional issue space but where the citizens all measure distance via the ℓ_∞ norm. Here we establish that the rotated two dimensional median (see Definition 3.2) is an equilibrium point (see Theorem 3.3). In short, this paper presents both a generalization in the formulation and the results in the spatial theory of majority decision making.

We now compare our results of location at a median to where a benevolent dictator would locate in a "similar" situation. First we consider the case where the citizens positions lie along some line L in R^n and where the dictator desires to pick a location θ^* that minimizes the sum of distances to the citizens (e.g., if every citizen has the same loss function which is linear in distance, then the dictator is essentially minimizing total utility loss to society [28]). If each citizen uses the same norm then the median along L will be the optimal location of the dictator. If the citizens do not all have the same norm, then the optimal dictator position might not even be on the line L. These results are analagous to our results in this paper. If, on the other hand, the citizens do not have collinear positions but if they all use the ℓ_1 norm, then the dictator (with the same criterion as discussed above) will again select the median as his optimal position. Again, this is analagous to our result in Theorem 3.2 except that Theorem 3.2 only applies to a two dimensional issue space.

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Similarly, an analogy exists for the ℓ_∞ case. We can summarize this comparison by concluding that a democracy and dictatorship will both arrive at the same decision in similar situations where this decision is characterized by the median.

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APPENDIX

Proof of Theorem 3.1

Suppose that θ^* is not a plurality equilibrium point. Then from Definition 1.1 we assert the existence of some $\theta \in R^n$ such that $q(\theta) > p(\theta)$ (i.e., the number of citizens strictly preferring θ over θ^* is greater than the number strictly preferring θ^* over θ).

Now if $\theta \notin L$ then by the Lemma we assert the existence of $\bar{\theta} \in L$ such that $||\bar{\theta}-x_i|| \leq ||\theta-x_i||$ for $i=1,\ldots,m$. If $\theta \in L$, let $\bar{\theta} = \theta$. Thus we have a position $\bar{\theta} \in L$ such that

$$||\bar{\theta}-x_i|| \leq ||\theta-x_i|| \quad \text{for } i=1,\ldots,m.$$

Consider two cases:

(i) $\bar{\theta} = \theta^*$;

(ii) $\bar{\theta} \neq \theta^*$.

In case (i) all citizens either strictly prefer $\bar{\theta} = \theta^*$ to θ or they are indifferent. This, however, contradicts our assertion that $q(\theta) > p(\theta)$. Thus case (i) is impossible.

We now consider case (ii). Since $\bar{\theta} \in L$ and since $\bar{\theta} \neq \theta^*$, the properties of a norm imply that at least one half of the citizens would strictly prefer θ^* to $\bar{\theta}$ (i.e., those citizens on L who are either located at θ^* or are on the opposite side of θ^* from $\bar{\theta}$). In particular, for these citizens (who compose at least one half of all the citizens m) we have

$$||\theta^*-x_i|| < ||\bar{\theta}-x_i|| .$$

For these same citizens we have

$$||\theta^*-x_i|| < ||\theta-x_i|| .$$

Thus, for these same citizens, we have from (3.1) that

$$L_i(\theta^*) < L_i(\theta) .$$

And again, since at least one half of the citizens have this preference, we conclude that $p(\theta) \geqq q(\theta)$. This is a contradiction to $q(\theta) > p(\theta)$ and we conclude that case (ii) is also impossible. Thus, by contradiction we have shown that a median θ^* must be a plurality equilibrium point.

To prove the uniqueness when m is odd we again consider two cases:

(i) $\bar{\theta} = \theta^*$;

(ii) $\bar{\theta} \neq \theta^*$.

In case (ii) we can use the same argument as before to argue that more than one half of the citizens will prefer θ^* to θ . Thus, in this case, we have $p(\theta) \geqq q(\theta)$ which would contradict $q(\theta) \leqq p(\theta)$.

In case (i) we have shown that the citizen located at the median will strictly prefer θ^* to θ while all the other citizens will either strictly prefer θ^* to θ or they will be indifferent. Thus, again, we have $p(\theta) \geqq q(\theta)$ which would contradict $q(\theta) \leqq p(\theta)$. Hence, we have proved the theorem.

Proof of Theorem 3.2

We exploit the fact that we are in R^2 by giving a simple geometric proof. In particular, we illustrate a typical situation in Figure A.1, where, for the six points $\{x_1,x_2,x_3,x_4,x_5,x_6\}$, θ^* is a point in M^*. Note that we have divided the R^2 space into "pie-shaped" regions via the lines AA', BB', CC', and DD'. (insert Figure A.1).

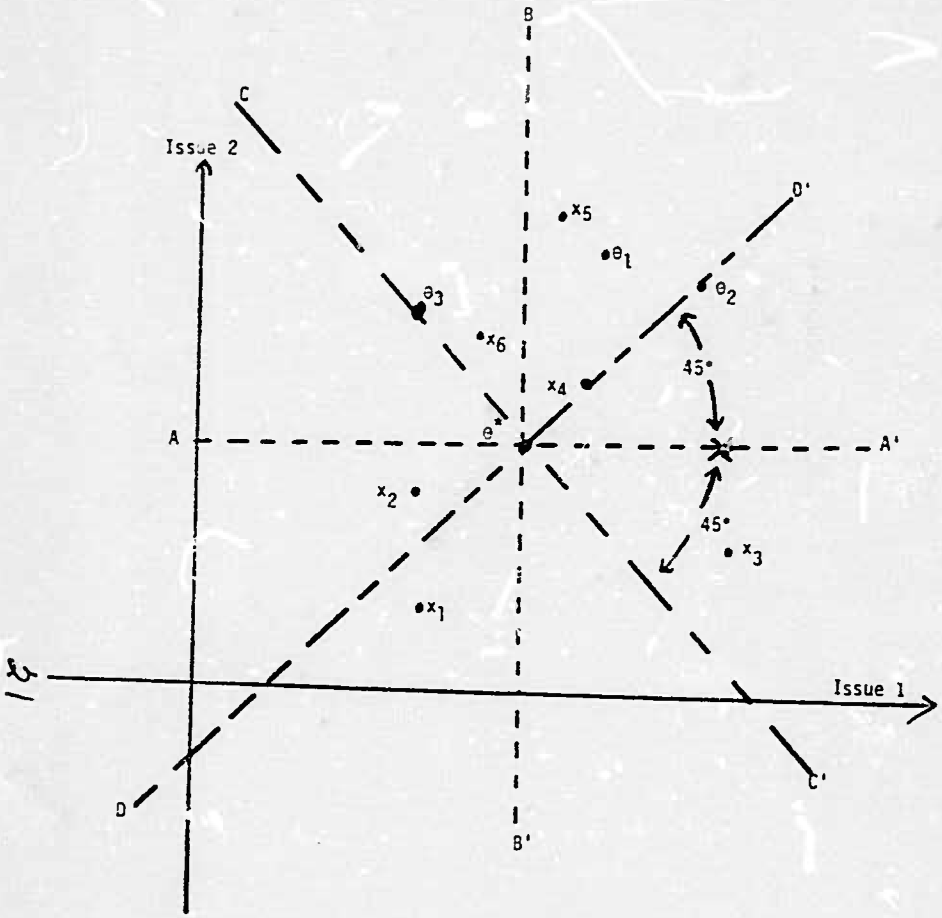

Figure A.1

Now suppose that θ^* is not a plurality equilibrium point. Then there exists some $\theta \epsilon R^2$ such that there are more citizens who strictly prefer θ than there are citizens who strictly prefer θ^* (i.e., $q(\theta) > p(\theta)$). Clearly θ will be in one of the four "pie-shaped" regions $C\theta^*D'$, $C'\theta^*D'$, $C'\theta^*D$, or $C\theta^*D$. First let's suppose that it is in the region $C\theta^*D'$. In that case there are only three possibilities:

(i) θ lies interior to the region $C\theta^*D'$ (as illustrated by the point θ_1);

(ii) θ lies on the half line θ^*D' (as illustrated by the point θ_2);

(iii) θ lies on the half line θ^*C (as illustrated by the point θ_3).

First we consider case (i). Since θ^* is a median position with respect to issue 2, at least one half of the citizens will have positions on or below the line AA'. By considering the indifference contours of each of these citizens (as illustrated in Figure 2.3), it is clear that each of these citizens would strictly prefer θ^* to θ. Therefore, in this case, we would have $p(\theta) \geq q(\theta)$. Since this contradicts $q(\theta) > p(\theta)$, case (i) is impossible.

We now consider case (ii). Define:

$Y_1 \equiv$ the number of citizens in the region $A\theta^*B'$;

$Y_2 \equiv$ the number of citizens in the interior of the region $A'\theta^*B'$ plus the number of citizens on the half line θ^*A' (with the exception of the point θ^*).

$Y_3 \equiv$ the number of citizens in the interior of the region $A\theta^*B$ plus the number of citizens on the half line θ^*B (with the exception of the point θ^*);

γ_4 = the number of citizens in the interior of the region $A'\theta^*B$. Note that the above classification divides R^2 into four non-overlapping areas. Furthermore, since θ^* is a median we have that

$$\gamma_1 + \gamma_2 \geqq \gamma_3 + \gamma_4$$

and

$$\gamma_1 + \gamma_3 \geqq \gamma_2 + \gamma_4 \ .$$

These inequalities imply that

$$(\gamma_1 + \gamma_2) - (\gamma_2 + \gamma_4) \geqq (\gamma_3 + \gamma_4) - (\gamma_1 + \gamma_3) \ .$$

This reduces to the inequality $\gamma_1 \geqq \gamma_4$.

We again appeal to the shape of the $\succsim_1$ indifference contours to assert that for θ on the half line θ^*D': the γ_1 citizens will strictly prefer θ^* over θ; the γ_2 and γ_3 citizens will either be indifferent between θ^* and θ or they will strictly prefer θ^* over θ; and the γ_4 citizens will strictly prefer θ over θ^*. But since $\gamma_1 \geqq \gamma_4$ we conclude that the number of citizens strictly preferring θ^* over θ is greater than the number strictly preferring θ over θ^*. Thus $p(\theta) \geqq q(\theta)$ which contradicts $p(\theta) > p(\theta)$ and we conclude that case (ii) is impossible. By symmetry we assert that case (iii) is impossible. Also by symmetry we assert that the same arguments can be made for any of the other regions $C'\theta^*D'$, $C'\theta^*D$, or $C\theta^*D$. Hence no such θ can exist and the first part of the theorem is proved by contradiction.

We now prove the assertion in the second part of the theorem that θ^* is the unique plurality equilibrium point when m is odd. If not true, then there exists some θ such that $q(\theta) \geqq p(\theta)$. Following the proof to the first part of the theorem, we first consider case (i). Since m is odd, we can assert that over one half of the citizens lie on or below line AA'. Thus

we get that $p(\theta) > q(\theta)$ which is a contradiction. If case (ii) applies one can see that

$$\gamma_1 + \gamma_2 > \gamma_3 + \gamma_4$$

and

$$\gamma_1 + \gamma_3 > \gamma_2 + \gamma_4$$

which imply that $\gamma_1 > \gamma_4$. Again we get that $p(\theta) > q(\theta)$ which is a contradiction. By symmetry we conclude that the second part of the theorem is also true.