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USE OF SYMBOLIC MANIPULATION TECHNIQUES TO EXAMINE GRAVITATIONAL AND UNIFIED FIELD THEORIES

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Perception Technology Corporation

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This is the six month technical report for the contract entitled "Use of Symbolic Manipulation Techniques to Examine Gravitational and Unified Field Theories". Following the report summary are seven appendices. The first is a published description of the New Theory of Gravitation and its ramification while the subsequent six are concerned with research performed during the first five months of the present contract.

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Research in the use of
Symbolic Manipulation Techniques
to Examine Gravitational and
Unified Field Theories

1. TECHNICAL PROBLEM

We are investigating the consequences of a new theory of gravitation by employing the symbolic manipulation system MACSYMA developed by the Mathematical Laboratory, Project MAC, MIT. The new theory was formulated by Dr. Huseyin Yilmaz (See Section 2) over the last twenty years and is believed to represent the only viable alternative to General Relativity, Einstein's gravitational theory.

There are three main objectives in the contract which are currently being investigated using MACSYMA. The theoretical question we shall answer concerns the verification that a conjectured relation satisfies the field equations of the theory to at least second order. Unlike the usual field theory in which field equations are written down and subsequently solved, the new theory presents a conjectured solution (Reference 5, Section 2, or Appendix 1) of the field equations to be verified in detail. A related theoretical question which we hope to answer is that the equations of motion of objects follow from the field equations and conservation laws. This feature has never been well implemented in the Einsteinian theory.

The new theory differs from General Relativity in many fundamental ways. We have shown, for example, that certain types of gravitational radiation believed to exist in General Relativity in fact are absent in that theory (Appendix 2). Also, it seems clear in Einstein's theory that gravitational waves (if they exist) cannot carry energy. On the other hand we have shown that gravitational radiation does exist in the New Theory and these waves do carry energy. In addition we are considering the New Theory as a unified field theory in which gravitation and/or electromagnetism arise from the field equations. It is hoped that these studies will answer the question of whether information may be transmitted via gravitational waves and whether gravitation modifies electromagnetic radiation to any practically significant extent.

2. LITERATURE REVIEW

The basic literature concerning the new theory consists essentially of the following 11 publications by Dr. Huseyin Yilmaz.

1. H. Yilmaz, Phys. Rev., 111, p. 1417 (1958).

2. H. Yilmaz, "Evidence for Gravitational Theories", ed. C. Moller, Academic Press, New York (1962).

3. H. Yilmaz, "Introduction to the Theory of Relativity", Blaisdell Publishing Company, New York (1965).

4. H. Yilmaz, Proceedings of the Rochester Meeting of the Division of Particles and Fields of the American Physical Society. ed. A. C. Melissinos and P. F. Slattery, American Institute of Physics, p. 259, 1971.

5. H. Yilmaz, "New Theory of Gravitation", Physical Review Letters, vol. 27, p. 1399, November 1971.

6. H. Yilmaz, Nuovo Cimento, 10B, p. 79, 1972.

7. H. Yilmaz, "On the New Theory of Gravitation", Lettere Al Nuovo Cimento, vol. 5, no. 4, September 1972.

8. H. Yilmaz, "Correspondence Paradox in General Relativity", Lettere Al Nuovo Cimento, vol. 7, no. 9, p. 337, June 1973.

9. H. Yilmaz, "Postulational Derivation of the New Theory of Gravitation", Lettere Al Nuovo Cimento, vol. 6, no. 5, p. 181, Feb. 1973.

10. H. Yilmaz, Physics Today, p. 15, March 1973.

11. H. Yilmaz, Annals of Physics, vol. 81, pp. 179-200, 1973.

3. TECHNICAL RESULTS TO NOVEMBER 30, 1973

In the following, we present an outline of our progress to date including new results we have obtained as well as the new capabilities of the MACSYMA system we have introduced. We have shown that plane gravitational waves in vacuum (contrary to current belief) do not exist in Einstein's theory (Appendix 2). Previous calculations proved the existence of the waves in Einstein's theory in first order only. Claims for exact plane wave solutions were advanced but never rigorously demonstrated. Our surprising result that such waves do not exist in Einstein's theory follows by computing the Einstein tensor to second order and beyond using the MACSYMA system.

In a similar way it is shown that such waves do indeed exist in the New Theory (Appendix 2, and Reference 11).

Using the MACSYMA system we have further found the <u>general static solutions</u> of the Einstein vacuum equations and Einstein's equations with a cosmical term in spherical coordinates (Appendix 3). We have also found the general solution of Einstein's vacuum equations in Cartesian-like coordinates (Appendix 4). It is doubtful that such solutions could be found by hand calculations. The enormous value of the MACSYMA system is here evident in helping us to formulate conjectures and making it possible to actually test them. The few previously known solutions are shown to be special cases of the general solution for appropriate choices of a free function which appears in the general solution. In addition, while investigating the Einsteinian theory we have discovered solutions which seem to violate the very foundations of the theory itself (Appendix 5).

We have implemented a first step in the MACSYMA system to allow indicial symbolic manipulation (Appendix 6). Using these new programs we have verified the first order functional expansion described above. This is a <u>major</u> development from the standpoint of symbolic manipulation (to our knowledge no other system has this facility). We shall gradually expand and improve this program.

We have made some progress in the understanding of matrices generated by expansions of the form $e^{\tilde{\lambda}} = \tilde{1} + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2!} + \ldots$ where $\tilde{\lambda}$ is a square matrix (Appendix 7). Such considerations are necessary for the study of unified field theory as well as the new theory in its present form. Expansions of this type are easily generated by the MACSYMA system and has allowed us to obtain many closed forms of the series for particular choices of $\tilde{\lambda}$.

We have begun to investigate stress-energy tensors generated by metrics with skew-symmetric parts. This is part of the effort directed toward a unified field theory.

The following list of capabilities of the MACSYMA system is directly attributable to the combined effort of Perception Technology Corporation and our program consultants at Project MAC. At the start of our contract the MACSYMA system was only capable of computing, for a given metric, the inverse metric, Christoffel symbols and Ricci tensors. The latter of these had a bug which was discovered and corrected by Perception Technology Corporation. The list of capabilities as of November 30, 1973 include:

1) For any metric tensor the Ricci tensors and Einstein tensors (covariant or mixed) can be computed in CRE (canonically rational expression) form.

2) For any metric tensor, the inverse metric, Christoffel symbols of both kinds, and mixed Einstein tensors can be computed in CRE or in a truncated power series to arbitrary order in an expansion parameter. We are limited only in the memory of the system.

3) For any metric tensor the components of the full Riemann tensor can be computed in CRE or in a truncated power series.

4) For any metric tensor the equations of motion can computed in CRE or in a truncated power series.

5) For any metric tensor the d'Alembertian and the t_{μ}^{ν} of the new theory, both applied to a scalar field, can be computed.

6) For any metric tensor the components of any second rank tensor in a transformed coordinate system can be computed.

7) For a function $\phi(z-t)$ one has the differential identities $\phi_z \equiv -\phi_t$, $\phi_{zt} \equiv -\phi_{tt}$, $\phi_{zz} \equiv +\phi_{tt}$ which are the wave conditions of certain specially chosen metrics. For components of the metric tensor containing functions of this kind the wave conditions may now be automatically implemented with MACSYMA thus tremendously shortening computations concerned with gravitational radiation.

8) The indicial manipulation program is now able to compute all Ricci and Einstein tensor components in first order. Soon the capability will extend to second order.

APPENDIX 1

New Theory of Gravitation

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(Received 2 June 1971)

A locally Lorentz-invariant curved-space theory of gravitation where the local field is a massless, spin-2 field $\varphi_\mu{}^\nu$ of Pauli-Fierz type is presented. In this theory $g_{\mu\nu}$ is no longer the gravitational potential itself but reduces to a functional $g_{\mu\nu}(\varphi_\alpha{}^\beta)$ of $\varphi_\mu{}^\nu$. Several rigorous and a general iteration solution of this type are exhibited. The static central body problem reduces exactly to author's 1958 theory in the special case $\varphi_\mu{}^\nu \to \varphi_0{}^0 = \varphi$ and $\partial^\mu \varphi_\mu{}^\nu \to \partial^0 \varphi = 0$ so that as in that theory the three crucial tests are satisfied.

About ten years ago there existed at least seven, more or less plausible, theories of gravitation. Three of these were the flat-space theories of Poincaré, Whitehead, and Birkhoff. In the curved-space category were the theories of Einstein, Jordan, Brans-Dicke, and the author, Einstein's theory being of course the most prominent among them. In 1961 with the measurement of gravitational red shift in earthbound laboratories it became clear that the flat-space theories have to be abandoned.[1] This situation led, since 1961, to a vigorous development of Jordan and Dicke type theories (Brans-Dicke theory in fact belongs to a class of Jordan theories) and until recently Brans-Dicke theory was considered quite seriously in many quarters.

In the fall of 1970, however, the experiment on the time delay of radar signals reflected from Venus and Mercury reached accuracies high enough to permit at least a preliminary judgement against the Jordan and Brans-Dicke type of theory.[2] If the present trend of results continues, the 1916 theory of Einstein and the largely undeveloped 1958 theory of the present author will probably be the only two generally known and experimentally viable theories of gravitation.[3] On the other hand the 1958 theory of the author, as published, has the theoretically objectionable feature that it does not possess time-dependent dynamic solutions. This of course casts serious doubts on the validity of this theory leaving Einstein's 1916 theory practically unrivaled. We are therefore motivated to present a natural extension of our 1958 theory which seems to open a new and interesting avenue of inquiry as well as avoiding the above mentioned objection.

Formulation of the theory.—One of the main differences between Einstein's theory and the present theory of gravitation is that in Einstein's theory the components of the metric tensor $g_{\mu\nu}$ are considered as the components of the gravitational potential and are therefore functions of space-time variables as $g_{\mu\nu}(x)$, whereas in the new theory the gravitational potentials form a different field $\varphi_\mu{}^\nu$ and $g_{\mu\nu}$ are functions or functionals of these as $g_{\mu\nu}(\varphi_\mu{}^\nu)$. Thus in a sense the new approach considers gravitation as a local field theory in a curved geometry $g_{\mu\nu}$ whose curvature arises from the existence of the fields themselves.

With this conception in mind it is possible to formulate the theory of gravitation directly as a functional relation between the fields $\varphi_\mu{}^\nu$ and the metric $g_{\mu\nu}$. We shall state this relation in the form of a functional differential equation,

$$dg_{\mu\nu}=2(g_{\mu\nu}d\varphi-g_{\mu\alpha}d\varphi_\nu{}^\alpha-g_{\nu\alpha}d\varphi_\mu{}^\alpha),\qquad(1)$$

where φ is the trace of $\varphi_\mu{}^\nu$. Equation (1) is here intended to represent a relation between the components of $\varphi_\mu{}^\nu$ and $g_{\mu\nu}$ in certain preferred coordinate system and therefore it is not *a priori* coordinate invariant. The preferred coordinates thus introduced will be seen to have an interpretation as gauge fields satisfying a sourceless wave equation. In accordance with our local requirements above, the $g_{\mu\nu}$ may be assumed to reduce to the special relativistic form $\eta_{\mu\nu}=(1,-1,-1,-1)$ when $d\varphi_\alpha{}^\beta=0$. In other words in the absence of the fields or their variations the metric is assumed to go over into a Lorentz form which can be done without loss of generality.[3] The fields $\varphi_\mu{}^\nu$ themselves will then arise via the distributions of matter according to the locally special relativistic equations

$$\varphi_\mu{}^\nu=\int(\sigma u_\mu u^\nu)'dV'/r',\qquad(2)$$

where the prime implies the usual retardation consideration on the integral. We shall therefore have the propagation equations

$$\Box^2\varphi_\mu{}^\nu=4\pi\sigma u_\mu u^\nu,\qquad(3)$$

where $\sigma u_\mu u^\nu$ is the usual "matter" tensor and $\Box^2$ is the general d'Alembertian $\Box^2=(\sqrt{-g})^{-1}\partial_\nu(\sqrt{-g}\times g^{\mu\nu}\partial_\mu)$ which under (7) below turns out to be $\Box^2=g^{\mu\lambda}\partial_\mu\partial_\lambda=\partial^\lambda\partial_\lambda$. The symbol ∂_μ denotes ordinary derivative in the coordinates in which (1) holds.

We may now point out that in such a gravitational field the energy-momentum "vector" of matter alone cannot be a conserved quantity and in fact it cannot be a true vector. We must add a "field" stress-energy term $t_\mu{}^\nu$ to $\sigma u_\mu u^\nu$ in order to obtain a conserved energy-momentum vector. We write

$$T_\mu{}^\nu=\sigma u_\mu u^\nu+(1/4\pi)t_\mu{}^\nu,\qquad(4)$$

where $t_\mu{}^\nu$ is not possible to determine uniquely from the usual special relativistic field theory arguments because the divergence of an arbitrary vector can be added to the Lagrangian and this will yield a different although equivalent expression. We shall, however, require that, as computed directly through (1), the geometric, divergence-free tensor $R_\mu{}^\nu-\frac{1}{2}\delta_\mu{}^\nu R$ will be equal to $8\pi T_\mu{}^\nu$. In other words we require the field equations to be of the form

$$R_\mu{}^\nu-\tfrac{1}{2}\delta_\mu{}^\nu R=2(\Box^2\varphi_\mu{}^\nu+t_\mu{}^\nu)=8\pi T_\mu{}^\nu.\qquad(5)$$

In this way we are able to guarantee the conservation of the energy-momentum vector identically and at the same time determine the form of $t_\mu{}^\nu$ essentially up to an additive cosmological constant Λ. We note again that under the combined effect of (1) and (7) the covariant derivative of the left-hand side of (5) is expressed in terms of the ordinary derivatives to yield the middle expression so that the conservation can hold in (5) with ordinary rather than covariant derivatives because of (1) and (7). In fact under (7) below the empty-space value of R ($\sigma=0$) gives the Lagrangian

$$L_f=\frac{R}{4}=-\partial^\lambda\varphi_\alpha{}^\beta\partial_\lambda\varphi_\beta{}^\alpha+\tfrac{1}{2}\partial^\lambda\varphi\partial_\lambda\varphi\qquad(6)$$

yielding, indeed, a familiar special relativistic $t_\mu{}^\nu$ corresponding to the fields $\varphi_\mu{}^\nu$, φ.

It is to be noted carefully that in order for the second-order derivatives of $\varphi_\mu{}^\nu$ to combine properly into $\partial^\lambda\partial_\lambda\varphi_\mu{}^\nu$ the field must satisfy a subsidiary condition

$$\partial^\mu\varphi_\mu{}^\nu=0,\qquad(7)$$

which, contrary to a first impression, does not imply a separate conservation law for the matter part alone. It rather implies that locally the gravitational field will propagate with the velocity of light c in all directions. Since the absolute values of $\varphi_\mu{}^\nu$ are unobservable and since under (1) and (7) $\Box^2$ no longer contains quadratic first-order derivatives of $\varphi_\mu{}^\nu$, the implied local coordinates are interpretable as sourceless gauge fields quite analogously to special relativity where x, y, z, and t satisfy a sourceless wave equation. It is, however, essential to keep (7) as a separate condition so that it can be dropped to allow general coordinate transformations.

Finally, we adopt the geodesic equations of motion

$$\frac{d^2x^\lambda}{ds^2}+\left\{{\lambda \atop \mu\nu}\right\}u^\mu u^\nu=0\qquad(8)$$

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as our equations governing the motions of test particles and light rays. We are able to do this because any line element that is a solution of (1) and (7) is, by construction, also a solution of (5). Since under certain general conditions the geodesic equations above are known to be a consequence of field equations (5), we are justified, as in Einstein's theory, in postulating (8) as our equations of motion. (See also our concluding remarks at the end of the paper.) The theory thus introduced goes over into our 1958 form in the limit of static fields $\varphi_\mu{}^\nu \to \varphi_0{}^0 \to \varphi$, $\partial^0\varphi = 0$.

Comparison with Einstein's theory.—It is now possible to start comparing the above theory with Einstein's theory to see that the two theories are very different conceptually and yet they give the same answers for the three classical tests of general relativity. For the time delay of radar signals echoed from Venus and also for the experiment on the isotropy of inertia there seem to be certain differences in prediction or interpretation. The theoretical differences are mainly related to the fact that in Einstein's theory one assumes the right-hand side of (5) to be the "matter" tensor alone, hence $t_\mu{}^\nu$ does not enter. Consequently in empty space where $\sigma = 0$ the matter tensor vanishes as well and one has the Einsteinian law

$$R_\mu{}^\nu = 0. \qquad (9)$$

By solving these equations for the static gravitational field of the sun one obtains the usual Schwarzschild line elements. In the present theory this static case corresponds via (3) to $\varphi_0{}^0 = \varphi = M/r$, and putting this into (1) we immediately have the line element

$$ds^2 = e^{-2\varphi}dt^2 - e^{2\varphi}(dx^2 + dy^2 + dz^2). \qquad (10)$$

The reader could check easily that upon computing the tensor $R_\mu{}^\nu - \frac{1}{2}\delta_\mu{}^\nu R$ with this line element one obtains for $t_\mu{}^\nu$ exactly the expression (6) with $\varphi_\mu{}^\nu \to \varphi_0{}^0 = \varphi$. He could also repeat this process for other solutions given below. We can see that the two theories are in fact very different. For example, in our theory we now have, instead of $R_\mu{}^\nu = 0$, the new empty space law

$$R_\mu{}^\nu = 2(t_\mu{}^\nu - \tfrac{1}{2}\delta_\mu{}^\nu t) \qquad (11)$$

exhibiting a local field theory form explicitly.

We may point out immediately that we do not have to solve the highly nonlinear equations (11) directly in order to obtain a line element. Instead we can simply deduce it from our functional equations (1) as we have just done to obtain the static line element (10). As other examples of such exact solutions let all components of $\varphi_\mu{}^\nu$ other than $\varphi_1{}^2(t,z)$ vanish. Then the solution is $g_{00} = -g_{33} = 1$, $g_{11} = g_{22} = -\cosh 4\varphi_1{}^2$, $g_{12} = g_{21} = \sinh 4\varphi_1{}^2$. If only $\varphi_0{}^3(x,y)$ exists then $g_{11} = g_{11} = -1$, $g_{00} = -g_{33} = \cos 4\varphi_0{}^3$, $g_{10} = g_{01} = \sin 4\varphi_0{}^3$, etc. The latter solution with a slight extension is found to provide an explanation (through an argument similar to that given by Lense and Thirring in 1918) for the rotating-disk and Sagnac type of experiments, and the former represents a plane gravitational wave which carries energy-momentum. There are other interesting exact solutions one of which seems to describe cylindrical gravitational waves.[5] A general iteration solution (1) with the boundary conditions that at the observation point $g_{\mu\nu}$ goes over into a Lorentz form $\eta_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + 2[\eta_{\mu\nu}\varphi - \eta_{\mu\alpha}\varphi_\nu{}^\alpha - \eta_{\nu\alpha}\varphi_\mu{}^\alpha] + 4\int[\eta_{\mu\nu}\varphi - \eta_{\mu\alpha}\varphi_\nu{}^\alpha - \eta_{\nu\alpha}\varphi_\mu{}^\alpha]\,d\varphi$$
$$- 4\int[\eta_{\mu\lambda}\varphi - \eta_{\mu\alpha}\varphi_\lambda{}^\alpha - \eta_{\lambda\alpha}\varphi_\mu{}^\alpha]\,d\varphi_\nu{}^\lambda - 4\int[\eta_{\nu\lambda}\varphi - \eta_{\nu\alpha}\varphi_\lambda{}^\alpha - \eta_{\lambda\alpha}\varphi_\nu{}^\alpha]\,d\varphi_\mu{}^\lambda + \cdots, \qquad (12)$$

which, in a case where $\varphi_0{}^0 = \varphi$ is the largest component, can be approximated, valid through second order, by the simpler form

$$g_{\mu\nu} = \eta_{\mu\nu}(1 + 2\varphi + 2\varphi^2) - 4\varphi_{\mu\nu}. \qquad (13)$$

Note that this approximation differs already by $\frac{1}{2}\varphi^2$ in g_{ii} from Einstein's theory. Other higher order terms are of the form $-\frac{4}{3}\eta_{00}\varphi^3$ for g_{00} and $-8\eta_{00}\varphi_i{}^0\varphi$ for g_{0i}.

Concerning the three classical tests of general relativity we may point out that the solution (10) agrees with the isotropic Schwarzschild line element in first and second order for g_{00} and in first order for g_{ii}. Since the three crucial tests are insensitive to higher terms in these components the present theory explains the three classical effects just as well as Einstein's theory.

With regard to the experiment on the isotropy of inertia the present theory seems to be in a better position than Einstein's theory because our line element is, under (7), isotropic. It may be argued that the principle of equivalence really implies isotropy of space because the gravitational energy is by definition only a function of position whereas the inertial energy could depend on direction if the velocity of light were different in different directions. Thus the equality of inertial and gravitational energy (mass) seems to imply the isotropy of the velocity of light, hence the automatically isotropic functional solution (10) could be said to represent more directly the experiment on the isotropy of inertia. For this interpretation to be tenable one must, however, assume that our usual definition of inertia corresponds to the locally special relativistic coordinates implied by (1) and (7). [Although the general covariance is not harmed by (7) and (1) they nevertheless seem to imply that the natural laws are restricted to those exhibiting locally Lorentz-invariant field-theory forms in the coordinates described.]

For the time delay of radar signals echoed from Venus there seems to be a curious difference in the predictions. This difference, however, does not arise from a difference in the terms or approximations of the line elements but rather from a difference in the interpretations of measure in the two theories.[4] In the usual theory one identifies the zeroth-order special relativistic time delay from the standard nonisotropic solution of Einstein's theory[6] whereas under (7) there are only isotropic solutions and our correspondence procedure seems to imply that the zeroth order delay should be taken from our isotropic solution (10). It is then found that the time delay seems to be $4M/c^3 \simeq 20$ μsec larger than the value given by Ross and Schiff.[6] We believe the problem is worthy of further clarification, especially since it is not clear whether the actual experiment uses the nonisotropic line element for correspondence or implicitly the isotropic one. The experiment at present seems to give ~5–10 μsec above the Ross-Schiff value but probable errors are of the order of 20 μsec. The situation should, however, be improved greatly within few years to an error of ~±3 μsec.

From a study of the vanishing covariant divergence of (5) it is concluded that, in this theory, in order to obtain the geodesic equations of motion (8), one must have in the limit of static fields the condition $\delta m + m\delta\varphi = 0$, which appears to be just the expression of variation of rest mass under the principle of equivalence (gravitation of energy). Thus the interpretation given here to the spin-2 field as the expression of gravity proper seems justifiable. (In Einstein's theory the corresponding condition is[7] $\delta m = 0$.) If under a more extensive analysis this interpretation of ours turns out to be impossible the $\varphi_\mu{}^\nu$ field will have to be interpreted as an extraneous field in interaction with the gravitational field $g_{\mu\nu}(x)$.

The present theory is also derivable from a general action principle.

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[1]A. Schild, in *Evidence for Gravitational Theories, International School of Physics "Enrico Fermi," Course XX*, edited by C. Møller (Academic, New York, 1962), pp. 69–115.

[2]J. Anderson, D. Muhleman, and P. Esposito, Jet Propulsion Laboratory, California Institute of Technology Report, 1970 (unpublished).

[3]The only other viable theory that the author is aware of is an unpublished work by W. T. Ni of California Institute of Technology (private communication).

[4]H. Yilmaz, Phys. Rev. 111, 1417 (1958).

[5]H. Yilmaz, unpublished.

[6]D. K. Ross and L. I. Schiff, Phys. Rev. 141, 1215 (1965).

[7]A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge U. Press, Cambridge, England, 1960), p. 127.

APPENDIX 2

PLANE GRAVITATIONAL WAVES

Let $g_{00} = 1$, $g_{33} = -1$, $g_{11} = -1 + 4Q$, $g_{22} = -1 - 4Q$, $g_{12} = 4R$ where R and Q are functions of t - z alone. This is the usual plane-wave solution in the linearized Einstein theory. It is also a first order solution for the new theory. Let us now investigate what happens in second order and beyond in Einstein's theory. In second order we let

$$g_{11} = -1 + 4\lambda Q + \lambda^2 U \qquad (1)$$

$$g_{22} = -1 - 4\lambda Q + \lambda^2 V \qquad (2)$$

$$g_{12} = 4\lambda R + \lambda^2 W \qquad (3)$$

where U, V, W are quadratic forms in R and Q and λ is an expansion parameter later to be set equal to unity. The Einsteinian equations

$$R_\mu^\nu = 0 \qquad (4)$$

plus the wave conditions lead to the single equation

$$(U + V)'' + 32(QQ'' + RR'') + 16(Q'^2 + R'^2) = 0 \qquad (5)$$

where prime denotes partial derivative relative to t or z. However, there exists no solution to (5) in the form $U + V = \alpha Q^2 + \beta QR + \gamma R^2$

except for the trivial cases of $R' = Q' = 0$ (space is flat) and $R = iQ$ (field is not real). In the latter case there is the additional objection that R and Q are not linearly independent. Other arguments show that no function

$$U + V = F(R,Q) \tag{6}$$

where R and Q are linearly independent can satisfy Equation (5). Of course, if the requirement of two linearly independent functions is dropped (spin becomes zero) there exists the possibility $-g_{11} = e^{4K-4Q}$ $-g_{22} = e^{4K+4Q}$, $U + V = -16Q^2 - 16K^2 \cdot 8K$, but the linearized equations already show that $K = At + Bz + C$. Then from (5) Q itself becomes prescribed (Q contains no information).

In the new theory this same problem goes as follows: The metric is

$$g_{\mu\nu} = \eta_{\mu\nu}\, e^{2(\phi - 2\tilde{\phi})} \tag{7}$$

$$\tilde{\phi} = \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & Q & R \\ \cdot & \cdot & R & -Q \end{array} \tag{8}$$

where $\phi = \phi^{\mu}_{\mu} = 0$, ${}^{2}\tilde{\phi} = 0$. Expansion of (7) yields

$$g_{11} = - \cosh 4\Omega + \frac{Q}{\Omega} \sinh 4\Omega \tag{9}$$

$$g_{22} = - \cosh 4\Omega - \frac{Q}{\Omega} \sinh 4\Omega \tag{10}$$

$$g_{12} = \frac{R}{\Omega} \sinh 4\Omega \tag{11}$$

$$g_{00} = - g_{33} = 1 \tag{12}$$

where $\Omega = (R^2 + Q^2)^{1/2}$. This metric is proven to be an exact solution of the field equations of the new theory. The wave equation is indeed satisfied, as stated, and the stress-energy tensor is the canonical stress-energy tensor of the Lagrangian

$$L_f = -\partial^\lambda \phi^\alpha_\beta \, \partial_\lambda \phi^\beta_\alpha + \frac{1}{2} \partial^\lambda \phi \, \partial_\lambda \phi \tag{13}.$$

APPENDIX 3

GENERAL STATIC CURVED SOLUTIONS OF EINSTEIN'S EQUATIONS FOR SPHERICALLY SYMMETRIC METRICS*

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For a general, static, spherically symmetric co-ordinate system we find the general curved solutions of the Einstein equations $G^{\alpha}_{\beta} = 0$ and $G^{\alpha}_{\beta} = \lambda \delta^{\alpha}_{\beta}$. From these general solutions the well known standard solutions are easily generated.

Submitted to Journal of Mathematical Physics, November, 1973

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INTRODUCTION

In a static spherically symmetric coordinate system there are a number of well known particular solutions of Einstein's equations corresponding to specific choices of the energy-momentum tensor. However, the problem of finding the general solution of the Einstein equations corresponding to a particular matter distribution does not seem to have been considered. It could be argued that in view of Birkhoff's theorem[1] this problem would be of marginal interest. This is not the case, however. On the one hand the various proofs of the Birkhoff theorem do not specify a method for constructing the most general form of the metric. Secondly, it is of mathematical interest to find general solutions for any system of non-linear differential equations as the existence proof might miss some particular solutions. Finally, although the general solution may itself be a transform of some particular solution[2], there is no a priori guarantee that all solutions, for a given matter distribution will, give the identical predictions of physical phenomena. This final point is discussed elsewhere[3] where we show that certain transforms of the Schwarzchild solution do not seem to lead, unambigously, to the same physical predictions.

Below, we solve the Einstein equations for a spherically symmetric coordinate system. We consider both the vacuum equations and the equations with the cosmical constant. From these general solutions certain well known solutions are easily generated.

All mathematical statements in this paper have been verified and intermediate steps simplified using the symbolic manipulation system, MACSYMA, developed by the Math Lab group at Project MAC, M.I.T.

1. VACUUM EQUATIONS

Consider the spherical coordinate system in which the line element may be written in the form[4]

$$ds^2 = -A\,dr^2 - F(d\theta^2 + \sin^2\theta d\phi^2) + D\,dt^2 \quad , \qquad (1.1)$$

where A, F, D are radial functions of class C^2. We wish to find the most general relation between A, F and D so that (1.1) satisfies the Einstein vacuum field equations. We shall not be interested in cases in which A, F or D themselves vanish everywhere.

It can be shown that for the metric (1.1) the only non vanishing components of the Einstein tensor are

$$G^1_1 = \frac{D[4AF - (F')^2] - 2FF'D'}{4AF^2D} \qquad (1.2)$$

$$G^2_2 = G^3_3 = \frac{D^2[A(2FF''-(F')^2)-A'FF']+D[A(D'FF'+2D''F^2)-A'D'F^2]-AF(D')^2}{4A^2F^2D^2} \qquad (1.3)$$

$$G^4_4 = \frac{4A^2F^2 + 2A'FF' + A[(F')^2 - 4FF'']}{4A^2F^2} \quad . \qquad (1.4)$$

These components set to zero yield the differential equations we wish to solve.

We approach this system of equations by noticing that $G_1^1 = 0$ may be solved for A as

$$A = \frac{F'(DF' + 2D'F)}{4DF} \quad . \qquad (1.5)$$

We substitute this into (1.3) and (1.4) (set to zero) to find

$$G_2^2 = G_3^3 = \frac{D'F(2D'FF'' - 3D'(F')^2 - 2D''FF')}{(F')^2(DF' + 2D'F)^2} = 0 \qquad (1.6)$$

and

$$G_4^4 = -\frac{2D(2D'FF'' - 3D'(F')^2 - 2D''FF')}{(F')(DF' + 2D'F)^2} = 0. \qquad (1.7)$$

It is clear that if $D \neq 0$ and $F \neq 0$ the most general way we may satisfy (1.6) and (1.7) simultaneously is by requiring

$$2D'FF'' - 3D'(F')^2 - 2D''FF' = 0 \quad . \qquad (1.8)$$

This differential equation is easily integrated to yield

$$D = K_1 - K_2F^{-1/2} \qquad (1.9)$$

where K_1 and K_2 are constants.

In (1.6) and (1.7) we note that $F' \neq 0$ and [illegible] $2D'F \neq 0$. These "restrictions" on the functions are, from (1.5), nothing more than the requirement $A \neq 0$ which we stipulated above.

We now substitute (1.9) into (1.5) to obtain

$$A = \frac{K_1 (F')^2}{4F \cdot (K_1 - K_2 F^{-1/2})} \tag{1.10}$$

From (1.9) and (1.10) in addition to a trivial coordinate transformation we then have[5]

<u>THEOREM:</u> <u>The metric tensor defined by the line element</u>

$$ds^2 = -\frac{(F')^2}{4F(1 - KF^{-1/2})} dr^2 - F(d\theta^2 + \sin^2\theta d\phi^2)$$

$$+ (1 - KF^{-1/2}) dt^2 , \tag{1.11}$$

<u>satisfies the Einstein vacuum equations identically where F is an arbitrary function of r of class C^2.</u>

We may transform (1.11) into a more intuitive form by letting $F \to B^2 r^2$ where $B = B(r)$ is arbitrary. With this new definition it is easily seen that

$$ds^2 = -\frac{[(Br)']^2}{1 - \frac{K}{Br}} dr^2 - B^2(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + (1 - \frac{K}{Br}) dt^2 . \tag{1.12}$$

The non-isotropic Schwarzchild solution[6] may now be obtained by setting $B = 1$ to find

$$ds^2 = -\frac{1}{1 - \frac{K}{r}}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) + (1 - \frac{K}{r})dt^2 \,. \qquad (1.13)$$

From (1.11) we may also obtain Fock's solution[7]. By choosing $F = (r + K/2)^2$ we find

$$ds^2 = -\frac{r + \frac{K}{2}}{r - \frac{K}{2}}\,dr^2 - (r + \frac{K}{2})^2(d\theta^2 + \sin^2\theta\,d\phi^2) + \frac{r - \frac{K}{2}}{r + \frac{K}{2}}\,dt^2. \qquad (1.14)$$

From (1.11) or (1.12) we may find the isotropic spherically symmetric line element by setting

$$\frac{(F')^2}{4F(1 - KF^{-1/2})} = \frac{F}{r^2} \qquad (1.15)$$

This differential equation may easily be solved for F and we find

$$F = \frac{(K_1 + \frac{K}{2r})^4 r^2}{4K_1^2} \qquad (1.16)$$

where K_1 is an integration constant. Thus, the isotropic line element takes the form

$$ds^2 = -\left(1 + \frac{K}{4r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + \frac{\left(1 - \frac{K}{4r}\right)^2}{\left(1 + \frac{K}{4r}\right)^2} dt^2 \qquad (1.17)$$

in accordance with the well known result.[8]

2. EQUATIONS WITH THE COSMICAL TERM

For the metric (1.1) it follows from (1.2), (1.3) and (1.4) that the component form of the Einstein equation

$$G^i_j = \lambda \delta^i_j \quad , \qquad (2.1)$$

where λ is the "cosmical constant" is

$$G^1_1 = \frac{D(4AF - (F')^2) - 2FF'D'}{4AF^2D} = \lambda \qquad (2.2)$$

$$G^2_2 = G^3_3 = \frac{D^2[A(2FF'' - (F')^2) - A'FF'] + D[A(D'FF' + 2D''F^2) - A'D'F^2] - AF^2(D')^2}{4A^2F^2D^2} = \lambda \quad (2.3)$$

$$G^4_4 = -\frac{4A^2F^2 + 2A'FF' + A((F')^2 - 4FF'')}{4A^2F^2} = \lambda \quad . \qquad (2.4)$$

As we did for the more simple case we solve (2.2) for A to find

$$A = -\frac{F'}{4FD}\left(\frac{DF' + 2D'F}{F\lambda - 1}\right) \qquad (2.5)$$

We substitute this into (2.3) and (2.4). In both cases we then subtract (the right hand side) from the left to obtain differential equations which turn out to be similar to each other. This procedure gives

$$G_2^2\Big|_{2.5} - \lambda = \frac{2D\{(2D'F^2F''+D(F')^3-D'F(F')^2-2D''F^2F')\lambda-2D'FF''+3D'(F')^2+2D''FF'\}}{F'(DF'+2D'F)^2} = 0 \qquad (2.6)$$

and

$$G_4^4\Big|_{2.5} - \lambda = -\frac{D'F\{(2D'F^2F''+D(F')^3-D'F(F')^2-2D''F^2F')\lambda-2D'FF''+3D'(F')^2+2D''FF'\}}{(F')^2(DF'+2D'F)^2} = 0 \qquad (2.7)$$

It is clear that to solve (2.6) and (2.7) we need to find the relation between F and D which causes the expression in curly brackets to vanish. For $\lambda = 0$ we know that the relation we seek must reduce to

$$D = K_1 - K_2F^{-1/2} \qquad (2.8)$$

hence we now seek a function N(F) such that

$$D = K_1 - K_2F^{-1/2} + \lambda N(F) \qquad (2.9)$$

satisfies (2.6) and (2.7).

Substitution of (2.9) into (2.6) results, after some laborious calculation, is the following differential equations for N:

$$N - N'F = 0$$
$$3N' - K_1 = 0 \qquad (2.10)$$

These are easily integrated to find

$$N = -\frac{K_1 F}{3} \qquad (2.11)$$

Thus, (2.9) becomes

$$D = K_1\left(1 - \frac{\lambda F}{3}\right) - K_2 F^{-1/2} \qquad (2.12)$$

With this expression for D we find

$$A = \frac{K_1 (F')^2}{4\left(K_1 - K_2 F^{-1/2} - \frac{\lambda}{3} K_1 F\right)F} \qquad (2.13)$$

From (2.13) and (2.14) and a trivial coordinate transformation to absorb an integration constant we have[5]

<u>THEOREM</u>: <u>The metric defined by</u>

$$ds^2 = -\frac{(F')^2}{4\cdot F\left(1 - KF^{-1/2} - \frac{\lambda}{3} F\right)} dr^2 - F(d\theta^2 + \sin^2\theta d\phi^2) + \left(1 - KF^{-1/2} - \frac{\lambda}{3} F\right)dt^2$$

(2.14)

satisfies

$$G^{\mu}_{\nu} = \lambda \delta^{\mu}_{\nu} \tag{2.15}$$

for any function F(r) of class C^2.

Let us now choose $F = r^2$ in (2.14). With this choice we find

$$ds^2 = -\frac{dr^2}{1 - \frac{K}{r} - \frac{\lambda}{3} r^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1 - \frac{K}{r} - \frac{\lambda}{3} r^2)dt^2. \tag{2.16}$$

This is the Kottler solution[9] and represents one of the very few known (if indeed others have been found) solutions of (2.1).

We now derive an expression for the isotropic solution corresponding to (2.14) by setting

$$\frac{(F')^2}{4F(1 - KF^{-1/2} - \frac{\lambda}{3} F)} = \frac{F}{r^2} \tag{2.17}$$

We see that F must satisfy

$$\int \frac{dF}{F(1 - KF^{-1/2} - \frac{\lambda}{3} F)^{1/2}} = \ln c \cdot r^{\pm 2} \tag{2.18}$$

where C is an integration constant. This integral does not appear to be expressible in closed form. However, one can obtain expression for particular cases of interest. By setting K = 0 in (2.18) we may perform the integration and solve for F to find

$$F = \frac{12\ CR^2}{(\lambda R^2 + C)^2} \tag{2.19}$$

Then

$$ds^2 = -\frac{12\ C}{(\lambda R^2 + C)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + \left(1 - \frac{\lambda CR^2}{(\lambda R^2 + C)^2}\right)dt^2 \tag{2.20}$$

Here we have the isotropic static form of the deSitter solution. Other solutions of possible cosmological interest could be found by performing a series expansion on the left hand side of (2.18). We have not considered such cases at the present time.

ACKNOWLEDGMENTS

We wish to thank Dr. Huseyin Yilmaz who first conjectured that (1.12) is a general solution of the Einstein vacuum equations. We also wish to thank the staff of Project MAC of the Massachusetts Institute of Technology and David Grabel for his help in preparing programs used in the MACSYMA system of symbolic manipulation to obtain our results. The MACSYMA system was developed by Project MAC with support from the Advanced Research Projects Agency under the Office of Naval Research Contract N00014-70-A-0362-001.

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1. See, for example, W. B. Bonner, "Recent Developments in General Relativity", Pergamon Press, (1962). Page 167 - The theorem is Bonner's form states that all spherically symmetric solutions of $G^i_j = \lambda\delta^i_j$ may be transformed into the appropriate Schwarzchild solution (2.1b).

2. There is no known method for generating the general solution from a particular solution via a coordinate transformation.

3. H. Yilmaz and R. Pavelle, to appear.

4. In this form the differential equations take a more simple form.

5. This theorem may also be proved by noting that since (1.11) may be transformed into the Schwarzchild solution (1.13), Birkhoff's theorem is satisfied (Reference 1) and it follows that (1.11) is a general solution in these coordinates. A SIMILAR ARGUMENT APPLIES TO (

6. R. C. Tolman, "Relativity, Thermodynamics and Cosmology", Oxford at the Clarendon Press, 1962, p. 82.

7. V. Fock, "The Theory of Space Time and Gravitation", Pergamon Press, 1964, p. 58.

8. Reference 6, (82.14).

9. F. Kottler, "Uber die physikalischen Grundlagen der Einsteinschen Gravitations theory., Ann. Physic (Ser. 4), p. 56, (1918), 401.

APPENDIX 4

UNIFORM ACCELERATION

Consider the Cartesian-like coordinate system defined by

$$ds^2 = -F(dx^2 + dy^2) - Hdz^2 + K\,dt^2 \tag{1.1}$$

where F, H and K are function of z. We wish to find the general relations between these functionals so that (1.1) satisfies Einstein's Vacuum Equation:

For (1.1) we find the components of the Einstein tensor (set to zero) to be

$$G_1^1 = G_2^2 = -\{K(H[2F^2K'' + FF'K'] - F^2H'K') - F^2H(K')^2$$

$$+ K^2(H(2FF'' - (F')^2) - FF'H')\} / 4F^2H^2K^2 = 0 \tag{1.2}$$

$$G_3^3 = -\frac{F'(2FK' + KF')}{4F^2HK} = 0 \tag{1.3}$$

$$G_4^4 = -\frac{H(F[2F^2F''+F(F')^2] + F^2(2FF''-(F')^2) - F^2(F')^2)-2F^3F'H'}{4F^4H^2} \tag{1.4}$$

We may satisfy (1.3) by requiring

$$2FK' + KF' = 0 \tag{1.5}$$

This diff... equation may be integrated and we find

$$F = C_1/K^2 \qquad (1.6)$$

where C_1 is a constant. This relation implies that (1.2) and (1.4) become

$$G^1_1 = G^2_2 = \frac{H(2KK'' - 5(K')^2) - KH'K'}{4H^2K^2} = 0 \qquad (1.7)$$

$$G^4_4 = \frac{H(2KK'' - 5(K')^2) - KH'K'}{H^2K^2} = 0 \qquad (1.8)$$

Hence, by solving (1.7) or (1.8) we will have a general solution of the Einstein Vacuum Equation. These are integrable and we find that

$$H = C_2 \frac{(K')^2}{K^5} \qquad (1.9)$$

satisfies (1.8) and (1.8). Thus, the metric tensor components defined by

$$ds^2 = -\frac{C_1}{K^2}(dx^2 + dy^2) - C_2 \frac{(K')^2}{K^5} dz^2 + K\, dt^2 \qquad (1.10)$$

is the general static solution of $G^i_j = 0$ for any function $K(z) \neq 0$.

On the other hand, we may satisfy (1.3) by choosing $F' = 0$. With this choice it is found that

$$G^1_1 = G^2_2 = - \frac{2KH\,K'' + -K\,H'K' - H(K')^2}{4H^2K} = 0 \qquad (1.11)$$

and

$$G^4_4 = 0 \qquad (1.12)$$

The solution of (1.11) is

$$H = \frac{C_1 (K')^2}{K} \qquad (1.13)$$

We thus find

$$ds^2 = - (dx^2 + dy^2) - C_1 \frac{(K')^2}{K} dt^2 + K\,dt^2 \qquad (1.14)$$

also satisfies $G^i_j = 0$.

However, it is not difficult to prove that for the metric (1.14), all components of the Riemann Christoffel Tensor vanish identically. Hence, the metric (1.14) is flat. It can also be shown that (1.14) is the only flat metric belonging to the general class (1.1).

From (1.10) we may seek the general isotropic line element by setting

$$\frac{C_1}{K^2} = \frac{C_2 (K')^2}{K^5} \qquad (1.15)$$

Then,

$$K' = C\,K^{3/2} \qquad (1.16)$$

or

$$K = (Cx + D)^2 \qquad (1.17)$$

Hence

$$ds^2 = -\,(Cx + D)^{-4}(dx^2 + dy^2 + dz^2) + (Cx + D)^2\,dt^2 \qquad (1.18)$$

is the isotropic line element.

APPENDIX 5

NEW SOLUTIONS FOR EINSTEIN'S THEORY[*]

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ABSTRACT

A general solution for Einstein's theory of gravitation corresponding to spherical symmetry is presented. It is shown that the usual Schwarchild metrics are members of this general solution. Some of the new solutions implied seem to make little or no physical sense. One that at the outset seems to be physically unobjectionable and interesting for possible astrophysical applications is described in some detail.

By the use of a well-known method of tensor transformation it is proven that the line element

$$ds^2 = fc^2dt^2 - \left(\frac{2m}{1-f}\right)^2 (d\theta^2 + \sin^2\theta d\varphi^2) - f^{-1} \left(\frac{2m}{1-f}\right)'^2 dr^2 \qquad (1)$$

where f is a twice differentiable function of r is a general solution of Einstein's field equations $R_{\mu\nu} = 0$. The usual Schwarzchild solutions are members of this general solution for[1]

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$$f = 1 - 2\phi \qquad , \; f = 1 - 2\phi(1 + \phi/2)^{-2} \tag{2}$$

where $\phi = m/r$ so that for large r the first order Newtonian correspondence is satisfied. Similarly $f = (1 - \phi)/(1 + \phi)$ yields the well-known Fock solution[2] satisfying the correspondence condition in first order. The solution (1) is new and offers enormous flexibility as we shall see below. It was originally conjectured during the performance of an ARPA contract directed to study the solutions of author's new theory of gravitation[3], and later explicitly verified by the use of the MACSYMA system of Mathematic's Laboratory, Project MAC, M.I.T. MACSYMA was also of definite value in the formulation of the conjecture itself.

Actually it is somewhat disturbing to find that f can be so general as in (1), because we seem to be able to set

$$f = \sum_n a_n \phi^n \tag{3}$$

and produce an arbitrary number of Schwarzchild type singularities, or choose,

$$f = e^{-2\phi} \tag{4}$$

and obtain a solution with no singularity at all, except at $r = 0$.

Note that some of the possibilities contained in (3) might be problematic for the experimental aspects of the theory. For example, by setting

$$f = 1 - 2\phi + b\phi^2 \tag{5}$$

$$ds^2 = (1 - 2\phi + b\phi^2)dt^2 - (1 - b\phi/2)^{-2}\,(r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) - \frac{1}{f}\,\frac{(1 - b\phi)^2}{(1 - b\phi/2)^4}\,dr^2 \tag{6}$$

where b is arbitrary, one can prove that the three crucial tests of red shift, light bending and perihelion advance are satisfied, but the radar-echo delay depends on b as[4]

$$\Delta t = 4M\,(\ln 4\,\frac{\ell\ell'}{R^2} + 1 + b) \tag{7}$$

Since b is arbitrary one may, for instance, set $b = -12$ and get a b-dependent part which cancels the whole gravitational effect expected from the theory. (Units are $c = G = 1$.)

From among the new solutions contained in (1) the possibility of setting or requiring $f = e^{-2\phi}$ is probably the most interesting. In this case we have the line element

$$ds^2 = e^{-2\phi}dt^2 - \left(\frac{2m}{1-e^{-2\phi}}\right)^2 (d\theta^2 + \sin^2\theta d\varphi^2) - \frac{(2m)^4 e^{-2\phi}}{r^4(1-e^{-2\phi})^4}\,dr^2 \tag{8}$$

This new line element has the following remarkable properties: a) For $r >> m$ it reduces to the usual isotropic Schwarzchild line element for all experimental purposes, hence it reproduces all four of the experimental tests. b) In the limit of slow motion of the test body the geodesic equations of motion reduce to

$$\frac{d^2x_\mu}{dt^2} = \left\{ {o \atop \mu o} \right\} u_o u^o = -\partial_\mu \phi \tag{9}$$

which are exactly the Newtonian equations. c) In the limit of slow motion of the source the field equations reduce to

$$\nabla^2\phi = -4\pi\sigma \tag{10}$$

which is exactly the Poisson equation. By contrast other line elements reduce to these Newtonian limits in an approximate sense only. d) It is the only solution of Einstein's equations which apply both to the exterior and the interior regions of a mass distribution without a change in form. e) It is harmonic in the sense that if the metric is written in the form, $ds^2 = Dc^2dt^2 - B(r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) - Adr^2$, that is as a factor to special relativity, then $\partial_\nu(\sqrt{-g}\, g^{\mu\nu}) = \partial_\nu(\sqrt{DB^2A}\, A^{-1}) = 0$. f) It has no singularity anywhere except at $r \to 0$ where it reduces to

$$ds^2 = -(2m)^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{11}$$

The latter represents the surface of an inpenetrable sphere of radius 2m. This spherical surface seems to be suspiciously similar to the Schwarzchild surface but note the difference that (8) has an essential singularity at the surface, not an ordinary one. Furthermore, there is really no interior of this sphere as r starts out at the surface. The only conceivable objection to (8) might be that, like all other solutions of Einstein's theory, it leads to a zero gravitational stress-energy tensor t_μ^ν and thereby seems to violate the Newtonian correspondence because we are missing[5]

$$t_\mu^\nu = -\partial_\mu \phi \, \partial^\nu \phi + \frac{1}{2} \delta_\mu^\nu \, \partial^\lambda \phi \, \partial_\lambda \phi \qquad (12)$$

As to a possible application of (8) other than the usual astronomical calculations one naturally thinks of the gravitational collapse. It can be seen that for (8) there does not exist the concept of a black hole in the usual sense. The collapsed body would here remain, at all times, as a naked object. Furthermore there is no singularity at finite r and the usual reversal of the signature of the metric does not take place.

It should be interesting to repeat the usual calculations of gravitational collapse and the dynamics of black holes using some of the new members of the general solution (1), including the unusual member (8).

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That the conjectural line element (1) is a general curved-space solution of Einstein's equations was proven by a [illegible] transformation on the metric. There are also purely flat-space solutions of the form $ds^2 = c^2dt^2 - F^2(d\theta^2 + \sin^2\theta d\phi^2) - F'^2dr^2$ which are not discussed in this paper. These solutions are checked by MACSYMA, a symbolic manipulation system, at Project MAC, MIT. The author is grateful to Dr. Joel Moses and Mr. David Grabel of Project MAC for making this verification possible. A direct proof of (1) is found by Dr. Richard Pavelle of Perception Technology Corporation by using the extensive manipulative power of MACSYMA system. Whether these are the only possible general solutions of the theory in spherical coordinates has not yet been established, although a reasoning based on Birkhoff's theorem seems to indicate that there are no others.

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APPENDIX 6

INDICIAL MANIPULATION IN THE MACSYMA SYSTEM

The MACSYMA system consists of over 250 functions for the manipulation of mathematical expressions. There are not only functions for performing mathematical computations but also those for many necessary miscellaneous manipulations such as extracting parts, simplifying, input output, etc. Each one of these is made up of one or as many as several hundred programs (in the case of integration) written in the LISP language. The expressions dealt with are composed of operators and arguments to them which are numbers, variables, functions references, arrays, lists, and matrices. The mathematical knowledge built for MACSYMA is continually expanding as more and more programs are written to provide additional capabilities. At present there are over 2000.

If one wishes to introduce a new mathematical entity into the system then he would not only have to cast it in terms of the existing syntax but also it would be necessary to provide the programs to perform the desired manipulations on it as it would be foreign to most of the existing programs. (MACSYMA does have a facility for syntax extension but currently this applies only to operators.) Perception Technology Corporation has embarked on an effort to create some facility for dealing with tensor manipulation as MACSYMA has nothing that would provide a convenient, efficient, and workable mechanism for indicial types of manipulations.

Since a tensor is defined not by the way it is written but by certain properties it has, we represent them as more general quantities which we call indexed objects. They possess a name and two lists of indices, the first list being the subscripts (covariant indices) and the second list being the superscripts (contravariant indices). These indexed objects are represented as functions of two arguments, which are the above mentioned lists. If either set of indices is absent it is represented by an empty list. Thus δ_i^j and A^{ijk} would be represented as δ([i],[j]) and A([ ],[i,j,k]).

Ordinary differentiation of an indexed object (say E) with respect to the coordinate X^k would be obtained by the command DIFF(E,k). Since MACSYMA wouldn't know that E depends on X^k it would give 0 but the differentiation program has been modified to assume that all indexed objects depend on all coordinates. If an indexed object is independent of all coordinates this could be stated by the command DECLARE (E,CONSTANT) and subsequent differentiation of E would yield 0. The derivative of an indexed object causes the coordinate with respect to which differentiation is carried out to be appended to the function which represents the indexed object in MACSYMA. These indices which denote differentiation with respect to a coordinate are sorted in alphabetic order to take advantage of the commutativity of differentiation in order to simplify expressions composed of those indexed objects.

At present two features have been implemented - symmetries and contractions. For the former purpose three functions are provided. One for declaring symmetries (e.g. G([ ],[j,i]) is equal to G([ ],[i,j])), One for removing them, and one for displaying them in case the user forgot which ones he declared. For the purpose of contracting products of indexed objects in expressions, four functions were provided. One to declare that some indexed object contracts with another to form a third, another function to remove these declarations, another to display them, and also a function to perform the contractions. This latter function is rather difficult to code as it is necessary to make several passes over the expression to make sure that all possible contractions have been utilized. The function also performs the substitutions implied by the Kronecker delta function if it occurs in an expression. In addition a function has been written to display expressions containing indexed objects in more natural notation than MACSYMA ordinarily would because MACSYMA assumes that A([i,j],[k],l,m) is a function of four arguments, and would display it as such, whereas we would like to see it displayed as $A^{k}_{ij,lm}$.

Using this equipment described above we were easily able to perform the calculation of the Ricci tensor to first order given the expressions for the covariant and contravariant forms of the metric tensor. The result in unsimplified form had 16 terms but these were reduced to 6 by using the contraction program.

To carry this calculation to second order will result in thousands of terms if the result was not simplified along the way. Although many times more complicated than the first order case, it will be possible to do but shall require more code than is available at present for simplifications.

APPENDIX 7

EXPONENTIAL MATRICES

In the New Theory, for particular physical problems, functional expansions of the form $e^{\tilde{\lambda}} = \tilde{1} + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2!} + \ldots$ where $\tilde{\lambda}$ is a square matrix are considered. Here $\tilde{\lambda}^2$ is the product matrix. The MACSYMA system is ideal for generating useful closed form expansions of this kind and we have been able to find many interesting cases. The convergence of the following forms may be proved by induction. In the following examples the dot represents a zero element.

$$\lambda = \begin{pmatrix} \cdot & A & B \\ -A & \cdot & C \\ -B & -C & \cdot \end{pmatrix} \qquad (1)$$

$$e^{\lambda} = \begin{pmatrix} 1 + \frac{(A^2+B^2)(\cos\rho-1)}{\rho^2} & \frac{A\sin\rho}{\rho} + \frac{BC(\cos\rho-1)}{\rho^2} & \frac{B\sin\rho}{\rho} - \frac{AC(\cos\rho-1)}{\rho^2} \\ -\frac{A\sin\rho}{\rho} + \frac{BC(\cos\rho-1)}{\rho^2} & 1 + \frac{(A^2+C^2)(\cos\rho-1)}{\rho^2} & \frac{C\sin\rho}{\rho} + \frac{AB(\cos\rho-1)}{\rho^2} \\ -\frac{B\sin\rho}{\rho} - \frac{AC(\cos\rho-1)}{\rho^2} & -\frac{C\sin\rho}{\rho} + \frac{AB(\cos\rho-1)}{\rho^2} & 1 + \frac{(B^2+C^2)(\cos\rho-1)}{\rho^2} \end{pmatrix} \qquad (2)$$

where $\rho^2 = A^2 + B^2 + C^2$.

$$\lambda = \begin{pmatrix} \cdot & \cdot & \cdot & R \\ \cdot & \cdot & \cdot & S \\ \cdot & \cdot & \cdot & T \\ -R & -S & -T & \cdot \end{pmatrix} \qquad (3)$$

$$e^{\lambda} = \begin{pmatrix} 1 + \frac{R^2}{\rho^2}(\cos\rho-1) & \frac{RS}{\rho^2}(\cos\rho-1) & \frac{RT}{\rho^2}(\cos\rho-1) & \frac{R}{\rho}\sin\rho \\ \frac{RS}{\rho^2}(\cos\rho-1) & 1 + \frac{S^2}{\rho^2}(\cos\rho-1) & \frac{ST}{\rho^2}(\cos\rho-1) & \frac{S}{\rho}\sin\rho \\ \frac{RT}{\rho^2}(\cos\rho-1) & \frac{ST}{\rho^2}(\cos\rho-1) & 1 + \frac{T^2}{\rho^2}(\cos\rho-1) & \frac{T}{\rho}\sin\rho \\ -\frac{R}{\rho}\sin\rho & -\frac{S}{\rho}\sin\rho & -\frac{T}{\rho}\sin\rho & \cos\rho \end{pmatrix} \qquad (4)$$

where $\rho^2 = R^2 + S^2 + T^2$.

$$\lambda = \begin{pmatrix} Q & R \\ R & Q \end{pmatrix} \qquad e^{\lambda} = \begin{pmatrix} \cosh(R) & \sinh(R) \\ \sinh(R) & \cosh(R) \end{pmatrix} \cdot e^{Q} \qquad (5)$$

$$\lambda = \begin{pmatrix} A & \cdot & \cdot & \cdot \\ \cdot & B & \cdot & \cdot \\ \cdot & \cdot & C & \cdot \\ \cdot & \cdot & \cdot & D \end{pmatrix} \qquad e^{\lambda} = \begin{pmatrix} e^{A} & \cdot & \cdot & \cdot \\ \cdot & e^{B} & \cdot & \cdot \\ \cdot & \cdot & e^{C} & \cdot \\ \cdot & \cdot & \cdot & e^{D} \end{pmatrix} \qquad (6)$$

$$\lambda = \begin{pmatrix} Q & R \\ R & -Q \end{pmatrix} \qquad (7)$$

$$
e^{\lambda} = \begin{pmatrix} \cosh(R^2 + Q^2)^{1/2} + \dfrac{Q \sinh(R^2 + Q^2)^{1/2}}{(R^2 + Q^2)^{1/2}} & \dfrac{R \sinh(R^2 + Q^2)^{1/2}}{(R^2 + Q^2)^{1/2}} \\ \dfrac{R \sinh(R^2 + Q^2)^{1/2}}{(R^2 + Q^2)^{1/2}} & \cosh(R^2 + Q^2)^{1/2} - \dfrac{Q \sinh(R^2 + Q^2)^{1/2}}{(R^2 + Q^2)^{1/2}} \end{pmatrix} \qquad (8)
$$