AD-776 022

DOUBLY AVERAGED EFFECT OF THE MOON AND SUN ON A HIGH ALTITUDE EARTH SATELLITE ORBIT

Michael E. Ash

Massachusetts Institute of Technology

Prepared for:

Air Force Systems Command

1 March 1974

DISTRIBUTED BY:

NTIS

National Technical Information Service
U. S. DEPARTMENT OF COMMERCE
5285 Port Royal Road, Springfield Va. 22151

UNCLASSIFIED
Security Classification

AD 776022

DOCUMENT CONTROL DATA – R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP
3. REPORT TITLE Doubly Averaged Effect of the Moon and Sun on a High Altitude Earth Satellite Orbit	
4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)* Technical Note	
5. AUTHOR(S) *(Last name, first name, initial)* Ash, Michael E.	
6. REPORT DATE 1 March 1974	7a. TOTAL NO. OF PAGES: 108 7b. NO. OF REFS: 19
8a. CONTRACT OR GRANT NO. F19628-73-C-0002 b. PROJECT NO. 649L c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Note 1974-5 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TR-74-45
10. AVAILABILITY/LIMITATION NOTICES Approved for public release; distribution unlimited.	
11. SUPPLEMENTARY NOTES None	12. SPONSORING MILITARY ACTIVITY Air Force Systems Command, USAF

13. ABSTRACT

The secular effect of the moon or sun on a high altitude earth satellite orbit is approximated by spreading the moon or sun into a circular ring of matter and averaging over one orbit of the satellite. Calculations are performed to first order in eccentricity for satellite radii inside and outside the lunar radius. The secular variations in semi-major axis a and inclination I to the ring plane are zero and those of the other orbital elements are given by infinite series. The secular motion of the ascending node Ω versus I and a is expressed graphically. The behavior of the eccentricity e and argument of perigee ω for orbits outside the ring is analogous to that caused by the earth's equatorial bulge. Inside the ring, ω goes through many revolutions and e has long periodic variations of $|I| < I_a$. If $|I| > I_a$, ω moves into the first or third quadrant and then e increases secularly without bound. The critical inclination I_a is 39°.2 for low earth satellites, 36°.2 at 1/3 the lunar radius and 27°.1 at 2/3 the lunar radius.

14. KEY WORDS

satellite lifetimes	lunar perturbations	secular perturbations
earth satellite orbit	solar perturbations	high altitude orbits

UNCLASSIFIED
Security Classification

108

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

DOUBLY AVERAGED EFFECT OF THE MOON AND SUN ON A HIGH ALTITUDE EARTH SATELLITE ORBIT

M. E. ASH
Group 67

TECHNICAL NOTE 1974-5

1 MARCH 1974

LEXINGTON MASSACHUSETTS

ib

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the Department of the Air Force under Contract F19628-73-C-0002.

ABSTRACT

The secular effect of the moon or sun on a high altitude earth satellite orbit is approximated by spreading the moon or sun into a circular ring of matter and averaging over one orbit of the satellite. Calculations are performed to first order in eccentricity for satellite radii inside and outside the lunar radius. The secular variations in semi-major axis a and inclination I to the ring plane are zero and those of the other orbital elements are given by infinite series. The secular motion of the ascending node Ω versus I and a is expressed graphically. The behavior of the eccentricity e and argument of perigee ω for orbits outside the ring is analogous to that caused by the earth's equatorial bulge. Inside the ring, ω goes through many revolutions and e has long periodic variations of $|I| < I_a$. If $|I| > I_a$, ω moves into the first or third quadrant and then e increases secularly without bound. The critical inclination I_a is 39°.2 for low earth satellites, 36°.2 at 1/3 the lunar radius and 27°.1 at 2/3 the lunar radius.

Accepted for the Air Force
Eugene C. Raabe, Lt. Col., USAF
Chief, ESD Lincoln Laboratory Project Office

TABLE OF CONTENTS

Preceding page blank

LIST OF FIGURES

I. Introduction

The secular or doubly averaged effect of the moon or sun on a high altitude earth satellite orbit is approximated by spreading the moon or sun into a circular ring of matter and averaging over one orbit of the satellite. In this note, computations are performed to first order in eccentricity for satellite radii inside and outside the lunar radius. Details of the calculations are given in Sections I through IX. Results are discussed in Sections X and XI.

The secular variations in semi-major axis a and inclination I to the ring plane are zero and those of the other orbital elements are given by infinite series in a/ρ inside the ring and ρ/a outside the ring, where ρ is the radius of the ring.

The satellite orbital plane stays at a constant inclination to the ring plane with the ascending node Ω moving on the ring plane. The motion of Ω both inside and outside the lunar radius is analogous to the well known effect of the earth's equatorial bulge: there is a secular progression of Ω if $90^\circ < I \leq 180^\circ$ and a retrogression if $0 \leq I < 90^\circ$. The secular motion of Ω is expressed graphically in Section X for the sun and moon effects separately and combined. Figures 6 and 7 giving the combined effects can be used to determine the motion of the satellite orbital plane with better than 10% accuracy out to 2/3 the lunar radius. The results might not be this precise beyond the lunar radius because of the longer averaging times.

The lunar secular effect on eccentricity e and argument of perigee ω outside the lunar radius is also analogous to the effect of the earth's equatorial bulge: there is a secular progression of the satellite perigee if $0 \leq I \lesssim 63^\circ.4$ or $116^\circ.6 \lesssim I \leq 180^\circ$, and a retrogression if $63^\circ.4 \lesssim I \lesssim 116^\circ.6$. The secular change in e is of higher order, and it is also long periodic away from the critical inclinations $I \approx 63^\circ.4$ or $116^\circ.6$. However, the change in eccentricity of a satellite orbit outside the lunar radius is mainly due to the effect of the sun.

Inside the lunar or solar ring of matter, there are critical inclinations I_a and $I_b = 180^\circ - I_a$ such that if $0 \leq I \lesssim I_a$ or $I_b \lesssim I \leq 180^\circ$, then ω goes through complete revolutions and e has long periodic oscillations with no secular buildup. If $I_a \lesssim I \lesssim I_b$, then ω remains in the first or third quadrant if it is initially there. If ω is initially in the second or fourth quadrant, it moves to the first or third quadrant. Since secular changes in e are proportional to $e\sin 2\omega$, e decreases while ω is in the second or fourth quadrant, but it will increase without bound when ω moves to the first or third quadrant. The critical inclination I_a is $39^\circ.2$ for low altitude satellites, $36^\circ.2$ at 1/3 the lunar radius and $27^\circ.1$ at 2/3 the lunar radius. The value $I_a = 39^\circ.2$ for the sun effect does not change appreciably because of the great distance of the sun from the earth compared to the distance at which a satellite gets into the solar sphere of influence.

A high altitude satellite with $I_a < I < I_b$ can have a useful lifetime if initial perigee location is chosen correctly, but eventually eccentricity will get very large, and the satellite will either enter the earth's atmosphere or be thrown out of the earth-moon system by a close lunar approach. A satellite near the ecliptic plane will have only periodic variations in eccentricity, but these can be of large amplitude at high altitudes.

As discussed in Appendix A, short period perturbations can move e enough away from zero to commence the long periodic variations or secular growth predicted by the double averaging calculations. Librations about uniform satellite motion are examined in Appendix B. Results from numerical integrations of the exact satellite equations of motion are presented in Appendix C. The most interesting fact revealed in the small sample examined was that retrograde satellites relative to the lunar plane suffer much smaller perturbations in eccentricity than direct satellites.

II. Spheres of Influence

A spacecraft is in the sphere of influence of the earth, moon or sun depending on which of these bodies has the strongest influence on its motion. To make this statement more quantitative, consider the following subscript notation:

e = earth

m = moon

s = sun

b = spacecraft.

For α, β = e, m, s, b let

$x^k_{\alpha\beta}$ = k^{th} cartesian coordinate of body α relative to body β (k = 1,2,3)

$r_{\alpha\beta}$ = distance between body α and body β

μ_α = gravitational constant times mass of body α

Then the equations of motion of the spacecraft relative to the earth as perturbed by the moon and sun are

$$\frac{d^2x^k_{be}}{dt^2} = -\mu_e \frac{dx^k_{be}}{r^3_{be}} + \Omega^k_{me} + \Omega^k_{se}, \quad k = 1,2,3 \tag{1}$$

where

$$\Omega^k_{me} = -\mu_m\left(\frac{x^k_{bm}}{r^3_{bm}} + \frac{x^k_{me}}{r^3_{me}}\right)$$

$$\Omega_{se}^{k} = -\mu_s \left(\frac{x_{bs}^{k}}{r_{bs}^{3}} + \frac{x_{se}^{k}}{r_{se}^{3}} \right)$$

The equations of motion relative to the moon or sun are similar with vector perturbing accelerations $\vec{\Omega}_{em}$, $\vec{\Omega}_{sm}$ in the relative to moon case and $\vec{\Omega}_{es}$, $\vec{\Omega}_{ms}$ in the relative to sun case.

Transition from the earth sphere of influence to the moon sphere of influence occurs when

$$\frac{|\vec{\Omega}_{me}|}{\mu_e/r_{be}^2} \approx \frac{|\vec{\Omega}_{em}|}{\mu_m/r_{bm}^2} \approx 0.5.$$

Transition from the earth sphere of influence to the sun sphere of influence occurs when

$$\frac{|\vec{\Omega}_{se}|}{\mu_e/r_{be}^2} \approx \frac{|\vec{\Omega}_{es}|}{\mu_s/r_{bs}^2} \approx 0.1.$$

The small value of this latter ratio is due to the large size of the sun's mass.

We now drop subscript notation for relative to earth quantities and take

$$r = r_{be}$$

$$x^{k} = x_{be}^{k}$$

$$\mu = \mu_e$$

The moon is at a mean distance of

$$\rho = r_{me} = 60.2665 \text{ earth radii.} \tag{2}$$

Assuming that the earth, moon, sun and spacecraft lie along a straight line, the earth's sphere of influence is in the region (see Fig. 1):

$$1 < r < 52, \; 72 < r < 125 \text{ earth radii} \tag{3}$$

We shall conservatively insist on retaining enough terms in infinite series to have formally valid results for

$$r/\rho < 0.9 \text{ or } \rho/r < 0.9 \tag{4}$$

even though we can only expect reasonably valid agreement with nature for a $\lesssim 0.7$ bound on the ratios r/ρ or ρ/r (see Appendix C).

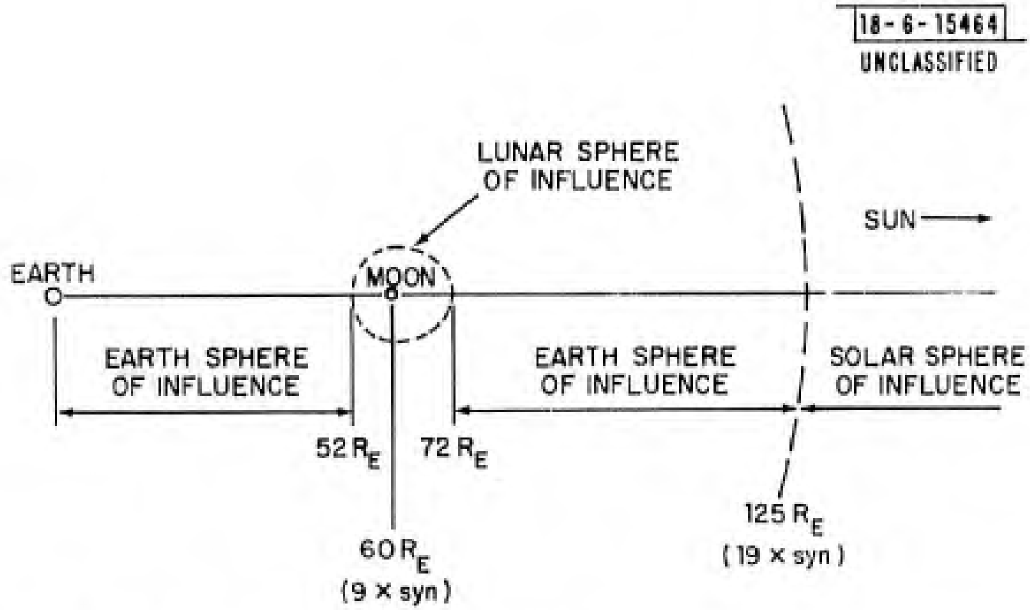

Fig. 1. Boundaries of the earth's sphere of influence.

III. Potential of a Ring

We assume that the mass of the moon is spread uniformly in a ring around its orbit and that the eccentricity of the lunar orbit is zero. Consider a reference system (x^1, x^2, x^3) with origin at the center of mass of the earth and with the (x^1, x^2) plane being the lunar orbital plane. A vector $\vec{\rho}$ pointing to a spot in the ring has components

$$\vec{\rho} = \rho(\cos\theta\,\vec{\epsilon}_1 + \sin\theta\,\vec{\epsilon}_2) \tag{5}$$

where $\vec{\epsilon}_j$ is the unit vector in the x^j direction. The gravitational potential at a point $\vec{r}$ due to the circular ring is

$$U = \frac{\mu_m}{2\pi}\int_0^{2\pi}\frac{d\theta}{|\vec{r}-\vec{\rho}\,|} \tag{6}$$

where μ_m is the mass of the ring times the gravitational constant. We employ the sign convention customary in celestial mechanics with the acceleration on a test particle at the point $\vec{r}$ being

$$\vec{F} = \overrightarrow{\text{grad } U} \tag{7}$$

The convention in physics is to put a minus sign in both (6) and (7).

We have[1]

$$U = \frac{\mu_m}{2\pi\rho}\int_0^{2\pi}\frac{d\theta}{\sqrt{1 - 2\frac{\vec{r}\cdot\vec{\rho}}{\rho^2} + \frac{r^2}{\rho^2}}}$$

$$= \frac{\mu_m}{2\pi\rho} \sum_{\ell=0}^{\infty} \left(\frac{r}{\rho}\right)^{\ell} \int_0^{2\pi} P_\ell(q) d\theta \qquad r < \rho \tag{8}$$

where

$P_\ell(q)$ = Legendre polynomial

$$q = \frac{\vec{r}}{r} \cdot \frac{\vec{\rho}}{\rho} = \frac{x^1}{r} \cos\theta + \frac{x^2}{r} \sin\theta$$

$$\vec{r} = x^1 \vec{\epsilon}_1 + x^2 \vec{\epsilon}_2 + x^3 \vec{\epsilon}_3 \tag{9}$$

We assume we are inside the ring, i.e., $r < \rho$. If we were outside the ring we could obtain a convergent expansion by reversing the roles of r and ρ in (8); see Section VIII.

If ℓ is odd, $P_\ell(q)$ has only odd powers of q, and hence odd powers of $\sin\theta$ and $\cos\theta$ with integrals 0 to 2π being zero. Thus we can write (6) as

$$U = \frac{\mu_m}{2\pi\rho} \sum_{\ell=0}^{\infty} \left(\frac{r}{\rho}\right)^{2\ell} \int_0^{2\pi} P_{2\ell}(q) d\theta \tag{10}$$

We have[1]

$$P_{2\ell}(q) = \sum_{k=0}^{\ell} (-1)^{k+\ell} \frac{1 \cdot 3 \cdot 5 \cdots (2k + 2\ell - 1)}{(\ell-k)!\,(2k)!\,2^{\ell-k}} q^{2k} \tag{11}$$

so that

$$U = \frac{\mu_m}{2\pi\rho} \sum_{\ell=0}^{\infty} \left(\frac{r}{\rho}\right)^{2\ell} \sum_{k=0}^{\ell} (-1)^{k+\ell} \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,(2k)!\,2^{\ell-k}} \cdot$$

$$\cdot \sum_{m=0}^{2k} \binom{2k}{m} \left(\frac{x^1}{r}\right)^m \left(\frac{x^2}{r}\right)^{2k-m} \int_0^{2\pi} \cos^m\theta \sin^{2k-m}\theta \, d\theta \qquad (12)$$

Since[2]

$$\int_0^{2\pi} \cos^{2m+1}\theta \sin^k\theta \, d\theta = \int_0^{2\pi} \cos^k\theta \sin^{2m+1}\theta \, d\theta = 0$$

$$\int_0^{2\pi} \cos^{2m}\theta \sin^{2k-2m}\theta \, d\theta = \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{2^k k!}(2\pi) \qquad (13)$$

we have

$$U = \frac{\mu_m}{\rho} \sum_{\ell=0}^{\infty} \left(\frac{r}{\rho}\right)^{2\ell} \sum_{k=0}^{\ell} (-1)^{k+\ell} \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+k}k!} \cdot$$

$$\cdot \sum_{m=0}^{k} \frac{1}{m!(k-m)!} \left(\frac{x^1}{r}\right)^{2m} \left(\frac{x^2}{r}\right)^{2k-2m} \qquad (14)$$

IV. Components of Acceleration Inside the Ring

Let $\vec{\epsilon}_r$ be a unit vector pointing from the earth to the satellite, so that $\vec{r} = r\,\vec{\epsilon}_r$; let $\vec{\epsilon}_s$ be a unit vector normal to $\vec{\epsilon}_r$ in the satellite orbital plane making an acute angle with the satellite velocity vector; and let $\vec{\epsilon}_w = \vec{\epsilon}_r \times \vec{\epsilon}_s$. Let

I = inclination of the satellite orbital plane on the lunar orbital plane

$\Omega \equiv 0$ = ascending node of the satellite orbital plane on the lunar orbital plane

η = angle from the ascending node to the satellite position measured along the satellite orbital plane. (15)

Since we are smearing the moon orbit out into a ring of matter, no loss of generality arises from the assumption that $\Omega = 0$. We have

$$\vec{\epsilon}_r = \cos\eta\,\vec{\epsilon}_1 + \sin\eta\cos I\,\vec{\epsilon}_2 + \sin\eta\sin I\,\vec{\epsilon}_3$$

$$\vec{\epsilon}_s = -\sin\eta\,\vec{\epsilon}_1 + \cos\eta\cos I\,\vec{\epsilon}_2 + \cos\eta\sin I\,\vec{\epsilon}_3$$

$$\vec{\epsilon}_w = -\sin I\,\vec{\epsilon}_2 + \cos I\,\vec{\epsilon}_3 \qquad (16)$$

The perturbing acceleration on the satellite is the difference between the acceleration on the satellite and the acceleration on the earth. By symmetry, the acceleration on the earth due to the circular ring is zero. Thus the perturbing accelerations on the satellite in the $\vec{\epsilon}_r$, $\vec{\epsilon}_s$, $\vec{\epsilon}_w$ directions are

$$R = \vec{\epsilon}_r \cdot \overrightarrow{\text{grad } U}$$

$$S = \epsilon_s \cdot \overrightarrow{\text{grad } U}$$

$$W = \vec{\epsilon}_w \cdot \overrightarrow{\text{grad } U} \tag{17}$$

Since

$$x^1 = r \cos\eta$$

$$x^2 = r \sin\eta \cos I$$

$$x^3 = r \sin\eta \sin I \tag{18}$$

formula (14) implies

$$R = \frac{\mu_m}{\rho^2} \sum_{\ell=1}^{\infty} 2\ell \left(\frac{r}{\rho}\right)^{2\ell-1} \sum_{k=0}^{\ell} (-1)^{k+\ell} \frac{1 \cdot 3 \cdot 5 \cdots (2k + 2\ell - 1)}{(\ell-k)!\, 2^{\ell+k} k!} \cdot$$

$$\sum_{m=0}^{k} \frac{(\cos I)^{2k-2m}}{m!(k-m)!} (\cos\eta)^{2m} (\sin\eta)^{2k-2m} \tag{19}$$

$$S = \frac{\mu_m}{\rho^2} \sum_{\ell=1}^{\infty} \left(\frac{r}{\rho}\right)^{2\ell-1} \sum_{k=1}^{\ell} (-1)^{k+\ell} \frac{1 \cdot 3 \cdot 5 \cdots (2k + 2\ell - 1)}{(\ell-k)!\, 2^{\ell+k} k!} \cdot$$

$$\sum_{m=0}^{k} \frac{(\cos I)^{2k-2m}}{m!(k-m)!} \left[(2k-2m)(\cos\eta)^{2m+1}(\sin\eta)^{2k-2m-1}\right.$$

$$\left. - 2m(\cos\eta)^{2m-1}(\sin\eta)^{2k-2m+1}\right] \quad (20)$$

$$W = -\frac{\mu_m \sin I}{\rho^2} \sum_{\ell=1}^{\infty} \left(\frac{r}{\rho}\right)^{2\ell-1} \sum_{k=1}^{\ell} (-1)^{k+\ell} \frac{1 \cdot 3 \cdot 5 \cdots (2k + 2\ell - 1)}{(\ell-k)!\, 2^{\ell+k} k!}$$

$$\sum_{m=0}^{k-1} \frac{(2k-2m)}{m!(k-m)!} (\cos I)^{2k-2m-1} (\cos\eta)^{2m} \cdot$$

$$(\sin\eta)^{2k-2m-1} \quad (21)$$

V. Equations for Osculating Elements

We employ the following osculating elliptic orbital elements for an earth satellite:

a = semi-major axis

e = eccentricity

I = inclination to lunar plane

Ω = ascending node on lunar plane ($\equiv 0$ in this note)

ω = argument of perigee

M = mean anomaly (22)

Let

μ = gravitational constant times mass of earth

$n = \mu^{1/2} a^{-3/2}$ = mean motion

ψ = true anomaly

$\eta = \psi + \omega$

ξ = eccentric anomaly

$p = a(1-e^2)$ = semi-latus rectum

$$r = \frac{p}{1+e\cos\psi} = \text{radius from earth} \tag{23}$$

The relations between the mean, eccentric and true anomalies are

$$\tan\frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\xi}{2}$$

$$M = \xi - e\sin\xi$$

Gauss' form for the equation for the osculating elements are[3]

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}} [R\, e \sin\psi + S\frac{p}{r}]$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na} [R \sin\psi + S(\cos\xi + \cos\psi)]$$

$$\frac{dI}{dt} = \frac{rW}{na^2\sqrt{1-e^2}} \cos(\omega + \psi)$$

$$\frac{d\Omega}{dt} = \frac{rW}{na^2\sqrt{1-e^2}} \frac{\sin(\omega+\psi)}{\sin I}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae} [-R \cos\psi + S(\frac{r}{p} + 1)\sin\psi]$$

$$- \cos I \frac{d\Omega}{dt}$$

$$\frac{dM}{dt} = n + R[- \frac{2r}{na^2} + \frac{1-e^2}{nae} \cos\psi]$$

$$-S \frac{\sin\psi}{nae} [1 - e^2 + \frac{r}{a}] \quad (24)$$

If we use η as the independent variable instead of t we have[4]

$$\frac{da}{d\eta} = \frac{2a^2r^2\gamma}{\mu p} [R\, e \sin\psi + S(1 + e \cos\psi)]$$

$$\frac{de}{d\eta} = \frac{r^2\gamma}{\mu}\left\{R\sin\psi + \frac{r}{p}\left[2\cos\psi + e(1+\cos^2\psi)\right]S\right\}$$

$$\frac{dI}{d\eta} = \frac{r^3\gamma}{\mu p}\cos\eta \cdot W$$

$$\frac{d\Omega}{d\eta} = \frac{r^3\gamma}{\mu p}\frac{\sin\eta}{\sin I} \cdot W$$

$$\frac{d\omega}{d\eta} = \frac{r^2\gamma}{\mu e}\left\{-R\cos\psi + S\left(1+\frac{r}{p}\right)\sin\psi\right\} - \cos I\frac{d\Omega}{d\eta}$$

$$\frac{dt}{d\eta} = \frac{r^2\gamma}{(\mu p)^{1/2}} \tag{25}$$

where

$$\gamma = \frac{1}{1 - \frac{r^3}{\mu p}\cos I\sin\eta \cdot W} \tag{26}$$

If β is one of the orbital elements (a, e, I, Ω, ω), we can see the average or secular change in β over one orbit by evaluating

$$\Delta\beta = \int_0^{2\pi}\frac{d\beta}{d\eta}\,d\eta \tag{27}$$

where the orbital elements are held constant in the integrand. We shall now determine the average effects due to the lunar ring of matter. Oi course, considering the moon spread out in a ring in evaluating (27) gives us a doubly averaged effect.

VI. First Order in Eccentricity

By (23) we have

$$r \approx a[1 - e\cos(\eta - \omega)]$$

$$\approx a[1 - e(\cos\eta\cos\omega + \sin\eta\sin\omega)]$$

$$r^j \approx a^j[1 - j\,e(\cos\eta\cos\omega + \sin\eta\sin\omega)] \qquad (28)$$

Thus, ignoring powers of e higher than the first, expressions (19), (20) and (21) for the acceleration components become

$$R = \frac{\mu_m}{\rho^2}\sum_{\ell=1}^{\infty} 2\ell\left(\frac{a}{\rho}\right)^{2\ell-1}\sum_{k=0}^{\ell}(-1)^{k+\ell}\frac{1\cdot3\cdot5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+k}k!}\,.$$

$$\sum_{m=0}^{k}\frac{(\cos I)^{2k-2m}}{m!(k-m)!}\left\{(\cos\eta)^{2m}(\sin\eta)^{2k-2m}\right.$$

$$-(2\ell-1)e\left[\cos\omega(\cos\eta)^{2m+1}(\sin\eta)^{2k-2m}\right.$$

$$\left.\left.+\sin\omega(\cos\eta)^{2m}(\sin\eta)^{2k-2m+1}\right]\right\} \qquad (29)$$

$$S = \frac{\mu_m}{\rho^2}\sum_{\ell=1}^{\infty}\left(\frac{a}{\rho}\right)^{2\ell-1}\sum_{k=1}^{\ell}(-1)^{k+\ell}\frac{1\cdot3\cdot5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+k}k!}\,.$$

$$\sum_{m=0}^{k}\frac{(\cos I)^{2k-2m}}{m!(k-m)!}\left\{(2k-2m)(\cos\eta)^{2m+1}(\sin\eta)^{2k-2m-1}\right.$$

$$-2m(\cos\eta)^{2m-1}(\sin\eta)^{2k-2m+1}$$

$$-(2\ell-1)e\ [(2k-2m)\cos\omega(\cos\eta)^{2m+2}(\sin\eta)^{2k-2m-1}$$

$$-2m\cos\omega(\cos\eta)^{2m}(\sin\eta)^{2k-2m+1}$$

$$+(2k-2m)\sin\omega(\cos\eta)^{2m+1}(\sin\eta)^{2k-2m}$$

$$-2m\sin\omega(\cos\eta)^{2m-1}(\sin\eta)^{2k-2m+2}]\Big\} \qquad (30)$$

$$W = -\frac{\mu_m \sin I}{\rho^2}\sum_{\ell=1}^{\infty}\left(\frac{a}{\rho}\right)^{2\ell-1}\sum_{k=1}^{\ell}(-1)^{k+\ell}\frac{1\cdot3\cdot5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+k}k!}\cdot$$

$$\sum_{m=0}^{k-1}\frac{(2k-2m)}{m!(k-m)!}(\cos I)^{2k-2m-1}\Big\{(\cos\eta)^{2m}\cdot$$

$$(\sin\eta)^{2k-2m-1}-(2\ell-1)\,e[\cos\omega(\cos\eta)^{2m+1}(\sin\eta)^{2k-2m-1}$$

$$+\sin\omega(\cos\eta)^{2m}(\sin\eta)^{2k-2m}]\Big\} \qquad (31)$$

By (26) and (31) we have to zeroth order in e

$$\gamma = \sum_{\ell=0}^{\infty}\frac{r^{3\ell}}{\mu^{\ell}p^{\ell}}(\cos I\sin\eta W)^{\ell}$$

$$\approx 1+\frac{a^2W}{\mu}\cos I\sin\eta+\cdots$$

$$\approx 1 - \left(\frac{\mu_m}{\mu}\right) \left(\frac{a}{\rho}\right)^3 \sin I \cos I \left\{ \frac{3}{2} (\sin\eta)^2 \right.$$

$$\left. + \left(\frac{a}{\rho}\right)^2 \left[-\frac{15}{4} (\sin\eta)^2 + \frac{105}{16} \left((\cos I)^2 (\sin\eta)^4 + (\sin\eta)^2 \right) \right] + \cdots \right\}$$

If $\mu/\mu_m = 81.301$ and $(a/\rho) = 0.9$, we have

$$\gamma - 1 \approx 8.96 \times 10^{-3} \sin I \cos I \left\{ 1.5 (\sin\eta)^2 \right.$$

$$\left. + \left[-2.28 (\sin\eta)^2 + 5.31 (\cos I)^2 (\sin\eta)^4 \right] + \cdots \right\}$$

The maximum possible value of $\sin I \cos I$ is 0.5. Thus assuming that $\gamma \equiv 1$ would seem to introduce errors no larger than a few percent even when $a/\rho = 0.9$. Actually, the higher order terms in the infinite series for $\gamma - 1$ are important, so we evaluated them with a computer program and found $|\gamma - 1| < 0.02$ when $a/\rho = 0.9$ for all values of I and η. It was only when η was near 90^o or 270^o that $|\gamma - 1|$ was so large as 0.02. Away from these values of η it was smaller than 0.01 or even 0.005. Also, when $a/\rho < 0.9$, $|\gamma - 1|$ got to be quite small.

Assuming that $\gamma = 1$ and ignoring powers of e higher than the first, equations (25) and (27) imply that the changes in the elements over one orbit are

$$\Delta a = \frac{2a^3}{\mu} \int_0^{2\pi} \left\{ S + Re(\sin\eta \cos\omega - \cos\eta \sin\omega) \right.$$

$$\left. - Se(\cos\eta \cos\omega + \sin\eta \sin\omega) \right\} d\eta \qquad (32)$$

$$\Delta e = \frac{a^2}{\mu} \int_0^{2\pi} \Big\{ R(\sin\eta \cos\omega - \cos\eta \sin\omega) + 2S(\cos\eta \cos\omega + \sin\eta \sin\omega)$$

$$- 2\,Re[\cos\eta \sin\eta\,(1-2\sin^2\omega) + \cos\omega \sin\omega\,(\sin^2\eta - \cos^2\eta)]$$

$$+ Se[1-5(\cos^2\eta \cos^2\omega + 2\sin\eta \cos\eta \cos\omega \sin\omega$$

$$+ \sin^2\eta \sin^2\omega)]\Big\} d\eta \qquad (33)$$

$$\Delta I = \frac{a^2}{\mu} \int_0^{2\pi} \Big\{ W \cos\eta - 3We(\cos^2\eta \cos\omega + \sin\eta \cos\eta \sin\omega)\Big\} d\eta \qquad (34)$$

$$\Delta\Omega = \frac{a^2}{\mu \sin I} \int_0^{2\pi} \Big\{ W\sin\eta - 3We(\cos\eta \sin\eta \cos\omega + \sin^2\eta \sin\omega)\Big\} d\eta \qquad (35)$$

$$\Delta\omega = \frac{a^2}{\mu e} \int_0^{2\pi} \Big\{ -R(\cos\eta \cos\omega + \sin\eta \sin\omega)$$

$$+ 2S(\sin\eta \cos\omega - \cos\eta \sin\omega) + 2Re(\cos^2\eta \cos^2\omega$$

$$+ 2\sin\eta \cos\eta \cos\omega \sin\omega + \sin^2\eta \sin^2\omega)$$

$$- 5Se[\cos\eta \sin\eta (\cos^2\omega - \sin^2\omega)$$

$$+ \cos\omega \sin\omega\,(1-2\cos^2\eta)]\Big\} d\eta - \cos I\,\Delta\Omega \qquad (36)$$

VII. Orbit Averaged Effects Inside the Ring

Integrals on η from 0 to 2π which involve odd powers of $\sin\eta$ and/or $\cos\eta$ are zero. Integrals involving even powers of $\sin\eta$ and/or $\cos\eta$ are given by (13). Thus to first order in e, equations (29), (30) and (31) imply

$$\int_0^{2\pi} R\,d\eta = \frac{2\pi\mu_m}{\rho^2}\sum_{\ell=1}^{\infty} 2\ell\left(\frac{a}{\rho}\right)^{2\ell-1}\sum_{k=0}^{\ell}(-1)^{k+\ell}\frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+2k}(k!)^2}\cdot \sum_{m=0}^{k}\frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(\cos I)^{2k-2m} \tag{37}$$

$$\int_0^{2\pi} S\,d\eta = 0 \tag{38}$$

$$\int_0^{2\pi} R\sin\eta\,d\eta = -\frac{\pi e\mu_m \sin\omega}{\rho^2}\sum_{\ell=1}^{\infty} 2\ell(2\ell-1)\left(\frac{a}{\rho}\right)^{2\ell-1}\sum_{k=0}^{\ell}(-1)^{k+\ell}\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+2k}k!(k+1)!}\cdot \sum_{m=0}^{k}\frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m+1)}{m!(k-m)!}(\cos I)^{2k-2m} \tag{39}$$

$$\int_0^{2\pi} R\cos\eta\,d\eta = -\frac{\pi e\mu_m \cos\omega}{\rho^2}\sum_{\ell=1}^{\infty} 2\ell(2\ell-1)\left(\frac{a}{\rho}\right)^{2\ell-1}\sum_{k=0}^{\ell}(-1)^{k+\ell}\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+2k}k!(k+1)!}\cdot \sum_{m=0}^{k}\frac{1\cdot 3\cdot 5\cdots(2m+1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(\cos I)^{2k-2m} \tag{40}$$

$$\int_0^{2\pi} S\sin\eta\, d\eta = -\frac{\pi e \mu_m \cos\omega}{\rho^2} \sum_{\ell=1}^{\infty} (2\ell-1)\left(\frac{a}{\rho}\right)^{2\ell-1} \sum_{k=1}^{\ell} (-1)^{k+\ell} \cdot$$

$$\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\, 2^{\ell+2k} k!(k+1)!} \cdot$$

$$\cdot \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!} (2k-4m)(\cos I)^{2k-2m} \qquad (41)$$

$$\int_0^{2\pi} S\cos\eta\, d\eta = -\frac{\pi e \mu_m \sin\omega}{\rho^2} \sum_{\ell=1}^{\infty} (2\ell-1)\left(\frac{a}{\rho}\right)^{2\ell-1} \sum_{k=1}^{\ell} (-1)^{k+\ell} \cdot$$

$$\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\, 2^{\ell+2k} k!(k+1)!} \cdot$$

$$\cdot \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!} (2k-4m)(\cos I)^{2k-2m} \qquad (42)$$

$$\int_0^{2\pi} W\sin\eta\, d\eta = -\frac{2\pi \mu_m \sin I \cos I}{\rho^2} \sum_{\ell=1}^{\infty} \left(\frac{a}{\rho}\right)^{2\ell-1} \sum_{k=1}^{\ell} (-1)^{k+\ell} \cdot$$

$$\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\, 2^{\ell+2k} (k!)^2} \cdot$$

$$\cdot \sum_{m=0}^{k-1} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!} (2k-2m)(\cos I)^{2k-2m-2} \qquad (43)$$

$$\int_0^{2\pi} W\cos\eta\, d\eta = 0 \qquad (44)$$

$$\int_0^{2\pi} R\sin^2\eta d\eta = \frac{\pi\mu_m}{\rho^2} \sum_{\ell=1}^{\infty} 2\ell\left(\frac{a}{\rho}\right)^{2\ell-1} \sum_{k=0}^{\ell} (-1)^{k+\ell} \cdot$$

$$\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\, 2^{\ell+2k} k!(k+1)!} \cdot$$

$$\cdot \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m+1)}{m!(k-m)!} (\cos I)^{2k-2m} \tag{45}$$

$$\int_0^{2\pi} R\cos^2\eta d\eta = \frac{\pi\mu_m}{\rho^2} \sum_{\ell=1}^{\infty} 2\ell\left(\frac{a}{\rho}\right)^{2\ell-1} \sum_{k=0}^{\ell} (-1)^{k+\ell} \cdot$$

$$\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\, 2^{\ell+2k} k!(k+1)!} \cdot$$

$$\cdot \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m+1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!} (\cos I)^{2k-2m} \tag{46}$$

$$\int_0^{2\pi} R\sin\eta \cos\eta d\eta = 0 \tag{47}$$

$$\int_0^{2\pi} S\sin^2\eta d\eta = 0 \tag{48}$$

$$\int_0^{2\pi} S\cos^2\eta d\eta = 0 \tag{49}$$

$$\int_0^{2\pi} S\sin\eta\cos\eta\, d\eta = \frac{\pi\mu_m}{\rho^2}\sum_{\ell=1}^{\infty}\left(\frac{a}{\rho}\right)^{2\ell-1}\sum_{k=1}^{\ell}(-1)^{k+\ell}\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+2k}k!(k+1)!}\cdot \sum_{m=0}^{k}\frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(2k-4m)\cdot(\cos I)^{2k-2m} \tag{50}$$

$$e\int_0^{2\pi} W\sin^2\eta\, d\eta = 0 \tag{51}$$

$$e\int_0^{2\pi} W\cos^2\eta\, d\eta = 0 \tag{52}$$

$$e\int_0^{2\pi} W\sin\eta\cos\eta\, d\eta = 0 \tag{53}$$

Inserting the above evaluations of the integrals into (32) to (36) we obtain to first order in e:

$$\Delta a = 0 \tag{54}$$

$$\Delta I = 0 \tag{55}$$

$$\Delta\Omega = -2\pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \cos I \sum_{\ell=1}^{\infty}\left(\frac{a}{\rho}\right)^{2\ell-2}\sum_{k=1}^{\ell}(-1)^{k+\ell}\cdot$$

$$\cdot\frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+2k}(k!)^2}\cdot$$

$$\cdot\sum_{m=0}^{k-1}\frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(2k-2m)(\cos I)^{2k-2m-2} \qquad (56)$$

$$\Delta e = -\pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 e\sin 2\omega \sum_{\ell=1}^{\infty}(\ell+1)(2\ell+3)\left(\frac{a}{\rho}\right)^{2\ell-2}\sum_{k=1}^{\ell}(-1)^{k+\ell}\cdot$$

$$\frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+2k}k!(k+1)!}\sum_{m=0}^{k}\frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}\cdot$$

$$\cdot(2k-4m)(\cos I)^{2k-2m} \qquad (57)$$

$$\Delta\omega = \pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3\sum_{\ell=1}^{\infty}\left(\frac{a}{\rho}\right)^{2\ell-2}\sum_{k=0}^{\ell}(-1)^{k+\ell}\frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+2k}k!(k+1)!}\cdot$$

$$\sum_{m=0}^{k}\frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(\cos I)^{2k-2m}\cdot$$

$$\{2\ell(2\ell+1)\left[(2m+1)\cos^2\omega + (2k-2m+1)\sin^2\omega\right]$$

$$-(4\ell+3)(2k-4m)[\cos^2\omega - \sin^2\omega]\} - \cos I\,\Delta\Omega \qquad (58)$$

The term $1\cdot 3\cdot 5\cdots(2m-1)$ is to be interpreted as 1 when $m=0$, since it arises from (13). Equations (54) to (58) give the average change in the orbital elements per orbital revolution inside the lunar radius. The average change per unit time can be obtained by dividing by the orbital period T:

$$T = \frac{2\pi a^{3/2}}{\mu^{1/2}} \tag{59}$$

We programmed formulas (56), (57) and (58) on a computer, but ran into numerical difficulties from having differences of large numbers. We therefore rearranged these formulas as follows. First, let us define

$$H_k = \sum_{m=0}^{k-1} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(2k-2m)(\cos I)^{2k-2m-2} \tag{60}$$

Then, formula (56) for $\Delta\Omega$ can be written as

$$\Delta\Omega = -2\pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \cos I \sum_{\ell=1}^{\infty} \left(\frac{a}{\rho}\right)^{2\ell-2} (-1)^{\ell+1} \cdot \frac{1\cdot 3\cdot 5\cdots(2\ell+1)}{(\ell-1)!\,2^{\ell+2}} \cdot \left[H_1 - \frac{(\ell-1)(2\ell+3)}{2^2\cdot 2^2}\left[H_2 - \frac{(\ell-2)(2\ell+5)}{2^2\cdot 3^2}\left[H_3 - \cdots - \frac{1\cdot(2\ell+2\ell-3)}{2^2\cdot(\ell-1)^2}\left[H_{\ell-1} - \frac{1\cdot(2\ell+2\ell-1)}{2^2\cdot \ell^2} H_\ell\right]\right]\cdots\right]\right]\right] \tag{61}$$

Next, let us define

$$\overline{H}_k = \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(2k-4m)(\cos I)^{2k-2m} \tag{62}$$

Then formula (57) for Δe can be written as

$$\Delta e = -\pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 e \sin 2\omega \sum_{\ell=1}^{\infty} (\ell+1)(2\ell+3)\left(\frac{a}{\rho}\right)^{2\ell-2} (-1)^{\ell+1} \cdot \frac{1\cdot 3\cdot 5\cdots(2\ell+1)}{(\ell-1)!\, 2^{\ell+2}\cdot 2} \cdot$$

$$\left[\overline{H}_k - \frac{(\ell-1)(2\ell+3)}{2^2\cdot 2\cdot 3}\left[\overline{H}_2 - \frac{(\ell-2)(2\ell+5)}{2^2\cdot 3\cdot 4}\left[\overline{H}_3 - \right.\right.\right.$$

$$\left.\left.\left.\cdots - \frac{1\cdot(2\ell+2\ell-3)}{2^2\cdot(\ell-1)(\ell)}\left[\overline{H}_{\ell-1} - \frac{1\cdot(2\ell+2\ell-1)}{2^2(\ell)(\ell+1)}\overline{H}_\ell\right]\cdots\right]\right]\right] \tag{63}$$

Finally, let us define

$$K_k = \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(\cos I)^{2k-2m}$$

$$\cdot\left[(2m+1)\cos^2\omega + (2k-2m+1)\sin^2\omega\right] \tag{64}$$

$$\overline{K}_k = \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!(k-m)!}(\cos I)^{2k-2m}$$

$$\cdot\ (2k-4m)\left[\cos^2\omega - \sin^2\omega\right] \tag{65}$$

Then formula (58) for $\Delta\omega$ can be written as

$$\Delta\omega = \pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \sum_{\ell=1}^{\infty} \left(\frac{a}{\rho}\right)^{2\ell-2} (-1)^{\ell} \frac{1\cdot 3\cdot 5\cdots(2\ell-1)}{\ell!\,2^{\ell}} \cdot$$

$$\Bigg\{ 2\ell(2\ell+1)\Bigg[K_0 - \frac{\ell(2\ell+1)}{2^2\cdot 1\cdot 2}\Bigg[K_1 - \frac{(\ell-1)(2\ell+3)}{2^2\cdot 2\cdot 3}\Bigg[K_2 -$$

$$\cdots - \frac{1\cdot(2\ell+2\ell-3)}{2^2\cdot(\ell-1)\cdot\ell}\Bigg[K_{\ell-1} - \frac{1\cdot(2\ell+2\ell-1)}{2^2\cdot\ell\cdot(\ell+1)} K_\ell\Bigg]\Bigg]\cdots\Bigg]\Bigg]\Bigg]$$

$$- (4\ell-3)\Bigg[\overline{K}_0 - \frac{\ell(2\ell+1)}{2^2\cdot 1\cdot 2}\Bigg[\overline{K}_1 - \frac{(\ell-1)(2\ell+3)}{2^2\cdot 2\cdot 3}\Bigg[\overline{K}_2 -$$

$$\cdots - \frac{1\cdot(2\ell+2\ell-3)}{2^2\cdot(\ell-1)\cdot\ell}\Bigg[\overline{K}_{\ell-1} - \frac{1\cdot(2\ell+2\ell-1)}{2^2\cdot\ell\cdot(\ell+1)} \overline{K}_\ell\Bigg]\Bigg]\cdots\Bigg]\Bigg]\Bigg]\Bigg\}$$

$$- \cos I\,\Delta\Omega \qquad (66)$$

The first few terms in the above formulas are

$$\Delta\Omega = -2\pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \cos I\Bigg\{\frac{3}{4} + \left(\frac{a}{\rho}\right)^2\left[-\frac{135}{128} + \frac{315}{128}\cos^2 I\right]$$

$$+ \left(\frac{a}{\rho}\right)^4\left[\frac{2625}{2048} - \frac{7875}{1024}\cos^2 I + \frac{17325}{2048}\cos^4 I\right] + \cdots\Bigg\} \qquad (67)$$

$$\Delta e = -\pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 e\sin 2\omega\Bigg\{-\frac{15}{4}\sin^2 I$$

$$+ \left(\frac{a}{\rho}\right)^2\left[\frac{315}{128} - \frac{315}{16}\cos^2 I + \frac{2205}{128}\cos^4 I\right]$$

$$+ \cdots\Bigg\} \qquad (68)$$

$$\Delta\omega = \pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \Bigg\{ 3 - \frac{15}{2}\sin^2\omega + \left(-\frac{3}{2} + \frac{15}{2}\sin^2\omega\right)\cos^2 I$$

$$+ \left(\frac{a}{\rho}\right)^2 \Bigg[-\frac{45}{32} + \frac{315}{64}\sin^2\omega + \left(\frac{585}{64} - \frac{315}{8}\sin^2\omega\right)\cos^2 I$$

$$+ \left(-\frac{315}{64} + \frac{2205}{64}\sin^2\omega\right)\cos^4 I \Bigg] + \cdots \Bigg\}$$

$$- \cos I \Delta\Omega$$

$$= \pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \Bigg\{ 3 - \frac{15}{2}\sin^2\omega \sin^2 I$$

$$+ \left(\frac{a}{\rho}\right)^2 \Bigg[-\frac{45}{32} + \frac{315}{64}\sin^2\omega + \left(\frac{225}{32} - \frac{315}{8}\sin^2\omega\right)\cos^2 I$$

$$+ \frac{2205}{64}\sin^2\omega \cos^4 I \Bigg] + \cdots \Bigg\} \qquad (69)$$

The leading terms for $\Delta\Omega$, Δe, $\Delta\omega$ agree to first order in e with the results in Ref. 5, which made no approximation about e but did ignore powers of (a/ρ) higher than 3.

The above formulas apply to the doubly orbit averaged effect of the gravitational attraction of the sun if we replace μ_m and ρ by the appropriate values for the sun and if we interpret I as the inclination to the ecliptic.

VIII. Orbit Averaged Effects Outside the Ring

If we are outside the ring the potential (8) has the following expansion:

$$U = \frac{\mu_m}{2\pi r}\int_0^{2\pi}\frac{d\theta}{\sqrt{1-2\frac{\vec{r}\cdot\vec{\rho}}{r^2}+\frac{\rho^2}{r^2}}}$$

$$= \frac{\mu_m}{2\pi r}\sum_{\ell=0}^{\infty}\left(\frac{\rho}{r}\right)^{\ell}\int_0^{2\pi}P_\ell(q)d\theta \qquad r > \rho \tag{70}$$

where q is given by (9). Computations similar to those used to derive (14) yield

$$U = \frac{\mu_m}{r}\sum_{\ell=0}^{\infty}\left(\frac{\rho}{r}\right)^{2\ell}\sum_{k=0}^{\ell}(-1)^{k+\ell}\frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+k}k!}$$

$$\cdot\sum_{m=0}^{k}\frac{1}{m!(k-m)!}\left(\frac{x_1}{r}\right)^{2m}\left(\frac{x_2}{r}\right)^{2k-2m} \tag{71}$$

Then to first order in e we have

$$R = -\frac{\mu_m}{a^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left(\frac{\rho}{a}\right)^{2\ell}\sum_{k=0}^{\ell}(-1)^{k+\ell}\frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+k}k!}$$

$$\sum_{m=0}^{k}\frac{(\cos I)^{2k-2m}}{m!(k-m)!}\left\{(\cos\eta)^{2m}(\sin\eta)^{2k-2m}\right.$$

$$+(2\ell+2)\,e\Big[\cos\omega(\cos\eta)^{2m+1}(\sin\eta)^{2k-2m}$$

$$\left.+\sin\omega(\cos\eta)^{2m}(\sin\eta)^{2k-2m+1}\Big]\right\} \tag{72}$$

$$S = \frac{\mu_m}{a^2} \sum_{\ell=1}^{\infty} \left(\frac{\rho}{a}\right)^{2\ell} \sum_{k=1}^{\ell} (-1)^{k+\ell} \frac{1 \cdot 3 \cdot 5 \cdots (2k+2\ell-1)}{(\ell-k)!\, 2^{\ell+k} k!} \cdot$$

$$\cdot \sum_{m=0}^{k} \frac{(\cos I)^{2k-2m}}{m!(k-m)!} \Big\{ (2k-2m)(\cos\eta)^{2m+1} (\sin\eta)^{2k-2m-1}$$

$$- 2m(\cos\eta)^{2m-1}(\sin\eta)^{2k-2m+1}$$

$$+ (2\ell+2)e \Big[(2k-2m)\cos\omega(\cos\eta)^{2m+2}(\sin\eta)^{2k-2m-1}$$

$$- 2m\cos\omega(\cos\eta)^{2m}(\sin\eta)^{2k-2m+1}$$

$$+ (2k-2m)\sin\omega(\cos\eta)^{2m+1}(\sin\eta)^{2k-2m}$$

$$- 2m\sin\omega(\cos\eta)^{2m-1}(\sin\eta)^{2k-2m+2} \Big] \Big\} \tag{73}$$

$$W = -\frac{\mu_m \sin I}{a^2} \sum_{\ell=1}^{\infty} \left(\frac{\rho}{a}\right)^{2\ell} \sum_{k=1}^{\ell} (-1)^{k+\ell} \frac{1 \cdot 3 \cdot 5 \cdots (2k+2\ell-1)}{(\ell-k)!\, 2^{\ell+k} k!} \cdot$$

$$\sum_{m=0}^{k-1} \frac{(2k-2m)}{m!(k-m)!} (\cos I)^{2k-2m-1} (\cos\eta)^{2m} \cdot$$

$$(\sin\eta)^{2k-2m-1} + (2\ell+2)\, e \Big[\cos\omega(\cos\eta)^{2m+1}(\sin\eta)^{2k-2m-1}$$

$$+ \sin\omega(\cos\eta)^{2m}(\sin\eta)^{2k-2m} \Big] \Big\} \tag{74}$$

Inserting these expressions into (32) to (36) we obtain to first order in e

$$\Delta a = 0 \tag{75}$$

$$\Delta I = 0 \tag{76}$$

$$\Delta\Omega = -2\pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{\rho}{a}\right)^2 \cos I \sum_{\ell=1}^{\infty} \left(\frac{\rho}{a}\right)^{2\ell-2} \sum_{k=1}^{\ell} (-1)^{k+\ell} \cdot$$

$$\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\,2^{\ell+2k}(k!)^2} \cdot \sum_{m=0}^{k-1} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!\,(k-m)!} \cdot$$

$$\cdot (2k-2m)(\cos I)^{2k-2m-2} \tag{77}$$

$$\Delta e = -\pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{\rho}{a}\right)^4 e \sin 2\omega \sum_{\ell=2}^{\infty} (\ell-1)(2\ell-1)\left(\frac{\rho}{a}\right)^{2\ell-4} \sum_{k=1}^{\ell} (-1)^{k+\ell} \cdot$$

$$\cdot \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\;2^{\ell+2k}k!\,(k+1)!} \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!\,(k-m)!} \cdot$$

$$\cdot\ (2k-4m)(\cos I)^{2k-2m} \tag{78}$$

$$\Delta\omega = \pi\left(\frac{\mu_m}{\mu}\right)\left(\frac{\rho}{a}\right)^2 \sum_{\ell=1}^{\infty} \left(\frac{\rho}{a}\right)^{2\ell-2} \sum_{k=0}^{\ell} (-1)^{k+\ell} \frac{1\cdot 3\cdot 5\cdots(2k+2\ell-1)}{(\ell-k)!\;2^{\ell+2k}k!\,(k+1)!}$$

$$\cdot \sum_{m=0}^{k} \frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2k-2m-1)}{m!\,(k-m)!}(\cos I)^{2k-2m} \cdot$$

$$\cdot \Big\{ 2\ell(2\ell+1)\left[(2m+1)\cos^2\omega + (2k-2m+1)\sin^2\omega\right]$$

$$+ (4\ell-1)(2k-4m)\left[\cos^2\omega - \sin^2\omega\right]\Big\}$$

$$-\ \cos I\ \Delta\Omega \tag{79}$$

Equations (75) through (79) give the average change in the orbital elements per orbital revolution outside the lunar radius. The average change per unit time can be obtained by dividing by the orbital period (59). Rearrangements of these formulas for numerical calculations are easily done in a manner analogous (60) through (66). The first few terms in (77) are similar to (67) with (a/ρ) replaced by (ρ/a) and with leading term $(\rho/a)^2$ rather than $(a/\rho)^3$, as can be seen by comparing (56) and (77). The lowest order terms in (78) and (79) are given in (152) and (153).

IX. Change to Earth Equatorial Coordinates

Let

I_m = inclination of the mean lunar orbital plane on the mean equator of the earth (80)

Ω_m = right ascension of the ascending node of the mean lunar orbital plane on the mean equator of the earth measured from the mean equinox (81)

The unit normal to the mean lunar orbital plane is then

$$\vec{N}_m = \sin\Omega_m \sin I_m \, \vec{e}_1 - \cos\Omega_m \sin I_m \, \vec{e}_2 + \cos I_m \, \vec{e}_3 \qquad (82)$$

where $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are unit vectors with $\vec{e}_3$ being perpendicular to the earth's equator pointing to the north, with $\vec{e}_1$ pointing to the vernal equinox, i.e., lying along the intersection of the equator and the ecliptic pointing towards Aries, and with $\vec{e}_2$ completing the right hand system.

Let ϵ_0 be the inclination between the mean equator of the earth and the mean ecliptic:[6]

$$\epsilon_0 = 23^\circ\!.452294 - 3^\circ\!.5626 \times 10^{-7}\tau - 1^\circ\!.23 \times 10^{-5}\tau^2 + 1^\circ\!.03 \times 10^{-20}\tau^3 \qquad (83)$$

where τ is measured in Ephemeris Days from the epoch 1900 January 0.5 E.T. (J.E.D. 2415020.0). The transformation between coordinates (u^1, u^2, u^3) referred to the mean equinox and equator of date and coordinates (v^1, v^2, v^3) referred to the mean equinox and ecliptic of date is

$$\left.\begin{aligned} v^k &= \sum_{\ell=1}^{3} E_{k\ell} u^\ell \\ u^k &= \sum_{\ell=1}^{3} E_{\ell k} v^\ell \end{aligned}\right\} \quad k = 1, 2, 3 \tag{84}$$

where the orthogonal matrix E is[7]

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon_o & \sin\epsilon_o \\ 0 & -\sin\epsilon_o & \cos\epsilon_o \end{bmatrix} \tag{85}$$

Now let[8]

$\overline{I}_m$ = inclination of the mean lunar orbital plane to the ecliptic

$$= 5\overset{\circ}{.}145396374 \quad (\sin\overline{I}_m/2 = 0.044886967) \tag{86}$$

$\bar{\Omega}_m$ = longitude of the ascending node of the mean lunar orbital plane on the mean ecliptic measured from the mean equinox

$$= 259^{\circ}.183275 - 0^{\circ}.0529539222\tau + 1^{\circ}.557 \times 10^{-12}\tau^2 + 5^{\circ} \times 10^{-20}\tau^3 \quad (87)$$

where τ is measured in ephemeris days from the epoch 1900 January 0.5 E.T. (J.E.D. 2415020.0). Let (w^1, w^2, w^3) be an earth-centered coordinate system with w^3 axis normal to the mean lunar orbital plane, with w^1 axis pointing towards the ascending node of the mean lunar orbit on the ecliptic, and with w^2 axis completing the right hand system. We have

$$\left.\begin{aligned} w^k &= \sum_{\ell=1}^{3} F_{\ell k} v^{\ell} \\ v^k &= \sum_{\ell=1}^{3} F_{k\ell} w^{\ell} \end{aligned}\right\} \quad k = 1, 2, 3 \quad (88)$$

where by standard Euler angle formulas

$$F_{11} = \cos\bar{\Omega}_m$$

$$F_{12} = -\sin\bar{\Omega}_m \cos\bar{I}_m$$

$$F_{13} = \sin\bar{\Omega}_m \sin\bar{I}_m$$

$$F_{21} = \sin\bar{\Omega}_m$$

$$F_{22} = \cos\overline{\Omega}_m \cos\overline{I}_m$$

$$F_{23} = -\cos\overline{\Omega}_m \sin\overline{I}_m$$

$$F_{31} = 0$$

$$F_{32} = \sin\overline{I}_m$$

$$F_{33} = \cos\overline{I}_m \tag{89}$$

We have

$$\begin{aligned}\vec{N}_m = {} & \sin\overline{\Omega}_m \sin\overline{I}_m \vec{e}_1 - (\sin\epsilon_o \cos\overline{I}_m \\ & + \cos\epsilon_o \cos\overline{\Omega}_m \sin\overline{I}_m)\vec{e}_2 \\ & + (\cos\epsilon_o \cos\overline{I}_m - \sin\epsilon_o \cos\overline{\Omega}_m \sin\overline{I}_m)\vec{e}_3\end{aligned} \tag{90}$$

Comparing (82) and (90) we see that

$$\cos I_m = (\cos\epsilon_o \cos\overline{I}_m - \sin\epsilon_o \cos\overline{\Omega}_m \sin\overline{I}_m)$$
$$0^o \leq I_m \leq 180^o \tag{91}$$

$$\left.\begin{aligned}\sin\Omega_m &= \frac{1}{\sin I_m}(\sin\overline{\Omega}_m \sin\overline{I}_m) \\ \cos\Omega_m &= \frac{1}{\sin I_m}(\sin\epsilon_o \cos\overline{I}_m + \cos\epsilon_o \cos\overline{\Omega}_m \sin\overline{I}_m)\end{aligned}\right\} 0 \leq \Omega_m < 360^o \tag{92}$$

The values of $\overline{\Omega}_m$, I_m, Ω_m over two 18.6-year cycle of $\overline{\Omega}_m$ are given in Fig. 2 for the period of interest to us.

Let

I_o = inclination of the satellite orbital plane on the mean equator of the earth (93)

Ω_o = right ascension of the ascending node of the satellite orbital plane on the mean equator of the earth measured from the mean equinox (94)

I = inclination of the satellite orbital plane on the mean lunar plane (95)

Ω = longitude of the ascending node of the satellite orbital plane on the mean lunar orbital plane measured from the intersection of the mean equator and the mean lunar orbital plane (96)

Let $\vec{\epsilon}_1$, $\vec{\epsilon}_2$, $\vec{\epsilon}_3$ be unit vectors with $\vec{\epsilon}_3$ being perpendicular to the mean lunar orbital plane pointing to the north, with $\vec{\epsilon}_1$ pointing along the intersection of the mean equator and the mean lunar orbital plane towards the lunar ascending node, and with $\vec{\epsilon}_2$ completing the right hand system. Note that $\vec{\epsilon}_1$, no longer points to the satellite ascending node on the lunar plane as it did in Section IV. We have

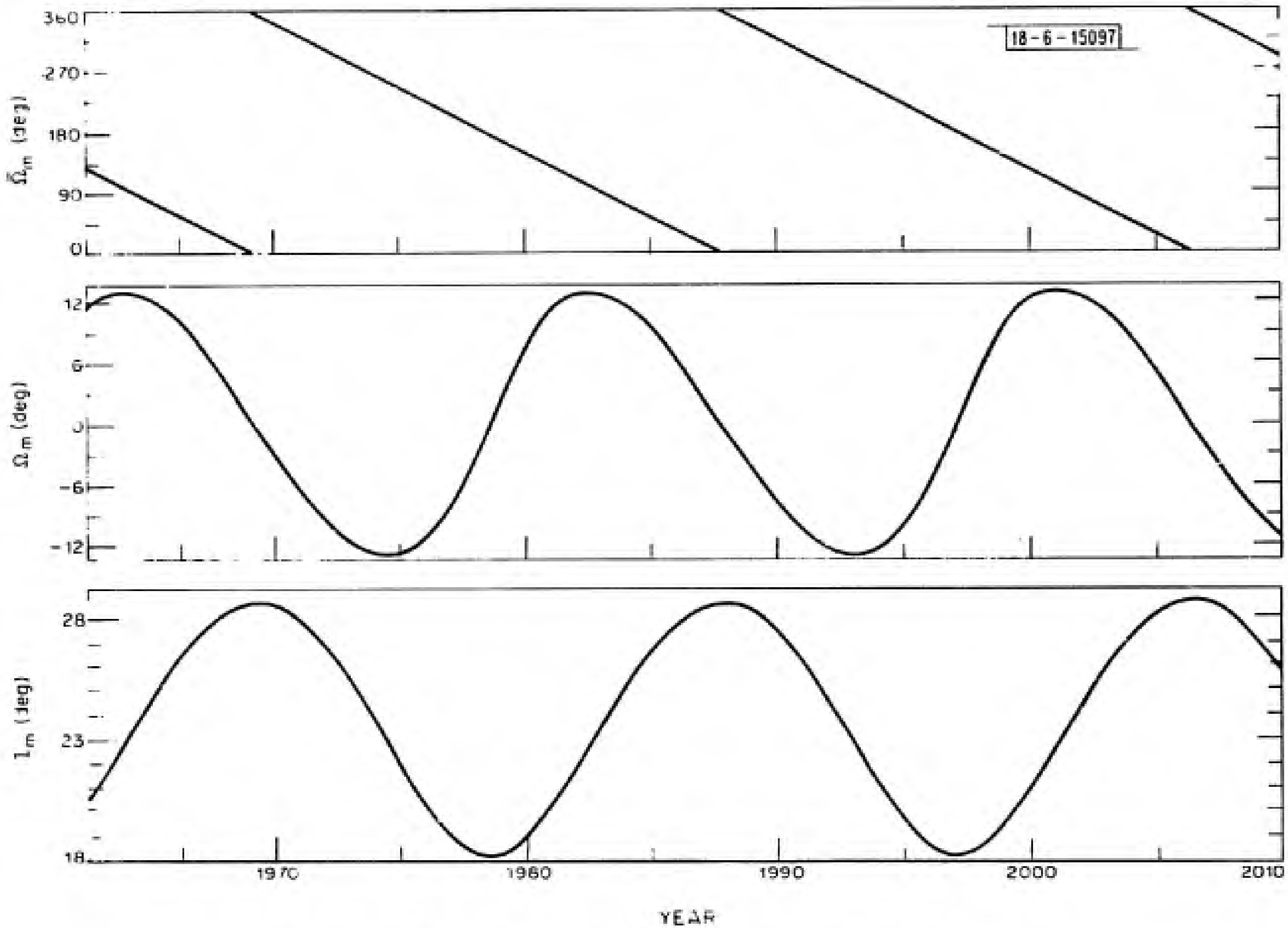

Fig. 2. Reference angles of the lunar orbital plane.

$$\vec{\epsilon}_1 = \cos\Omega_m \vec{e}_1 + \sin\Omega_m \vec{e}_2$$

$$\vec{\epsilon}_2 = -\sin\Omega_m \cos I_m \vec{e}_1 + \cos\Omega_m \cos I_m \vec{e}_2 + \sin I_m \vec{e}_3$$

$$\vec{\epsilon}_3 = \sin\Omega_m \sin I_m \vec{e}_1 - \cos\Omega_m \sin I_m \vec{e}_2 + \cos I_m \vec{e}_3 \; . \qquad (97)$$

$$\vec{e}_1 = \cos\Omega_m \vec{\epsilon}_1 - \sin\Omega_m \cos I_m \vec{\epsilon}_2 + \sin\Omega_m \sin I_m \vec{\epsilon}_3$$

$$\vec{e}_2 = \sin\Omega_m \vec{\epsilon}_1 + \cos\Omega_m \cos I_m \vec{\epsilon}_2 - \cos\Omega_m \sin I_m \vec{\epsilon}_3$$

$$\vec{e}_3 = \sin I_m \vec{\epsilon}_2 + \cos I_m \vec{\epsilon}_3 \; . \qquad (98)$$

The unit normal $\vec{N}$ to the satellite orbital plane can be expressed as

$$\begin{aligned}
\vec{N} &= \sin\Omega_o \sin I_o \vec{e}_1 - \cos\Omega_o \sin I_o \vec{e}_2 + \cos I_o \vec{e}_3 \\
&= (\cos\Omega_m \sin\Omega_o \sin I_o - \sin\Omega_m \cos\Omega_o \sin I_o)\vec{\epsilon}_1 \\
&+ (-\sin\Omega_m \cos I_m \sin\Omega_o \sin I_o - \cos\Omega_m \cos I_m \cos\Omega_o \cdot \\
&\quad \cdot \sin I_o + \sin I_m \cos I_o)\vec{\epsilon}_2 \\
&+ (\sin\Omega_m \sin I_m \sin\Omega_o \sin I_o + \cos\Omega_m \sin I_m \cos\Omega_o \cdot \\
&\quad \cdot \sin I_o + \cos I_m \cos I_o)\vec{\epsilon}_3
\end{aligned} \qquad (99)$$

or as

$$\begin{aligned}
\vec{N} &= \sin\Omega \sin I \vec{\epsilon}_1 - \cos\Omega \sin I \vec{\epsilon}_2 + \cos I \vec{\epsilon}_3 \\
&= (\cos\Omega_m \sin\Omega \sin I + \sin\Omega_m \cos I_m \cos\Omega \sin I \\
&\quad + \sin\Omega_m \sin I_m \cos I)\vec{e}_1
\end{aligned}$$

$$+ (\sin \Omega_m \sin \Omega \sin I - \cos \Omega_m \cos I_m \cos \Omega \sin I$$

$$- \cos \Omega_m \sin I_m \cos I) \vec{e}_2$$

$$+ (- \sin I_m \cos \Omega \sin I + \cos I_m \cos I) \vec{e}_3 \; . \qquad (100)$$

Comparing (99) and (100) we see that

$$\cos I_o = (-\sin I_m \cos \Omega \sin I + \cos I_m \cos I) \qquad 0 \leq I_o \leq 180^o \qquad (101)$$

$$\left.\begin{aligned} \cos \Omega_o &= - \frac{1}{\sin I_o} (\sin \Omega_m \sin \Omega \sin I \\ &\quad - \cos \Omega_m \cos I_m \cos \Omega \sin I \\ &\quad - \cos \Omega_m \sin I_m \cos I) \\ \sin \Omega_o &= \frac{1}{\sin I_o} (\cos \Omega_m \sin \Omega \sin I \\ &\quad + \sin \Omega_m \cos I_m \cos \Omega \sin I \\ &\quad + \sin \Omega_m \sin I_m \cos I) \end{aligned}\right\} \; 0 \leq \Omega_o < 360^o \qquad (102)$$

$$\begin{aligned} \cos I &= (\sin \Omega_m \sin I_m \sin \Omega_o \sin I_o \\ &\quad + \cos \Omega_m \sin I_m \cos \Omega_o \sin I_o \\ &\quad + \cos I_m \cos I_o) \qquad 0 \leq I \leq 180^o \end{aligned} \qquad (103)$$

$$\left.\begin{aligned} \cos \Omega &= - \frac{1}{\sin I} (-\sin \Omega_m \cos I_m \sin \Omega_o \sin I_o \\ &\quad - \cos \Omega_m \cos I_m \cos \Omega_o \sin I_o \\ &\quad + \sin I_m \cos I_o) \\ \sin \Omega &= \frac{1}{\sin I} (\cos \Omega_m \sin \Omega_o \sin I_o \\ &\quad - \sin \Omega_m \cos \Omega_o \sin I_o) \end{aligned}\right\} \; 0 \leq \Omega < 360^o \qquad (104)$$

Given I_o, Ω_o we can determine I, Ω from (103) and (104). Given I, we determine $\Delta\Omega$ from (56) if we are inside the moon's orbit or by (77) if we are outside the moon's orbit. By (101) and (102) a change in Ω implies the following changes in I_o, Ω_o:

$$\Delta I_o = \frac{\Delta\Omega}{\sin I_o} \sin I_m \sin\Omega \sin I$$
$$= \Delta\Omega \sin I_m \sin(\Omega_o - \Omega_m) \qquad (105)$$

$$\Delta\Omega_o = \frac{\Delta\Omega}{\sin I_o} [\sin\Omega_o (\sin\Omega_m \cos\Omega \sin I + \cos\Omega_m \cos I_m \sin\Omega \sin I)$$
$$+ \cos\Omega_o(\cos\Omega_m \cos\Omega \sin I - \sin\Omega_m \cos I_m \sin\Omega \sin I)]$$
$$= \Delta\Omega[\cos I_m - \sin I_m \cot I_o \cos(\Omega_o - \Omega_m)] \ . \qquad (106)$$

The only quantity in the final results above which depends on the satellite orbital plane angles (specifically, I) relative to the lunar plane is $\Delta\Omega$, so we simplify expression (103) for cos I:

$$\cos I = \sin I_m \sin I_o \cos(\Omega_o - \Omega_m) + \cos I_m \cos I_o \ . \qquad (107)$$

We can also simplify the expressions for $\sin\Omega$, $\cos\Omega$:

$$\cos\Omega = \frac{1}{\sin I} [\cos I_m \sin I_o \cos(\Omega_o - \Omega_m) - \sin I_m \cos I_o]$$
$$\sin\Omega = \frac{\sin I_o}{\sin I} \sin(\Omega_o - \Omega_m) \ . \qquad (108)$$

Let

ω = argument of perigee of the satellite orbit measured along the orbital plane from the ascending node on the mean lunar orbital plane. (109)

ω_o = argument of perigee of the satellite orbit measured along the orbital plane from the ascending node on the mean equatorial orbital plane. (110)

The unit vector $\vec{M}$ pointing from the center of the earth to the satellite perigee can be expressed as

$$\vec{M} = (\cos\Omega_o \cos\omega_o - \sin\Omega_o \sin\omega_o \cos I_o)\,\vec{e}_1$$
$$+ (\sin\Omega_o \cos\omega_o + \cos\Omega_o \sin\omega_o \cos I_o)\,\vec{e}_2$$
$$+ \sin\omega_o \sin I_o\,\vec{e}_3$$
$$= [\cos\Omega_m (\cos\Omega_o \cos\omega_o - \sin\Omega_o \sin\omega_o \cos I_o)$$
$$+ \sin\Omega_m (\sin\Omega_o \cos\omega_o + \cos\Omega_o \sin\omega_o \cos I_o)]\,\vec{\epsilon}_1$$
$$+ [-\sin\Omega_m \cos I_m(\cos\Omega_o \cos\omega_o - \sin\Omega_o \sin\omega_o \cos I_o)$$
$$+\cos\Omega_m \cos I_m(\sin\Omega_o \cos\omega_o + \cos\Omega_o \sin\omega_o \cos I_o)$$
$$+ \sin I_m \sin\omega_o \sin I_o]\,\vec{\epsilon}_2$$
$$+ [\sin\Omega_m \sin I_m(\cos\Omega_o \cos\omega_o - \sin\Omega_o \sin\omega_o \cos I_o)$$
$$-\cos\Omega_m \sin I_m(\sin\Omega_o \cos\omega_o + \cos\Omega_o \sin\omega_o \cos I_o)$$
$$+ \cos I_m \sin\omega_o \sin I_o]\,\vec{\epsilon}_3 \qquad (111)$$

or as

$$\vec{M} = (\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos I)\,\vec{\epsilon}_1$$
$$+ (\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos I)\,\vec{\epsilon}_2$$
$$+ \sin\omega \sin I\,\vec{\epsilon}_3$$
$$= [\cos\Omega_m (\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos I)$$
$$- \sin\Omega_m \cos I_m (\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos I)$$

$$+ \sin\Omega_m \sin I_m \sin\omega \sin I] \vec{e}_1$$

$$+ [\sin\Omega_m (\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos I)$$

$$+ \cos\Omega_m \cos I_m (\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos I)$$

$$- \cos\Omega_m \sin I_m \sin\omega \sin I] \vec{e}_2$$

$$+ [\sin I_m (\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos I)$$

$$+ \cos I_m \sin\omega \sin I] \vec{e}_3 \ . \qquad (112)$$

Comparing (111) and (112) we have

$$\left.\begin{aligned} \sin\omega_o &= \frac{1}{\sin I_o} [\sin I_m (\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos I) \\ &\quad + \cos I_m \sin\omega \sin I] \\ \cos\omega_o &= (\cos\Omega_o \cos\Omega_m + \sin\Omega_o \sin\Omega_m). \\ &\quad .(\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos I) \\ &\quad +(-\cos\Omega_o \sin\Omega_m \cos I_m + \sin\Omega_o \cos\Omega_m \cos I_m). \\ &\quad .(\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos I) \\ &\quad +(\cos\Omega_o \sin\Omega_m - \sin\Omega_o \cos\Omega_m) \sin I_m \sin\omega \sin I \end{aligned}\right\} 0 \le \omega_o < 360^0 \qquad (113)$$

$$\left.\begin{aligned} \sin\omega &= \frac{1}{\sin I} [\sin\Omega_m \sin I_m (\cos\Omega_o \cos\omega_o - \sin\Omega_o \sin\omega_o \cos I_o) \\ &\quad -\cos\Omega_m \sin I_m (\sin\Omega_o \cos\omega_o + \cos\Omega_o \sin\omega_o \cos I_o) \\ &\quad +\cos I_m \sin\omega_o \sin I_o] \\ \cos\omega &= (\cos\Omega \cos\Omega_m - \sin\Omega \sin\Omega_m \cos I_m). \\ &\quad .(\cos\Omega_o \cos\omega_o - \sin\Omega_o \sin\omega_o \cos I_o) \\ &\quad +(\cos\Omega \sin\Omega_m + \sin\Omega \cos\Omega_m \cos I_m). \\ &\quad .(\sin\Omega_o \cos\omega_o + \cos\Omega_o \sin\omega_o \cos I_o) \\ &\quad + \sin\Omega \sin I_m \sin\omega_o \sin I_o \ . \end{aligned}\right\} 0 \le \omega < 360^0 \qquad (114)$$

Expression (114) for $\sin\omega$, $\cos\omega$ can be simplified:

$$\left.\begin{aligned}
\sin\omega &= \frac{1}{\sin I}\,[-\sin I_m \cos\omega_o \sin(\Omega_o-\Omega_m) \\
&\quad - \sin I_m \sin\omega_o \cos I_o \cos(\Omega_o-\Omega_m) \\
&\quad + \cos I_m \sin\omega_o \sin I_o] \\
\cos\omega &= \frac{1}{\sin I}\,[\sin I_m \sin\omega_o \sin(\Omega_o-\Omega_m) \\
&\quad - \sin I_m \cos\omega_o \cos I_o \cos(\Omega_o-\Omega_m) \\
&\quad + \cos I_m \cos\omega_o \sin I_o]\ .
\end{aligned}\right\} 0 \le \omega < 360^\circ \qquad (115)$$

Expressions (113) for $\sin\omega_o$, $\cos\omega_o$ can be put in the form

$$\left.\begin{aligned}
\sin\omega_o &= \frac{1}{\sin I_o}\,[\cos I_m \sin\omega \sin I + \sin I_m (\sin\Omega \cos\omega \\
&\quad + \cos\Omega \sin\omega \cos I)] \\
\cos\omega_o &= \frac{1}{\sin I_o}\,[\cos I_m \cos\omega \sin I + \sin I_m (-\sin\Omega \sin\omega \\
&\quad + \cos\Omega \cos\omega \cos I)]\ .
\end{aligned}\right\} 0 \le \omega_o < 360^\circ \qquad (116)$$

Finally, differentiating (116) we obtain

$$\begin{aligned}
\Delta\omega_o = \frac{1}{\sin I_o}\,\{ &\Delta\omega\,[\cos I_m \sin I \cos(\omega_o-\omega) \\
&\quad + \sin I_m \sin\Omega \sin(\omega_o-\omega) \\
&\quad + \sin I_m \cos I \cos\Omega \cos(\omega_o-\omega)] \\
&+ \Delta\Omega\,[\sin I_m \cos\Omega \cos(\omega_o-\omega) \\
&\quad + \sin I_m \cos I \sin\Omega \sin(\omega_o-\omega)]\}\ .
\end{aligned} \qquad (117)$$

The above formulas apply to the transformation of the doubly averaged effects of the sun from the ecliptic system to the equatorial system if we set $\Omega_m = 0$, $I_m = \epsilon_o$.

X. Secular Motion of the Orbital Plane

According to the doubly averaging model for lunar perturbations, the satellite orbital plane stays at a constant inclination to the lunar plane with its ascending mode moving on the lunar plane. If the moon were really spread out in a ring, this behavior would be very exactly followed.

Namely, we checked our series expansions by performing numerical integrations with the Planetary Ephemeris Program (PEP)[9] of satellite motion perturbed by a circular lunar ring of matter and no other forces except the earth central attractions. This coding exists in PEP because it was added at M.I.T. to analyze the effect of Saturn's rings on a spacecraft flyby.[10] We considered four near circular cases: 40 and 80 earth radii semi-major axes at 30^{o} and 60^{o} inclination. The secular behavior over two years of the ascending node on the lunar ring plane was almost exactly that predicted by the series expansions. There were no secular variations in inclination to the lunar ring plane, only oscillations of magnitude a few tenths of a degree and period the orbital period of the satellite. Therefore, we can be quite certain that the series expansions are correct.

Of course, the moon is not spread out in a ring of matter and the lunar orbit is eccentric ($e \approx 0.0549$). Also, the lunar plane is not fixed in space, but rotates on the ecliptic one complete revolution in 18.6 years. Because of the small lunar inclination to the ecliptic ($\approx 5.^{o}14$) we can approximately consider the formulas for the variations in the satellite orbital elements to apply to the fixed ecliptic rather than the moving lunar plane. Alternately, we can calculate the exact formulas for the changes in inclination as well as ascending node on the ecliptic by formulas (105) and (106) with I_m, Ω_m changed to $\overline{I}_m$, $\overline{\Omega}_m$. For high altitude orbits where the earth gravitational potential harmonic effects are small it is simpler to visualize orbital variations relative to the ecliptic or lunar plane, rather than relative to the equatorial plane.

Figure 3 gives the doubly averaged effect of the sun on a satellite's ecliptic ascending node. The ascending node moves backward on the ecliptic for direct orbits ($0^\circ \leq I < 90^\circ$) and forward for retrograde orbits ($90^\circ < I \leq 180^\circ$). When $I=90^\circ$, there is no secular motion of the orbital plane. The magnitude of the rotation rate for $I > 90^\circ$ is the same as the magnitude of the rotation rate for an inclination of $180^\circ - I$.

Figures 4 and 5 give the doubly averaged effect of the moon on a satellite's ascending node on the lunar plane inside and outside the lunar radius. Figures 6 and 7 assume that the lunar plane coincides with the ecliptic and give the combined doubly averaged effect of the moon and sun on a satellite's ecliptic ascending node.

In order to evaluate the lunar infinite series within 10 earth radii of the moon, it was not only necessary to rearrange formulas (56) and (77) into the form (61), but 32 decimal places had to be carried along in adding the alternating sign terms together. Up to 44 terms in the infinite series were evaluated to generate Figs. 3 to 7. The last ℓ-term used in the series was 10^{-4} times the size of the leading term.

Figure 6 agrees with the results of PEP numerical integrations within 5-10% out to 2/3 the distance to the moon. For example, numerical integration of a 12 day period satellite inclined 36° to the ecliptic yielded $\Delta I = -1\overset{\circ}{.}5$/year,

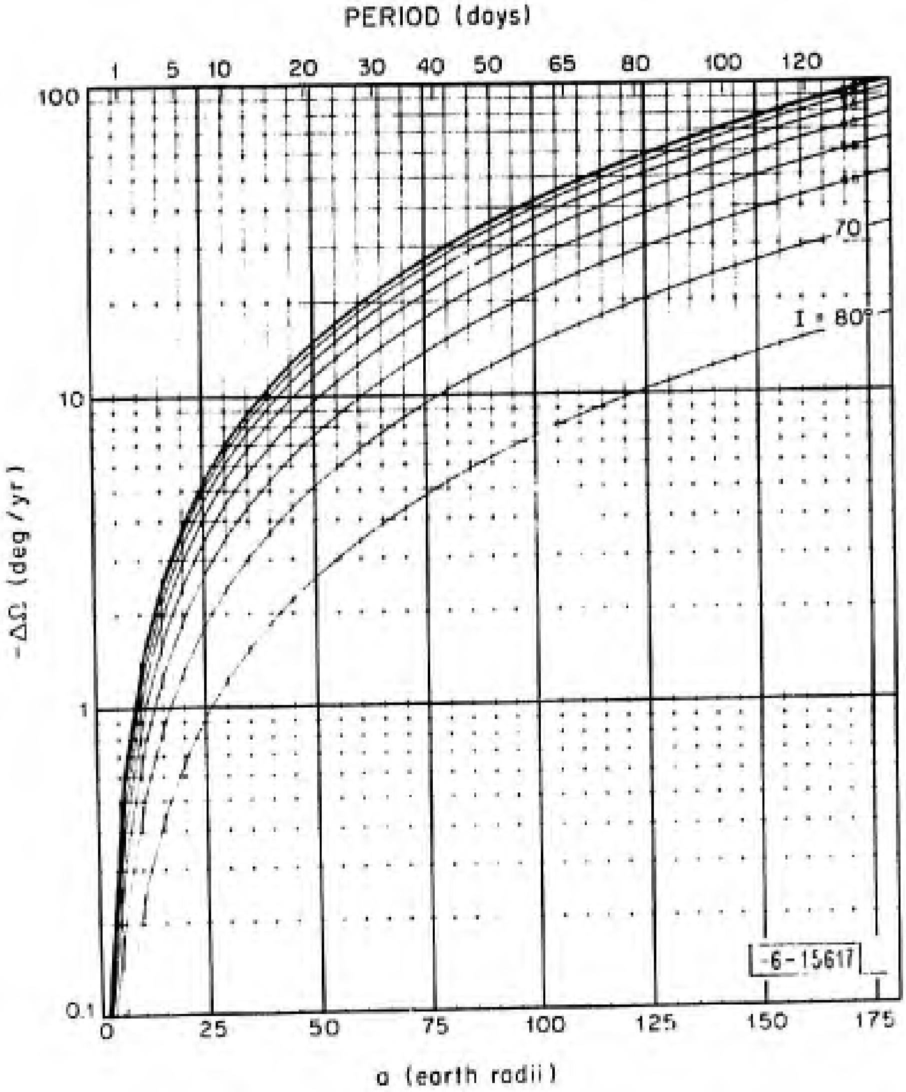

Fig. 3. Solar secular effect on a satellite's orbital plane.

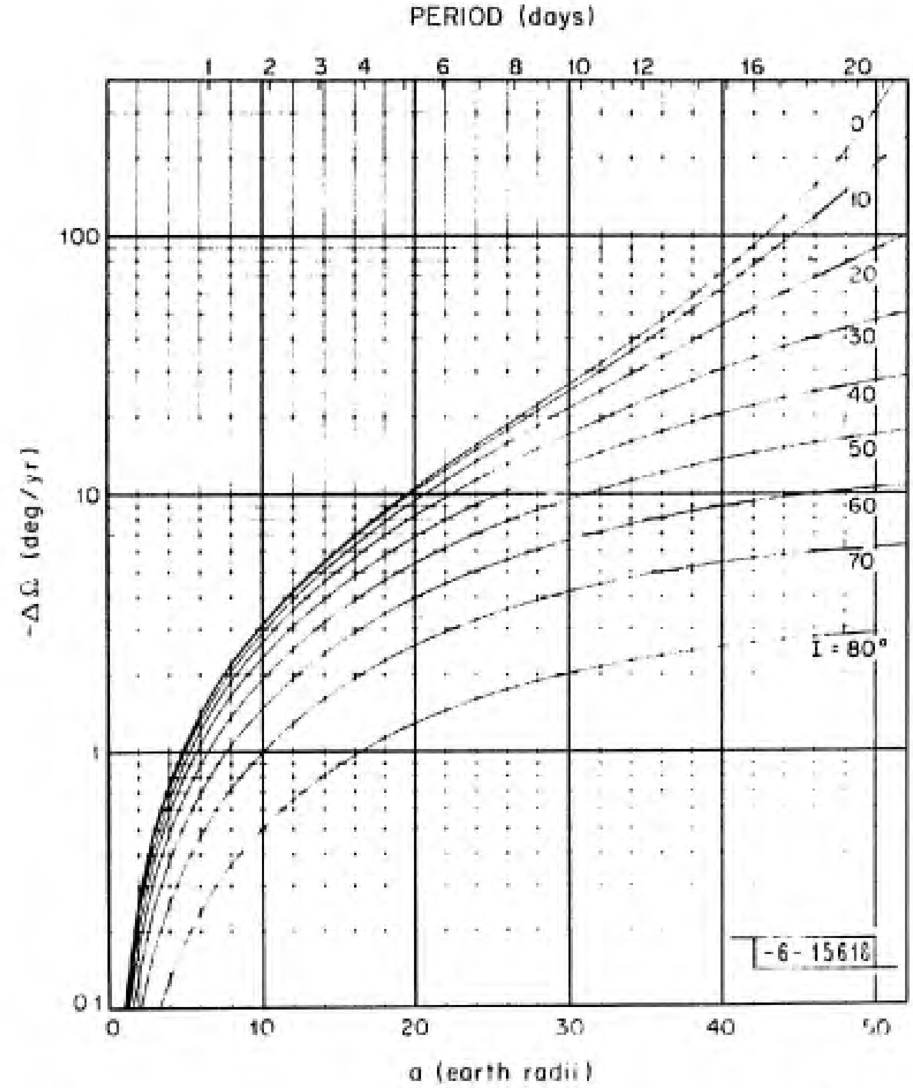

Fig. 4. Lunar secular effect on a satellite's orbital plane (inside lunar orbit).

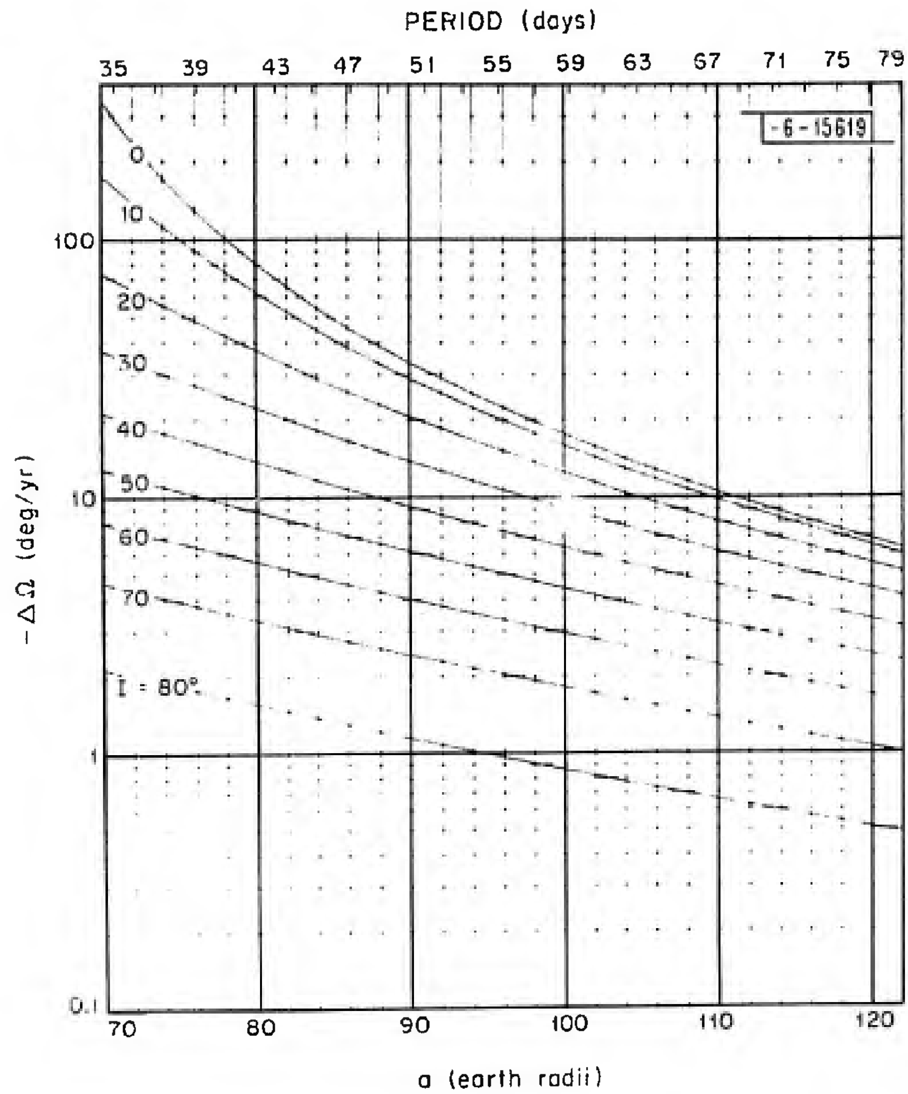

Fig. 5. Lunar secular effect on a satellite's orbital plane (outside lunar orbit).

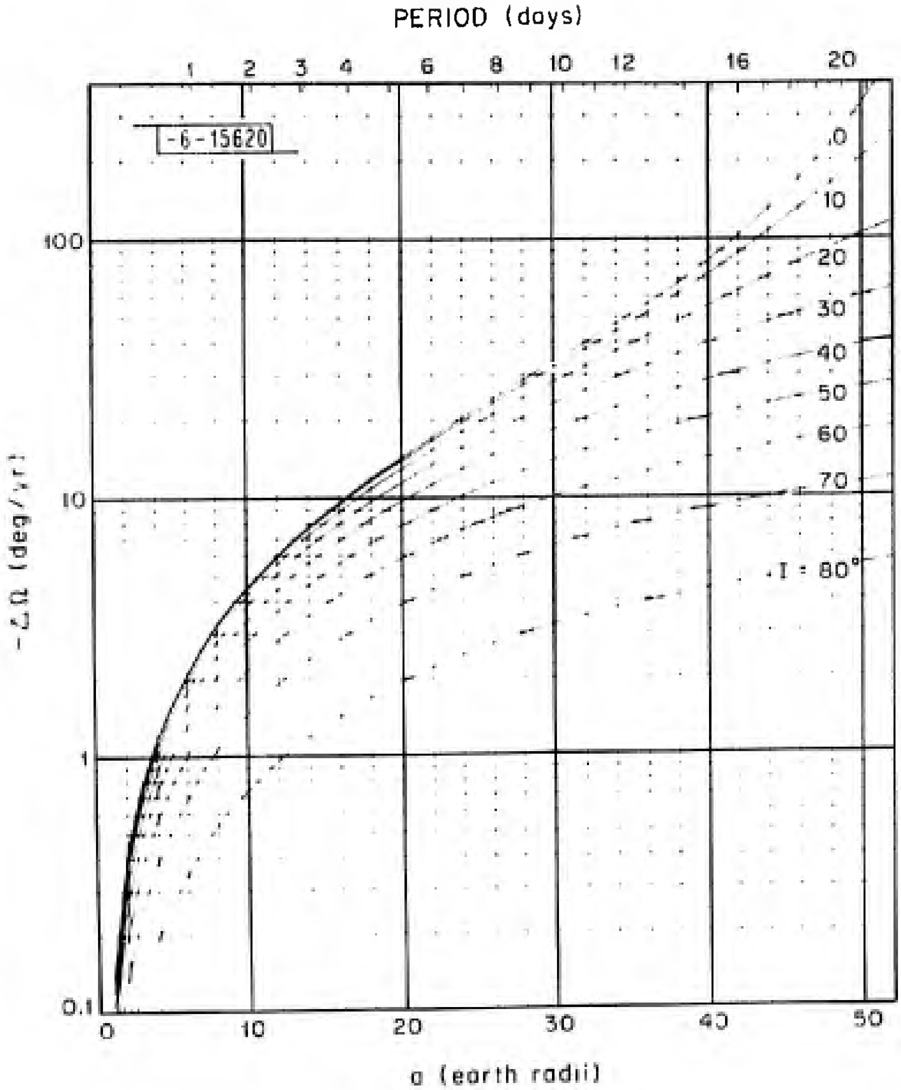

Fig. 6. Combined solar and lunar secular effects on a satellite's orbital plane (inside lunar orbit).

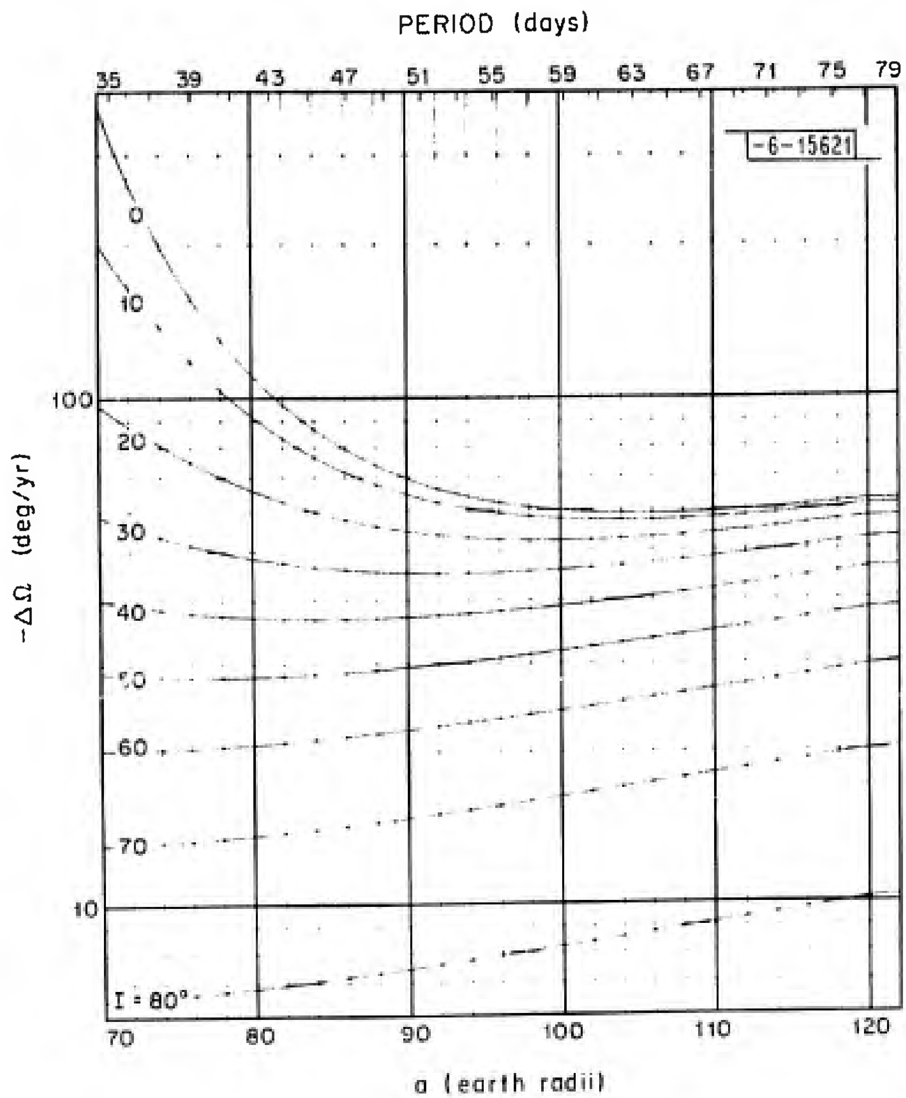

Fig. 7. Combined solar and lunar secular effects on a satellite's orbital plane (outside lunar orbit).

$\Delta\Omega = -22^{\circ}\!.5$/year, whereas the doubly averaged lunar-solar results are $\Delta I = 0$, $\Delta\Omega = -24^{\circ}$/year. Periodic variations in the satellite orbital plane are small (see Appendix A).

XI. Secular and Long Periodic Change of Orbital Shape

The doubly averaged time rate of change of e and ω are

$$\frac{de}{dt} = De \sin 2\omega \tag{118}$$

$$\frac{d\omega}{dt} = E \cos^2 \omega + F \sin^2 \omega \tag{119}$$

where D, E and F are given by (57) and (58) inside the lunar (or solar) orbit and by (78) and (79) outside the lunar orbit, all divided by the orbital period T. The coefficients D, E and F are functions of a/ρ (or ρ/a) and I, and hence are constant as e and ω vary (to first order in eccentricity). As shown later, inside the lunar or solar orbit

$D \geq 0$

$D = 0$ only if $I=0, 180^\circ$

$E > 0$

F is positive or negative, depending on I and a. (120)

The outside lunar orbit case will be handled later. For now we assume conditions (120) are valid.

Let the initial conditions of (118) and (119) be

$$e = e_o, \ \omega = \omega_o \text{ when } t = t_o. \tag{121}$$

If F > 0, integration of (118) and (119) yields[11]

$$e = e_o \left[\frac{E \cos^2 \omega_o + F \sin^2 \omega_o}{E \cos^2 \omega + F \sin^2 \omega}\right]^{\frac{D}{E-F}} \tag{122}$$

$$\tan \omega = \sqrt{\frac{E}{F}} \tan\left[\sqrt{EF} \ (t-t_o) + \arctan\left(\sqrt{\frac{E}{F}} \tan \omega_o\right)\right] \tag{123}$$

As $t \to \infty$, $\omega \to \infty$ and e oscillates. The period of the oscillation is the time for ω to increase by 2π radians:

$$T_e = \frac{2\pi}{\sqrt{EF}} \tag{124}$$

If F = 0, integration of (118) and (119) yields

$$e = e_o \left(\frac{\cos \omega_o}{\cos \omega}\right)^{2D/E} \tag{125}$$

$$\omega = \arctan \left(E(t-t_o) + \tan \omega_o\right) \tag{126}$$

As $t \to \omega$,

$$\omega \to \frac{\pi}{2} \text{ if } -\frac{\pi}{2} < \omega_o \leq \frac{\pi}{2} \tag{127}$$

$$\text{or } \omega \to \frac{3\pi}{2} \text{ if } \frac{\pi}{2} < \omega_o \leq \frac{3\pi}{2} \tag{128}$$

and the doubly averaged value of e gets arbitrarily large.

If F < 0, integration of (119) yields[11]

$$\tan \omega = \sqrt{\left|\frac{E}{F}\right|} \tanh \left[\sqrt{|EF|}\ (t-t_o) + \operatorname{arctanh}\left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_o\right)\right]$$

$$\text{if } \sin^2 \omega_o < \frac{E}{E-F},\ \sin^2 \omega \leq \frac{E}{E-F} \tag{129}$$

$$\tan \omega = \sqrt{\left|\frac{E}{F}\right|} \coth \left[\sqrt{|EF|}\ (t-t_o) + \operatorname{arccoth}\left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_o\right)\right]$$

$$\text{if } \sin^2 \omega_o > \frac{E}{E-F},\ \sin^2 \omega \geq \frac{E}{E-F} \tag{130}$$

$$\omega = \omega_o \quad \text{if } \sin^2 \omega_o = \frac{F}{E-F}\ . \tag{131}$$

Integration of (118) yields[11]

$$e = e_o \left[\frac{E \cos^2 \omega_o + F \sin^2 \omega_o}{E \cos^2 \omega + F \sin^2 \omega} \right]^{\frac{D}{E-F}} \quad \text{if } \sin^2 \omega_o < \frac{E}{E-F}, \quad \sin^2 \omega \le \frac{E}{E-F}, \tag{132}$$

$$e = e_o \left[\frac{E \cos^2 \omega_o + F \sin^2 \omega_o}{E \cos^2 \omega + F \sin^2 \omega} \right]^{\frac{D}{E-F}} \quad \text{if } \sin^2 \omega_o > \frac{E}{E-F}, \quad \sin^2 \omega \ge \frac{E}{E-F} \tag{133}$$

$$e = e_o \exp \left[D(t-t_o) \sin 2\, \omega_o \right] \quad \text{if } \sin^2 \omega_o = \frac{E}{E-F} \tag{134}$$

As depicted in Fig. 8, the equation

$$\sin^2 \omega_i = \frac{E}{E-F} < 1 \tag{135}$$

has 4 solutions modular 2π such that ω_i is in quadrant i and

$$\sin^2 \omega > \frac{E}{E-F} \quad \text{if } \omega_1 < \omega < \omega_2 \quad \text{or} \quad \omega_3 < \omega < \omega_4 \tag{136}$$

$$\sin^2 \omega < \frac{E}{E-F} \quad \text{if } \omega_4 - 2\pi < \omega < \omega_1 \quad \text{or} \quad \omega_2 < \omega < \omega_3 \tag{137}$$

Note that

$$\omega_3 = \omega_1 + \pi, \quad \omega_4 = \omega_2 + \pi. \tag{138}$$

With reference to Fig. 8, a number of different cases must be considered.

(1) $\omega_4 - 2\pi < \omega_o \le \omega_1$. ω moves to ω_1 in time

$$(t-t_o) = \frac{1}{\sqrt{|EF|}} \left[\operatorname{arctanh} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_1 \right) - \operatorname{arctanh} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_o \right) \right] \tag{139}$$

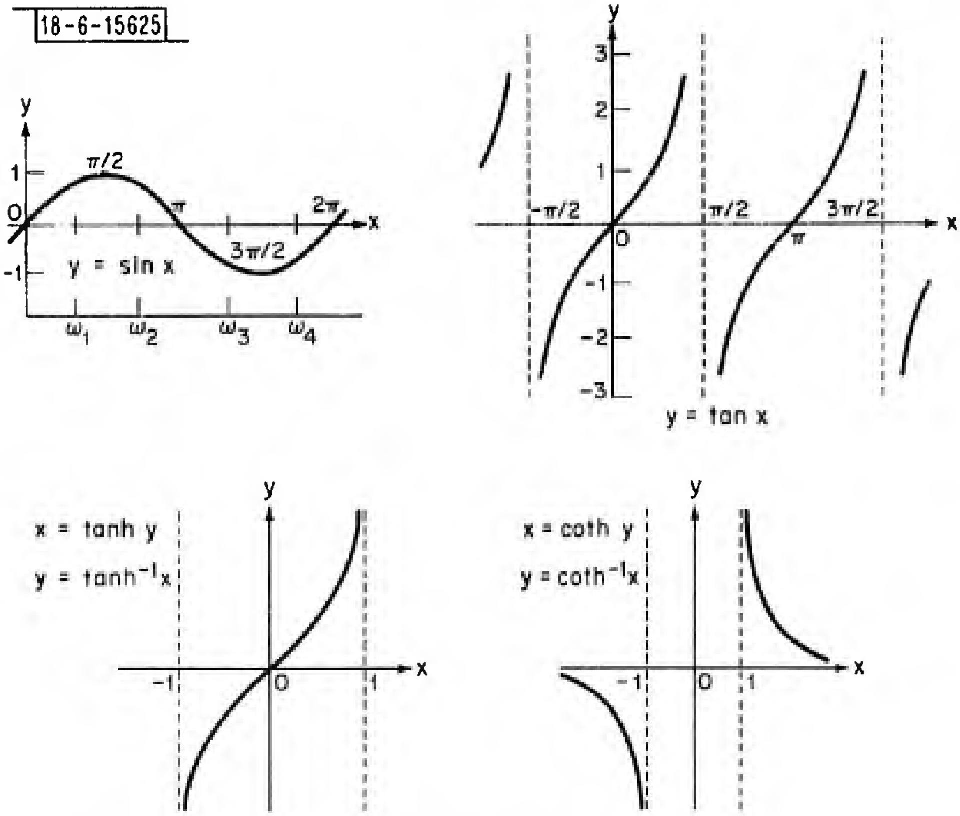

Fig. 8. Trigonometric and hyperbolic functions.

(2) $\omega_1 \leq \omega_o \leq \frac{\pi}{2}$. ω moves to ω_1 in time

$$(t-t_o) = \frac{1}{\sqrt{|EF|}} \left[\text{arccoth} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_1\right) - \text{arccoth} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_o\right) \right] \tag{140}$$

(3) $\frac{\pi}{2} \leq \omega_o < \omega_2$. ω moves to $\frac{\pi}{2}$ in time

$$(t-t_o) = \frac{1}{\sqrt{|EF|}} \left[-1 - \text{arccoth} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_o \right] \tag{141}$$

and then moves from $\pi/2$ to ω_1 in time (140).

(4) $\omega_2 < \omega_o \leq \omega_3$. ω moves to ω_3 in time

$$(t-t_o) = \frac{1}{\sqrt{|EF|}} \left[\text{arctanh} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_3\right) - \text{arctanh} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_o\right) \right] \tag{142}$$

(5) $\omega_3 \leq \omega_o \leq \frac{3\pi}{2}$. ω moves to ω_3 in time

$$(t-t_o) = \frac{1}{\sqrt{|EF|}} \left[\text{arccoth} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_3\right) - \text{arccoth} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_o\right) \right] \tag{143}$$

(6) $\frac{3\pi}{2} \leq \omega_o < \omega_4$. ω moves to $\frac{3\pi}{2}$ in time

$$(t-t_o) = \frac{1}{\sqrt{|EF|}} \left[-1 - \text{arccoth} \left(\sqrt{\left|\frac{F}{E}\right|} \tan \omega_o\right) \right] \tag{144}$$

and then moves from $3\pi/2$ to ω_3 in time (143).

Summary of (1)-(6).

$$\omega_4 - 2\pi < \omega_o < \omega_2 \rightarrow \omega \text{ moves to } \omega_1 \tag{145}$$

$$\omega_2 < \omega_o < \omega_4 \rightarrow \omega \text{ moves to } \omega_3 \tag{146}$$

Since

$$\sin 2\,\omega_1 = \sin 2\,\omega_3 > 0 \tag{147}$$

and D > 0 (I $\neq$ 0, 180^o since F < 0), equation (134) implies that e grows without bound once ω reaches ω_1 or ω_3.

(7) If $\omega_o = \omega_2$ or $\omega_o = \omega_4$, ω stays at ω_2 or ω_4. Since

$$\sin 2\,\omega_2 = \sin 2\,\omega_4 < 0 \tag{148}$$

equation (134) implies that e decreases. However, ω_2 and ω_4 are unstable equilibrium points, so ω is almost sure to move to ω_1 or ω_3, and e will eventually still increase without bound.

Reference 5 goes through a similar analysis inside the perturbing body orbit radius with an e^2 dependence in E and F which leads to more complicated integrals. We have ignored powers of e higher than the first, but do consider higher powers of (a/ρ) than Ref. 5 does.

By (68) and (69) the first two terms for D, E, F inside the lunar (or solar) radius are

$$D \approx \frac{15}{4}\,\frac{\pi}{T}\,\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \sin^2 I \left\{1 + \frac{21}{32}\left(\frac{a}{\rho}\right)^2 \left[6-7\,\sin^2 I\right]\right\} \quad ,\ a < \rho \tag{149}$$

$$E \approx \frac{3\,\pi}{T}\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \left\{1 + \frac{15}{32}\left(\frac{a}{\rho}\right)^2 \left[1-5\,\sin^2 I\right]\right\}, \quad a < \rho \tag{150}$$

$$F \approx \frac{3\,\pi}{T}\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{\rho}\right)^3 \left\{1 - \frac{5}{2}\sin^2 I + \frac{15}{64}\left(\frac{a}{\rho}\right)^2 \cdot \left[8-52\,\sin^2 I + 49\,\sin^4 I\right]\right\}, \quad a < \rho \tag{151}$$

By (78) and (79) the leading terms for D, E, F outside the lunar radius are

$$D \approx \frac{45}{128} \frac{\pi}{T} \left(\frac{\mu_m}{\mu}\right) \left(\frac{\rho}{a}\right)^4 \sin^2 I \,(6-7 \sin^2 I), \quad \rho > a \tag{152}$$

$$E \approx F \approx \frac{3\pi}{4} \left(\frac{\mu_m}{\mu}\right) \left(\frac{\rho}{a}\right)^2 \left[5 \cos^2 I - 1\right], \quad \rho > a \tag{153}$$

$\sin^2 I$ is a factor in all the terms of the expansions (57) and (78) for D.

Outside the lunar orbit, the motion of ω caused by a lunar ring of matter is exactly analogous, to lowest order, to the well known effects of the earth's equatorial bulge (i.e., the effect of the second zonal harmonic of the gravitational potential). Namely, there is a secular progression of the satellite perigee if $0 \leq I \lesssim 63.^\circ4$ or $116.^\circ6 \lesssim I \leq 180^\circ$, and a retrogression if $63.^\circ4 \lesssim I \lesssim 116.^\circ6$. The secular change in e is of higher order, and it is also long periodic away from the critical inclination $I \approx 63.^\circ4$ or $116.^\circ6$. However, the change in eccentricity of a satellite orbit outside the lunar radius is mainly due to the effect of the sun.

Inside the lunar (or solar) radius, F changes sign when

$$\sin^2 I \approx \frac{2}{5} - \frac{93}{200} \left(\frac{a}{\rho}\right)^2 \tag{154}$$

which has solutions

$$I_a \approx 39.^\circ2 - 27.^\circ2 \left(\frac{a}{\rho}\right)^2 \tag{155}$$

$$I_b \approx 120.^\circ8 + 27.^\circ2 \left(\frac{a}{\rho}\right)^2 \tag{156}$$

For $0 \leq I \lesssim I_a$ or $I_b \lesssim I \leq 180^\circ$, ω goes through complete revolutions and e will oscillate with (essentially) no secular buildup. If $I_a \lesssim I \lesssim I_b$, ω will remain in the first or third quadrant if it is initially there. If ω is initially in the second or fourth quadrant, it will move to the first or third quadrant. Thus, if ω is initially in the second or fourth quadrant, e will decrease, but eventually e will increase without bound if $I_a \lesssim I \lesssim I_b$ and it has an initial non-zero average value. Short periodic variations (see Appendix A) can move e away from zero enough to commence secular growth.

At low altitudes the transition from long periodic to secular growth in e occurs at $I = \pm 39^\circ\!.2$ ($120^\circ\!.8 \equiv -39^\circ\!.2$ inclination). This transition occurs at $I = \pm 36^\circ\!.2$ at 1/3 the lunar radius and at $I = \pm 27^\circ\!.1$ at 2/3 the lunar radius. This latter value can have further small corrections from the still higher order terms left out of (151). In additon, the 5° inclination of the lunar plane to the ecliptic and its motion on the ecliptic causes a $\pm 5^\circ$ ambiguity in the value of the critical inclination (155) and (156) to the ecliptic when considering combined lunar and solar effects. The boundaries for the sun effect do not change appreciably from $\pm 39^\circ\!.2$ because of the great distance of the sun from the earth compared to the distance at which a satellite gets into the solar sphere of influence. The results for the sun outside the lunar radius are less reliable because of the longer averaging times.

All of the manipulations so far is this report have assumed that $e^2 \approx 0$. Thus the prediction of continuous secular growth in e between the limits (155) and (156) could break down when e gets appreciable. However, numerical integration of a number of cases has shown no sign of a stop in eccentricity growth. If e does grow with the average value of a staying constant, perigee eventually dips into the earth's atmosphere. Alternately, a close lunar approach could throw the satellite out of the earth-moon system (see Appendix C).

A third possibility is that perigee is inside and apogee outside the lunar radius, which situation is not included in the formulas developed in this report. In the sun-Neptune-Pluto three-body problem, Pluto's perihelion is inside and aphelion outside the Neptune orbit radius, but there is never a close Neptune-Pluto approach, and the configuration remains stable for millions of years of numerical integration.[12] In the analogous earth-moon satellite configuration, stability might be destroyed by solar perturbations. The five equilibrium points of the restricted three-body problem are other possibilities for high altitude orbits.[13]

The coefficient D is plotted in Figs. 9 through 11 for the perturbations due to the sun alone and moon alone inside and outside the lunar radius. These plots can be regarded as giving the relative size of the perturbing effects on e at various radii and inclinations.

The series expansions for D are zero when $I = 0$. The magnitude of D is largest for high inclinations, except for a curious reversal of roles for the lunar effect between about 40 and 80 earth radii. The infinite series were evaluated in the factored form (63) using 32 decimal place computations. Up to 40 terms were used in the series. The last ℓ-term used in the series was 10^{-4} times the size of the leading term, except in some cases greater than 50 earth radii where it was no larger than 10^{-3} times the size of the leading term. No more than 40 terms could be used in the expansion, because 32 decimal places were not enough to add together the alternating sign terms beyond that order. The behavior in Figs. 10 and 11 between 40 and 80 earth radii could be due to an error in the formulas or in the computer program that evaluated them, except that very thorough checks did not find any errors. Another possibility is that the series for Δe have a smaller radius of convergence than those for $\Delta\Omega$. Alternately, the doubly averaged model might actually behave in the manner depicted in Figs. 10 and 11 near the moon.

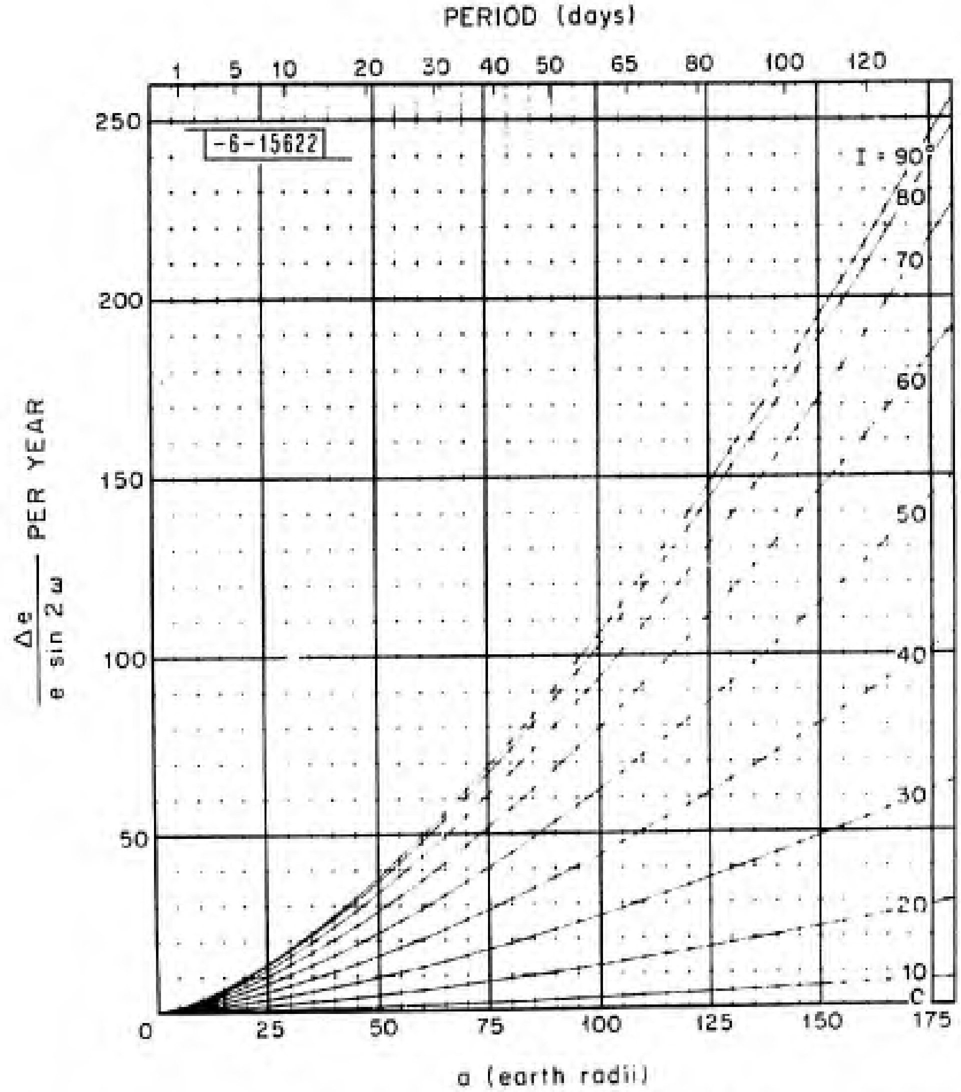

Fig. 9. Solar secular effect on a satellite's eccentricity.

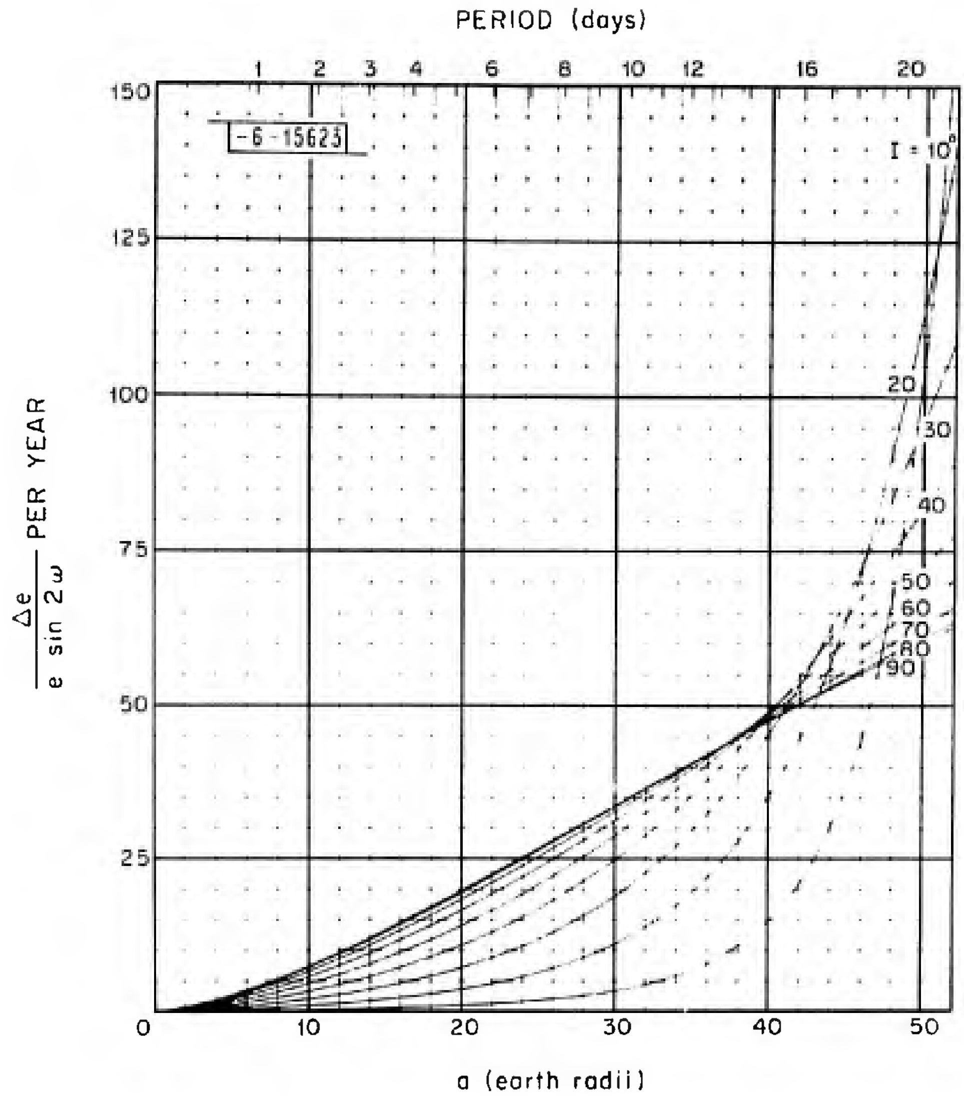

Fig. 10. Lunar secular effect on a satellite's eccentricity (inside lunar orbit).

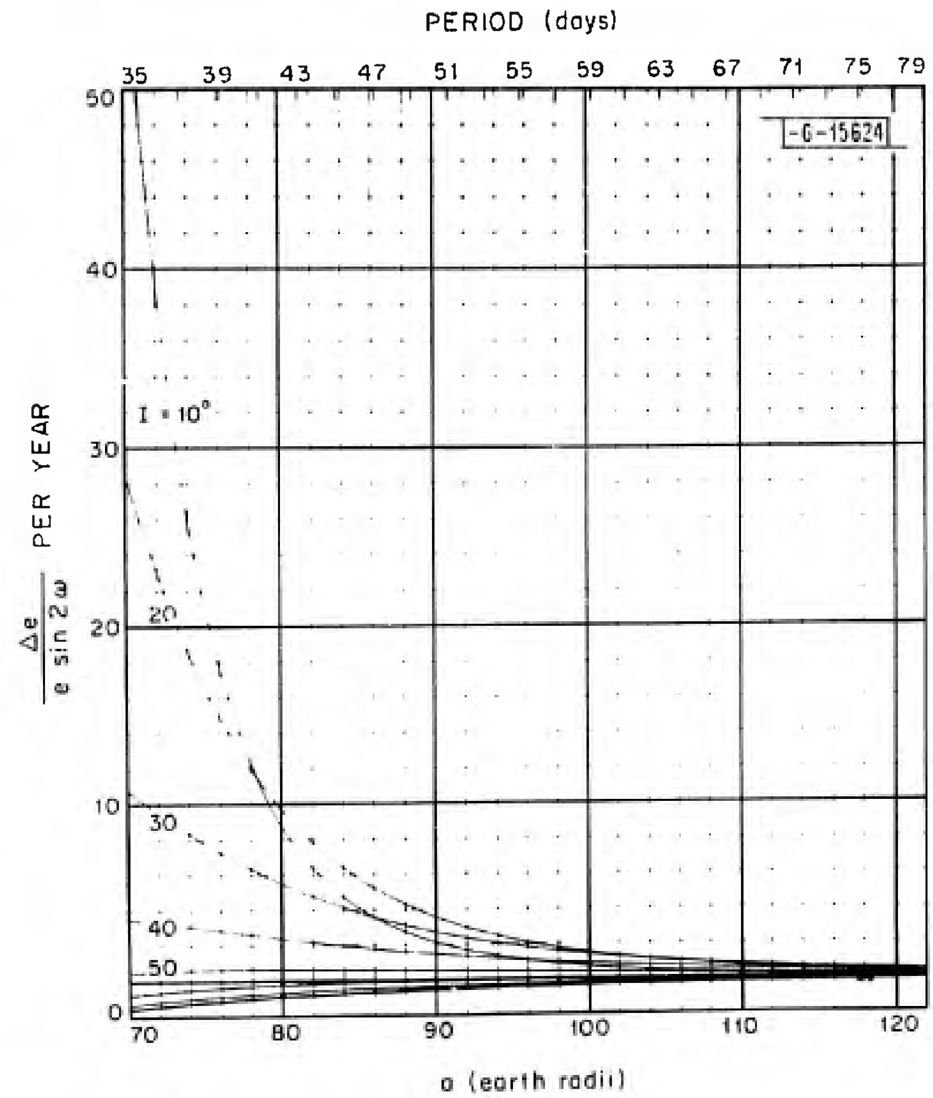

Fig. 11. Lunar secular effect on a satellite's eccentricity (outside lunar orbit).

APPENDIX A

Low Order Short Period Lunar and Solar Perturbations

As a complement to the secular perturbations discussed in the earlier part of this note, we now discuss the periodic perturbations in a satellite's orbital elements. Our purpose is to get a feel for these periodic perturbations, so we assume $e \approx 0$ and only make expansions to the lowest possible order. To do otherwise quickly leads to very complicated expressions.[14,15] We also restrict the discussions to inside the lunar radius when the manipulations begin to get intricate.

The derivation of short period perturbations cannot use the technique of orbit averaging. Rather, we start with the exact equations for the osculating elements[16]

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial \bar{R}}{\partial M}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial \bar{R}}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \bar{R}}{\partial \omega}$$

$$\frac{dI}{dt} = \frac{\cos I}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial \bar{R}}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial \bar{R}}{\partial \Omega}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial \bar{R}}{\partial I}$$

$$\frac{d\omega}{dt} = -\frac{\cos I}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial\overline{R}}{\partial I} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial\overline{R}}{\partial e}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e}\frac{\partial\overline{R}}{\partial e} - \frac{2}{na}\frac{\partial\overline{R}}{\partial a} \qquad \text{(A-1)}$$

where we have utilized the partial derivatives of the disturbing function $\overline{R}$ instead of the radial, tangential and normal components of perturbing acceleration as in (24). If we define

x^1, x^2, x^3 = coordinates of satellite relative to earth

x_m^1, x_m^2, x_m^3 = coordinates of moon relative to earth

$$r = \left[\sum_{j=1}^{3} (x^j)^2\right]^{\frac{1}{2}}$$

$$r_m = \left[\sum_{j=1}^{3} (x_m^j)^2\right]^{\frac{1}{2}}$$

$$\Delta = \left[\sum_{j=1}^{3} (x^j - x_m^j)^2\right]^{\frac{1}{2}}$$

μ_m = gravitational constant times mass of moon

the disturbing function for the effect of the moon is[17]

$$\overline{R} = \mu_m\left(\frac{1}{\Delta} - \frac{r \cos S}{r_m^2}\right) \tag{A-2}$$

where S is the angle between the vectors pointing from the earth to the satellite and moon:

$$\cos S = \frac{1}{r\, r_m} \sum_{j=1}^{3} x^j\, x_m^j \tag{A-3}$$

The next step is to express the disturbing function in terms of the orbital elements of the satellite and the moon, evaluate the right sides of (A-1), and integrate keeping constant all the orbital elements of the satellite except the mean anomaly. To carry the perturbation analysis to second order, we would insert the just derived time variations in the elements into the right sides of (A-1) and integrate again. Secular variations in Ω and ω can be included in the first-order calculations to make it less necessary to carry the analysis to second order.

Inside the lunar radius we have

$$\frac{1}{\Delta} = \frac{1}{\left[r + r_m^2 - 2r\, r_m \cos S\right]^{1/2}}$$

$$= \frac{1}{r_m\left[1-2\left(\frac{r}{r_m}\right) \cos S + \left(\frac{r}{r_m}\right)^2\right]^{1/2}}$$

$$= \frac{1}{r_m} \sum_{j=0}^{\infty} P_j(\cos S)\left(\frac{r}{r_m}\right)^j, \quad r < r_m \tag{A-4}$$

where P_j is the jth Legendre polynomical. Outside the lunar radius we have

$$\frac{1}{\Delta} = \frac{1}{r} \sum_{j=0}^{\infty} P_j(\cos S) \left(\frac{r_m}{r}\right)^j, \quad r > r_m \tag{A-5}$$

Since[1]

$$P_0(\cos S) = 1, \; P_1(\cos S) = \cos S,$$

$$P_2(\cos S) = \frac{3}{2}\cos^2 S - \frac{1}{2} \tag{A-6}$$

we obtain

$$\overline{R} \approx \frac{\mu_m}{r_m}\left[1 + \left(\frac{3}{2}\cos^2 S - \frac{1}{2}\right)\left(\frac{r}{r_m}\right)^2\right], r < r_m \tag{A-7}$$

$$\overline{R} \approx \frac{\mu_m}{r_m}\left[\left(\frac{r_m}{r}\right) + \cos S\left(\frac{r_m}{r}\right)^2 + \left(\frac{3}{2}\cos^2 S - \frac{1}{2}\right)\left(\frac{r_m}{r}\right)^3 - \cos S\left(\frac{r}{r_m}\right)\right], \quad r > r_m \tag{A-8}$$

Let

I = inclination of the satellite orbital plane on the (x^1, x^2) lunar orbital plane

Ω = longitude of the ascending node of the satellite orbital plane on the lunar plane (not necessarily zero as assumed in deriving the lunar ring of matter effects)

$\eta = \omega + \psi$ = angle measured from the ascending node of the satellite on the lunar plane to the true position of the satellite.

Then standard Euler angle formulas imply

$$x^1 = a\,(\cos\Omega\ \cos\eta - \sin\Omega\ \sin\eta\ \cos I)$$

$$x^2 = a(\sin\Omega\ \cos\eta + \cos\Omega\ \sin\eta\ \cos I)$$

$$x^3 = a\ \sin\eta\ \sin I$$

$$r = a \tag{A-9}$$

If the lunar coordinates are given by

$$x_m^1 = a'\ \cos\eta'$$

$$x_m^2 = a'\ \sin\eta'$$

$$x_m^3 = 0$$

$$r_m = a' \tag{A-10}$$

then by (A-3)

$$\cos S = \cos(\Omega+\eta-\eta') + 2\sin(\Omega-\eta')\sin\eta \ \sin^2\frac{I}{2}$$

$$= \cos^2\frac{I}{2}\cos(\Omega+\eta-\eta') + \sin^2\frac{I}{2}\cos(\Omega-\eta-\eta') \qquad \text{(A-11)}$$

Here and in the following we make use of

$$2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta) \qquad \text{(A-12)}$$

Since

$$\cos^2 S = (\frac{1}{2} - \frac{1}{4}\sin^2 I) + \frac{1}{4}\sin^2 I\left[\cos 2\eta + \cos(2\Omega - 2\eta')\right]$$

$$+ \frac{1}{2}\cos^4\frac{I}{2}\cos(2\Omega+2\eta-2\eta')$$

$$+ \frac{1}{2}\sin^4\frac{I}{2}\cos(2\Omega-2\eta-2\eta') \qquad \text{(A-13)}$$

equations (A-7) and (A-8) can be written as

$$\overline{R} \approx \frac{\mu_m}{r_m} \{1 + \{(\frac{1}{4} - \frac{3}{8}\sin^2 I) + \frac{3}{8}\sin^2 I \left[\cos 2\eta\right.$$

$$\left. + \cos(2\Omega - 2\eta')\right] + \frac{3}{4}\cos^4\frac{I}{2}\cos(2\Omega + 2\eta - 2\eta')$$

$$+ \frac{3}{4}\sin^4\frac{I}{2}\cos(2\Omega - 2\eta - 2\eta')\}(\frac{r}{r_m})^2\}$$

$$r < r_m \qquad \text{(A-14)}$$

$$\overline{R} \approx \frac{\mu_m}{r_m} \{\frac{r_m}{r} + \left[\cos^2\frac{I}{2}\cos(\Omega + \eta - \eta')\right.$$

$$\left. + \sin^2\frac{I}{2}\cos(\Omega - \eta - \eta')\right]\left[(\frac{r_m}{r})^2 - \frac{r}{r_m}\right]$$

$$+ \{(\frac{1}{4} - \frac{3}{8}\sin^2 I) + \frac{3}{8}\sin^2 I\left[\cos^2\eta\right.$$

$$\left. + \cos(2\Omega - 2\eta')\right] + \frac{3}{4}\cos^4\frac{I}{2}\cos(2\Omega + 2\eta - 2\eta')$$

$$+ \frac{3}{4}\sin^4\frac{I}{2}\cos(2\Omega - 2\eta - 2\eta')\}(\frac{r_m}{r})^3\}$$

$$r > r_m \qquad \text{(A-15)}$$

The relation between the tree anomaly ψ, eccentric anomaly ξ, mean anomaly M and time t are

$$\tan\frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\xi}{2}$$

$$M = \xi - e\sin\xi$$

$$M = n(t - t_0) + M_0 \qquad \text{(A-16)}$$

where M_0 is the mean anomaly at the initial epoch t_0. The radius in an elliptic orbit is given by

$$r = a(1-e\cos\xi) = \frac{a(1-e^2)}{1+e\cos\psi} \tag{A-17}$$

The partial derivatives of r and $\eta=\omega+\psi$ are to first order in e[18]

$$\frac{\partial r}{\partial a} = \frac{r}{a} \approx 1 - e\cos(\eta-\omega)$$

$$\frac{\partial r}{\partial e} = -a\cos\xi + \frac{a^2 e\sin^2\xi}{r} \approx -a\cos(\eta-\omega)$$

$$\frac{\partial r}{\partial I} = 0,\ \frac{\partial r}{\partial \Omega} = 0,\ \frac{\partial r}{\partial \omega} = 0$$

$$\frac{\partial r}{\partial M} = \frac{a^2 e\sin\xi}{r} \approx a\, e\sin(\eta-\omega) \tag{A-18}$$

$$\frac{\partial \eta}{\partial a} = 0$$

$$\frac{\partial \eta}{\partial e} = \left[\frac{a}{r} + \frac{1}{1-e^2}\right]\sin\psi \approx \left[2+e\cos(\eta-\omega)\right]\sin(\eta-\omega)$$

$$\frac{\partial \eta}{\partial I} = 0,\ \frac{\partial \eta}{\partial \Omega} = 0,\ \frac{\partial \eta}{\partial \omega} = 1,$$

$$\frac{\partial \eta}{\partial M} = \sqrt{1-e^2}\left(\frac{a}{r}\right)^2 \approx 1 + 2e\cos(\eta-\omega) \tag{A-19}$$

The expressions for $\partial r/\partial a$, $\partial \eta/\partial a$ would be more complicated if we used M_o instead of M as an orbital element.[18]

By (A-1), (A-14), (A-18) and (A-19) we have inside the lunar radius to zeroth order in e

$$\frac{da}{dt} \approx \frac{3a\mu_m}{nr_m^3} \{-\frac{1}{2}\sin^2 I \sin 2\eta - \cos^4\frac{I}{2}\sin(2\Omega+2\eta-2\eta')$$

$$+ \sin^4\frac{I}{2}\sin(2\Omega-2\eta-2\eta')\} \qquad \text{(A-20)}$$

$$\frac{de}{dt} \approx \frac{\mu_m}{2nr_m^3} \{(1-\frac{3}{2}\sin^2 I)\sin(\eta-\omega)$$

$$-\frac{9}{4}\sin^2 I \sin(\eta+\omega) - \frac{3}{4}\sin^2 I \sin(3\eta-\omega)$$

$$-\frac{3}{4}\sin^2 I \sin(2\Omega-\eta+\omega-2\eta')$$

$$+\frac{3}{4}\sin^2 I \sin(2\Omega+\eta-\omega-2\eta')$$

$$-\frac{9}{2}\cos^4\frac{I}{2}\sin(2\Omega+\eta+\omega-2\eta')$$

$$-\frac{3}{2}\cos^4\frac{I}{2}\sin(2\Omega+3\eta-\omega-2\eta')$$

$$+\frac{3}{2}\sin^4\frac{I}{2}\sin(2\Omega-3\eta+\omega-2\eta')$$

$$+\frac{9}{2}\sin^4\frac{I}{2}\sin(2\Omega-\eta-\omega-2\eta')\} \qquad \text{(A-21)}$$

$$\frac{dI}{dt} \approx \frac{3\mu_m \sin I}{4n\ r_m^3} \{- \cos I \sin 2\eta + \sin(2\Omega-2\eta')$$

$$+ \cos^2 \frac{I}{2} \sin(2\Omega+2\eta-2\eta')$$

$$+ \sin^2 \frac{I}{2} \sin(2\Omega-2\eta-2\eta')\} \qquad \text{(A-22)}$$

$$\frac{d\Omega}{dt} \approx \frac{3\mu_m}{4nr_m^3 e} \{\cos I \left[-1 + \cos 2\eta + \cos(2\Omega-2\eta')\right]$$

$$- \cos^2 \frac{I}{2} \cos(2\Omega+2\eta-2\eta')$$

$$+ \sin^2 \frac{I}{2} \cos(2\Omega-2\eta-2\eta')\} \qquad \text{(A-23)}$$

$$\frac{d\omega}{dt} \approx \frac{\mu_m}{2nr_m^3 e} \{(-1 + \frac{3}{2} \sin^2 I) \cos(\eta-\omega)$$

$$- \frac{9}{4} \sin^2 I \cos(\eta+\omega) + \frac{3}{4} \sin^2 I \cos(3\eta-\omega)$$

$$- \frac{3}{4} \sin^2 I \cos(2\Omega-\eta+\omega-2\eta')$$

$$- \frac{3}{4} \sin^2 I \cos(2\Omega+\eta-\omega-2\eta')$$

$$- \frac{9}{2} \cos^4 \frac{I}{2} \cos(2\Omega+\eta+\omega-2\eta')$$

$$+ \frac{3}{2} \cos^4 \frac{I}{2} \cos(2\Omega+3\eta-\omega-2\eta')$$

$$+ \frac{3}{2} \sin^4 \frac{I}{2} \cos(2\Omega-3\eta+\omega-2\eta')$$

$$- \frac{9}{2} \sin^4 \frac{I}{2} \cos(2\Omega-\eta-\omega-2\eta')\} - \cos I \frac{d\Omega}{dt} \qquad \text{(A-24)}$$

$$\frac{d(M+\omega)}{dt} \approx n - \frac{\mu_m}{n r_m^3} \{(1 - \frac{3}{2} \sin^2 I) + \frac{3}{2} \sin^2 I \left[\cos 2\eta \right.$$

$$\left. + \cos (2\Omega - 2\eta')\right] + 3 \cos^4 \frac{I}{2} \cos (2\Omega + 2\eta - 2\eta')$$

$$+ 3 \sin^4 \frac{I}{2} \cos (2\Omega - 2\eta - 2\eta')\}$$

$$- \cos I \frac{d\Omega}{dt} \tag{A-25}$$

We consider the equation for $(M+\omega)$ rather than M in order to eliminate the singularity when e=0. This singularity is inherent in the equation for ω, but it nicely dropped out of the equation for e. The singularity for I=0 in (A-1) no longer exists in (A-22) and (A-23).

We now assume that for $e \approx 0$

$$\eta = M + \omega = n(t-t_o) + \eta_o$$

$$\eta' = n'(t-t_o) \tag{A-26}$$

where

$$n = \mu^{1/2} a^{-3/2}$$

$$n' = \mu^{1/2} (a')^{-3/2} \quad . \tag{A-27}$$

We also assume that

$$\Omega = \Omega_o + \dot{\Omega}(t-t_o)$$

$$\omega = \omega_o + \dot{\omega}(t-t_o) \tag{A-28}$$

Let $\delta\beta$ be the change in orbital element β from its value at time t_o. Integrating equations (A-20) to (A-25) we obtain

$$\frac{\delta a}{a} \approx \left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{a'}\right)^3 \Bigg\{ \frac{3}{4}\sin^2 I\cos(2\eta)$$

$$+ \frac{3n}{2(\dot{\Omega}+n-n')}\cos^4\frac{I}{2}\cos(2\Omega+2\eta-2\eta')$$

$$- \frac{3n}{2(\dot{\Omega}-n-n')}\sin^4\frac{I}{2}\cos(2\Omega-2\eta-2\eta')\Bigg\} \qquad \text{(A-29)}$$

$$\delta e \approx \left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{a'}\right)^3 \Bigg\{ -\frac{n}{2(n-\dot{\omega})}\left(1-\frac{3}{2}\sin^2 I\right)\cos(\eta-\omega)$$

$$+ \frac{9n}{8(n+\dot{\omega})}\sin^2 I\cos(\eta+\omega)$$

$$+ \frac{3n}{8(3n-\dot{\omega})}\sin^2 I\cos(3\eta-\omega)$$

$$+ \frac{3n}{8(2\dot{\Omega}-n+\dot{\omega}-2n')}\sin^2 I\cos(2\Omega-\eta+\omega-2\eta')$$

$$- \frac{3n}{8(2\dot{\Omega}+n-\dot{\omega}-2n')}\sin^2 I\sin(2\Omega+\eta-\omega-2\eta')$$

$$+ \frac{9n}{4(2\dot{\Omega}+n+\dot{\omega}-2n')}\cos^4\frac{I}{2}\cos(2\Omega+\eta+\omega-2\eta')$$

$$+ \frac{3n}{4(2\dot{\Omega}+3n-\dot{\omega}-2n')}\cos^4\frac{I}{2}\cos(2\Omega+3\eta-\omega-2\eta')$$

$$- \frac{3n}{4(2\dot{\Omega}-3n+\dot{\omega}-2n')}\sin^4\frac{I}{2}\cos(2\Omega-3\eta+\omega-2\eta')$$

$$- \frac{9n}{4(2\dot{\Omega}-n-\dot{\omega}-2n')}\sin^4\frac{I}{2}\cos(2\Omega-\eta-\omega-2\eta')\Bigg\}$$

(A-30)

$$\delta I \approx \left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{a'}\right)^3 \sin I \left\{ \frac{3}{8} \cos I \cos 2\eta - \frac{3n}{8(\dot{\Omega}-n')} \cos(2\Omega-2\eta') \right.$$

$$- \frac{3n}{8(\dot{\Omega}+n-n')} \cos^2 \frac{I}{2} \cos(2\Omega+2\eta-2\eta')$$

$$\left. - \frac{3n}{8(\dot{\Omega}-n-n')} \sin^2 \frac{I}{2} \cos(2\Omega-2\eta-2\eta') \right\} \qquad \text{(A-31)}$$

$$\delta\Omega \approx \left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{a'}\right)^3 \left\{ -\frac{3}{4} \cos I \; n(t-t_o) + \frac{3}{8} \cos I \sin 2\eta \right.$$

$$+ \frac{3n}{8(\dot{\Omega}-n')} \cos I \sin(2\Omega-2\eta')$$

$$- \frac{3n}{8(\dot{\Omega}+n-n')} \cos^2 \frac{I}{2} \sin(2\Omega+2\eta-2\eta')$$

$$\left. + \frac{3n}{8(\dot{\Omega}-n-n')} \sin^2 \frac{I}{2} \sin(2\Omega-2\eta-2\eta') \right\} \qquad \text{(A-32)}$$

$$e\delta\omega \approx \left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{a'}\right)^3 \left\{ -\frac{n}{2(n-\dot{\omega})} \left(1-\frac{3}{2} \sin^2 I\right) \sin(\eta-\omega) \right.$$

$$- \frac{9n}{8(n+\dot{\omega})} \sin^2 I \sin(\eta+\omega)$$

$$+ \frac{3n}{8(3n-\dot{\omega})} \sin^2 I \sin(3\eta-\omega)$$

$$- \frac{3n}{8(2\dot{\Omega}-n+\dot{\omega}-2n')} \sin^2 I \sin(2\Omega-\eta+\omega-2\eta')$$

$$- \frac{3n}{8(2\dot{\Omega}+n-\dot{\omega}-2n')} \sin^2 I \ \sin(2\Omega+\eta-\omega-2\eta')$$

$$- \frac{9n}{4(2\dot{\Omega}+n+\dot{\omega}-2n')} \cos^4 \frac{I}{2} \ \sin(2\Omega+\eta+\omega-2\eta')$$

$$+ \frac{3n}{4(2\dot{\Omega}+3n-\dot{\omega}-2n')} \cos^4 \frac{I}{2} \ \sin(2\Omega+3\eta-\omega-2\eta')$$

$$+ \frac{3n}{4(2\dot{\Omega}-3n+\dot{\omega}-2n')} \sin^4 \frac{I}{2} \ \sin(2\Omega-3\eta+\omega-2\eta')$$

$$- \frac{9n}{4(2\dot{\Omega}-n-\dot{\omega}-2n')} \sin^4 \frac{I}{2} \ \sin(2\Omega-\eta-\omega-2\eta')\Big\}$$

$$- e \cos I \delta\Omega \qquad \text{(A-33)}$$

$$\delta\tilde{\eta} \equiv \delta(M+\omega+\Omega \cos I - n(t-t_o))$$

$$\approx -\left(\frac{\mu_m}{\mu}\right)\left(\frac{a}{a'}\right)^3 \Big\{(1-\frac{3}{2}\sin^2 I)\ n(t-t_o)$$

$$+ \frac{3}{4} \sin^2 I \ \sin 2\eta$$

$$+ \frac{3n}{4(\dot{\Omega}-n')} \sin^2 I \ \sin(2\Omega-2\eta')$$

$$+ \frac{3n}{2(\dot{\Omega}+n-n')} \cos^4 \frac{I}{2} \sin(2\Omega+2\eta-2\eta')$$

$$+ \frac{3n}{2(\dot{\Omega}-n-n')} \sin^4 \frac{I}{2} \ \sin(2\Omega-2\eta-2\eta')\Big\} \qquad \text{(A-34)}$$

The first secular term in (67) appears in (A-32). In order to get the first secular or long periodic terms in (68) and (69) we would have to carry the above development of e and ω to the next order in e. The secular term in (A-34) arises because lunar perturbations alter Kepler's third law relating mean semi-major axis and the orbital period (see Appendix B). Expressions for solar perturbations are obtained if we change μ_m, r_m to μ_s, r_s.

For a given satellite, some terms in (A-29) to (A-34) are important and others are insignificant. If the orbital elements are altered, important terms can become small and other terms pop-up to take their place.

A resonant situation occurs (perhaps in a higher order expansion than we used) when

$$i\, n + i'\, n' + j\, \dot{\Omega} + k\, \dot{\omega} \approx 0 \tag{A-35}$$

where i, i', j, k are positive or negative integers. Small values of i, i', j, k for which (A-35) is true can yield important effects, such as the well known long period inequality in the motions of Jupiter and Saturn.

The orbital period versus radius is plotted in Fig. 12 with possible lunar resonances being indicated. Any orbital period can be in resonance with the moon for some values of i and i', but we need only be concerned with small values of i and i'.

Also plotted in Fig. 12 are the direct and retrograde synodic periods. Let

T = satellite orbital period

T' = lunar orbital period

T_s = synodic period (time between close lunar approaches).

Then for a low inclination direct satellite

$$\frac{1}{T_s} = \frac{1}{T} - \frac{1}{T'} \tag{A-36}$$

and for a low inclination retrograde satellite

$$\frac{1}{T_s} = \frac{1}{T} + \frac{1}{T'} \tag{A-37}$$

The lunar perturbations on high altitude satellite are much less for retrograde orbits than for direct orbits, since a satellite passes through close lunar approach much more slowly in a direct orbit than in a retrograde orbit (see Appendix C).

Close lunar approaches can wreck havoc on analytical models. For highly inclined satellites, it might be possible to avoid such close approaches. But then we would have the secular buildup in eccentricity as discussed in Section XI, except perhaps for Neptune-Pluto type resonances.[12]

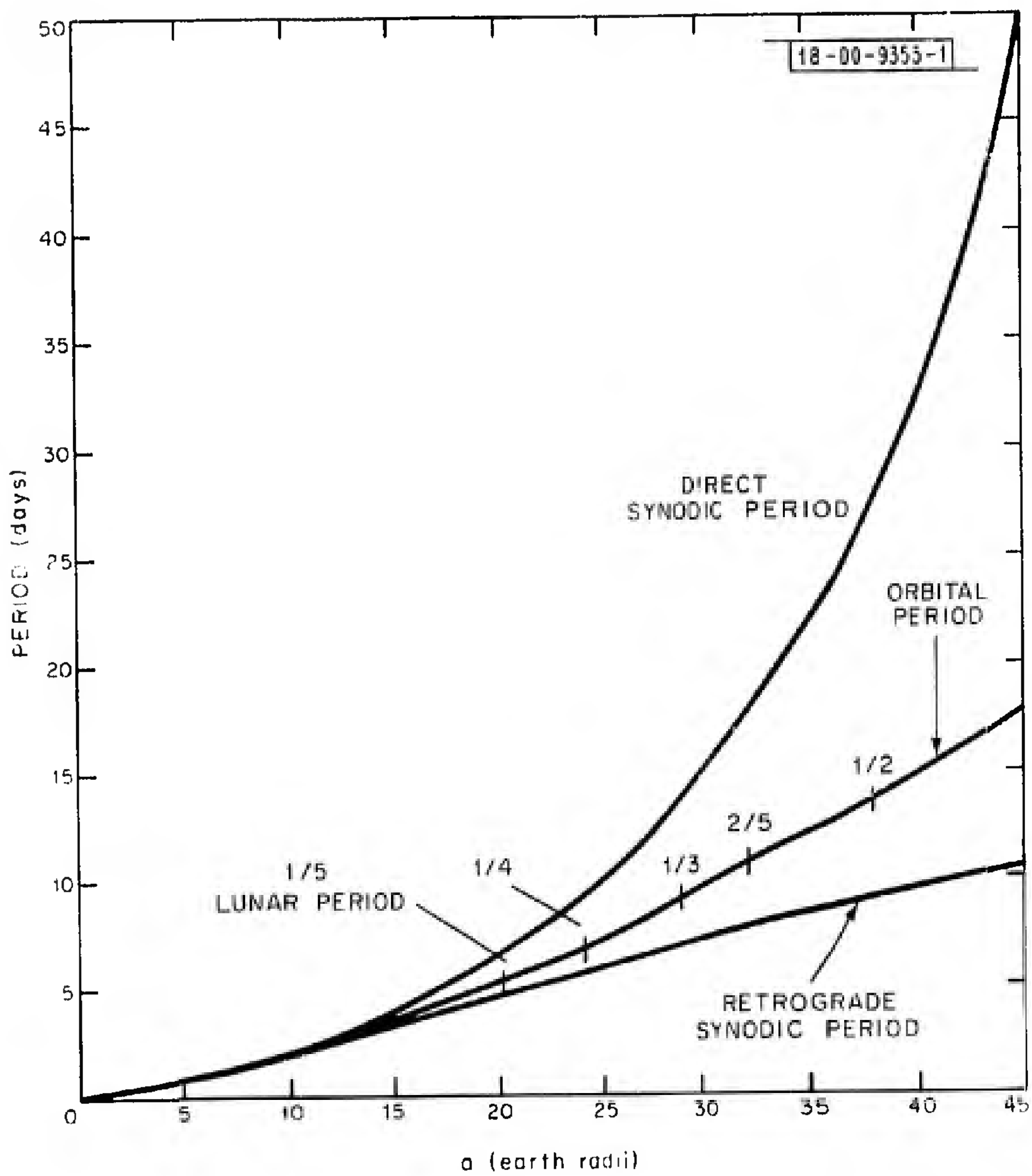

Fig. 12. Orbital and synodic periods.

APPENDIX B

Libraations about the Mean Position in Orbit

As a satellite orbits the earth it librates in its orbital plane about the mean position it would have if it traveled uniformly in its orbit with its average mean motion. This libration is due to (i) the eccentricity e of the orbit and (ii) the direct effect of lunar, solar and other perturbations.

The true anomaly ψ can be expanded in terms of the mean anomaly M as follows:[19]

$$\psi = M + \left(2e - \frac{1}{4}e^3 + \frac{5}{96}e^5 + \frac{107}{4608}e^7\right)\sin M$$

$$+ \left(\frac{5}{4}e^2 - \frac{11}{24}e^4 + \frac{17}{192}e^6\right)\sin 2M$$

$$+ \left(\frac{13}{12}e^3 - \frac{43}{64}e^5 + \frac{95}{512}e^7\right)\sin 3M$$

$$+ \left(\frac{103}{96}e^4 - \frac{451}{480}e^6\right)\sin 4M + \left(\frac{1097}{960}e^5 - \frac{5957}{4608}e^7\right)\sin 5M$$

$$+ \frac{1223}{960}e^6 \sin 6M + \frac{47273}{32256}e^7 \sin 7M$$

$$+ \ldots \qquad \text{(B-1)}$$

The maximum difference of $(\psi - M)$ is plotted in Fig. 13 versus eccentricity.

The direct forced solar and lunar librations are given by the trigonometric terms in (A-34) inside the lunar orbit to zeroth order in the eccentricities of the satellite and perturbing bodies and to the lowest order in (a/a'), where a' is the semi-major axis of the perturbing body. The most important term is

$$(\frac{\mu_m}{\mu})(\frac{a}{a'})^3 \frac{3n}{2(\dot{\Omega}+n-n')} \cos^4\frac{I}{2} \sin(2\Omega+2\eta-2\eta')$$

because the amplitude gets very large when $a \to a'$, $n \to n'$. However, the combined lunar and solar forced libration for the I=0 worst case is 0°.8 at a=40 earth radii and 2°.7 at a=50 earth radii. The libration caused by a satellites forced eccentricity growth is much larger.

The secular term in the right side of (A-34) implies the following modification of Kepler's third law (59) due to lunar perturbations:

$$T \approx \frac{2\pi \bar{a}^{3/2}}{\mu^{1/2}} \left[1 + (\frac{\mu_m}{\mu})(\frac{\bar{a}}{a'})^3 (1 - \frac{3}{2}\sin^2 I)\right] \qquad \text{(B-2)}$$

A bar has been placed over the semi-major axis a to emphasize that it is the average value over an orbital revolution. The orbital period T in (B-2) should be interpreted as the sidereal period, as opposed to, say, the nodal period.

The fractional change in orbital periods for given $\bar{a}$ is plotted in Figs. 13 and 14 for the solar and lunar effects. The fractional change for $I > 90^\circ$ is the same as that for an inclination of $180^\circ - I$. There is no fractional change if I=54°.7 or 125°.3. The fractional change in mean semi-major axis for given T is -2/3 the value in Figs. 13 and 14.

If a satellite were to be inserted into a high altitude orbit with a given desired orbital period, Figs. 13 and 14 can be used to determine the difference between $\bar{a}$ and the value determined from Kepler's third law (59). Then given the orbital elements ($\bar{a}$, $e \approx o$, I, Ω, n), the required instantaneous a at insertion can be determined from the moon + similar sun trigometric terms in (A-29). Below about 2X synchronous radius, the influence of the second harmonic of the earth's gravitational potential would also have to be considered.

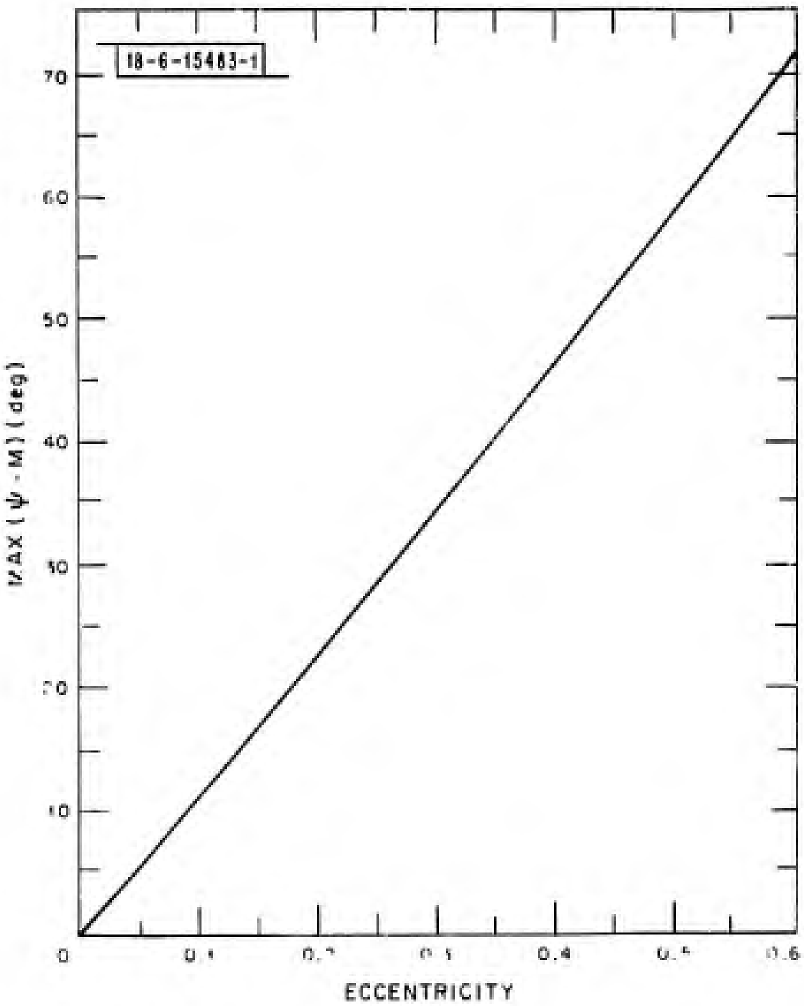

Fig. 13. Maximum difference between true and mean anomalies.

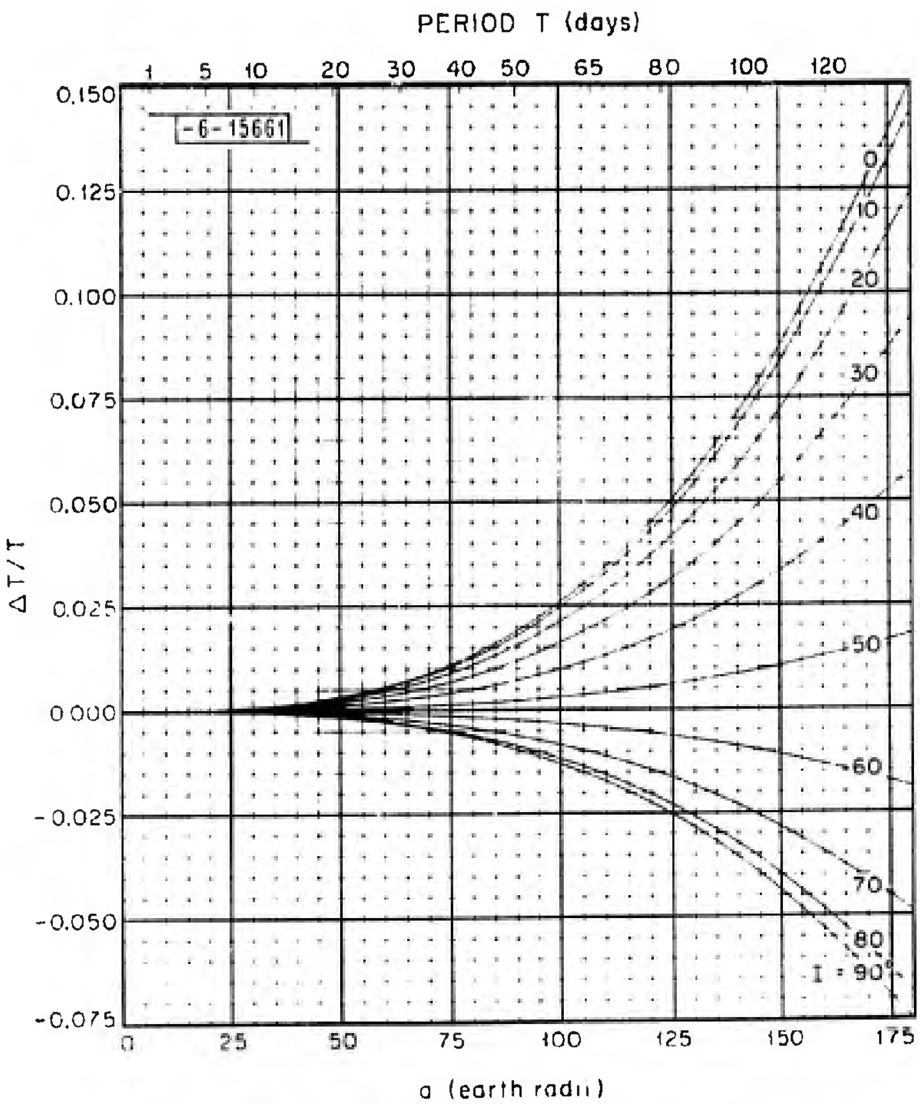

Fig. 14. Fractional change in orbital period due to solar perturbations.

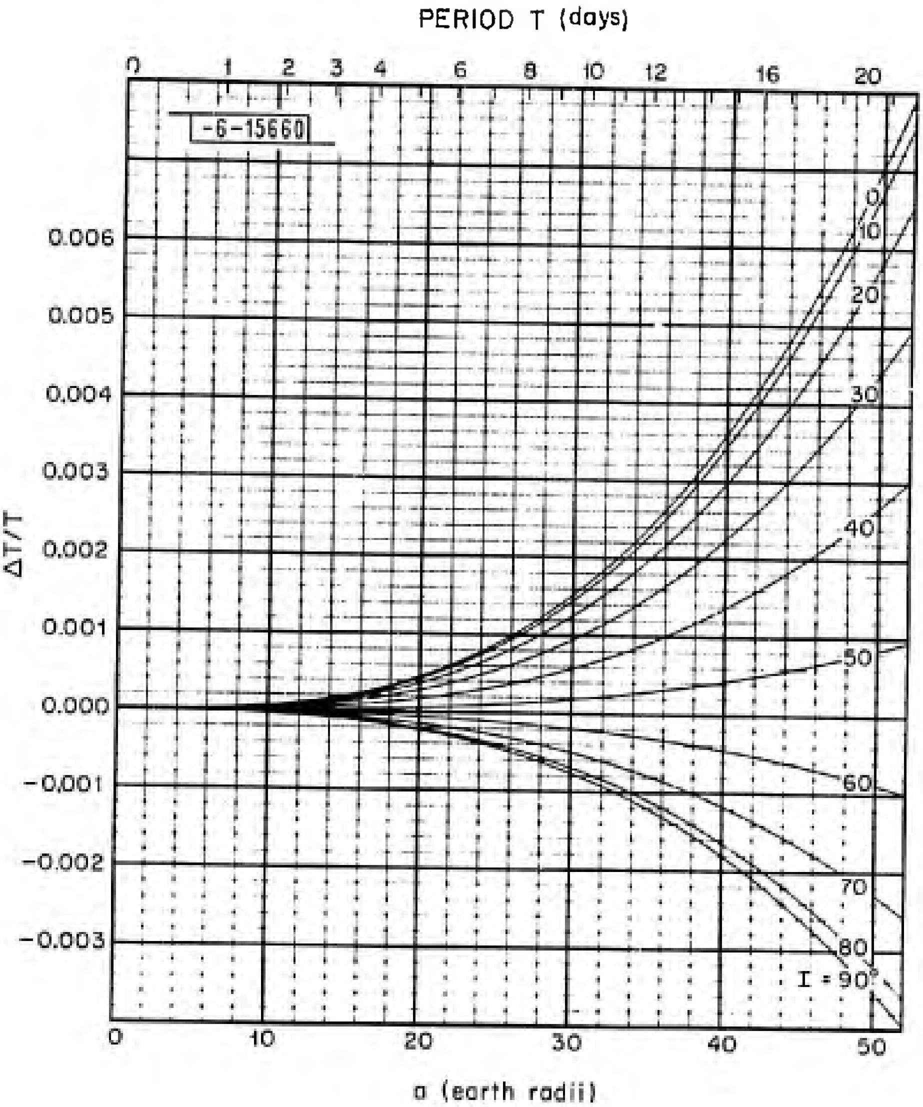

Fig. 15. Fractional change in orbital period due to lunar perturbations.

APPENDIX C

Sample Numerical Integrations

The Earth's sphere of influence extends to 0.9 the lunar orbit radius (see Section II), but a satellite orbit would not necessarily remain stable over a long period of time at that distance.

Figure 16 gives the eccentricity behavior of two 15 day period satellites 2/3 the way to the moon. The satellite motion was generated by numerical integration[9]. Both satellites are in the lunar plane 120° apart. One satellite has a close lunar approach, and is thrown out of the earth-moon system, whereas the other just has periodic variations in e. These two satellite orbits are exactly the same initially, except for the phase angle along the orbit.

If both the moon and a 15 day period satellite were in circular orbits, the close approach distance of the satellite to the moon would be 130,000 km. Taking into account the moon's 0.05 eccentricity, this close approach distance ranges between 110,000 and 150,000 km. The satellite's eccentricity could cause even more variation. Thus it is clear why different phasing can cause wildly different behavior in a high altitude satellite orbit.

A retrograde satellite in the lunar plane could come closer to the moon than a direct satellite without being thrown out of the earth-moon system. A satellite inclined to the lunar plane would not necessarily be near the moon when it passes through the lunar plane, so an inclined satellite could perhaps be safely higher than 2/3 the lunar orbital radius. However, outside a $\pm 39.^{\circ}2$ ($\pm 27.^{\circ}1$ or less at high altitudes) inclination band, the doubly averaged secular growth in eccentricity will eventually cause orbital degeneracy even if a close lunar approach is avoided.

Figures 17 and 18 depict the eccentricity variations of 11, 12 and 13 day period satellites inclined $+45^{\circ}$ and -45° (= 135°) to the lunar plane. There is secular growth because we are outside the $\pm 39\overset{\circ}{.}2$ inclination band. In addition there are periodic variations that change significantly with just a small change in altitude. The retrograde orbits had a factor of 5 less eccentricity variation than did the direct satellite orbits, even though the doubly averaging technique predicts the same rate of secular growth for a -45° inclined orbit as for a $+45^{\circ}$ inclined orbit with the same initial eccentricity and perigee values. Since all cases in this Appendix started with e=0 and secular changes in e are proportional to e, any secular growth in e must arise from initial periodic variations in e. These periodic variations are smaller for retrograde satellites than for direct satellites, which explains the difference between Figs. 17 and 18.

Analytic investigations are necessary to give a full understanding of satellite orbital behavior. However, they must be supplemented by numerical integrations of special cases, especially where the analytic assumptions do not hold exactly.

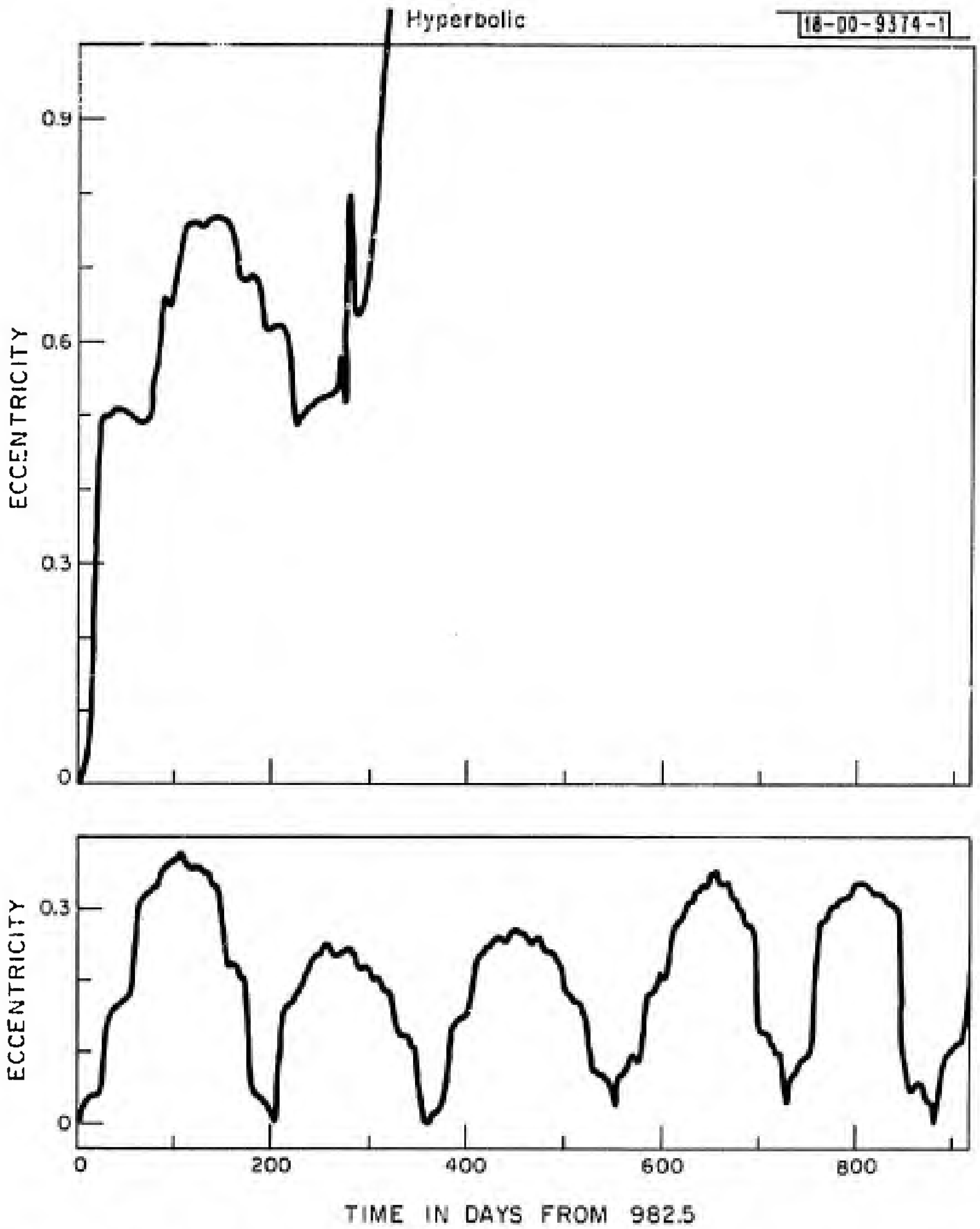

Fig. 16. Two 15-day direct satellites 120° apart in the lunar plane.

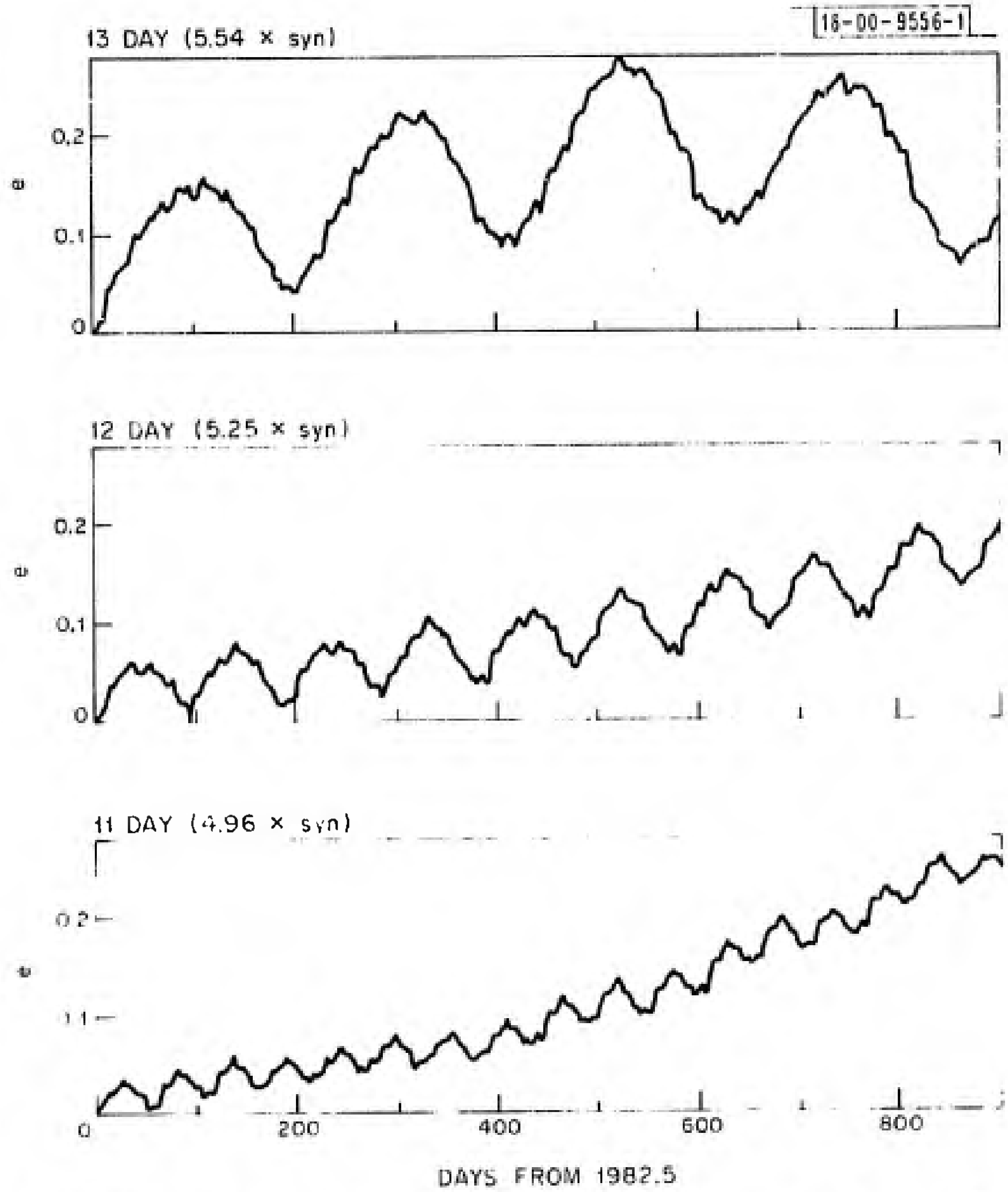

Fig. 17. Eccentricity variation of direct satellites inclined 45° to the lunar plane.

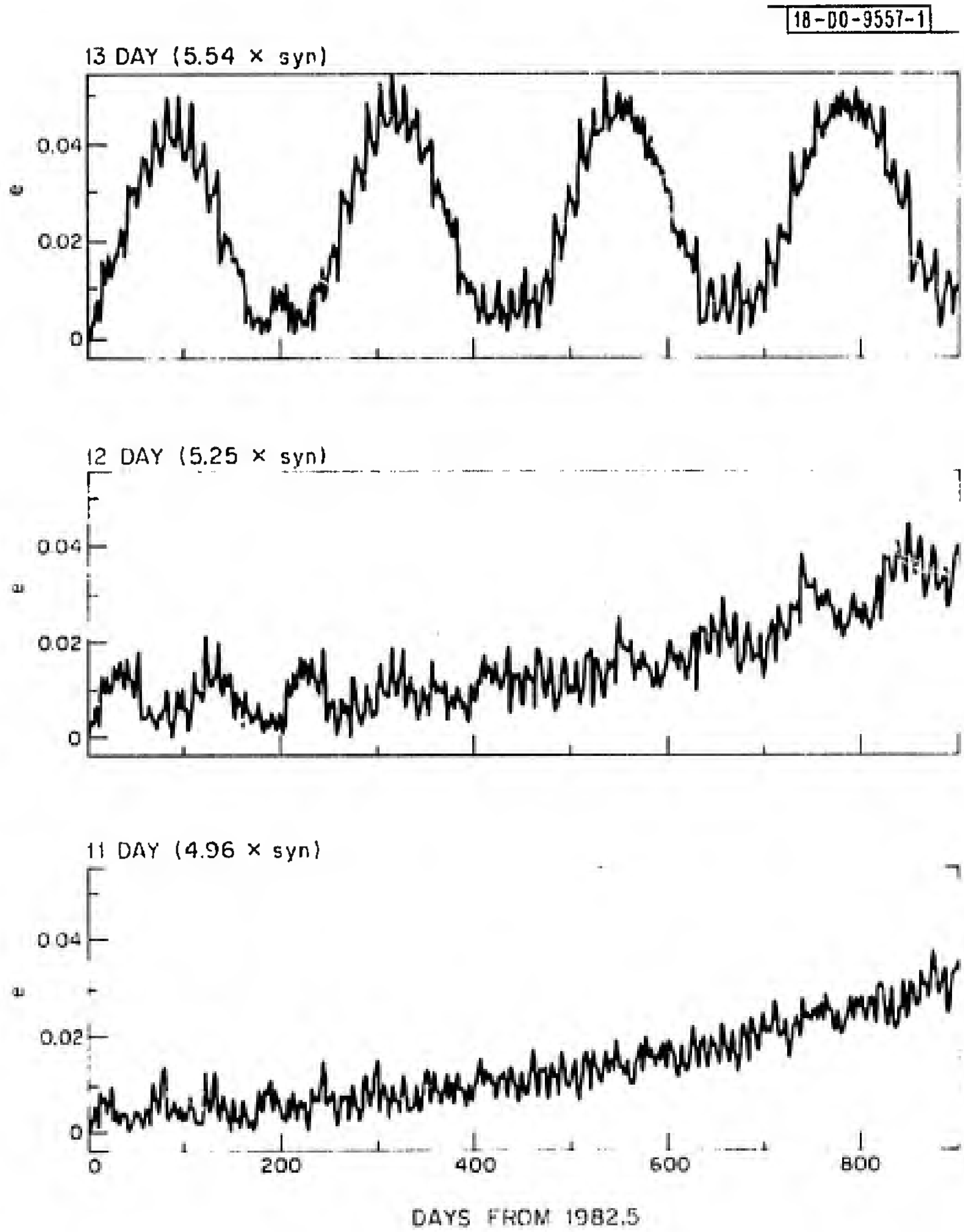

Fig. 18. Eccentricity variation of retrograde satellites inclined 135° to lunar plane.

NOMENCLATURE

a	semi-major axis of satellite orbit
a'	semi-major axis of lunar orbit
b	subscript denoting spacecraft
D	coefficient in equation (118) for de/dt
e	eccentricity of satellite orbit
e	subscript denoting earth
$\vec{e}_1, \vec{e}_2, \vec{e}_3$	unit vectors in the coordinate directions referred to the mean equinox and equator (of the earth) of date
E	transformation matrix (85)
E	coefficient in equation (119) for $d\omega/dt$
F	coefficient in equation (119) for $d\omega/dt$
F	transformation matrix (89)
$\vec{F}$	perturbing acceleration on satellite
I	inclination of satellite orbital plane to the lunar or solar ring plane
I_o	inclination of the satellite orbital plane on the mean equator of the earth
I_a	critical inclination when F=0 (see equation (155))
$I_b=180^o-I_a$	other solution of equation F=0 (see equation (156))
I_m	inclination of the mean lunar plane on the mean equator of the earth
$\bar{I}_m$	inclination of the mean lunar plane on the mean ecliptic
m	subscript denoting moon
M	mean anomaly
M_o	initial mean anomaly
$\tilde{M}$	$M + \omega + \Omega$
$\vec{M}$	unit vector from center of earth to satellite perigee
$n=\mu^{1/2}a^{-3/2}$	mean motion of satellite
n'	mean motion of moon

$p=a(1-e^2)$	semi-latus rectum
P_ℓ	Legendre polynomial of order ℓ
q	argument of Legendre polynomial (equation (9))
r	distance of satellite from earth
$\vec{r}$	vector from center of earth to satellite
r_m	distance of moon from earth
r_s	distance of sun from earth
$r_{\alpha\beta}$	distance between body α and body β (α,β=e,m,s,b)
R	component of perturbing acceleration in the radial $\vec{\varepsilon}_r$ direction
$\overline{R}$	disturbing function
R_E	equatorial earth radius (6378.16 km)
s	subscript denoting sun
S	component of perturbing acceleration in the tangential $\vec{\varepsilon}_s$ direction
syn	synchronous radius (42,164.2 km)
t	time
t_o	initial epoch
$T=2\pi/n$	orbital period
T_e	period of oscillation in e for $\lvert I\rvert<I_a$ (see equation (124))
T'	lunar orbital period
T_s	synodic period
u^1,u^2,u^3	coordinates referred to the mean equinox and equator (of the earth) of date
U	gravitational potential due to a ring of matter
v^1,v^2,v^3	coordinates referred to the mean equinox and ecliptic of date
w^1,w^2,w^3	coordinates with w^3 normal to the mean lunar orbital plane and with w^1 pointing towards the mean lunar ascending node on the ecliptic
W	component of perturbing acceleration in the normal $\vec{\varepsilon}_w$ direction

x^k	k^{th} cartesian coordinate of the satellite relative to the earth with (x^1,x^2) being the lunar or solar ring plane. (k=1,2,3)
x_m^k	k^{th} cartesian coordinate of the moon relative to the earth with (x^1,x^2) being the lunar plane (k=1,2,3)
$x_{\alpha\beta}^k$	k^{th} cartesian coordinate of body α relative to body β ($k=1,2,3$; $\alpha,\beta=e,m,p,b$).
β	an orbital element (a,e,I,Ω,ω plus M,ω,η in appendix A)
γ	quantity ≈ 1 defined in equation (26)
Δ	distance between satellite and moon
$\delta\beta$	change in orbital element β (especially short periodic changes)
$\Delta\beta$	average change in orbital element β over one orbit (ascending node to ascending node)
ε_o	mean obliquity of the ecliptic on the equator of the earth
$\vec{\varepsilon}_1,\vec{\varepsilon}_2,\vec{\varepsilon}_3$	unit vectors in the coordinate directions (x^1,x^2,x^3)
$\vec{\varepsilon}_r,\vec{\varepsilon}_s,\vec{\varepsilon}_w$	unit vectors in the radial, tangential and normal directions
η	$\Psi + \omega$
η_o	initial value of η in Appendix A
η'	angle η for moon
$\tilde{\eta}$	$\eta + \Omega \cos I - n(t-t_o)$ in Appendix A
θ	angle around the lunar or solar ring plane
μ	gravitational constant times mass of earth (3.986013×10^5 km^3/sec^2)
μ_α	gravitational constant time mass of body α(α=e,m,s)
μ_e/μ_m	81.301
$\mu_s/(\mu_e+\mu_m)$	328900.1
ξ	eccentric anomaly
ρ	radius of lunar or solar ring of matter
ρ_m	mean distance of moon from earth = 60.2665 earth radii (earth radius $\equiv$ 6378.16 km)

ρ_s	mean distance of sun from earth = 23455 earth radii
$\vec{\rho}$	vector from center of earth to a point in the ring of matter
τ	days from Julian Ephemeris Date 2415020.0
ψ	true anomaly
ω	argument of perigee of satellite orbit measured along the satellite orbital plane from its ascending node on the lunar or solar ring plane
$\tilde{\omega}$	$\omega + \Omega$
ω_o	argument of perigee of satellite orbit measured along the satellite orbital plane from its ascending node on the lunar or solar ring plane
ω_o	initial value of ω in Appendix A
Ω	longitude of ascending node of the satellite orbital plane on the lunar or solar ring plane
Ω_o	right ascension of ascending node of the satellite orbital plane of the mean equator of the earth measured from the mean equinox
Ω_o	initial value of Ω in Appendix A
Ω_m	right ascension of ascending node of the mean lunar plane on the mean equator of the earth measured from the mean equinox
$\overline{\Omega}_m$	longitude of ascending node of the mean lunar plane on the mean ecliptic measured from the mean equinox
$\Omega^k_{\alpha\beta}$	perturbing effect of body α on motion of a spacecraft relative to body β (α,β=e,m,s)

References

1. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I (Interscience Publishers, New York, 1953), pp. 82-87.

2. Handbook of Chemistry and Physics (Chemical Rubber Publishing Co., Cleveland, Ohio, 1963), pp. 275 & 283 (integrals 275 & 411).

3. D. Brouwer and G. M. Clemence, Methods of Celestial Mechanics (Academic Press, New York and London, 1961), p. 301.

4. I. I. Shapiro, "The Prediction of Satellite Orbits," International Union of Theoretical and Applied Mechanics Symposium on the Dynamics of Satellites, Paris, 28-30 May 1962, pp. 257-312.

5. R. R. Williams and J. Lorell, "The Theory of Long-Term Behavior of Artificial Satellite Orbits Due to Third Body Perturbations," C.I.T. Jet Propulsion Laboratory Technical Report 32-916 (15 February 1966).

6. Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac (Her Majesty's Stationery Office, London, 1961), p. 98.

7. Ibid., p. 27.

8. Ibid., p. 107

9. M. E. Ash, "Determination of Earth Satellite Orbits," Technical Note 1972-5, Lincoln Laboratory, M.I.T. (19 April 1972), DDC AD-742301.

10. G. N. Sherman, "Detectability of Relativisitic and Other Small Effects on Solar System Dynamics through Spacecraft Tracking," M.I.T. Ph.D. thesis (September 1973).

11. I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products (Academic Press, New York and London, 1965), pp. 152-153.

12. J. G. Williams and G. S. Benson, "Resonances in the Neptune-Pluto System," Astron.J. 76, 167-176 (1971).

13. V. Szebehely, Theory of Orbits (Academic Press, New York and London, 1967), pp. 133-144.

14. P. Musen, A. Bailie and E. Upton, "Development of the Lunar and Solar Perturbations in the Motion of an Artificial Satellite," NASA Goddard Spaceflight Center Technical Note D-494 (January 1961).

15. D. Fisher and J. P. Murphy, "Short Period Lunar and Solar Perturbations for Artificial Satellites," NASA Goddard Spaceflight Center Technical Note X-547-66-493 (October 1966).

16. W. M. Kaula, Theory of Satellite Geodesy (Blaisdell Publishing Company, Waltham, Massachusetts, 1966) p. 29.

17. D. Brouwer and G. M. Clemence, op. cit., p. 465.

18. M. E. Ash, "Generation of Planetary Ephemerides on an Electronic Computer," Technical Report 391, Lincoln Laboratory, M.I.T. (2 June 1965), pp. 9-13, DDC AD-629130.

19. D. Brouwer and G. M. Clemence, op. cit., p. 77.