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A SPECTRAL TRANSFER MODEL OF OCEAN WAVE SPECTRA. I. FORMULATION

J. Alex Thomson, et al

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A one-dimensional spectral transfer equation is developed for ocean surface waves in the presence of non-uniform surface currents. The formalism is phenomenological and includes several parameters which are determined by comparison with data. The model contains phenomenological descriptions of wave breaking, wind generation and amplification, radiation stress, viscous damping, capillary effects, wave-wave interaction, non-linear wave group velocities and convective and refraction terms. The free parameters are

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determined by fitting the model to the Pierson-Stacy representation of a fully developed sea. Numerical calculations are carried out following wave packets along trajectories in k,x space. The effects of wave reflections and trappings by moving current patterns are included.

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This technical report has been reviewed and is approved

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A SPECTRAL TRANSFER MODEL OF OCEAN WAVE SPECTRA

I. INTRODUCTION

In this report we develop a dynamical description of ocean wave spectra that can be applied to both equilibrium and developing seas. Our principal objective is to develop a model that can be applied to fairly general situations so that it will be possible to correlate data taken under a variety of conditions.

Although the most casual observation of ocean waves shows that the normal ocean is closer to being isotropic than one-dimensional, we will develop here a strictly one-dimensional model. The extension to two dimensions will be treated in a later report.

To discuss any model of the general characteristics of the ocean surface as we do in the following sections, one must have a clear picture in mind of the variety of wave structures to be found there. For the sake of clarity of presentation, the ocean waves are separated into four regions of interest. These regions are determined by wavelength and have quite different characteristics. The longest waves are the tides and tsunamis ("tidal waves") which have a wave period of from 10^3 to 10^4 seconds and have a phase velocity on the order of 500 mph. The extremely long wavelengths makes these waves uninteresting in the present discussion in which we are concerned with rather localized wave interactions. The longest waves of interest to us are called swells and have periods on the order of 10 seconds and wavelengths between 60 and 400 meters. These waves are sinusoidal and can travel, without loss of energy, for thousands of miles in deep water. It is through these long wave length gravity waves that the effect of storms are found at great distances from storm centers.

The surface gravity waves of intermediate size ride on these much longer waves and can interact strongly with them. These waves are seen to be sharply peaked and in large number. They are generated by the wind and can be seen to grow, increase in slope and eventually break under the wind's influence. The generation of surface waves by wind is an incompletely understood phenomenon, but the mechanism for the waves breaking seems to be clear. A wave becomes unstable when its steepness, i.e., the ratio of its height to its length, exceeds one to seven (Stokes 1847). Therefore a wave which is seven meters long will break when its height exceeds one meter. The quantity H/λ , where H is the wave height and λ the wave length, measures how close a given wave is to breaking. A quantity we will use frequently is the wave slope (Θ) defined as $\Theta = ka$, where k is the wavenumber and a the wave amplitude.

The smallest waves present on the ocean surface are ripples or capillaries, which are fractions of a centimeter long. These waves can interact strongly with the surface gravity waves in the neighboring regime. The waves in this region are strongly damped by molecular viscosity and are generated both by the wind and by the gravity waves.

To round out our present classification of the ocean surface effects, we must introduce the notion of a variable surface current which can arise from various sources: large scale ocean current systems, tides, river outlets, wind-driven currents, and internal waves. An internal wave is unique in that it gives rise to a travelling current pattern. The effect of this wave at the ocean's surface is to increase the flow of water above the internal wave crest and decrease the flow above the internal wave trough, thereby giving rise to a position dependent translating current system at the ocean's surface. Weak small wavelength surface gravity waves can interact quite strongly with such current patterns (see Thomson and West, 1972).

To capture the sequence in which the preceding wave types enter a region of the ocean, let us consider a glassy ocean surface in which there are no currents and no swells from distant regions. If a uniform wind is slowly turned on, one observes the generation of ripples on the water. In principle, it is the shearing stress between the wind and the water which produces this initial growth of the waves. In a fairly short time interval, these ripples grow in amplitude and become intermediate size gravity waves. This time is dependent on the velocity of the wind. Waves of many sizes are generated; those with small wavelength are, however, quickly damped by viscosity. Those with longer wavelength, as they grow, become more efficient in extracting energy from the wind since their amplitude acts as a sail on which the wind can push. The waves which are the most efficient in extracting energy from the wind are those which have a phase comparable to the wind speed.

The continued growth of these primary gravity waves is of course limited by the stability criteria mention earlier, i.e., $H/\lambda \approx 1/7$. The energy being supplied by the wind must, therefore, be redistributed to other wavenumbers. The interaction between gravity waves is weak, however, so energy can be fed in by the wind more rapidly than it can be redistributed by the interaction between waves. This excess energy is dissipated in two ways: (i) the formation of "white caps" in which the gravity wave becomes unstable and breaks and (ii) the generation of "parasitic capillaries", i.e., short waves which are radiated from the crest of a sharply-peaked gravity wave and are subsequently viscously damped.

If the wind speed is insufficient to cause the sharp cresting of the gravity waves which lead to (i) and (ii) above, these effects may still be induced by means of a number of different mechanisms. The "capping" or breaking of waves can be caused by two fully-developed waves running together so that their superposition forms a sharply-crested peak which is unstable. Breaking can also occur by relatively short gravity waves passing over the crest of a swell so that, again, the crest of the combined waves is unstable. It should be pointed out that breaking and the formation of "parasitic capillaries" are competitive mechanisms and where one is found so generally is the other. It is probably the case, however, that the latter mechanism is operative when one exceeds the stability limit by only a small amount.

As one increases the wind speed, longer wavelength waves are generated. Those waves already present become saturated and begin breaking. The energy pumped into the waves is rapidly dissipated by "capping" and the fraction of those waves which are breaking increases rapidly.

There have been a number of attempts to develop a statistical description of the air-sea interface. The most extensive modeling of this type has been carried out in a series of papers by Hasselman (1962, 1963).

The basic concept behind Hasselman's approach stems from the fact that, although the equations describing the wave motion at the sea surface are non-linear, these interactions are remarkably weak except under extreme circumstances. Perturbation theory is then applicable. Under these conditions transfer of energy occurs most rapidly between various modes that satisfy a resonance condition. Hasselman's model has been applied successfully to the description of wave spectra at relatively long wavelengths where the very strong non-linearities that result in breaking waves and white caps have only an indirect effect.

However, in the normal sea, a significant fraction of the wave spectrum is usually assumed to be "saturated". In this region the continued energy input from the wind is balanced by highly non-linear effects eventually resulting in dissipation of the energy. These breaking waves or white caps are not described by the Hasselman model (or any model based on perturbation theory.)

The approach we use here is to construct a "transfer" equation in which different interaction effects are identified and models constructed for each of these effects. This transfer equation and the derivation of the individual terms are described in detail in the next section.

2. THE TRANSFER EQUATION

We wish to develop a phenomenological equation which describes how energy is fed into ocean waves, how it gets redistributed among these waves and how it is dissipated. To realize this goal, it is necessary to summarize in a single equation all the effects discussed qualitatively in the preceding section. In general, we need to construct an equation for the rate of change of wave energy at a given point in space $(\vec{x})$, time (t), and wavenumber $(\vec{k})$. Since we are particularly interested in constructing a model that allows for spatial gradients resulting from a travelling current field, we think of a transfer equation that will allow waves to be born at some point, to convect to a different point in space, and to decay or dissipate. To this end we have divided the present section into a number of subsections, each one dealing with a single term in our equation. Figure 1 gives a pictoral summary of the effects we include.

In Figure 1 we divide the waves on the ocean surface into three regimes as mentioned previously. The arrows in the figure indicate the direction of energy flow between different wavenumbers; the labels on the arrows indicate the mechanism by which the energy is transferred. The solid arrows indicate that the mechanism is important in the wavenumber region considered; a dashed arrow indicates a relatively unimportant

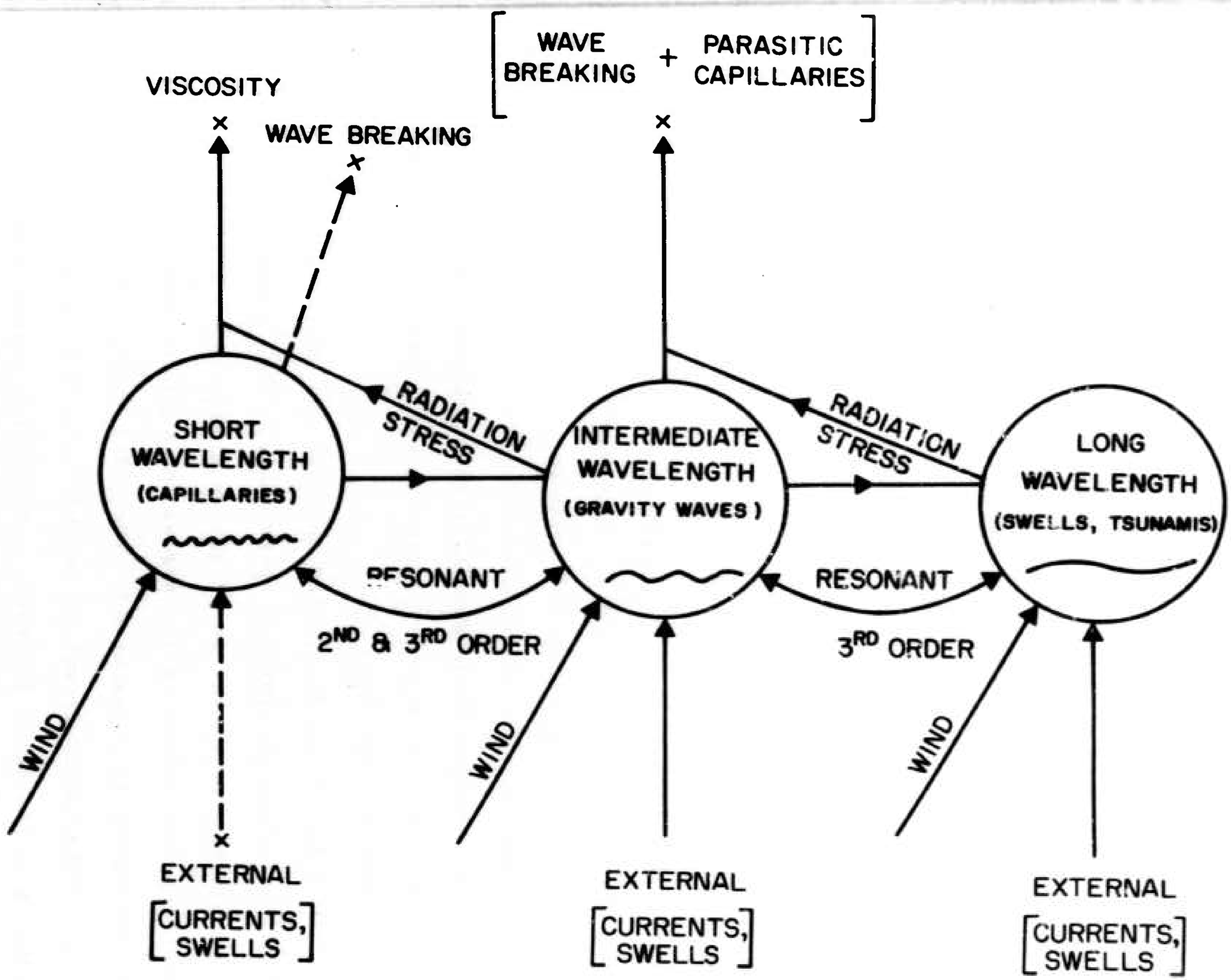

FIGURE 1. Interaction Regimes in Wavenumber Space and the Interaction Mechanisms.

mechanism. For example, viscosity is an important dissipative mechanism for high wavenumber (k_H) waves but not for low wavenumber waves (k_L).

We define the quantity $F(\vec{k},\vec{x},t)$, [sometimes written simply as $F(\vec{k})$ for brevity], as the spectral density of the wave amplitude at the position $\vec{x}$ at time t for waves with wavenumber $\vec{k}$. This is the quantity that describes the processes represented in Figure 1. The total rate of change in $F(\vec{k},\vec{x},t)$ is dependent on the five processes labeled in Figure 1: (1) wind, (2) resonant interactions between gravity waves, (3) the radiation stress of a wave from one $\vec{k}$-regime passing over a wave of another regime, (4) viscous damping, (5) formation of parasitic capillaries and wave breaking. The overall form of the one-dimensional equation which we will arrive at is shown below:

$$\frac{\partial F}{\partial t} = -(c_g + U)\frac{\partial F}{\partial x} - \frac{\partial k}{\partial U}\frac{dU}{dx}\frac{\partial F}{\partial k} + G\frac{dU}{dx}F + \frac{\partial}{\partial k}\left(D_k \frac{\partial F}{\partial k}\right) + \left(\frac{\partial F}{\partial t}\right)_{wind} + \left(\frac{\partial F}{\partial t}\right)_{breaking} + \left(\frac{\partial F}{\partial t}\right)_{dissipation} .$$

The first four terms on the right hand side comprise, respectively, the effects of convection, refraction, radiation stress and wave-wave interaction. In the remainder of this section we discuss each of the terms of this equation in detail.

2.1 Model of the Total Time Derivative

We will restrict our attention in the following discussions to the one-dimensional case. We write the energy density at a point on our one-dimensional ocean surface, where there exists a current [U(x)], as a function of the wavenumber (k), position (x), and time (t). To determine the total rate of change in the energy density at a point in space one must consider how the energy density changes with respect to all its arguments in an increment of time Δt.

The quantity F(k,x,t) is the probable value of the power spectrum of the wave amplitude at the phase point k,x,t and is proportional to the energy density. Its rate of change with time dF/dt combines several effects. In the following, we will derive models for the rate at which this energy density will change as viewed by an observer who is being convected along with the individual wave packets. The trajectory of a packet in our two-dimensional phase space is specified in terms of the local dispersion relation $\omega(k,x)$ (we only consider here the special case of a time independent dispersion relation). The generalization to the time dependent case [$\omega = \omega(k,x,t)$] will be made subsequently. The trajectory along which energy naturally propagates is then given by

$$\frac{d\vec{x}}{dt} = \nabla_{\vec{k}}\,\omega \text{ or in one dimension } \frac{dx}{dt} = \frac{\partial\omega}{\partial k}$$

and (2.1.1)

$$\frac{d\vec{k}}{dt} = -\nabla_{\vec{x}}\,\omega \text{ or in one dimension } \frac{dk}{dt} = -\frac{\partial\omega}{\partial x} \; .$$

When ω is explicitly a function of a local surface current [U(x)] this latter equation becomes

$$\frac{dk}{dt} = -\frac{\partial\omega}{\partial U}\frac{dU}{dx} \text{ in one dimension.} \qquad (2.1.2)$$

For the sake of maintaining a consistent convention with other writers, we express dx/dt in the form

$$\frac{dx}{dt} = C_g + U \qquad (2.1.3)$$

where C_g is the group velocity relative to the local water surface ($C_g = \partial\omega_o/\partial k$ where, for gravity waves, $\omega_o = \sqrt{gk}$). Also for a steady current

$$\omega = Uk + \omega_o(k) = \text{constant} \qquad (2.1.4)$$

and using Equation (2.1.2) we have

$$\frac{dk}{dt} = -k\,\frac{dU}{dx} \qquad (2.1.5)$$

and from Equation (2.1.4),

$$k(C+U) = \omega = \text{constant}$$

where C is the phase velocity in the ocean coordinate system ($C = \omega_o/k$).

The total rate of change of F(k,x,t) following individual wave packets is related to the local space-time gradients according to

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{dx}{dt}\frac{\partial F}{\partial x} + \frac{dk}{dt}\frac{\partial F}{\partial k} = \dot{F}_{gain} - \dot{F}_{loss}$$

or, using Equations (2.1.3) and (2.1.5), (2.1.6)

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (C_g + U)\frac{\partial F}{\partial x} - k\frac{dU}{dx}\frac{\partial F}{\partial k} = \dot{F}_{gain} - \dot{F}_{loss} \ .$$

The first two terms on the left-hand side of Equation (2.1.6) are familiar, being the variation in F with time and that due to convection at a point in physical space. The third term is a little more interesting, since it results from the variation in wave number due to the gradient of the ocean current (refraction). This term is analogous to the action of an external force on a group of particles undergoing collisions (F would be the number of particles). Equation (2.1.6) is the wave analog of the Boltzmann equation.

In problems connected to the surface currents produced by propagating internal waves, the coordinate system in which the wave pattern is steady is that moving with the internal wave at its phase velocity (C_I). It will be convenient in these cases to write the total current seen in this coordinate system in the form

$$U = \Delta U - C_I$$

where ΔU is the flow perturbation produced by the internal wave.

2.2 Model for Wave Breaking

As discussed previously, there are three types of non-resonant interactions that can give rise to wave breaking: (i) the superposition of two fully-developed waves, (ii) the passage of a packet of short waves over a fully-developed wave of low wavenumber and (iii), the continued action of the wind on a wave. The first two of these interactions occurs rapidly, typically on a time scale of the order of the wave period or less. The action of the wind occurs over a longer time period. In this section we develop a model for the breaking of fully-developed waves.

The basic model we will use is dependent on there being some critical value for the ratio of wave height to wave length. The analysis of Stokes established that for gravity waves this value was approximately 1/7 after which gravitational instability would set in. Referring to this ratio as the wave slope, we assume that if the wave slope exceeds some critical value (Θ_o), strong interactions occur which radically alter the energy distribution in wavenumber space. These mechanisms could be wave breaking or the generation of parasitic capillaries at the wave crest. Exactly how these effects occur is irrelevant here; we simply assume that whenever the slope (Θ) attempts to exceed Θ_o, some strongly dissipative process occurs to keep Θ at Θ_o. The problem which remains is to estimate the proper statistics of the wave slope.

We assume that the wave field of the ocean is a random superposition of modes; further, that these waves have random phases and are distributed over a broad range of wavenumbers. If we write the slope of the surface Θ as the sum of the contributions from each mode, then

$$\Theta = \sum_k \Theta_k \quad . \tag{2.2.1}$$

By assumption each of the Θ_k is a random variable, so that if no non-linear process existed in the interaction between the surface waves, the law of large numbers would result in a Gaussian distribution for the surface slope. In that case it would be possible to write for the probability that the surface has a slope greater than the critical value (Θ_o) as

$$P(|\Theta| > \Theta_o) = \frac{2}{\sqrt{\pi\overline{\Theta^2}}} \int_{\Theta_o}^{\infty} d\Theta \exp\left\{-\Theta^2/\overline{\Theta^2}\right\} \quad . \tag{2.2.2}$$

In the asymptotic region $\left[\Theta_o^2 >> \overline{\Theta^2}\right]$, we may write the approximate expression

$$P(|\Theta| > \Theta_o) \approx \frac{\sqrt{\overline{\Theta^2}}}{\pi\Theta_o} \int_{\Theta_o^2/\overline{\Theta^2}}^{\infty} dx \exp\left\{-x\right\} = \sqrt{\frac{\overline{\Theta^2}}{\pi}} \; \frac{\exp\left\{-\Theta_o^2/\overline{\Theta^2}\right\}}{\Theta_o} \tag{2.2.3}$$

to replace Eq. (2.2.2). Eq. (2.2.3) is the fraction of the ocean surface in the wave field whose slope exceeds the critical value Θ_o and would form white caps according to this model.

The coherence time or duration of the breaking process is of the order of the reciprocal of the wave period, i.e., the wave formed by the superposition of the surface modes. If a fraction ε of the wave energy is lost in one such breaking, then the mean rate of loss of the energy density is of the order

$$\left(\frac{\partial F}{\partial t}\right)_{\text{breaking}} = -\omega\varepsilon\, P(|\Theta| > \Theta_o)\, F \quad . \tag{2.2.4}$$

Equation (2.2.4) is the probable fraction of energy lost per unit time interval due to wave breaking. Using (2.2.3), we may rewrite (2.2.4) as

$$\left(\frac{\partial F}{\partial t}\right)_{\text{breaking}} = -\omega\varepsilon\, \frac{F}{\Theta_o} \sqrt{\frac{\overline{\Theta^2}}{\pi}}\, \exp\left\{-\Theta_o^2 / \overline{\Theta^2}\right\} \tag{2.2.5}$$

in terms of the mean square slope of the surface $\overline{\Theta^2}$.

The mean square slope $\overline{\Theta^2}$ may be expressed in terms of the energy density by recalling that

$$\int_{-\infty}^{\infty} F(k)dk = \overline{a^2} \tag{2.2.6}$$

where $\overline{a^2}$ is the mean square elevation of the ocean surface. We may use this to write the mean square slope as,

$$\overline{\Theta^2} = \overline{(\nabla\zeta)^2} = \int_{-\infty}^{\infty} k^2\, F(k)dk \quad . \tag{2.2.7}$$

In order to develop the concept of a spectral model for non-linearly interacting waves, it is necessary to invoke the concept of a correlation interval in wavenumber space. We assume that waves within the interval $k \pm \Delta k/2$ are completely (phase) correlated and act as a single non-linear wave. Waves separated by a wavenumber interval greater than Δk are assumed to be uncorrelated with no consistent phase relationships. Since for a well-developed sea we expect Δk to be proportional to k, we take the mean square slope of waves having wavenumber k to be

$$\overline{\theta^2(k)} = \text{constant} \cdot k^3 F(k) \ . \qquad (2.2.8)$$

The constant can be absorbed into the factor θ_o^2 and we can write the order of magnitude expression,

$$\left(\frac{\partial F}{\partial t}\right)_{\text{breaking}} \approx -\varepsilon\, \omega(k) \left(\frac{k^3 F^3(k)}{\pi\, \theta_o^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{\theta_o^2}{k^3 F(k)}\right\} . \qquad (2.2.9)$$

We may extend this result to the case where we treat the right and left traveling waves separately. This separation occurs when we view the ocean in a coordinate system in which the local current is steady. In this coordinate system, the probability of the net slope exceeding θ_o in the wave should depend on the total energy density $F = F^{(+)} + F^{(-)}$. Thus, we have the set of coupled equations,

$$\left[\frac{\partial F^{(+)}}{\partial t}\right]_{\text{breaking}} = \omega(k)\,\varepsilon F^{(+)} \frac{\sqrt{k^3\left(F^{(+)} + F^{(-)}\right)}}{\sqrt{\pi}\,\Theta_o} \exp\left\{-\frac{\Theta_o^2}{k^3\left(F^{(+)} + F^{(-)}\right)}\right\}$$

and (2.2.10)

$$\left[\frac{\partial F^{(-)}}{\partial t}\right]_{\text{breaking}} = \frac{F^{(-)}}{F^{(+)}} \left[\frac{\partial F^{(+)}}{\partial t}\right]_{\text{breaking}} \quad (2.2.11)$$

2.3 Non-Linear Wave-Wave Interaction

In one sense the non-linear interaction between surface gravity waves is the most clearly defined problem discussed in this report in that there exists a well-defined system of equations which describes the interaction process. These equations are the kinematic boundary condition at the ocean surface and Bernoullis' equation. Solving these equations for a surface profile, for any but the simplest situation, is a notoriously difficult task. One must, therefore turn to suitable approximation techniques to solve these equations. The approach which is the most compatible with the spectral modeling developed in this report is a modal description of the interaction process.

The mode oriented models treat the ocean surface as a superposition of waves (in the unperturbed ocean this would be a superpositon of sine waves). The non-linear terms in the interaction equations for the surface waves will allow coupling of the different modes. The first systematic development of such a theory is that due to Hasselmann (1962) and (1966). A similar approach has been used by West, Watson and Thomson (1973) to construct an interaction equation in terms of the slope of a wave q_k ,

$$\frac{dq_k^{(s)}}{dt} = \sum_{k_1+k_2=k} \Gamma^{k}_{k,k_2} q_{k_1}^{(s_1)} q_{k_2}^{(s_2)} e^{i(S_1\omega_1+S_2\omega_2-S\omega_k)t}$$

$$+ \sum_{k_1+k_2=k_3+k} \Gamma^{kk_3}_{k_1k_2} q_{k_1}^{(s_1)} q_{k_2}^{(s_2)} q_{k_3}^{(s_3)} e^{i\Omega_{k_1k_2kk_3}t} \qquad (2.3.1)$$

where

$$\Omega_{k_1k_2kk_3} = S_1\omega_{k_1} + S_2\omega_{k_2} + S_3\omega_{k_3} - S\omega_k \qquad (2.3.2)$$

and s=± corresponds to a right (+) or left (-) travelling wave and

$$S_i = +1 \quad \text{if } s_i = +$$

$$S_i = -1 \quad \text{if } s_i = -$$

in the exponentials. In the language of Hasselmann (1966), Eq. (2.3.1) includes three and four wave scatterings in a perturbation expansion of the wave amplitudes.

The structure of the interaction terms in Eq. (2.3.1) indicate that strong interactions and energy transfers occur between modes which together form a resonance, i.e.,

$$\left.\begin{aligned} \omega_1 + \omega_2 &= \omega \\ k_1 + k_2 &= k \end{aligned}\right\} \text{ 2nd order}$$

or

$$\left.\begin{aligned} \omega_1 + \omega_2 &= \omega_3 + \omega \\ k_1 + k_2 &= k_3 + k \end{aligned}\right\} \text{ 3rd order .}$$

For gravity waves ($\omega_i = \sqrt{gk_i}$), it is well known that the only resonant solutions at 2nd order are the uninteresting ones k_1 or $k_2 = 0$. At third order, however, non-zero solutions are possible. In order to have a perfect frequency match, the only solutions permissible are

$$k_1 = k_3 \quad \text{and} \quad k_2 = k$$

or

$$k_1 = k \quad \text{and} \quad k_2 = k_3$$

where k ($= k_4$) is the fourth member of this interacting set of waves. The near resonance condition is depicted in Figure (2) where a four wave interaction is represented schematically in (2a) and the location of the waves on the linear dispersion curve is indicated in (2b).

We see in Figure (2b) that a slight detuning has been allowed to occur. Fortunately, for finite energy transfer times it is not necessary to demand a perfect frequency match, and modes which couple together to give a frequency mismatch $\delta\omega < 1/T$

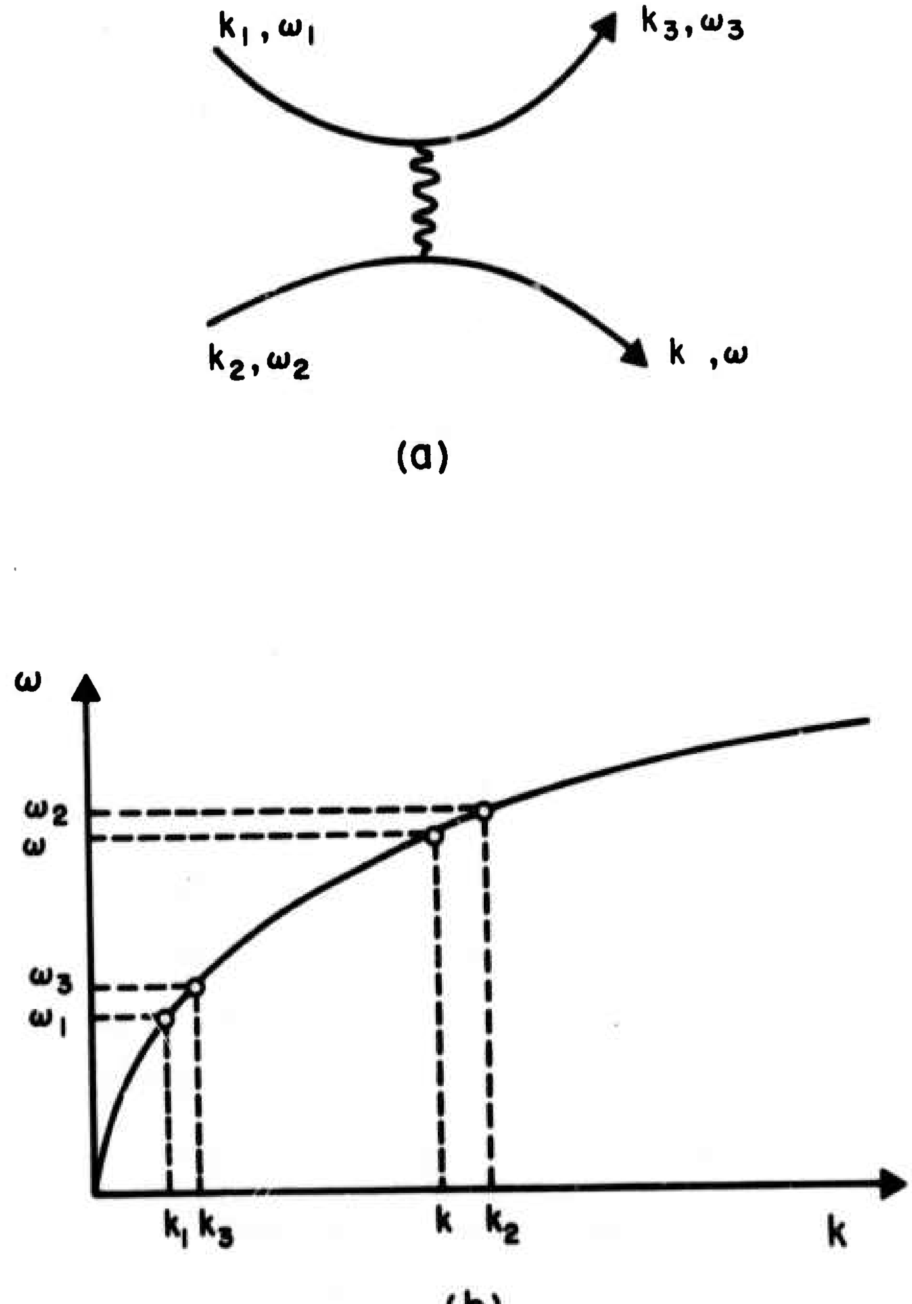

FIGURE 2. The Linear Dispersion Curve and Near Resonant Four Wave Interactions

(where T is the effective lifetime for the energy transfer) will form effectively a resonant group. The magnitude of the mismatch frequency may be expressed in terms of the deviation of the wavenumbers k_1 and k_2 from our reference wavenumber k:

$$k_1 = k(1 + \delta_1)$$
$$k_2 = k(1 + \delta_2) \quad . \tag{2.3.3}$$

The wavenumber k_3 is a function of k_1 and k_2:

$$k_3 = k_1 + k_2 - k \quad . \tag{2.3.4}$$

Then, to second order in δ_1 or δ_2, we have for the mismatch

$$\delta\omega = \omega_1 + \omega_2 - \omega_3 - \omega$$
$$= \omega \frac{\delta_1 \delta_2}{4} = \frac{\omega}{4}\left(\frac{k_1 - k}{k}\right)\left(\frac{k_2 - k}{k}\right) \quad . \tag{2.3.5}$$

The region of wavenumber space that contributes to the resonance ($\delta\omega < 1/T$) is sketched in Figure 3 for $\omega T \sim 250$ (this value of ωT is of the order expected in a near saturated sea). Thus, for the small values of $\delta\omega$ that contribute to the integral, the strongly-interacting modes lie close together. This fact can be used to derive a relatively simple description of the wave-wave interaction. To show this we start with the non-linear mode interaction formalism of Hasselman (1962). He uses perturbation theory to describe the resonant energy transfer be-

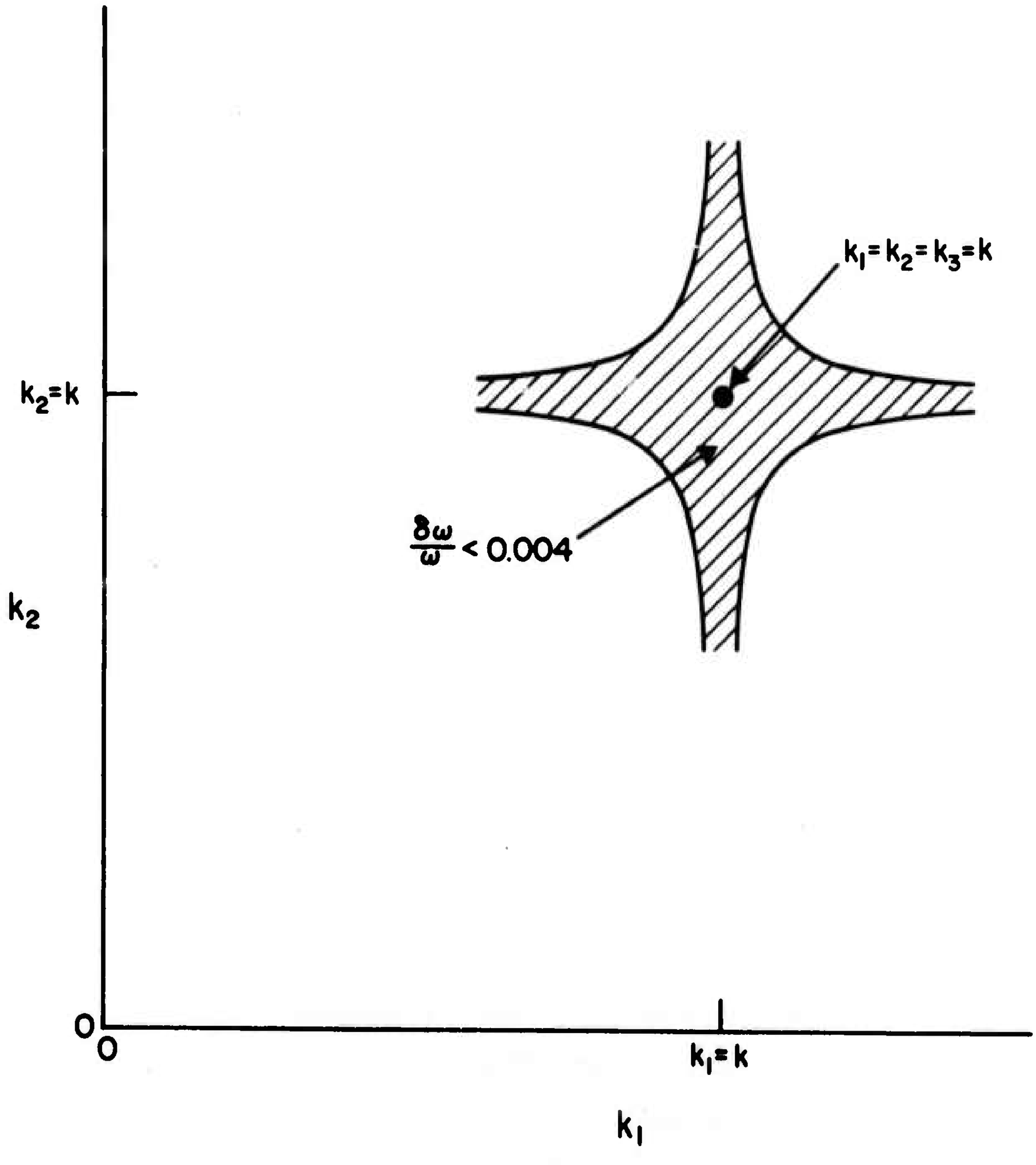

FIGURE 3. The Region of Wavenumber Space that Contributes to the Resonance ($\delta\omega < 1/T$) is Sketched for $\omega T \sim 250$

tween different modes of the ocean spectrum. By assuming the statistical distribution of wave amplitudes on the ocean surface to be Gaussian, he was able to derive an expression for the rate of increase of energy of a particular mode (k_4) in terms of the product of the energy density in three other modes

$$\frac{\partial F_4}{\partial t} = \int A\left(\omega_4 F_1 F_2 F_3 + \omega_3 F_1 F_2 F_4 - \omega_2 F_1 F_3 F_4 - \omega_1 F_2 F_3 F_4\right) d^2k_1\, d^2k_2 \tag{2.3.6}$$

where

$$A = \frac{9\pi g^2 D^2 \omega_4}{4\rho^2 (\omega_1 \omega_2 \omega_3 \omega_4)^2}\ \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

and we have required that

$$\vec{k}_3 = \vec{k}_1 + \vec{k}_2 - \vec{k}_4$$

In Eq. (2.3.6) F_i represents the power spectrum of the wave height at the wavenumber $\vec{k}_i$ [$F_i = F(\vec{k}_i)$]. According to our earlier discussion, the major contribution to the integral will come from wavenumber $\vec{k}_1$ and $\vec{k}_2$ that are close to $\vec{k}_4$ (see Figure 3). If this interval is small enough, we are able to expand $F(\vec{k}_i)$ in a Taylor series about $\vec{k}_i = \vec{k}_4$ and express the integral in terms of the derivatives of $F(\vec{k})$ at $\vec{k} = \vec{k}_4$. To do this explicitly, we would need to proceed beyond the limits of perturbation theory to derive specific procedures for allowing for the frequency mismatch. However, the form of the result can be obtained without requiring knowledge of such methods.

In general, we will have

$$\frac{\partial F}{\partial t} = G\left(k, F, \frac{\partial F}{\partial k}, \frac{\partial^2 F}{\partial k^2}, \ldots\right) \qquad (2.3.7)$$

We include only terms up to the second derivative and note that this interaction can only rearrange the energy distribution, leaving the total energy constant. Thus

$$\int \frac{\partial F}{\partial t} dk = \frac{d}{dt} \int F dk = 0 \qquad (2.3.8)$$

Since this relation must hold for arbitrary distributions, $F(k)$, $\frac{\partial F}{\partial t}$ must be a derivative of the form

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial k}\left[g\left(k, F, \frac{\partial F}{\partial k}\right)\right] \qquad (2.3.9)$$

Using the general structure of Eq. (2.3.6) we have that the right-hand side of Eq. (2.3.9) must be cubic in F, so that Eq. (2.3.9) must be of the general form

$$\frac{\partial F}{\partial t} = (\text{constant}) \frac{\partial}{\partial k}\left[k^p F^2 \frac{\partial (F k^s)}{\partial k}\right] \qquad (2.3.10)$$

Dimensional analysis can be used to further limit the parameters for gravity waves. Since the acceleration of gravity is the only available parameter and since F has dimensions of $(\text{length})^3$, we can only have

$$\frac{\partial F}{\partial t} = C_1 \pi \sqrt{g} \frac{\partial}{\partial k} \left[k^{8.5-s} F^2 \frac{\partial}{\partial k} (F k^s) \right] \qquad (2.3.11)$$

where C_1 is a dimensionless number.

Finally we argue that, since there is no characteristic scale to the problem, an equilibrium condition must correspond to one in which the wave slopes are independent of scale, i.e., that

$$(Fk^3)_{\text{equilibrium}} \rightarrow \text{constant} \quad . \qquad (2.3.12)$$

Thus, we choose s = 3 and write

$$\left(\frac{\partial F}{\partial t}\right)_{\text{wave-wave}} = C_1 \pi \sqrt{g} \frac{\partial}{\partial k} \left[\frac{(k^3 F)^2}{\sqrt{k}} \frac{\partial}{\partial k} (k^3 F) \right] \cdot \qquad (2.3.13)$$

The value of C_1 can only be determined from a more detailed theory or from experiment, such as to that carried out by Benjamin and Feir (1967). Here a finite amplitude traveling wave was set up in a wave tank and its decay or break up observed as a function of distance from the wave maker.* In the transfer model, the wave height should obey in this case

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial k} \left[C_1 \pi \, k^6 F^2 \sqrt{\frac{g}{k}} \frac{\partial}{\partial k} (F k^3) \right] \quad . \qquad (2.3.14)$$

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* The Benjamin-Feir study consisted of the observation of the breakup of a single coherent non-linear wave. The appropriate experiment to test a diffusion model should be somewhat different and would observe the decay of a coherent wave in the presence of an already fully developed sea.

It is convenient for calculational purposes to define a (dimensionless) quantity $\gamma = Fk^3$

$$\frac{\partial \gamma}{\partial t} = k^3 \frac{\partial}{\partial k} \left(C_1 \pi \sqrt{\frac{g}{k}}\, \gamma^2 \frac{\partial \gamma}{\partial k} \right) \quad . \tag{2.3.15}$$

This equation can be interpreted as a diffusion equation (in wavenumber space) having a non-linear diffusion coefficient defined by

$$D_k = C_1 \pi \sqrt{\frac{g}{k}}\, k^3 \gamma^2 \quad . \tag{2.3.16}$$

An effective measure of the time (T) for energy to spread out of a wavenumber interval Δk (i.e., a lifetime) is

$$T \sim \Delta k_o^2 \Big/ D_k \quad . \tag{2.3.17}$$

We argue that the permissible mismatch of frequency $\delta\omega$ must be of order T^{-1}. Thus we may write

$$\frac{\delta\omega}{\omega} = \frac{(k_1-k_4)(k_3-k_4)}{4\,k_4^2} \sim \frac{1}{\omega T} \sim \frac{D_k}{\omega \Delta k_o^2} \quad . \tag{2.3.18}$$

Since both k_1-k_4 and k_3-k_4 are each of order Δk_o, we find that the width of the wavenumber regime into which the mode decays is of order

$$\begin{aligned} \frac{\Delta k}{k} &\simeq \left(\frac{4D_k}{\omega k^2} \right)^{1/4} \\ &\simeq \left(4C_1 \pi \right)^{1/4} \gamma^{1/2} \quad . \end{aligned} \tag{2.3.19}$$

The lifetime is given by

$$\omega T = \frac{2}{\sqrt{C_1 \pi}\, \gamma} \quad . \tag{2.3.20}$$

For a saturated sea, the value of $\gamma (= \gamma_{sat})$ is known from experiment:

$$\gamma_{sat} \simeq 0.00585 \pm 10\% \quad . \tag{2.3.21}$$

Thus the lifetime (N, measured in wave periods) of a monochromatic wave in a near saturated ocean is expected to be given by

$$N = \frac{\omega T}{2\pi} \simeq \frac{1}{\sqrt{\pi^3 C_1}\,(.00585)} \simeq \frac{30.5}{\sqrt{C_1}} \quad . \tag{2.3.22}$$

Thus, experiments of this type could be used to determine the coefficient C_1, as well as simply to test the diffusion model.

2.4 Linear Interaction Theory

2.4A Model of the Surface Wave-Current Interaction

In this section we will be concerned with constructing a model to describe the effect of long waves on short waves; i.e., the interaction between widely separated regions of wavenumber space. These interactions may be separated into a number of distinct physical situations, such as swells passing through a region of developing ocean, surface currents produced by internal waves interacting with surface waves, etc. To establish a model of these phenomena, we will review some aspects of the linear theory for these processes. We consider the short waves to be infinitesimal surface waves in the development of the linear theory and the long waves to be describable in terms of a surface current. The analysis centers on how energy is transferred between the two wave length regimes. The model is an outgrowth of a series of studies made by Longuet-Higgins and Stewart (1961) and (1964) and more recently by Phillips (1966, pages 50 and 136).

The model ocean used in this section is both homogeneous and irrotational. This allows us to use a potential description of the velocity field, i.e.

$$\nabla \times \vec{u} = 0 \quad \text{which implies} \quad \vec{u} = \nabla\Phi \quad . \qquad (2.4.1)$$

The velocity potential also satisfies Laplace's equation

$$\nabla^2\Phi = 0 \qquad (2.4.2)$$

since the fluid is assumed to be incompressible ($\nabla \cdot \vec{u} = 0$). Using the integral of the momentum equation, we may write Bernoulli's equation at the ocean surface as

$$\Phi_t + \frac{1}{2} \nabla\Phi \cdot \nabla\Phi + gz = 0 \qquad \text{at } z = \zeta \qquad (2.4.3)$$

with the condition that hydrostatic equilibrium must prevail at infinity, and $z = \zeta$ is the free surface of the ocean. The rate of increase in the wave height following a fluid element is the vertical component of the fluid velocity:

$$\frac{d\zeta}{dt} = \frac{\partial \Phi}{\partial z} \qquad \text{at } z = \zeta \qquad (2.4.4)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \nabla\Phi \cdot \nabla \qquad (2.4.5)$$

is the Eulerian derivative.

We intend to deal with the linearized forms of Equations (2.4.3) and (2.4.4) and so we separate the velocity potential and surface displacement into a long and short wave component;

$$\Phi = \Psi + \phi \qquad (2.4.6)$$

and

$$\zeta = \chi + h \quad .$$

The terms Ψ and χ satisfy the free surface equations independently of the surface wave height (h) and velocity potential (ϕ), that is

$$\Psi_t + \frac{1}{2} \nabla\Psi \cdot \nabla\Psi + g\chi = 0 \qquad \text{at } z = \chi \qquad (2.4.7a)$$

and

$$\chi_t + \nabla\Psi\cdot\nabla\chi = \frac{\partial\Psi}{\partial z} \qquad \text{at } z = \chi\ . \qquad (2.4.7b)$$

These equations refer to the unperturbed ocean surface; that is, unperturbed by the short surface waves and may be identified with swells, surface currents produced by internal waves or the effect of other generating mechanisms of interest.

Equations (2.4.7a) and (2.4.7b) are valid at the unperturbed surface $z = \chi$, but the more general expression given by Equations (2.4.3) and (2.4.4) are valid at the free surface $z = \chi + h$. To determine the interaction between the surface waves and current*, we substibute Equations (2.4.6) into (2.4.3) obtaining,

$$\phi_t + \frac{1}{2} \nabla\phi\cdot\nabla\phi + \nabla\phi\cdot\nabla\Psi + gh = - [\Psi_t + \frac{1}{2} \nabla\Psi\cdot\nabla\Psi + g\chi]$$

$$\text{at } z = \chi + h \quad . \qquad (2.4.8)$$

To simplify this equation, we could implement (2.4.7a); however, Equation (2.4.7a) is applicable only at $z = \chi$. We therefore expand the right hand side of this equation about the unperturbed

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* We will use the term current to describe the gradient of the long wave velocity potential regardless of the generating mechanism.

surface. We obtain the following expressions at the unperturbed surface

$$\nabla\Psi\Big|_{z=h+\chi} = \left\{\nabla\Psi + h\frac{\partial}{\partial z}(\nabla\Psi) + \ldots\right\}\Big|_{z=\chi}$$

and

$$\Psi_t\Big|_{z=h+\chi} = \left\{\Psi_t + h\frac{\partial}{\partial z}(\Psi_t) + \ldots\right\}\Big|_{z=\chi}$$

in which we will keep only terms linear in the surface wave height (h). Substituting these expansions into Equation (2.4.8), we obtain for the interaction equation

$$\phi_t + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \left[g + \frac{\partial V}{\partial t} + \nabla V\cdot\nabla\phi + \nabla\Psi\cdot\nabla V\right]h + \nabla\Psi\cdot\nabla\phi = 0$$

which to terms linear in ϕ and h is

$$\phi_t + \nabla\Psi\cdot\nabla\phi + \left[g + \nabla\Psi\cdot\nabla V + \frac{\partial V}{\partial t}\right]h = 0 \quad \text{at } z=\chi \,. \qquad (2.4.9)$$

We have introduced the notation in the above equations that the surface current may be written as

$$\nabla\Psi = \hat{e}_x U(x,t) + \hat{e}_z V(x,t) \qquad (2.4.10)$$

where U(x,t) and V(x,t) are the horizontal and vertical components of the current, respectively.

Equation (2.4.9) is seen to retain a linear interaction between the current given by Equation (2.4.10) and the surface waves. A second aspect worthy of note is the modification in the gravitational acceleration due to the finite value of the vertical current component which induces an additional vertical acceleration.

We may write Equation (2.4.9) in the alternate form

$$\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial z}\right\}\phi + \left\{g + \frac{dV}{dt}\right\}h = 0 \quad \text{at } z = \chi. \tag{2.4.11}$$

The generating mechanism for the current determines the relative magnitude of the unperturbed horizontal and vertical velocity components. For example, if the current is produced by an internal wave, then $|U| >> |V|$ so that only the horizontal velocity component need be retained in Equation (2.4.11). If, however, the current is produced by a swell passing under the region of interest, the vertical component of the velocity may be comparable to the horizontal. In this case, both components must be considered. The particular mechanism being investigated will therefore determine the form of the equation used in the analysis.

To determine the linear form of the kinematic boundary condition [Equation (2.4.4)], we encounter the same difficulty as with (2.4.3), that is, going from the free surface to the unperturbed surface. To obtain our expression, we expand the right-hand side of Equation (2.4.4) about $z=\chi$ as follows:

$$\left.\frac{\partial \Psi}{\partial z}\right|_{z=h+\chi} \cong \left\{\frac{\partial \Psi}{\partial z} + h\,\frac{\partial^2 \Psi}{\partial z^2} + \ldots\right\}\Bigg|_{z=\chi}$$

and using the definition of the current components in Equation (2.4.10) obtain,

$$\left.\frac{\partial \Psi}{\partial z}\right|_{z=h+\chi} \cong \left\{\frac{\partial \Psi}{\partial z} + h\,\frac{\partial V}{\partial z}\right\}\Bigg|_{z=\chi}$$

to terms linear in h. Of course, we must also expand the appropriate terms on the left side of Equation (2.4.4) and use the expansion for $\nabla\Psi$. The inclusion of these expansions and using Equation (2.4.8) to simplify, we obtain

$$\left(\frac{\partial}{\partial t} + \nabla\phi\cdot\nabla + \nabla\Psi\cdot\nabla + \nabla V\cdot\nabla\right) h = \frac{\partial \phi}{\partial z} + h\,\frac{\partial V}{\partial z} - \nabla\phi\cdot\nabla\chi - \nabla V\cdot\nabla\chi$$

or to terms linear in ϕ and h

$$\left(\frac{\partial}{\partial t} + \nabla\Psi\cdot\nabla\right) h = \frac{\partial \phi}{\partial z} + h\,\frac{\partial V}{\partial z} - \nabla\phi\cdot\nabla\chi \qquad \text{at } z = \chi. \quad (2.4.12)$$

We may use the incompressibility condition ($\nabla\cdot\vec{u} = 0$) to replace $\frac{\partial V}{\partial z}$ with $-\frac{\partial U}{\partial x}$ and write (2.4.12) as

$$\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial z} + \frac{\partial U}{\partial x}\right\} h = \frac{\partial \phi}{\partial z} - \nabla\phi\cdot\nabla\chi \quad \text{at } z=\chi. \qquad (2.4.13)$$

If we restrict our analysis to a single horizontal dimension, we may write (2.4.13) as

$$\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial z} + \frac{\partial U}{\partial x}\right\} h = \frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial \chi}{\partial x} \quad \text{at } z=\chi \qquad (2.4.14)$$

which is the vertical velocity of the surface waves. We note that this equation was assumed to have the form

$$\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right\} h = \frac{\partial \phi}{\partial z}$$

in PD-72-023 at the perturbed surface. The effect of calculating at the unperturbed surface is to modify the equation by the gradient of the horizontal surface current and the horizontal variation in the unperturbed surface height.

We may replace the horizontal gradient of the unperturbed surface in Equation (2.4.14) by taking the horizontal derivative of Equation (2.4.8) to obtain

$$\frac{\partial \chi}{\partial x} = -\frac{1}{g}\left\{\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial V}{\partial x}\right\} .$$

Also, using the fact that the ocean is irrotational, i.e., $\nabla \times \vec{u} = 0$, we may replace $\frac{\partial V}{\partial x}$ with $\frac{\partial U}{\partial z}$ so that

$$\frac{\partial \chi}{\partial x} = -\frac{1}{g}\left\{\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial z}\right\}$$

$$= -\frac{1}{g}\frac{dU}{dt} \quad . \qquad (2.4.15)$$

Substituting Equation (2.4.15) into Equation (2.4.14) yields

$$\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial z} + \frac{\partial U}{\partial x}\right\}h = \frac{\partial \phi}{\partial z} + \frac{1}{g}\frac{\partial \phi}{\partial x}\frac{dU}{dt} \qquad (2.4.16)$$

as our second linearized dynamic equation for the surface waves. Equation (2.4.16) has the same structure as the expression obtained by Zachariasen (1972), however, it is evaluated at $z=\chi(x,t)$ and not $z=0$ as are his expressions.

Milder (1972) has shown that the simplicity of these equations can be maintained by noting that the right-hand side of Eq. (2.4.14) can be written as

$$\frac{\partial \phi}{\partial z} - \frac{\partial \chi}{\partial x}\frac{\partial \phi}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial \phi}{\partial n} \qquad (2.4.17)$$

where s is the distance along the surface $z=\chi(x,t)$ and $\frac{\partial}{\partial n}$ is the derivative normal to that surface. Since the velocity potential satisfies the two dimensional Laplacian,

$$\nabla^2\phi = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z}\right) = 0$$

Milder also shows that

$$\frac{\partial s}{\partial x}\frac{\partial \phi}{\partial n} = -i\frac{\partial \phi}{\partial x} \quad . \qquad (2.4.18)$$

We premultiply Eq. (2.4.11) by the operator

$$\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}$$

and use Eqs. (2.4.16) and (2.4.18) to obtain

$$\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\right\}\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right\}\phi = ig_{eff}\frac{\partial \phi}{\partial x} \tag{2.4.19}$$

where we have neglected the vertical velocity V compared with U and $g_{eff} = g + \frac{dV}{dt}$. We obtain the term by term expansion of Eq. (2.4.19)

$$\left\{\frac{\partial^2}{\partial t^2} + U^2\frac{\partial^2}{\partial x^2} + \left[\frac{\partial U}{\partial t} + 2U\frac{\partial}{\partial t} + U\frac{\partial U}{\partial x} - ig_{eff}\right]\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\frac{\partial}{\partial t}\right\}\phi = 0 \tag{2.4.20}$$

which is our linearized dynamic equation for the surface velocity potential at the unperturbed surface $z=\chi(x,t)$.

2.4B Linear Energy Transfer Equation

To investigate Equation (2.4.20), we assume a solution of the form

$$\phi = A(x,t)\, e^{i\Gamma(x,t)} \tag{2.4.21}$$

where the frequency and wavenumbers for the wavetrain are defined by

$$k = \frac{\partial \Gamma}{\partial x} \quad \text{and} \quad \omega = -\frac{\partial \Gamma}{\partial t}. \tag{2.4.22}$$

Substituting Equation (2.4.21) into Equation (2.4.20) and using the definition of the frequency and wavenumber, we obtain

$$(\omega - kU)^2 - k\,g_{eff} = \frac{1}{A}\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\right\}\left\{\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right\} A \tag{2.4.23}$$

when $|U| >> |V|$, for the real part of Equation (2.4.20). The imaginary part of (2.4.20) yields the equation

$$- \frac{2}{A} \sqrt{k\, g_{eff}} \left\{ \frac{\partial}{\partial t} + \left[U + \frac{1}{2} \sqrt{g_{eff}/k} \right] \frac{\partial}{\partial x} \right\} A$$

$$- \left\{ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right\} (\omega - kU) - \sqrt{k\, g_{eff}} \; \frac{\partial U}{\partial x} = 0. \qquad (2.4.24)$$

All the terms on the right-hand side of Equation (2.4.23) are of second order and in the WKB approximation we may write the dispersion relation

$$(\omega - kU)^2 - k\, g_{eff} \approx 0 \quad . \qquad (2.4.25)$$

If we introduce the following definitions of a velocity (c) and energy (E),

$$c = \sqrt{g_{eff}/k} \qquad (2.4.26)$$

and

$$E = kA^2 = k|\phi|^2 \qquad (2.4.27)$$

we obtain the energy conservation equation

$$\frac{\partial}{\partial t} [cE] + \frac{\partial}{\partial x} \left[\left(U + \frac{c}{2} \right) cE \right] = 0 \qquad (2.4.28)$$

from (2.4.24) using (2.4.25). Equation (2.4.28) may be written in the alternate form

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\left(U + \frac{c}{2}\right) E\right] + \tfrac{1}{2} E \frac{\partial U}{\partial x} = 0 \qquad (2.4.29)$$

where $\frac{1}{2} E \frac{\partial U}{\partial x}$ is interpreted at the "radiation stress".

2.5 Wave Packet-Surface Current Interaction

In this section, we restrict our attention to the case of a surface current whose horizontal velocity component is much greater than the vertical component; e.g., a current generated by a non-dispersive internal wave. If we consider a surface wave packet moving into a region of non-uniform but temporally-steady current, we can write the average value of the square of the wave packet amplitude in terms of the energy density F(k) as

$$\overline{a^2} = \int_{\Delta k} F(k)\, dk \quad . \tag{2.5.1}$$

The wave packet is considered to be localized in a region Δx of physical space and must therefore be concentrated in a region Δk of wavenumber space. The integral Equation (2.5.1) is restricted to this region of wavenumber space (Δk). If the width of the wave packet is narrow; i.e., $\Delta k \ll k$, then the initial wave train is nearly monochromatic and we may write

$$\overline{a^2} \sim \overline{F}(k)\, \Delta k \quad . \tag{2.5.2}$$

The quantity $\overline{F}(k)$ in Equation (2.5.2) is the average energy density of the wave packet in k-space. We will drop the bar for convenience in the following discussion.

Let us consider the case of a steady horizontal current $U = U(x)$. The total time variation (following the wave packet) in the squared amplitude may be written using Equation (2.5.2) as

$$\frac{d}{dt} a^2 = \Delta k \frac{dF}{dt} + F \frac{d\Delta k}{dt} . \qquad (2.5.3)$$

It is clear from Equation (2.5.3) that as the wave packet traverses the current region its width changes markedly. If we define the packet width as $\Delta k = k_1 - k_2$, then

$$\frac{d\Delta k}{dt} = \frac{dk_1}{dt} - \frac{dk_2}{dt} \qquad (2.5.4)$$

but we know from the dispersion relation of the preceding section that

$$\frac{dk}{dt} = -k \frac{\partial U}{\partial x}$$

so that the time variation in the packet width given by Equation (2.5.4) can be written as

$$\frac{d\Delta k}{dt} = -\Delta k \frac{\partial U}{\partial x} \left(\frac{4U + 3c}{4U + 2c}\right) \qquad (2.5.5)$$

Using this equation in Equation (2.5.3) and dividing the resulting expression by Equation (2.5.2) yields

$$\frac{1}{a^2} \frac{da^2}{dt} = \frac{1}{F} \frac{dF}{dt} - \frac{\partial U}{\partial x} \left(\frac{4U + 3c}{4U + 2c}\right) \qquad (2.5.6)$$

as the percent variation in the packet amplitude as it moves into a region of non-uniform current.

2.5A Wave Packet Distortion

An alternative expression to Equation (2.5.6) may be obtained by using the energy conservation equation

$$\frac{\partial}{\partial t}\left(cE\right) + \frac{\partial}{\partial x}\left[\left(U + \frac{c}{2}\right)cE\right] = 0 \qquad (2.5.7)$$

of the preceding section. For a steady non-uniform current, we have

$$c = \sqrt{g/k}$$

and

$$v_G = U + \frac{c}{2}$$

for the phase velocity in the absence of the current and group velocities of the surface waves, respectively. For these conditions Longuet-Higgins (1961, 1964) among others, has shown that Equation (2.5.7) has the solution

$$c\,v_G E = \text{const.} \qquad (2.5.8)$$

To express Equation (2.5.8) in terms of the wavepacket amplitude, we introduce the quantities

$$\phi = Ae^{i\Gamma} \quad \text{and} \quad h = ae^{i\Gamma} \qquad (2.5.9)$$

where the wavenumber and frequency is defined by Equation (2.4.23). The surface elevation and velocity potential defined by Equation (2.5.9) clearly obey Equation (2.4.11),

$$\left\{\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right\} \phi + gh \approx 0 \tag{2.5.10}$$

where we have neglected the effects of the finite vertical velocity V. Using Equation (2.5.9) in (2.5.10), we obtain

$$\frac{1}{A} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) A - i\,(\omega - kU) + g \frac{a}{A} \approx 0 \tag{2.5.11}$$

which using the dispersion relation Equation (2.4.26) becomes

$$-\frac{1}{gA} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) A + i \sqrt{k/g} = \frac{a}{A} . \tag{2.5.12}$$

If the fractional change in the amplitude is much smaller than the acceleration of gravity

$$\frac{1}{A}\frac{dA}{dt} << g$$

in the WKB limit, we obtain from Equation (2.5.12)

$$A \approx c\,a \quad . \tag{2.5.13}$$

We may use Equation (2.5.13) in the energy conservation Equation (2.5.8) and using Equation (2.4.28) obtain

$$kc^3 v_G a^2 = \text{constant}$$

but $kc^2 = g$ so that

$$c v_G a^2 = \text{constant} \quad . \qquad (2.5.14)$$

Returning now to our original problem, i.e., finding a second expression for the time variation in the wave packet amplitude, we take the derivative of Equation (2.5.14) as follows

$$\frac{d}{dt} \left\{ c \left(U + \frac{c}{2}\right) a^2 \right\} = 0 \quad . \qquad (2.5.18)$$

For a steady current, i.e., $\frac{\partial U}{\partial t} = 0$, and using

$$\frac{dc}{dt} = - \frac{c}{2k} \frac{dk}{dt} = \frac{c}{2} \frac{\partial U}{\partial x}$$

we obtain

$$\frac{1}{a^2} \frac{da^2}{dt} = - \left\{ 1 + \frac{1}{2} \frac{U + c}{U + \frac{c}{2}} \right\} \frac{\partial U}{\partial x} \qquad (2.5.19)$$

as our second expression. Equating Equation (2.5.6) with Equation (2.5.19) yields

$$\frac{1}{F}\frac{dF}{dt} - \frac{\partial U}{\partial x}\left(\frac{4U + 3c}{4U + 2c}\right) = -\left\{1 + \frac{1}{2}\frac{U + c}{U + \frac{c}{2}}\right\}\frac{\partial U}{\partial x}$$

or

$$\frac{1}{F}\frac{dF}{dt} + \frac{1}{2}\frac{\partial U}{\partial x} = 0 \qquad (2.5.20)$$

which accounts for the non-dispersive effects associated with compression of the wave packet.

2.5.3 The Effect of Capillary Waves on Wave Packets

The effect of capillary waves may be included in the radiation stress obtained in the preceding section by modifying the appropriate terms in Eq. (2.5.7). As will be discussed in the following section, if we define the parameter $\gamma = T/\rho g$ where T is the surface tension and ρ the density of the ocean, the frequency of surface waves becomes,

$$\omega = \sqrt{gk(1 + \gamma k^2)} \quad . \qquad (2.5.21)$$

The phase velocity c is

$$c = \frac{\omega}{k} = \sqrt{g/k}\,\sqrt{1 + \gamma k^2} \qquad (2.5.22)$$

and group velocity becomes

$$c_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{g/k}\,\frac{1 + 3\gamma k^2}{\sqrt{1 + \gamma k^2}} \qquad (2.5.23)$$

which becomes $c/2$ as $\gamma \to 0$.

We can use these definitions in the conservation expression Eq. (2.5.18) to yield

$$\frac{d}{dt}\left\{c(U+c_g)a^2\right\} = 0 \quad . \tag{2.5.24}$$

For a steady current, i.e., $\frac{\partial U}{\partial t} = 0$, and using

$$\frac{1}{c}\frac{dc}{dt} = \left(\frac{1}{2} - \frac{\gamma k^2}{1+\gamma k^2}\right)\frac{\partial U}{\partial x} \tag{2.5.25}$$

and

$$\frac{1}{c_g}\frac{dc_g}{dt} = \left(\frac{1}{2} - \frac{6\gamma k^2}{1+3\gamma k^2} + \frac{\gamma k^2}{1+\gamma k^2}\right)\frac{\partial U}{\partial x} \tag{2.5.26}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (U+c_g)\frac{\partial}{\partial x}$$

we obtain from Eq. (2.5.24)

$$\frac{1}{a^2}\frac{da^2}{dt} = -\left[1 + \frac{1}{2}\left\{(2+U/c_g) - \frac{2\gamma k^2}{1+\gamma k^2}\left(\frac{U}{c_g}\right) - \frac{12\gamma k^2}{1+3\gamma k^2}\right\}\left(1 + \frac{U}{c_g}\right)^{-1}\right]\frac{\partial U}{\partial x} \tag{2.5.27}$$

in place of Eq. (2.5.19). Substituting Eq. (2.5.27) into Eq. (2.5.6) we arrive at

$$\frac{1}{F}\frac{dF}{dt} + \frac{1}{2}\left[\frac{1 - \gamma k^2}{1 + \gamma k^2}\right]\frac{\partial U}{\partial x} = 0 \qquad (2.5.28)$$

as the modified radiation stress term due to the action of capillary waves.

2.5C Non-Linear Amplitude Effects on Effective Group Velocity

The dispersion relation used in the last section was of course that given for a linear gravity wave modified to include capillary waves. We now wish to model the change in the dispersion relation due to finite amplitude effects by incorporating in our equations a modified group velocity. A more systematic approach to this problem would be to modify the energy balance equation introduced in Section 2.4 to include the finite amplitude effects. This approach is presently being explored and will be reported at a later time.

To second order the dispersion relation of an isolated non-linear gravity wave, i.e., Stokes wave, is

$$\omega = kc = \sqrt{gk}\ (1 + k^2a^2)^{1/2} \qquad (2.5.29)$$

where ka is the slope of the wave. The group velocity of this wave is

$$c_g = \frac{d\omega}{dk} = c_g^{(o)}\left(1 + 2\,k^2a^2\right) \Big/ \sqrt{1 + k^2\,a^2} \qquad (2.5.30)$$

where $c_g^{(o)}$ is the linear group velocity $\frac{1}{2}\sqrt{g/k}$ or that given by Eq. (2.5.23). To generalize Eq. (2.5.30) to the situation where there is a spectrum of waves, we introduce the notion of a correlation interval. We assume that waves in the interval

$$k - \frac{\Delta k}{2} \leq k \leq k + \frac{\Delta k}{2}$$

are correlated. Waves outside this interval are assumed uncorrelated and not to alter the wave phases in any systematic way. Thus we write the dispersion relation

$$\omega' = \frac{d\phi}{dt} = \sqrt{gk} + \text{effect of correlated waves}$$
$$= \sqrt{gk}\left(1 + \frac{1}{2}\int_{k-\frac{\Delta k}{2}}^{k+\frac{\Delta k}{2}} k^2 F(k)dk\right). \qquad (2.5.31)$$

The group velocity can also be written

$$c_g = \frac{d\omega'}{dk} = c_g^{(o)}\left(1 + \frac{1}{2}\int_{k-\frac{\Delta}{2}}^{k+\frac{\Delta k}{2}} k^2 F(k)dk\right)$$
$$+ \frac{1}{2}\left\{k^2 F(k)\Big|_{k+\frac{\Delta k}{2}} - k^2 F(k)\Big|_{k-\frac{\Delta k}{2}}\right\}\sqrt{gk}$$

$$c_g \simeq c_g^{(o)}\left\{1 + 2k^2 F(k)\Delta k + \frac{1}{2}\int_{k-\frac{\Delta k}{2}}^{k+\frac{\Delta k}{2}} k^2 F(k)dk\right\} \qquad (2.5.32)$$

If the spectrum is sharply peaked we write

$$c_g \approx c_g^{(o)} \left\{ 1 + \frac{5}{2} k^2 F(k) \Delta k \right\} \tag{2.5.33}$$

The model we use in our calculation parameterizes the effect of Δk in Eq. (2.5.33) so that we have

$$c_g = c_g^{(o)} \left(1 + \lambda \, k^3 F(k) \right) \tag{2.5.34}$$

where λ is a parameter on the order of unity.

2.6 Model for Generation of Waves by Wind: Effects Independent of Wave Slope

In this section we wish to outline the initial growth of surface waves due to the statistical fluctuation in air pressure due to turbulence, eddies, wind gusts, etc. To construct this model, we consider the linear form of the interaction equation for infinitesmal surface gravity waves discussed in Section 2.4. We modify the previous equation by the inclusion of the incremental pressure with respect to ambient (Δp), that is

$$\frac{d\phi}{dt} + gh + \frac{1}{\rho} \Delta p = 0 \qquad \text{at } z = 0 \tag{2.6.1}$$

and

$$\frac{dh}{dt} = \frac{\partial\phi}{\partial z} \qquad \text{at } z = 0 \quad . \tag{2.6.2}$$

We note that the above equations are taken about the $z = 0$ position, which implies that there is no ambient current present, i.e., $U = V = 0$ from Section 2.4.

The expression for the pressure increment may be obtained by considering the interface between two fluids. Consider the vertical forces acting on a strip of surface of width δx. If p is the pressure just below the water surface, p_a the pressure of the air and T the surface tension, then

$$(p - p_a)\delta x + \delta\left[T \frac{\partial h}{\partial x}\right] = 0 \qquad (2.6.3)$$

is the force balance acting on the strip δx. The pressure is therefore given by

$$p = p_a - T \frac{\partial^2 h}{\partial x^2} \quad . \qquad (2.6.4)$$

The air pressure p_a can be modeled by a sum of these terms; (i) the direct coupling of the wind to the water, (ii) the local statistical variation in the surface pressure, and (iii) the surrounding or ambient pressure at the surface,

$$p_a = \rho \alpha \frac{\partial h}{\partial x} + P + p_{ambient} \quad . \qquad (2.6.5)$$

The first term on the right of Equation (2.6.5) gives the in-phase pressure variation at the surface due to the action of wind. This term will be discussed more fully in Section 2.7. Substituting Equation (2.6.5) into Equation (2.6.1) yields

$$\frac{d\phi}{dt} + gh + \alpha \frac{\partial h}{\partial x} - \frac{T}{\rho} \frac{\partial^2 h}{\partial x^2} = -\frac{P}{\rho} \qquad \text{at } z = 0 \quad . \qquad (2.6.6)$$

To obtain a solution to Equation (2.6.6) and (2.6.1), we decompose the vector potential and surface waves into their fourier components and thereby obtain a modal description of our linear problem. The surface elevation is

$$h = \sum_k h_k(t) \frac{e^{ikx}}{\sqrt{L}} \qquad (2.6.7)$$

and since ϕ obeys Laplace's equation $[\nabla^2\phi = 0]$ the vector potential is

$$\phi = \sum_\kappa \phi_k(t)\, e^{|k|z} \frac{e^{ikx}}{\sqrt{L}} \quad . \qquad (2.6.8)$$

Using Equation (2.6.2) we obtain the relation between the mode amplitudes

$$\phi_k = \dot{h}_k \frac{e^{-|k|z}}{k} \qquad (2.6.9)$$

and because our dynamic equation is taken at z = 0, we obtain

$$\ddot{h}_k + \left\{ gk + \gamma k^3 + ik^2 \alpha \right\} h_k = -\frac{k}{\rho} P(k,t) \qquad (2.6.10)$$

when Equations (2.6.7) - (2.6.9) are substituted into Equation (2.6.6). The quantity P(k,t) in Equation (2.6.10) is the statistical fluctuation of the pressure in k-space and $\gamma = T/\rho$.

The structure of Equation (2.6.10) is clearly that of a driven harmonic oscillator

$$\ddot{h}_k + \Omega_k^2 h_k = -\frac{k}{\rho} P(k,t) \quad . \qquad (2.6.11)$$

The driving force in this case is the statistical function $P(k,t)$. This function is uncorrelated with the gravity wave amplitudes $[h_k(t)]$ and acts to randomize the growth of the mode amplitudes.

The solution to Equation (2.6.11) in the homogeneous case, with an initial amplitude and slope of zero, i.e., $h_k(t = 0) = \dot{h}_k(t = 0) = 0$, is

$$h_{hom.}(k,t) = A_k \sin \Omega_k t \quad . \qquad (2.6.12)$$

The solution to the inhomogeneous equation can, of course, be written in terms of Equation (2.6.12) as

$$h_k(t) = -\frac{k}{\rho\Omega_k} \int_0^t P(k,t) \sin \Omega_k (t-t') \, dt' \qquad (2.6.13)$$

which determines the statistical character of the mode amplitudes. We can smooth over these fluctuations by means of an ensemble average and concern ourselves with the spectrum of surface displacements. Using Equation (2.5.2), we obtain

$$F(k,t) = \frac{\langle h_k(t) \; h^*_k(t) \rangle}{\Delta k} \qquad (2.6.14)$$

as the quantity of interest, rather than the mode amplitudes themselves. The brackets in Equation (2.6.14) denote an ensemble average and Δk is the element of wavenumber space occupied by the wave packet of interest.

Substituting Equation (2.6.13) into (2.6.14), we obtain for the surface spectrum

$$F(k,t) = \frac{k^2}{\rho\, 2\Omega\Omega^*}\int_0^t dt'' \int_0^t dt' \,\frac{\langle P(k,t')P(k,t'')\rangle}{\Delta k}$$

$$\times \sin \Omega(t-t') \sin \Omega^*(t-t'') \qquad (2.6.15)$$

where $\langle P(k,t')P(k,t'')\rangle$ is the spectrum of the pressure fluctuations. If we approximate the complex frequency as

$$\Omega_k = \sqrt{\omega^2 + ik^2\alpha} = \omega_k\sqrt{1 + i\alpha/c^2} \approx \omega_k(1 + i\alpha/2c^2) \qquad (2.6.16)$$

and assume that the statistical pressure field is stationary, Equation (2.6.15) can be simplified. It is not necessary to reproduce here the detailed reduction of this equation [see Phillips (1966), p. 85]; we merely note that when the time interval is greater than the time over which pressure components of wavenumber k are correlated, Equation (2.6.15) can be written in the form

$$F(k,t) = \frac{\pi Q(k,\omega)}{\rho^2 c^2}\left\{\frac{\sinh\left[\frac{\alpha\omega}{c^2}t\right]}{\alpha\omega/c^2}\right\}. \qquad (2.6.17)$$

The quantity $\mu = \alpha/c^2$ is the coupling parameter between the wind and water.

In the regime where the hyperbolic sine may be expanded, that is, for time $t << \frac{1}{\mu\omega}$, Equation (2.6.17) may be written as

$$F(k,t) = \frac{\pi}{\rho^2 c^2} Q(k,\omega)\, t \quad . \qquad (2.6.18)$$

Alternatively, we may write for the rate of growth of the energy spectrum close to the time t = 0, i.e., the initial wave growth

$$\frac{\partial F}{\partial t} = \frac{\pi}{\rho^2 c^2} Q(k,\omega) \quad . \qquad (2.6.19)$$

The functional form we use for $Q(k,\omega)$ is that proposed by Barnett (1968) which is based on the work of Priestly (1965)

$$Q(k,\omega) = \frac{6.13\times 10^{-4} W^6}{\pi^2 \omega^2} \; \frac{\nu_2}{\nu_2^2 + k^2 \sin^2\theta} \; \frac{\nu_1}{\nu_1^2 + (k\cos\theta - H)^2}$$

(2.6.20)

where the following definitions are used:

ρ is the density of water (kg/m^2)

c is the surface wave phase velocity (= ω/k)

θ is the angle between the wave and wind vectors

W is the wind speed (m/sec)

H is ω/W (1/m)

ν_1 is $0.33\ H^{1.28}$

ν_2 is $0.52\ H^{0.95}$.

These values of ν_1 and ν_2 are taken from Collins (1972).

2.7 Model for the Generation of Waves by Wind: Wave Slope Dependent Effects

The wind blowing over the surface of the ocean is the primary mechanism for the generation of waves. The wind sets up a pressure distribution at the ocean surface and it is this variation in pressure which stimulates the growth of waves. Phillips divides these pressure fluctuations into two broad categories: those induced by the coupling of the wind to the irregular ocean surface and those produced by turbulent eddies in the air flow. The first effect is treated as a direct coupling of the ocean wave field to the average wind field. The second effect is treated as a random disturbance on the ocean wave amplitudes of different wavenumbers. Both these terms feed energy into the surface waves: the direct term into a narrow band around $\bar{k}$, which corresponds to the matching of the average wind speed and phase velocity of the surface wave, and the turbulent term into a broad spectral band around $\bar{k}$.

Assume that the strength of the coupling between the wind and water is given by the complex parameters $(\nu + i\gamma)$. It has been determined that the real part of the parameter (ν) is generally negative and models the dissipation of energy due to the formation of spray at wave crests. The negative value of ν produces a decrease in pressure (suction) at a wave crest and an increase in pressure at the trough, thereby aiding in the deformation of the wave. This model of spindrift has been included in principle within the wave-breaking terms and will not be further considered here. The imaginary part of the coupling parameter (γ) models the rate at which new waves are generated in the ocean wave field and is discussed more fully below.

The coupling of the wind to water waves determined by the above parameter produces a shearing force at the ocean surface. Phillips (1966, Section 4.3) discusses two contributions to this stress: one, a resonant contribution from the region in the boundary which moves at the same speed as the wave under consideration (the matched layer), and the other, the contribution of the boundary layer. The matched layer contributions dominate in the region of strong interaction and is the only effect we consider here. For this contribution, Phillips adapted a model developed by Miles (1957) for quasi-laminar flow. In this model the Reynolds stress is written in the form

$$\tau_W(0) = \frac{1}{2}\gamma\rho_W c^2 k^2 a_k^2 \quad , \tag{2.7.1}$$

where ρ_w is the density of water and γ is the imaginary part of the coupling parameter. We may relate the mean momentum flux $\Phi(k)$ to the mean energy flux $\Psi(k)$ by the equation

$$\Phi(k) = \frac{k}{\omega(k)} \Psi(k) \tag{2.7.2}$$

so that

$$\left(\frac{\partial F}{\partial t}\right)_{wind} = \gamma \; \omega(k) \; F(k) \tag{2.7.3}$$

where we have used the relation

$$\Psi(k) = \rho_w g k \left(\frac{\partial F}{\partial t}\right) \tag{2.7.4}$$

between the energy density and energy flux.

The coupling parameter γ is modeled by considering the mean flow over a (moving) wavy surface. The primary contribution to γ is from the region where the wind speed is just equal to the wave speed, i.e., at a height z_m. Phillips calculates the contribution to γ from this matched layer to be

$$\gamma = \frac{\rho_a}{\rho_w} \frac{A_m \Gamma^2 k^3}{c^2} \left[\frac{-U''}{(U')^3}\right]_{z_m} \left(\int_{z_m}^{\infty} [U(z)-c]^2 \, e^{-kz} \, dz\right)^2 . \tag{2.7.5}$$

The equilibrium wind profile is approximately of the form

$$U(z) - c \approx \frac{U^*}{\kappa_o} \log \left(\frac{z}{z_m}\right) \qquad (2.7.6)$$

where $U^* = (\tau_o/\rho_a)^{1/2}$ is the wind friction velocity and $\kappa_o \simeq 0.42$ is Karman's constant. With this substitution into the above equation, we have as the coupling parameter

$$\gamma = \frac{\rho_a}{\rho_w} \left(\frac{U^*}{c\kappa_o}\right)^2 A_m \Gamma^2 \beta \qquad (2.7.7)$$

where

$$\beta = (kz_m)^3 \left\{ \int_1^\infty \log^2 u \exp(-kz_m u)\, du \right\}^2 . \qquad (2.7.8)$$

If from the equilibrium wind profile the point where $U(z_o) = 0$ is taken as the origin, then for an aerodynamically-rough flow

$$kz_m \approx \frac{B_o U^{*2}}{c^2} \exp \left\{ \frac{\kappa_o c}{U^*} \right\} \qquad (2.7.9)$$

where $z_o \approx B_o U^{*2}/g$ and $B_o \simeq 1.1 \times 10^{-2}$ as suggested by Charnock (1955). This empirical relation seems to be a good representation at all but the lowest wind speeds, where the flow is aerodynamically smooth.

Miles found A_m to be equal to π and $\Gamma^2 \sim 1/3$ for $z > z_m$, and that these values are insensitive to fluid viscosity in quasi-laminar flow. In the present calculation then, these terms are constants in the above equations. In Figure 4 the quantity β is plotted as a function of the ratio C/U_*. It is clear from this figure that the coupling of the wind to the waves is restricted to values of $C/U_* \lesssim 20$. Waves having phase speeds greater than 20 U_* effectively "out run" the wind and are not amplified significantly.

Measured in wave periods, the growth time $[T = (\omega\gamma)^{-1}\text{sec}]$ is given by

$$N = \frac{\omega T}{2\pi} = \frac{\rho_\omega}{2\pi\rho_a} \left(\frac{cK_o}{U_*}\right)^2 \Big/ A_m\Gamma^2\beta$$

$$\simeq 26.8 \ (c/U_*)^2/\beta$$

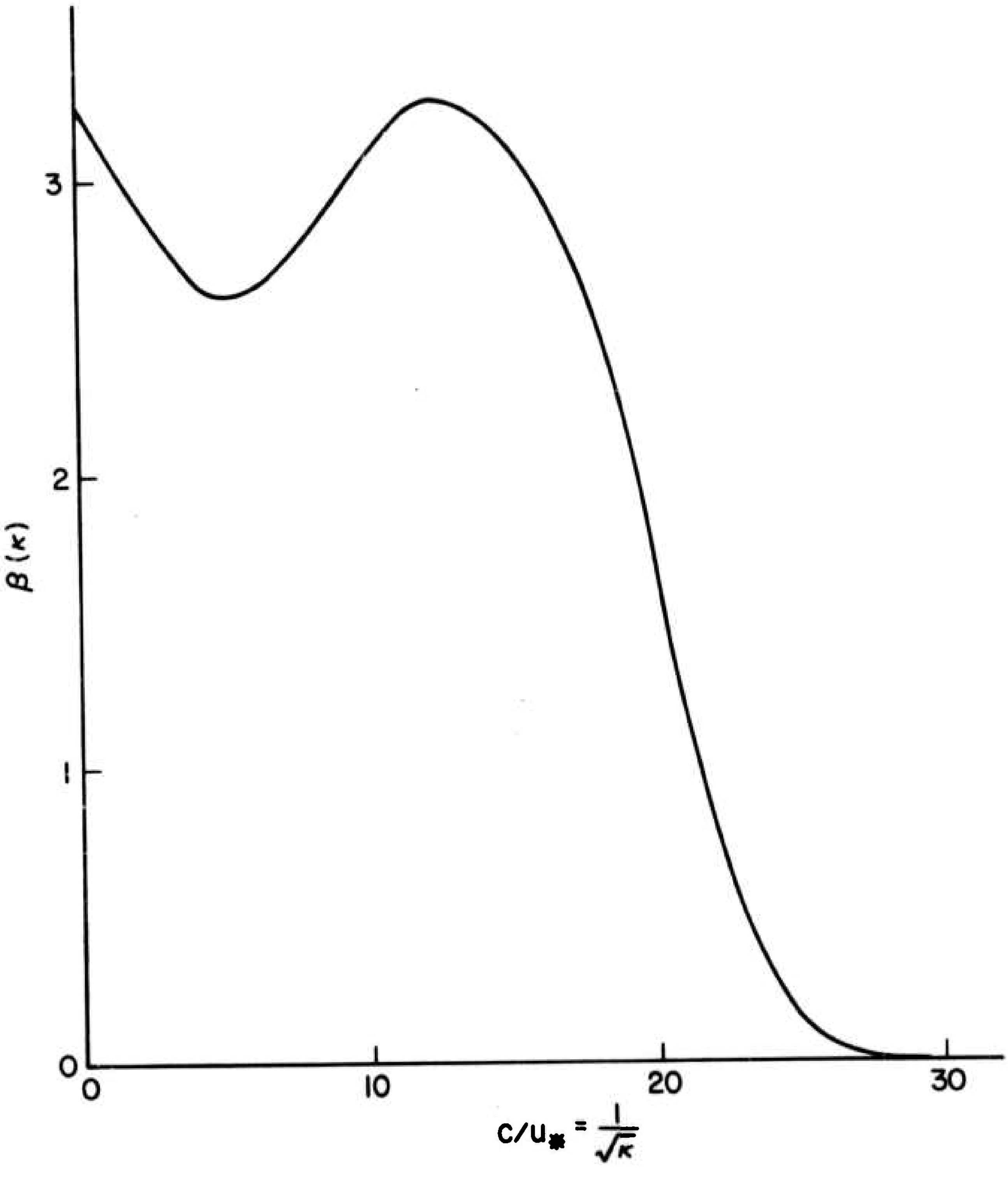


FIGURE 4. Coupling of the Wind to the Ocean Surface as a Function of C/U_*

2.8 Wave Cresting Effects

As the spectrum develops into a fully saturated state, interactions between waves having widely differing wave-lengths alter the wave-breaking phenomena discussed in Section 2.2. Phillips (1963) has identified this mechanism as being primarily responsible for the attenuation of long gravity waves passing through a saturated wave region. Here as the long wave-length waves or swells pass under the short gravities, the wave breaking at the crests of the long waves is enhanced whereas in the troughs the waves are calmed. Phillips estimates the attenuation rate of the long waves by considering the time dependent exchange of energy between the short and long waves. In the absence of breaking or wind input, this exchange, which results from the radiation stress term, is oscillatory and averages to zero. In the presence of wave breaking, there is a net transfer of energy from the long waves to the short waves, which in a steady sea is balanced by the wind input.

The same phenomena should be responsible for a calming effect of long waves on short waves. Here, when the sea has developed to the point where long waves have appreciable amplitude, there is a net enhancement of breaking of the short gravities over that which occurred before the long waves had developed. The net result then is that the mean wave height or slope of the short waves in the saturated region tends to decrease as the long wave-length portion of the spectrum develops.

In this section we construct models for both these effects.

2.8A Effect of Long Waves on Short Waves

We consider the alteration of the wave breaking term induced by a long wave-length wave. The wave-breaking term has the form (from Section 2.2)

$$\left(\frac{\partial F}{\partial t}\right)_{\text{breaking}} = -\omega\varepsilon F\sqrt{\frac{k^3 F}{\pi\Theta_o^2}}\ \exp\left(-\Theta_o^2 / k^3 F\right) \quad . \quad (2.8.1)$$

We may write F in the form

$$F(k,x) = \bar{F}(k) + F_1(k,x) \quad (2.8.2)$$

where $\bar{F}(k)$ is the mean value of the wave height spectrum averaged over the long waves and $F_1(k,x)$ is the perturbation of the short wave-length spectrum induced by the long waves.

To estimate the effect of the long waves, we expand the right-hand side of Equation (2.8.1) to second order in F_1. It is convenient for this purpose to define a function G(Y) according to

$$G(Y) = Y^{3/2}\ e^{-1/Y} \quad . \quad (2.8.3)$$

Then

$$\left(\frac{\partial F}{\partial t}\right)_{\text{breaking}} = \frac{\omega\varepsilon\,\Theta_o^2}{k^3\sqrt{\pi}}\ G(k^3F/\Theta_o^2) \quad (2.8.4)$$

and approximately

$$\left(\frac{\partial F}{\partial t}\right)_{\text{breaking}} = \frac{\omega\varepsilon\Theta_o^2}{k^3\sqrt{\pi}} \; G(\bar{Y}) \left\{1 + \left(\frac{\bar{Y}\,G_Y}{G}\right)\frac{Y_1}{\bar{Y}} + \left(\frac{\bar{Y}^2 G_{YY}}{2G}\right)\frac{Y_1^2}{\bar{Y}^2}\right\} \tag{2.8.5}$$

where

$$Y_1 = k^3 F_1/\Theta_o^2 \tag{2.8.6}$$

$$\bar{Y} = k^3\bar{F}/\Theta_o^2 \tag{2.8.7}$$

$$\bar{Y}G_Y/G = (1 + \tfrac{3}{2}\,\bar{Y})/\bar{Y} \tag{2.8.8}$$

and

$$\bar{Y}^2 G_{YY}/G = (1 + \bar{Y} + \tfrac{3}{4}\,\bar{Y}^2)/\bar{Y}^2 \quad . \tag{2.8.9}$$

For linear gravity waves riding on a steady current $U(x)$, we indicated in Sections (2.4) and (2.5) that the following conservation rules apply.

$$k\,(\sqrt{g/k} + U) = \text{constant} \tag{2.8.10}$$

$$a^2\,\sqrt{g/k}\left(U + \tfrac{1}{2}\,\sqrt{g/k}\right) = \text{constant} \quad . \tag{2.8.11}$$

Thus a small change in current δU produces an amplitude change δa according to

$$\frac{\delta a}{a} = -\,\frac{2\sqrt{g/k} + 3U}{(\sqrt{g/k} + 2U)^2}\,\delta U \quad , \tag{2.8.12}$$

which can also be obtained from Equation (2.5.19) by equating δa^2 with $\left(U + \frac{c}{2}\right) \frac{\partial a^2}{\partial x} \cdot \delta t$. The mean current U in this case is to be taken as the phase velocity of the swell or long wavelength gravity $U \sim \sqrt{g/k_s}$. Since most of the long wavelength waves are moving much faster than the short gravities, we can take $U >> \sqrt{g/k}$ and write

$$\frac{\delta a}{a} \simeq -\frac{3}{4}\frac{\delta U}{U} \tag{2.8.13}$$

Now using the relation $F_1 \sim 2a\delta a$ and $\bar{F} \sim a^2$, we find

$$\frac{Y_1}{\bar{Y}} = -\frac{3}{2}\frac{\delta U}{U} \quad . \tag{2.8.14}$$

For a long wavelength gravity wave, U is the phase velocity ($\sqrt{g/k_s}$) and δU is the orbital velocity

$$\delta U = \omega_s a_s \cos (k_s x - \omega_s t) \quad . \tag{2.8.15}$$

Thus

$$\frac{Y_1}{\bar{Y}} = \frac{3}{2} a_s k_s \cos (k_s x - \omega_s t) \quad . \tag{2.8.16}$$

Averaged over the period of the long wave, the self-breaking term becomes

$$\left(\frac{\partial F}{\partial t}\right)_{\text{breaking}} = -\frac{\omega\varepsilon\Theta_o^2}{k^3\sqrt{\pi}} G\left(\frac{k^3\bar{F}}{\Theta_o^2}\right)\left\{1 + \left[\frac{1+\bar{Y}+\frac{3}{4}\bar{Y}^2}{\bar{Y}^2}\right]\frac{9}{8} a_s^2 k_s^2\right\}$$

(2.8.17)

This calculation assumed a single long wavelength wave of amplitude a_s. For a spectrum of waves, it is appropriate to replace the term $a_s^2 k_s^2$ by

$$a_s^2 k_s^2 \rightarrow \int_0^k F(k)\, k^2\, dk \qquad (2.8.18)$$

since only waves having wavelengths greater than that of the gravity wave will contribute to the wave-cresting effect.

The final form of the wave-breaking expression now becomes

$$\frac{\partial F(k)}{\partial t} = \frac{-\varepsilon\omega\Theta_o^2}{k^3\sqrt{\pi}}\, G(Y) \left\{ 1 + \frac{9}{8}\left[\frac{1+Y+\frac{3}{4}Y^2}{Y^2}\right] \int_0^k F(k) k^2 dk \right\}$$

with

$$Y = k^3 F/\Theta_o^2 \qquad (2.8.19)$$

The term in the curly brackets is the enhancement of the breaking of short waves due to the presence of the long waves.

When the spectrum is saturated above a wavenumber k_{min} (i.e., $Fk^3 \sim$ constant for $k > k_{min}$), we may approximate

$$\int_0^k F(k)\, k^2 dk \sim \overline{Fk^3}\, \ell n\, k/k_{min} \quad .$$

Thus the wave-breaking term takes the approximate form for a partially saturated sea

$$\frac{1}{\tau_{\text{breaking}}} = -\frac{\omega}{F}\frac{\partial F}{\partial t} = \frac{\varepsilon}{\Theta_o}\sqrt{\frac{k^3F}{\pi}}\left\{1 + \frac{9\Theta_o^4}{8k^3F}\,\ln\left(\frac{k}{k_{min}}\right)\right\}\exp\left(-\Theta_o^2/k^3F\right)$$

Since the magnitude of the enhancement term is strongly dependent on the choice of the parameter Θ_o, careful comparison with data will be required to arrive at a useful model.

2.8B Effect of Short Waves on Long Waves

As pointed out in the introduction to this section, Phillips has constructed a model of the interaction between swells and short gravity waves in which the short waves act as a damping mechanism for the longer swells. The notion employed is that a swell passing through a region of saturated sea will transfer energy to the gravity waves by means of the "radiation stress." This transfer enhances the breaking of surface waves at the crests of the swell and calms the surface in the swells' trough. During this process, the wind continually feeds energy into the gravity waves, so that although energy is lost at the swell crest, the wind builds the waves back up after the breaking.

Following Phillips (1966, Section 4.9) we use the energy balance equation from Section 2.4:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\left(U + \frac{c}{2}\right)E\right] + \frac{1}{2} E \frac{\partial U}{\partial x} = \alpha\omega E \qquad (2.8.20)$$

where U is the surface current produced by the swells and the right-hand side of Equation (2.8.20) is the energy fed into the surface waves by the wind. We recall that the phase velocity in Equation (2.8.20) is given by

$$c = \sqrt{\left(g + \frac{dV}{dt}\right)/k} \qquad (2.8.21)$$

since the vertical acceleration of the swell is finite. Because the surface pattern is progressing with the phase velocity of the swell (c_s) we may replace $\frac{\partial}{\partial x}$ with $-\frac{1}{c_s}\frac{\partial}{\partial t}$ in Equation (2.8.20). To determine the effect of the short waves on the swell, we need only consider the interaction between the surface waves and current.

During a single cycle of the swell, the variation in the energy density E of the short waves is small relative to the total energy density. We can therefore approximate the interaction term from Equation (2.8.20) by

$$\frac{\partial}{\partial t}\left\{\frac{3}{2}\frac{UE}{c_s}\right\} \qquad (2.8.22)$$

which is correct to second order. We can integrate Equation (2.8.22) to find the net energy transfer from the swell to the short waves over a cycle, that is

$$\delta E_s = \frac{3}{2} \frac{U}{c_s} \delta E_W \tag{2.8.23}$$

where δE_W is the change in energy of the short waves due to the direct action of the wind.

If T is the time interval necessary for successive crests of the swell to overtake a packet of surface gravity waves, we may write for the change in energy of the swell during this time to be

$$\frac{dE(k_s)}{dt} = - \frac{\delta E_s}{T} \quad . \tag{2.8.24}$$

During this same time interval, we can write for the energy fed into the surface waves

$$\delta E_W = \alpha\omega\bar{E}T \tag{2.8.25}$$

where $\bar{E}$ is the average energy of the surface waves. Using Equations (2.8.25), (2.8.24) and (2.8.23), we can write for the change in swell energy

$$\frac{dE(ks)}{dt} = - \frac{3}{2} \frac{U}{c_s} \alpha(k) \omega(k) \bar{E}(k) \tag{2.8.26}$$

over a cycle.

We write the velocity potential for the swell in one dimension as

$$\chi = -c_s\, a_s \cos\,[k_s x - \omega_s t] \tag{2.8.27}$$

so that the current is

$$U = k_s c_s a_s \sin\,[k_s x - \omega_s t] \quad . \tag{2.8.28}$$

We can write for the maximum change in the swell energy, using Equation (2.8.28)

$$\frac{dE(k_s)}{dt} = -\frac{3}{2}\, k_s a_s\, \alpha(k)\, \omega(k)\, \overline{E}(k) \quad , \tag{2.8.29}$$

which is the result derived by Phillips (pg. 152).

If we now wish to generalize Equation (2.8.29), we could consider a spectrum of surface gravity waves, that is, since from Equation (2.5.2)

$$\overline{E}(k) \approx \rho g F(k)\, \Delta k$$

for a nearly monochromatic wave train ($\Delta k << k$), we might make the replacement

$$\alpha(k)\, \omega(k)\, F(k)\, \Delta k \rightarrow \int_{k_s}^{\infty} \alpha(k)\, \omega(k)\, F(k)\, dk \tag{2.8.30}$$

so that Equation (2.8.29) becomes

$$\frac{dE(k_s)}{dt} = -\frac{3}{2}\, k_s a_s \rho g \int_{k_s}^{\infty} \alpha(k)\, \omega(k)\, F(k)\, dk \quad . \tag{2.8.31}$$

For a single swell, we may write $E(k_s) = \frac{1}{2} \rho g a_s^2$ for the energy density and $E(k_s) \approx \rho g F(k_s)$ so that Equation (2.8.31) becomes

$$\frac{dF(k_s)}{dt} = -\frac{3}{2} k_s \sqrt{2F(k_s)} \int_{k_s}^{\infty} \alpha(k)\, \omega(k)\, F(k)\, dk \qquad (2.8.32)$$

as the rate of dissipation of the swell energy density.

2.9 Wind and Wave Induced Surface Currents

In a fully developed wind driven sea, the turbulent stress acting on the ocean surface continuously transfers momentum from the atmosphere into the water. In a developing sea, part of this momentum appears as increasing wave momentum primarily among the longer developing waves. In a fully developed sea, however, this momentum eventually appears as a wind driven mean current. At steady state, a turbulent boundary layer develops in the water near the sea surface. The turbulent stress created by the atmosphere is

$$\tau_a = \rho_a U_{*a}^2 \qquad (2.9.1)$$

and is resisted by the turbulent stress transferred through the water

$$\tau_w = \rho_w U_{*w}^2 \qquad (2.9.2)$$

The U_* is the friction velocity; the remaining notation is obvious. In steady state $\tau_a \approx \tau_w$.

Using a similarity argument for a fluid flowing over a smooth surface, the velocity gradient can be written as

$$\frac{\partial u(z)}{\partial z} = \frac{u}{\kappa_o z}$$

where $\kappa_o \approx 0.42$ is Kármán's constant. We may therefore write the velocity profiles (see Figure 5)

$$u_a(z) = \frac{U_{*a}}{\kappa_o} \ln\left[\frac{z+z_a^{(o)}}{z_a^{(o)}}\right] + u_D \quad \text{in air} \tag{2.9.3}$$

and

$$u_w(z) = -\frac{U_{*w}}{\kappa_o} \ln\left[\frac{|z|+z_w^{(o)}}{z_w^{(o)}}\right] + u_D \quad \text{in water} \tag{2.9.4}$$

where $z^{(o)}$ is a roughness length and u_D is the velocity of the water surface. We may relate the friction velocities in air and water by taking the steady state condition $\tau_a \approx \tau_w$, so that

$$U_{*w} = \left(\frac{\rho_a}{\rho_w}\right)^{\frac{1}{2}} U_{*a} \approx \frac{U_{*a}}{30} \tag{2.9.5}$$

corresponding to typical friction velocities in water of the order of centimeters per second.

We take the total current to be zero at some large water depth D. The surface drift velocity is then given in terms of D:

$$u_D = \frac{U_{*w}}{\kappa_o} \ln\left(\frac{D+z_w^{(o)}}{z_w^{(o)}}\right) ,$$

$$= \frac{U_{*a}}{\kappa_o} \left(\frac{\rho_a}{\rho_w}\right)^{\frac{1}{2}} \ln\left[\frac{D+z_w^{(o)}}{z_w^{(o)}}\right] , \tag{2.9.6}$$

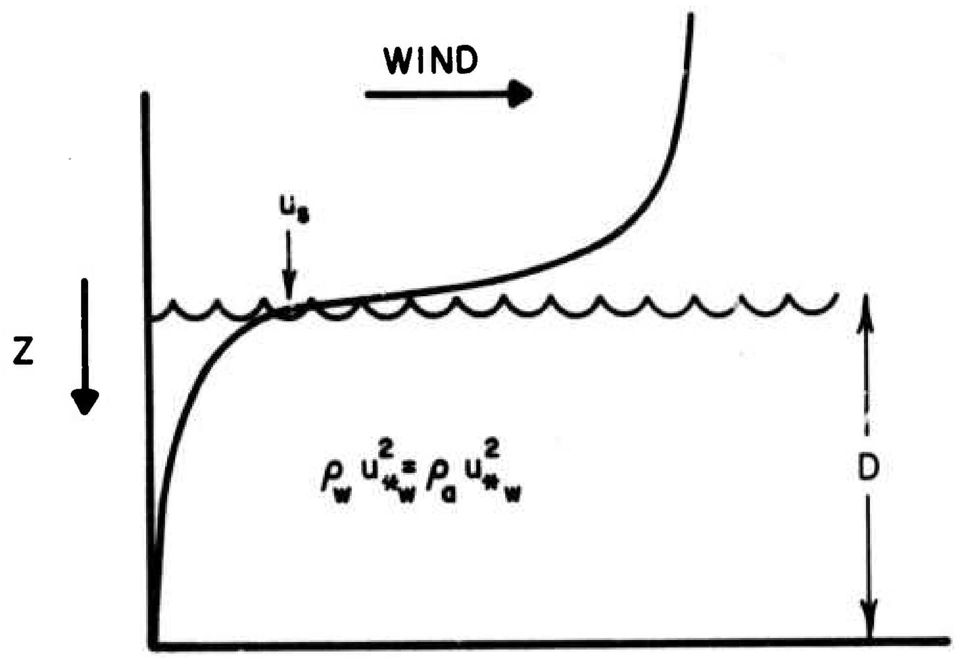


FIGURE 5. The wind and Water Shear Profiles Matched at the Ocean Surface by Equating the Shearing Stresses

by setting $u_w(z) = 0$ in Eq. (2.9.4). At the depth $|z|$ the induced current is given by Eq. (2.9.4) which, using Eq. (2.9.6), becomes

$$u_w(z) = \frac{U_{*a}}{\kappa_o} \left(\frac{\rho_a}{\rho_w}\right)^{\frac{1}{2}} \ell n \left[\frac{D+z_w^{(o)}}{|z|+z_w^{(o)}}\right] \quad . \qquad (2.9.7)$$

A proper analysis would treat the wave motion in a shearing layer in detail [such an approach is discussed by Phillips (1973)]. Consistent with the semi-empirical approach of the current model and in view of the very weak sensitivity of the result to our assumptions about the profile we adopt a very simple approach. We assume that the wave motion in the presence of a shearing current is essentially the same as that of a wave in a uniform current. The value of this effective uniform current should depend on the depth to which the wave orbital velocities penetrate. To find the effective mean current $\left(\bar{u}(k)\right)$ for a given wave (k), we weight this profile [Equation (2.9.7) with a suitable power of the amplitude of the wave orbital velocity,

$$\begin{aligned} \bar{u}_w(k) &= \alpha k \int_{z^{(o)}}^{\infty} e^{-\alpha k|z|} \; u_w(z) \; d|z| \\ &= \frac{U_{*a}}{\kappa_o} \left(\frac{\rho_a}{\rho_w}\right)^{\frac{1}{2}} \int_{\alpha k z^{(o)}}^{\infty} e^{-y} \; \ell n \left[\frac{y_{max}}{y}\right] dy \end{aligned} \qquad (2.9.8)$$

where $y_{max} = \alpha kD >> 1$ and $y = \alpha kz$. In general, the quantity $kz^{(o)}$ is extremely small and the lower limit of the integral may be replaced by zero, yielding

$$\bar{u}_w(k) \approx \frac{U_{*a}}{\kappa_o}\left(\frac{\rho_a}{\rho_w}\right)^{\frac{1}{2}}\left\{\ell n \alpha kD + \gamma\right\} \qquad (2.9.9)$$

where γ is Euler's constant (γ = .5772...). Substituting the appropriate values for the parameters in Eq. (2.9.9) yields

$$\bar{u}_w(k) \approx 0.0744\ U_{*a}\ \ell n(1.78\alpha kD) \quad . \qquad (2.9.10)$$

Equation (2.9.10) gives us an expression for the average drift current felt by a surface wave of wavenumber k. This weak dependence on the power α gives us confidence in this method of replacing the actual sheared boundary layer by a single effective mean current.

2.10 Assembly of the Transfer Equation

The various terms modeled in the preceding sections may now be assembled into the overall transfer equation. Thus we have

$$\frac{\partial F}{\partial t} = -(c_g + U')\frac{\partial F}{\partial x} \qquad : \text{convection}$$

$$+ k\frac{\partial U'}{\partial x}\frac{\partial F}{\partial k} \qquad : \text{refraction}$$

$$+ \pi c_1 \sqrt{g}\frac{\partial}{\partial k}\left[\frac{(k^3F)^2}{\sqrt{k}}\frac{\partial}{\partial k}(k^3F)\right] \qquad : \text{wave-wave interaction}$$

$$- \frac{1}{2}\left(\frac{1-\gamma k^2}{1+\gamma k^2}\right)\frac{\partial U'}{\partial x}F \qquad : \text{radiation stress with capillaries plus k-compression}$$

$$-\epsilon\sqrt{\frac{gk}{\pi\Theta_o}}(kF)^{3/2}(1+\Delta)\exp\left[-\frac{\Theta_o^2}{k^3F}\right] \qquad : \text{wave breaking}$$

$$+ c_2\sqrt{gk}\left(\frac{U_*}{c_o}\right)^2\beta F \qquad : \text{matched layer wind generation}$$

$$- 4\nu_T k^2F \qquad : \text{turbulent and viscous damping}$$

$$+ 1.18\times 10^{-11}\frac{W^6}{H^{2.23}}\left[1+\left(\frac{1-c/w}{.33H^{1.28}}k\right)^2\right]^{-1}$$

: turbulent wind (in m^3/sec)

(2.10.1)

where we have employed the following definitions

$$U' = U + \bar{u}(k) \qquad \text{from Section 2.9}$$

$$c_g = \tfrac{1}{2}\sqrt{g/k}\,\frac{1+3\gamma k^2}{\sqrt{1+\gamma k^2}}\,(1+\lambda k^3 F) \qquad \text{from Section 2.5B,C}$$

$$\Delta = \frac{9}{8}\left(\frac{1 + Y + \frac{3Y^2}{4}}{Y^2}\right)\int_0^k k^3 F \,\frac{dk}{k} \qquad \text{from Section 2.8}$$

$$(Y = k^3 F/\Theta_o^2)$$

$$H = \sqrt{gk}/W \text{ in meters}^{-1} \qquad \text{from Section 2.6}$$

and

$$W = 22.5\, U_* \text{ in meters/sec.}$$

The transfer equation may be put in a more convenient form by introducing a number of dimensionless variables:

wavenumber:	$\kappa = kU_*^2/g$
distance:	$\xi = x\, g/U_*^2$
time:	$\tau = t\, g/U_*^2$
surface tension:	$\bar{\gamma} = \frac{g^2}{U_*^4}\,\gamma$
current:	$u = U'/U_*$
power spectrum:	$Y = F(k)k^3/\Theta_o^2$
angular frequency:	$\Omega = \omega\, U_*/g$

In this system of units we have the relations

$$\kappa = (U_*/c_o)^2$$

and

$$U' + c_g = \frac{U_*}{2\sqrt{\kappa}}\left[A(\kappa) + 2u\sqrt{\kappa}\right]$$

where

$$A(\kappa) = \frac{1+3\bar{\gamma}\kappa^2}{\sqrt{1+\bar{\gamma}\kappa^2}}\,(1 + c_4 Y)$$

Using these relations we may write the dimensionless form of the transfer equation as,

$$\begin{aligned}
2\sqrt{\kappa}\,\frac{\partial Y}{\partial \tau} &+ \left(A(\kappa) + 2u\sqrt{\kappa}\right)\frac{\partial Y}{\partial \xi} - 2\kappa^{3/2}\,\frac{du}{d\xi}\,\frac{\partial Y}{\partial \kappa} \\
&= 2\pi c_1 \Theta_o^4\,\kappa^{7/2}\,\frac{\partial}{\partial \kappa}\left[\frac{Y^2}{\sqrt{\kappa}}\,\frac{\partial Y}{\partial \kappa}\right] \\
&- \sqrt{\kappa}\,Y\left(\frac{7 + 5\bar{\gamma}\kappa^2}{1 + \bar{\gamma}\kappa^2}\right)\frac{du}{d\xi} \\
&- \frac{2}{\sqrt{\pi}}\,\varepsilon\kappa Y^{3/2}\,(1+\Delta)\,\exp(-1/Y) \\
&+ c_2\kappa^2\beta(\kappa)Y \\
&- (8g\nu_T/U_*^3)\,\kappa^{5/2}\,Y \\
&+ c_3\left(\frac{U_*}{U_{ref}}\right)^{5.46}\frac{\kappa^{2.385}}{\Theta_o^2}\left[1 + \left(\frac{1-1/22.5\sqrt{\kappa}}{s_1}\right)^2\right]^{-1}
\end{aligned} \qquad (2.10.2)$$

where

$$s_1 = \frac{0.625}{(22.5\sqrt{\kappa})^{1.28}}\left(\frac{U_{ref}}{U_*}\right)^{0.56}\kappa^{0.28} \quad ,$$

$$\Delta = \frac{9}{8}\left(\frac{1+Y+\frac{3}{4}Y^2}{Y^2}\right)\Theta_o^2 \int_o^{\kappa} Y \frac{d\kappa}{\kappa}$$

and $c_1 \sim 1$, $c_2 = 0.0119$, $c_3 = 1.89$ and c_4 is variable.

An examination of Eq.(2.10.2) shows that, in the absence of viscous and surface tension effects, the value of the wind speed can be scaled out of the problem by expressing the data in terms of the reduced variables just given. The general scaling laws are

$$F_k = k^{-3} f\left(\frac{ku^{*2}}{g}, \frac{gt}{u^*}\right)$$

in the open ocean, and

$$F_k = k^{-3} f\left(\frac{ku^{*2}}{g}, \frac{gx}{u^{*2}}\right)$$

at steady state in coastal waters (x is the distance from the lee shore: the fetch). A "fully developed" sea develops after a long time in the open ocean

$$F_{k_\infty} \to k^{-3} f\left(\frac{ku^{*2}}{g}, \infty\right) .$$

To show self consistency of the present model, we compare in the next section the predicted saturated and nonsaturated spectrum with measured data.

3. INTEGRATION TECHNIQUE AND SAMPLE CALCULATIONS

3.1 Method of Integration

The computer code that has been developed solves the one dimensional wave spectral transfer equation (2.10.2) in a dimensionless phase space whose coordinates are ξ and κ. The code uses a nonlinear explicit forward marching difference procedure (in time) to solve the equation. These equations are written in a coordinate system that may be moving relative to the ocean and in which it is assumed that there is a temporally steady but spatially variable current $[u(\xi)]$. The velocity at which energy propagates in this coordinate system is taken to be,

$$U' + c_g = c_g^{(o)}\left[A(\kappa) + 2\sqrt{\kappa}(\xi)\right] \qquad (3.1.1)$$

relative to the group velocity of gravity waves in a static ocean.

In the general case the current at $\xi = \pm\infty$ will have a finite value $[u_\infty]$. In the case that $u_\infty < 0$, the energy propagation velocity will be positive (to the right) for sufficiently small wavenumbers but negative for very large wavenumbers. The critical wavenumber (in dimensionless units) that divides these two regions is

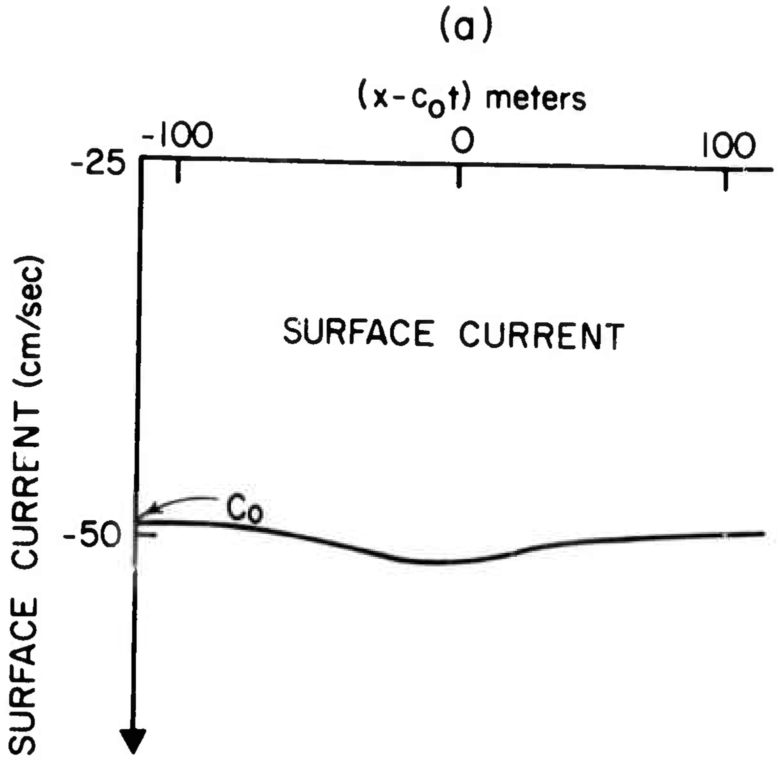
$$\kappa_c = \frac{1}{4} u_\infty^2 A^2(\kappa_c) \quad \text{for } u_\infty < 0 \text{ only} \tag{3.1.2}$$

Boundary conditions when $u_\infty < 0$ must be given at both $\xi = +\infty$ and $\xi = -\infty$. At the boundary $\xi = -\infty$ values for the spectrum Y must be given only for $\kappa<\kappa_c$; whereas at $\xi = +\infty$ values are to be given for $\kappa>\kappa_c$. In order to obtain a convergent solution it is necessary to ensure that the marching direction is always in the direction of energy propagation.

At any station ξ, in the κ,ξ-phase space, there will be a critical wavenumber which separates right from left propagating waves,

$$\kappa_{crit}(\xi) = \frac{1}{4} u^2(\xi)\ A^2(\kappa_{crit}) \tag{3.1.3}$$

(see Figure 6). Energy is transferred across this boundary in two ways: one, by local gravity wave-wave interaction (modeled by the diffusive term in Eq. (2.10.2) and two, by convection. Since this critical boundary corresponds to zero net group velocity in physical space, the trajectories of wave packets in the ξ,κ plane will have a vertical slope at this boundary. The trajectories (characteristics) along which energy propagates (in the absence of the diffusive wave-wave term) are given (in dimensionless terms) by,



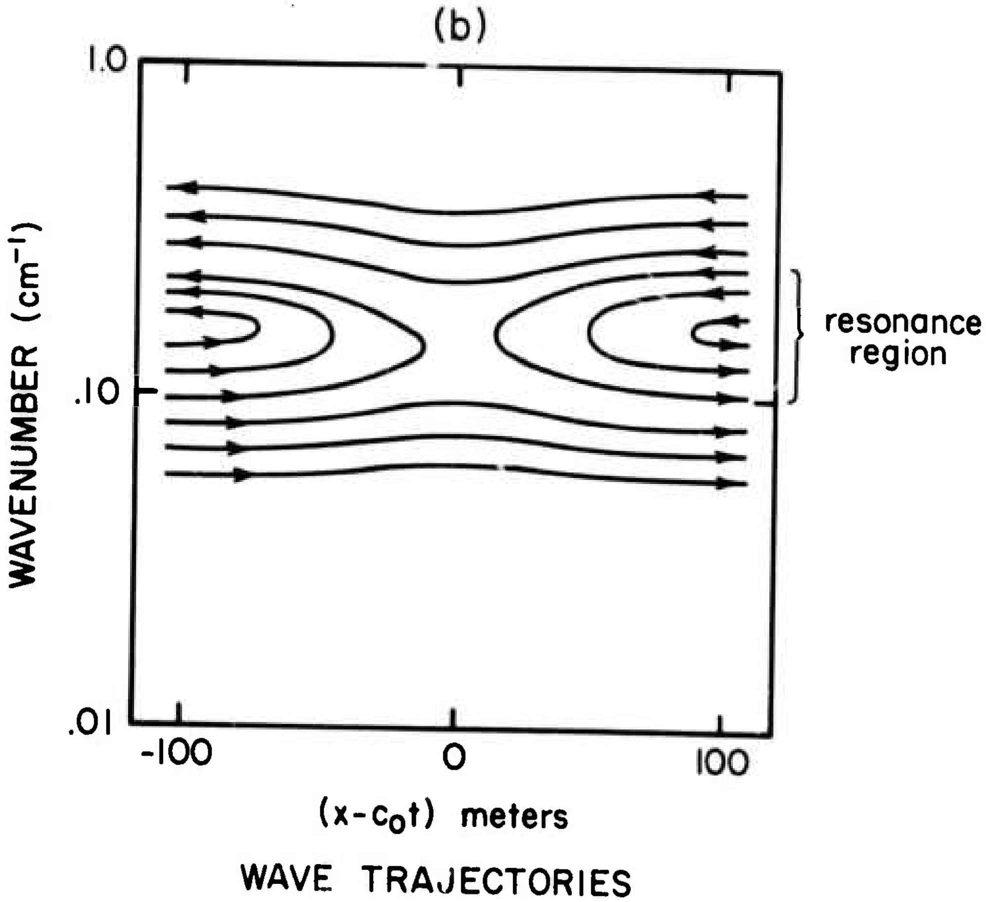


FIGURE 6. Wave Trajectories in the Presence of a Variable Current

$$\frac{d\xi}{d\tau} = \frac{A(\kappa)}{2\sqrt{\kappa}} + u(\xi)$$

$$\frac{d\kappa}{d\tau} = -\kappa \frac{du(\xi)}{d\xi} \tag{3.1.4}$$

The family of trajectories given by Eq. (3.1.4) for the particular current pattern

$$u(\xi) = -\left[1 + 0.04 \exp(-\xi^2/\Delta^2)\right] \tag{3.1.5}$$

is depicted in Figure (6b) where the current is shown in (6a).

In the absence of local wave-wave interactions these curves Fig. (6b) represent the direction of flow of energy in the (κ,ξ)-phase space. A wave packet generated at some initial point ξ_1,κ_1 will travel along the particular trajectory which passes through the point ξ_1,κ_1. The value of the normalized spectrum Y along this trajectory is obtained by integrating Eq. (2.10.2) in the form,

$$\frac{dY}{d\tau} = G(Y,\kappa,\xi) \tag{3.1.6}$$

with

$$\frac{dY}{d\tau} = \frac{\partial Y}{\partial \tau} + \frac{d\xi}{d\tau}\frac{\partial Y}{\partial \xi} + \frac{d\kappa}{d\tau}\frac{\partial Y}{\partial \kappa} \tag{3.1.7}$$

and $\frac{d\xi}{d\tau}$ and $\frac{d\kappa}{d\tau}$ are the trajectories given by Eq. (3.1.4).

Neglecting the diffusion term in Eq. (2.10.2) has essentially decoupled the various trajectories except for the term involving Δ from Section 2.8. Integration along independent characteristics can then proceed according to

$$\frac{dY}{d\tau} = -\frac{Y}{2}\left(\frac{7 + 5\bar{\gamma}\kappa^2}{1 + \bar{\gamma}\kappa^2}\right)\frac{du}{d\xi}$$

$$- \varepsilon\sqrt{\frac{\kappa}{\pi}}\, Y^{3/2} \exp\left(-\left[1/Y\right]\right)$$

$$+ \left[\tfrac{1}{2}c_2\ \kappa^{3/2}\beta(\kappa) - \frac{4g\nu_T}{U_*^3}\kappa^2\right] Y$$

$$+ c_3\left(\frac{U_*}{U_{ref}}\right)^{5.46} \frac{\kappa^{1.885}}{2\Theta_o^2}\left[1 + \left(\frac{1-1/22.5\sqrt{\kappa}}{s_1}\right)^2\right]^{-1} .$$

As mentioned, to ensure stability, the direction of integration must be the same as that of energy flow. Thus the integration starts at $\xi = -\infty$ for $\kappa<\kappa_{crit}$ and follows the energy trajectories in Figure (6); while for $\kappa>\kappa_{crit}$ the integration must start at $\xi = +\infty$.

3.2 Pierson and Stacy Analytic Fit to Data

The report of W.J. Pierson and R.A. Stacy (1972) attempts to synthesize the experimental data on the development of surface waves as a function of wind friction velocity (U_*). One major difficulty in any attempt to represent the spectrum of the ocean surface is that there are thirteen orders of magnitude in frequency (six in wavenumber) which must be described. The approach taken by Pierson and Stacy (PS) is to partition the wavenumber spectrum into five distinct regions and to fit the data by an appropriate spectral function in each of these regions. These functions are then equated at the boundaries of the spectral regions. This phenomenological approach attempts to identify the major contributor to each part of the surface spectrum just as the spectral transfer equation attempts to identify the interactive mechanism in the equation for the spectral evolution (2.10.2). The empirical expressions of PS model the effects, whereas Eq. (2.10.2) is an attempt to model the causes. Comparison of calculations with the empirical expressions for a given wind speed will assist us in determining the veracity of our models of the individual interactive mechanisms.

The analytic expression constructed to represent the spectrum of a wind roughened sea is taken directly from the (PS) report. We first define,

$$S(k,\Phi) = S(k)\ F(k,\Phi) \qquad 0 < k < \infty\ ;\ -\frac{\pi}{2} < \Phi < \frac{\pi}{2}$$

$$S(k,\Phi) = 0 \qquad 0 < k < \infty\ ;\ \frac{\pi}{2} < \Phi < \pi\ ;\ -\pi < \Phi < \frac{\pi}{2} \tag{3.2.1}$$

where S(k) is the one dimensional spectrum discussed in this report and $F(k,\Phi)$ contains all the angular information,

$$F(k,\Phi) = \frac{8}{3\pi} \cos\Phi \left\{1 - \exp\left[-g^2/2k^2\left[U(U_*)\right]^4\right]\right\} + \frac{1}{\pi}\left[1 - \frac{A}{2} + A\cos^2\Phi\right] \exp\left[-\ g^2/2k^2\left[U(U_*)\right]^4\right] \tag{3.2.2}$$

where $0<A<2$ and A could be a function of k and U_* , $S(k,\Phi)$ is the wavenumber spectrum with Φ the angle between $\vec{k}$ and the wind. For comparison with our one dimensional theory, i.e., the result of integrating Eq. (2.10.2), we set $F(k,\Phi) = 1$.

The partitioning of the wavenumber specturm is done as follows;

$$S(k) = \begin{cases} S_1(k) & ;\ 0 < k < k_1 & \text{(i)} \\ S_2(k) & ;\ k_1 < k < k_2 & \text{(ii)} \\ S_3(k) & ;\ k_2 < k < k_3 & \text{(iii)} \\ S_4(k) & ;\ k_3 < k < k_\nu & \text{(iv)} \\ S_5(k) & ;\ k_\nu < k < \infty & \text{(v)} \end{cases} \tag{3.2.3}$$

The analytic forms constructed to fit the experimental data in each of these regions are:

(i) Gravity wave-gravity wave equilibrium range

$$S_1(k) = \frac{a}{2k^3} \exp\left\{-\beta g^2 / \left[U_{19.5}(U_*)\right]^4 k^2\right\} \qquad (3.2.4a)$$

$$0 < k < k_1 = \frac{k_2 U_{*m}^2}{U_*^2}$$

(ii) isotropic turbulence range

$$S_2(k) = \frac{a}{2k_1^{1/2} k^{5/2}} \qquad k_1 < k < k_2 \qquad (3.2.4b)$$

(iii) the connecting range due to Leykin and Rosenberg

$$S_3(k) = \frac{a\, D(U_*)}{2k_3^p k^{3-p}} \qquad k_2 < k < k_3 \qquad (3.2.4c)$$

(iv) capillary range

$$S_4(k) = \frac{a\, D(U_*)}{2k^3} \qquad k_3 < k < k_\nu \qquad (3.2.4d)$$

(v) Viscous cutoff range

$$S_5(k) = \frac{E\, U_*^3 k_m^6}{\nu g k^9} \qquad k_\nu < k < \infty \qquad (3.2.4e)$$

The data utilized in each of these regions has not been restricted to open ocean experiments; use has also been made of wind-water tank experiments. The bibliography of the experimental literature is too extensive to list here; see the PS report for this information.

In addition to fitting the experimental data, the functions in Eq. (3.2.4) were selected to fit smoothly at the boundaries between the spectral regions, i.e., $S_1(k_1) = S_2(k_1)$, $S_2(k_2) = S_3(k_2)$, $S_3(k_3) = S_4(k_3)$ and $S_4(k_\nu) = S_5(k_\nu)$. The parameters in Eq. (3.2.4) were determined to have the following values:

$$\begin{aligned} a &= 8.1\cdot 10^{-3} & k_2 &= k(6\pi) \approx 0.35 \\ \beta &= 0.74 & k_3 &= k(10\pi) \approx 0.92 \\ E &= 9.08\ 10^{-3} & k_m &= (g\rho/T)^{\frac{1}{2}} \approx 3.63 \\ E/\nu g &= 9.26\ 10^{-4} & U_{*m} &= 12 \text{ cm/sec} \end{aligned} \qquad (3.2.5)$$

and functional forms

$$p = \log_{10}\left[D(U_*)\ \frac{U_{*m}}{U_*}\right] / \log_{10}(k_3/k_2)$$

and (3.2.6)

$$D(U_*) = \left[1.274 + 0.0268\ U_* + 6.03\ 10^{-5} U_*^2\right]^2$$

$U_{19.5}(U_*)$ is the wind speed at 19.5 meters for a friction velocity of U_* in a neutral atmosphere. A complete discussion relating the values of the above coefficients to the appropriate experiments may be found in the (PS) report.

The quantity $S(k)$ defined by Eqs. (3.2.4) corresponds to the equilibrium value of the one dimensional spectrum $F(k)$

calculated from the spectral transfer equation (2.10.2). In Figures (7) and (8) we graph the spectral distribution for two values of the wind friction velocity $U_* = 24$ cm/sec and 48 cm/sec, using both the empirical expressions of Pierson and Stacy and the spectrum obtained by integrating Eq. (2.10.2).* The slope spectrum (Fk^3) has been graphed vs. the log of the wavenumber in Figures (7) and (8) for a number of reasons: firstly, because the spectra span decades in wavenumber; secondly, because the Phillips' saturated spectrum appears as a horizontal line in this representation; thirdly, the separation between spectra for different wind friction velocities is most clearly made and finally, because of the clear separation between the regions of the spectrum as modeled by Pierson and Stacy, it is an easy matter to compare our results with theirs.

In Figure (7) we see that the low wavenumber ($k \lesssim .01$) part of the spectrum is modeled equivalently by PS or the transfer equation for a U_* of 24 cm/sec. The gravity wave-gravity wave equilibrium range $\left[k^3 S_1(k)\right]$ essentially rises to the value a/2 where the PS spectrum becomes the saturated spectrum of Phillips. The transfer equation, however, deviates from the horizontal giving some slope to this region - the angle of which is dependent on U_*. The saturation is therefore enhanced above the value given by PS. The transfer equation rises slowly through the regions given by

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* Values of the free parameters Θ_o, E and C_1 were chosen to be 0.16, 0.5 and 0.0 for this comparison.

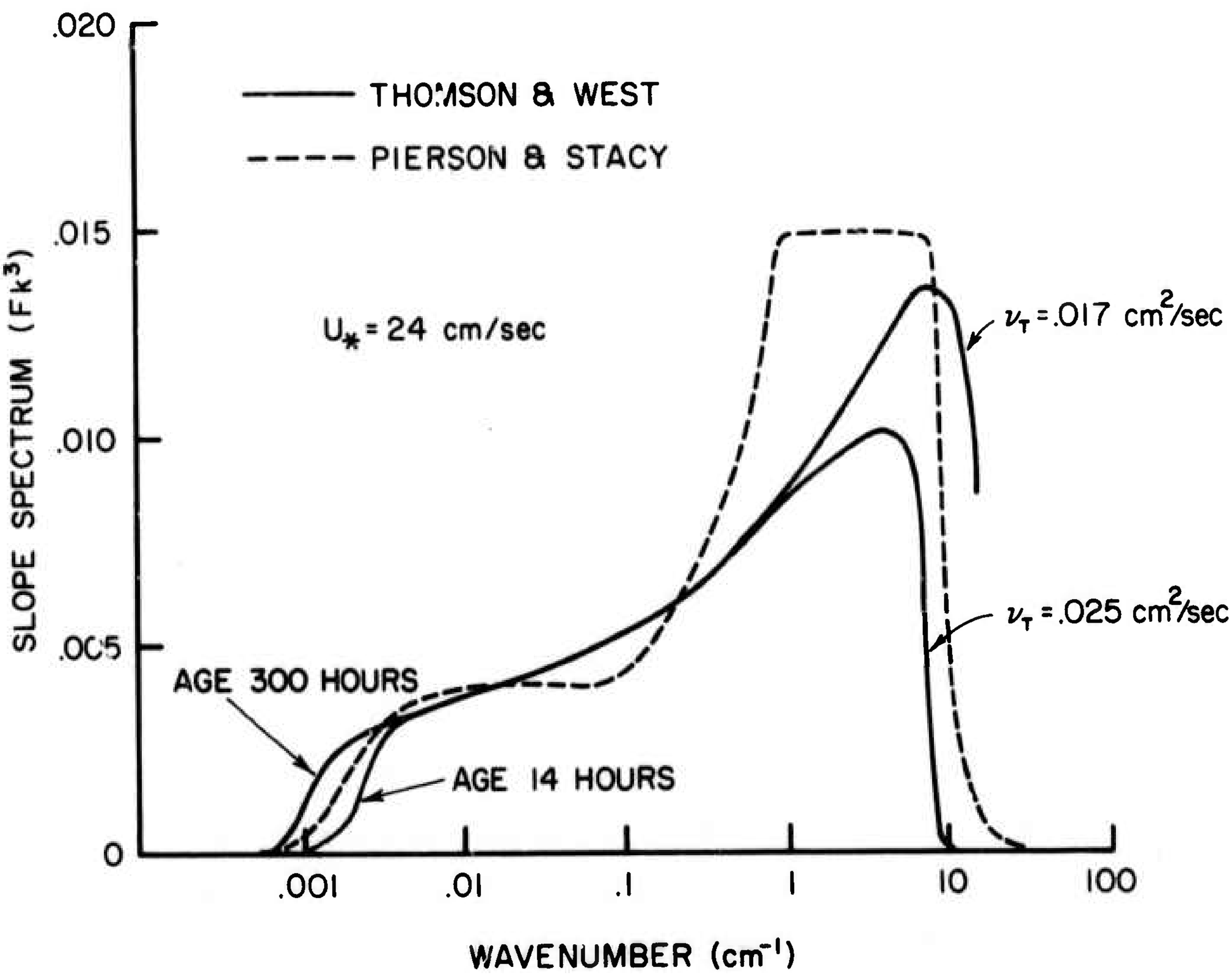

FIGURE 7. Fully-Developed Spectrum for a Friction Velocity of 24 cm/sec.

$\left[k^3S_2(k)\right]$ and $\left[k^3S_3(k)\right]$ and is therefore suppressed below the PS spectrum at the start of the capillary range. If the value of the effective viscosity coefficient (ν_T) is taken to be that of molecular viscosity the transfer spectrum becomes too large in the capillary regime. This growth represents the sensitive balance between the energy input of the wind and energy dissipation due to viscous damping in Equation (2.10.2). With modified values of the viscosity coefficient $\nu_T = (1.7\text{-}2.5)\nu_{molec}$, the spectrum is bounded in the capillary region. It is, however, clear that the capillary region is not saturated.

Figure (8) allows us to make two separate comparisons: first, the PS spectra for two values of the wind friction velocity, i.e., $U_* = 24$ cm/sec and 48 cm/sec and second, how the transfer spectra compare with the PS spectra. We see, first of all, that the height of the capillary region in the PS spectrum scales linearly with U_*, that is, for twice the wind friction velocity one has twice the level of capillary waves. In fact it appears that outside the gravity wave equilibrium range that one could scale the PS spectrum directly with U_*. Again, for the transfer spectrum, (at $U_* = 48$ cm/sec) the balance between the energy input of the wind and the energy drain of viscosity in the capillary $\left[k^3S_4(k)\right]$ and cutoff $\left[k^3S^5(k)\right]$ regions produces and un-

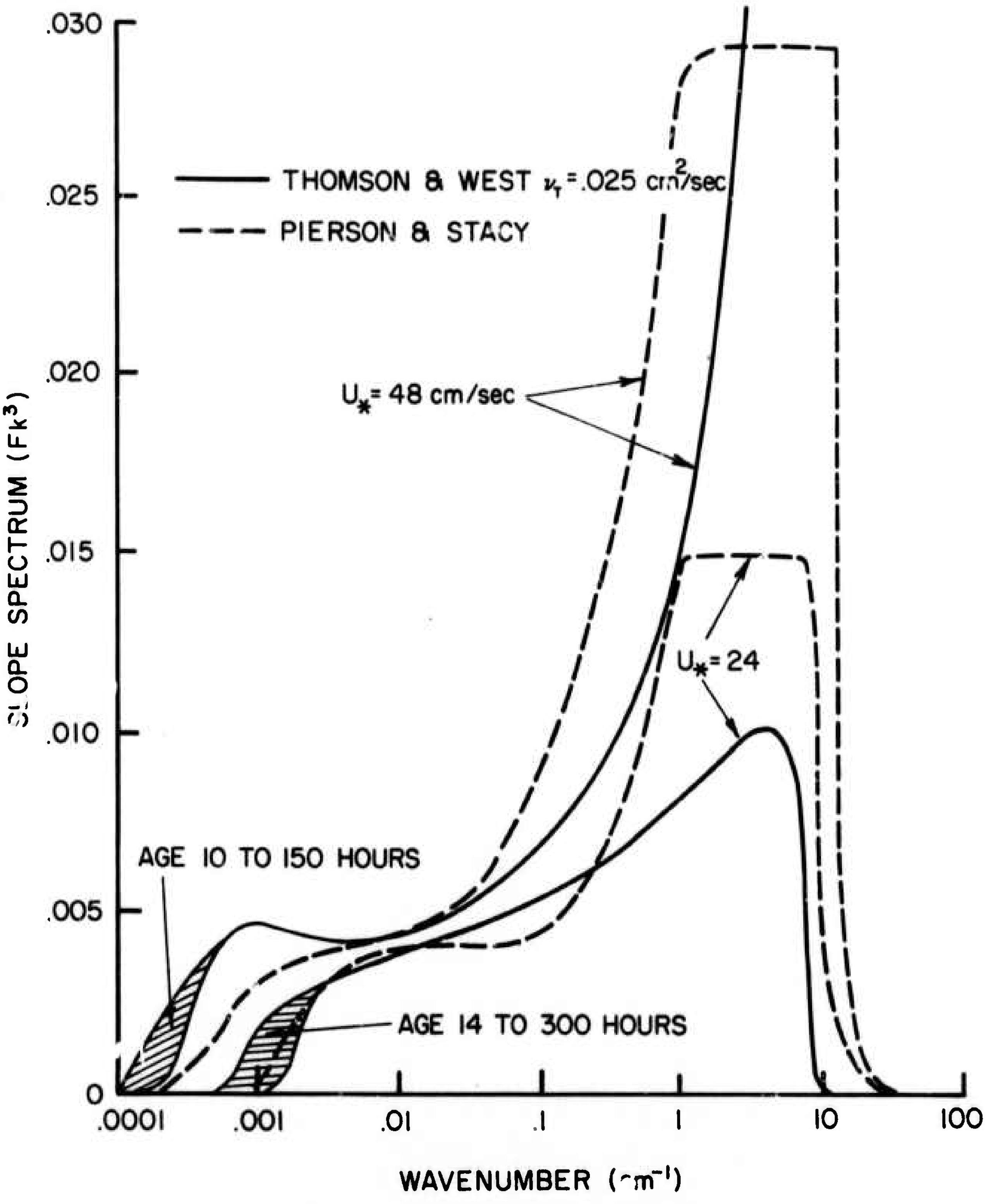

FIGURE 8. Fully-Developed Spectra for Friction Velocities of 24 and 28 cm/sec

bounded growth. This growth stresses the importance of constructing a viable model for the viscosity coefficient which includes a dependence on the wind friction velocity U_*. An example of such a theory might be the application of the theory of breaking waves by Banner and Phillips (1973) to determine the enhanced viscosity produced by turbulent flow in the breaking wave region.

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