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OPTIMAL CONTROL OF A GLIDING PARACHUTE
SYSTEM

Allan E. Pearson

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Natick, Massachusetts

August 1972

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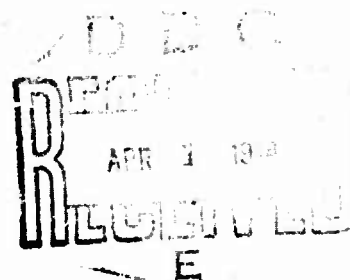
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Optimal Control of a Gliding Parachute System

by

Allan E. Pearson



August 1972

**Airdrop Engineering Laboratory
U. S. ARMY NATICK LABORATORIES
Natick, Massachusetts 01760**

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FOREWORD

The flight of a gliding parachute in a uniform wind field is examined using the methods of Optimal Control Theory. This report presents this analysis and is part of a continuing effort directed toward investigating methods which will improve the accuracy and dispersion characteristics of airdrop systems.

This study was conducted under Department of the Army Project No. 1F162203AA33 Drop Zone Dispersion Studies by Dr. Allan E. Pearson, Professor, Division of Engineering, Brown University while he was assigned to the Airdrop Engineering Laboratory as a visiting scientist.

Appreciation is extended Mr. Thomas F. Goodrich, Mr. Arthur L. Murphy, Jr., and Dr. Edward W. Ross of the U. S. Army Natick Laboratories for their help during this study.

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1. Introduction

The automatic control of gliding parachute systems has been considered for the recovery of sounding rockets and in various military airborne operations. Control can be effected by a servo-motor pulling on the shroud lines of the parachute which causes a banked turn of the parachute and a change in the direction of flight. Flight tests of an automatic-homing control system have been reported [1] which detect the angular error between the line of flight and the target by means of a ground based transmitter and a pair of directional antennas located on board the parachute. Referring to Fig. 1 where p is the position vector of the parachute relative to the fixed target in the horizontal plane, v is the horizontal component of the velocity vector of the parachute relative to air, and w is the wind velocity vector, the automatic-homing control system attempts to align v with $-p$, i.e. to keep the angle γ near zero at all times. Goodrick [2] analyzed the wind effects on a bang-bang (full-on or full-off) version of this control system and showed that increasing the horizontal system velocity relative to air tends to decrease the terminal distance error from the target in the presence of wind distortion. Murphy [3] analyzed the performance characteristics of the automatic-homing control system for the ideal situation in which the angle γ in Fig. 1 is maintained null at all time. Under the condition that the magnitude of the wind velocity vector w be less than the magnitude of the system horizontal velocity vector v relative to air, i.e. $\|w\| < \|v\|$, the parachute possesses a wind penetrating capability and the potential of reaching the target under arbitrary wind directions. The assumption is made that the initial altitude is properly chosen in accordance with the wind angle $\angle^w$ and the ratio $\|v\|/\|w\|$.

In this report the control of a gliding parachute is viewed in the context of optimal control theory. The basic philosophy is taken that at some time t_0 intermediate between the launch time 0 and the terminal time T , $0 < t_0 < T$, an estimate $\hat{w}$ of the wind vector is made available based on measurements taken in the interval $0 \leq t \leq t_0$. Under the assumption that the wind remains constant over $t_0 \leq t \leq T$, an optimal control problem is formulated which minimizes the terminal distance error from the target. The solution to this problem involves integrating a system of four nonlinear differential equations with mixed boundary conditions, i.e., a two point boundary value problem. An approximate solution to this problem is obtained which results in a sub-optimal feedback control law. Since the sub-optimal solution is a closed-loop control law, this feedback controller can be used on-line as new measurements of the wind vector w become available. The filtering problem of estimating the wind based on available measurements is not considered in this report.

The equations of motion and optimal control system formulations are presented in Section II. The sub-optimal control solution is presented in Section III. Various extensions of this work are indicated in Section IV.

II. Optimal Control Problem Formulations

Assuming a constant wind w in the horizontal plane, a constant rate of descent ν , and an initial altitude h_0 at launch time, the equations of motion governing the parachute can be represented in the horizontal plane according to

$$\dot{p} = v + w, \quad 0 \leq t \leq T = h_0/\nu \quad (1)$$

In this report the velocity vector $v(t)$ will be assumed to be constant in magnitude

$$\|v(t)\| = \left[|v_1(t)|^2 + |v_2(t)|^2 \right]^{1/2} = a = \text{constant.}$$

Then $v(t)$ can be represented by

$$\begin{aligned} v_1(t) &= a \cos \omega(t) \\ v_2(t) &= a \sin \omega(t) \end{aligned} \quad (2)$$

where the angle $\omega = \angle v$ is related to the bank angle ϕ of the parachute via the well known relation

$$\dot{\omega} = \frac{g}{\|v\|} \tan \phi. \quad (3)$$

Since the bank angle ϕ can be directly manipulated by changes in the servo-motor connecting the shroud lines, it is sufficient to summarize (1) - (3) by

$$\begin{aligned} \dot{p}_1 &= a \cos \omega + w_1 \\ \dot{p}_2 &= a \sin \omega + w_2 \\ \dot{\omega} &= u \quad t_0 \leq t \leq T \end{aligned} \quad (4)$$

where u is regarded as the control variable.

Let $p(t_0)$, $\omega(t_0)$ and w be given at some initial time t_0 in the interval $0 < t_0 < T$. A performance index which reflects a number of desirable features for this problem is

$$P = \frac{1}{2} \|p(T)\|^2 + \frac{q_1}{2} |\omega(T) - \omega^w - \pi|^2 + \frac{q_2}{2(T-t_0)} \int_{t_0}^T |u(t)|^2 dt \quad (5)$$

where q_1 and q_2 are non-negative weighting parameters. The first term in (5), $1/2 \|p(T)\|^2$, reflects the desirability of minimizing the Euclidean distance from the target at the terminal time T . The second term with weighting parameter q_1 reflects the desirability of having the parachute point upwind at the terminal time in order to reduce the total horizontal velocity at touch-down. The third term with weighting parameter q_2 reflects the cost of control effort in terms of the "average power" expended over the interval $t_0 \leq t \leq T$.

Before proceeding with the necessary conditions for a minimum of (5), some further comments regarding the role of the weighting parameters is in order. In the absence of any other constraints on the control variable u , it is necessary to choose $q_2 > 0$ in order to be assured of a meaningful solution. Physically, an optimal control solution with q_2 small will lead to relatively smaller values of the terminal distance error $\|p(T)\|$ but will generally require relatively larger bank angles ϕ of the parachute, in contrast with larger values for q_2 . In the interest of making $\|p(T)\|$ as small as possible, a meaningful solution does result for $q_2 = 0$ provided a constraint on u is assumed of the form $|u(t)| \leq M$, i.e. the bank angle ϕ is effectively limited by some upper bound. However, the analytical solution to this problem is less tractable than the unconstrained problem with $q_2 > 0$. As will be indicated in Section IIB, the solution to the constrained optimization problem is bang-bang followed by a possible null interval, i.e. $u(t) = \pm M$ or 0 , which implies instantaneous reversal of bank angles. Since the dynamics of the servo-motor and bank angle rates are not reflected in the equations of motion (4) the value of this problem ($q_2 = 0$) is probably of less practical importance although it is of theoretical interest.

A further comment regarding the performance index (5) refers to the second term with weighting parameter q_1 . There might be some motivation for choosing $q_1 = 0$ and adding the terminal constraint that $\omega(T) = \omega^w + \pi$. However, as in the case with potential terminal constraint on $p(T)$, it will not generally be possible to satisfy such constraints in the fixed interval (t_0, T) for any but a small set of initial conditions. In a practical implementation of an optimal control law, q_1 would probably be chosen as zero with a programmed command that with ΔT seconds to go the parachute execute a rapid turn into the wind regardless of the distance from the target.

A. Transformation to Normalized Variables

Let a time-varying transformation of the origin be made according to

$$y = p + (T-t) w, \quad t_0 \leq t \leq T.$$

Then minimizing the terminal distance $\|p(T)\|$ is equivalent to minimizing $\|y(T)\|$. In addition, let the independent variable be transformed via

$$\tau = \frac{t-t_0}{T-t_0} \quad (7)$$

and define new dependent variables via

$$x_1 = \frac{y_1}{a(T-t_0)}, \quad x_2 = \frac{y_2}{a(T-t_0)}, \quad x_3 = \omega. \quad (8)$$

In terms of these variables, the equations of motion (4) become

$$\begin{aligned} x_1' &= \cos x_3 \\ x_2' &= \sin x_3 \\ x_3' &= (T-t_0)u \quad 0 \leq \tau \leq 1 \end{aligned} \quad (9)$$

where prime denotes differentiation with respect to τ .

The performance index (5) becomes

$$P = \frac{a^2(T-t_0)^2}{2} \left[|x_1(1)|^2 + |x_2(1)|^2 + Q_1 |x_3(1) - \Delta^w - \pi|^2 + Q_2 \int_0^1 |u(\tau)|^2 d\tau \right] \quad (10)$$

where

$$Q_1 = \frac{q_1}{a^2(T-t_0)^2}, \quad Q_2 = \frac{q_2}{a^2(T-t_0)^3}. \quad (11)$$

Hence minimizing (5) or (10) is equivalent to minimizing

$$\tilde{P} = \frac{1}{2} \left[|x_1(1)|^2 + |x_2(1)|^2 + Q_1 |x_3(1) - \angle^w - \pi|^2 + Q_2 \int_0^1 |u(\tau)|^2 d\tau \right] \quad (12)$$

The major advantage to reformulating the problem in terms of the x and τ variables relates to the calculations needed in developing the sub-optimal control law in Section III.

B. Necessary Conditions for Optimal Control

The Hamiltonian for the unconstrained optimal control problem of minimizing (12) ($Q_2 > 0$) with the equations of motion (9) is given by

$$H(\lambda, x, u) = \lambda_1 \cos x_3 + \lambda_2 \sin x_3 + (T - t_0) \lambda_3 u + \frac{1}{2} Q_2 u^2 \quad (13)$$

where $(\lambda_1, \lambda_2, \lambda_3)$ are the co-state or adjoint variables for this problem which satisfy the differential equations

$$\begin{aligned} \lambda_1' &= - \frac{\partial H}{\partial x_1} = 0 \\ \lambda_2' &= - \frac{\partial H}{\partial x_2} = 0 \\ \lambda_3' &= - \frac{\partial H}{\partial x_3} = \lambda_1 \sin x_3 - \lambda_2 \cos x_3 \end{aligned} \quad (14)$$

Applying the "Minimum Principle" of optimal control theory [4], the necessary conditions for (12) to achieve a minimum are that $H(\lambda, x, u)$ be minimized over u , i.e.

$$u^* = - \frac{T - t_0}{Q_2} \lambda_3, \quad (15)$$

and that the following transversality conditions be satisfied:

$$\begin{aligned} \lambda_1(1) &= x_1(1), \quad \lambda_2(1) = x_2(1) \\ \lambda_3(1) &= Q_1 [x_3(1) - \angle^w - \pi]. \end{aligned} \quad (16)$$

Combining (9) with (14) - (16), and defining

$$x_4 = \lambda_3, \quad (17)$$

it follows that the necessary conditions for the unconstrained optimal control problem involve solving the two point boundary value problem:

$$\begin{aligned} x_1' &= \cos x_3 \\ x_2' &= \sin x_3 \\ x_3' &= - \frac{(T-t_0)^2}{Q_2} x_4 \\ x_4' &= x_1(1) \sin x_3 - x_2(1) \cos x_3 \end{aligned} \quad (18)$$

subject to the given initial conditions

$$\begin{aligned} x_1(0) &= \frac{1}{a(T-t_0)} [p_1(t_0) + (T-t_0) w_1] \\ x_2(0) &= \frac{1}{a(T-t_0)} [p_2(t_0) + (T-t_0) w_2] \\ x_3(0) &= \omega(t_0) \end{aligned} \quad (19)$$

and the terminal condition

$$x_4(1) = Q_1 [x_3(1) - \angle^w - \pi]. \quad (20)$$

By a reduction of (18) to three differential equations involving dx_1/dx_3 , dx_2/dx_3 and dx_4/dx_3 , it is possible to derive three algebraic equations in the three unknowns $x_1(1)$, $x_2(1)$ and $x_4(0)$ which depend on the given initial conditions in (19). However, these equations involve Elliptic integrals of the first and second kind with arguments depending on the mixed boundary data (see Appendix). Hence an exact analytical solution to (18) - (20) via this approach does not appear promising, although some approximations to these equations might be fruitful. This has not been explored in any detail.

In the case of the constrained optimal control problem in which $Q_2 = 0$ and u is constrained via $|u(t)| \leq M$, the Hamiltonian becomes

$$H(\lambda, x, u) = \lambda_1 \cos x_3 + \lambda_2 \sin x_3 + (T - t_0) \lambda_3 u$$

where $(\lambda_1, \lambda_2, \lambda_3)$ satisfy the same differential equations as in (14). However, in this case the minimum principle leads to

$$u^* = \begin{cases} -M & \text{if } \lambda_3 > 0 \\ +M & \text{if } \lambda_3 < 0 \\ \text{no conclusion} & \text{if } \lambda_3 = 0 \end{cases}$$

The appropriate value for u^* over a singular interval during which $\lambda_3 = 0$ can be resolved by the condition that $\lambda_3 = 0 \Rightarrow \lambda_3' = 0 \Rightarrow x_3 = \text{constant} \Rightarrow x_3' = 0 \Rightarrow u^* = 0$. Moreover, if a singular control interval is achieved on some time interval contained within the given interval $t_0 \leq t \leq T$, the control remains singular, i.e. null, throughout the remainder of the interval. This follows because $u^* = 0 \Rightarrow x_3' = 0$ which means x_4 remains at zero for the remainder of the interval. Hence the optimal control is bang-bang-*cif* for the constrained problem with no intermediate singular intervals. It is difficult to say how many switches will occur during the interval $t_0 \leq t \leq T$, although presumably more switches will occur when the parachute is close to the target with a large amount of time to go until touch down.

In summary, the solution to the constrained optimal control problem involves solving the two point boundary value problem:

$$x_1' = \cos x_3$$

$$x_2' = \sin x_3$$

$$x_3' = -(T - t_0) M \operatorname{Sgn}(\lambda_3)$$

$$x_4' = x_1(1) \sin x_3 - x_2(1) \cos x_3$$

subject to the initial and terminal data in (19) and (20), where the "Sgn" function is defined by

$$\text{Sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The open-loop optimal control solution $u^*(\tau, p(t_0), \omega(t_0))$, $0 \leq \tau \leq 1$, regarded as a function of the initial conditions $p(t_0)$ and $\omega(t_0)$, becomes a closed-loop or feedback, control law for either of the above problems simply by choosing $\tau = 0$ and using the continuous variables $p(t)$ and $\omega(t)$ in place of $p(t_0)$ and $\omega(t_0)$. In this way $u^*(0, p(t), \omega(t))$ is the optimal control for every t based on continuous measurements in $p(t)$ and $\omega(t)$.

III. A Sub-Optimal Feedback Control Law

In the interest of obtaining a feedback, i.e. closed-loop, control law, an approximate solution will be developed in this section to the necessary conditions, (18) - (20), for the unconstrained optimal control problem. Let a parameter ϵ be defined by

$$\epsilon = \frac{(T-t_0)^2}{Q_2} = \frac{a^2(T-t_0)^4}{Q_2} \quad (21)$$

and consider (18) - (20) in vector notation:

$$x' = f(x, \tau, \epsilon) = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ -\epsilon x_4 \\ x_1(1) \sin x_3 - x_2(1) \cos x_3 \end{bmatrix} \quad (22)$$

$$A x(0) + B x(1) = \zeta \quad (23)$$

where the matrices A , B and ζ are given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -Q_1 & 1 \end{bmatrix}, \quad \zeta = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ -Q_1 \delta \end{bmatrix} \quad (24)$$

and the constants $(\alpha, \beta, \gamma, \delta)$ are defined as

$$\begin{aligned}\alpha &= x_1(0) = \frac{1}{a(T-t_0)} [p_1(t_0) + (T-t_0)w_1] \\ \beta &= x_2(0) = \frac{1}{a(T-t_0)} [p_2(t_0) + (T-t_0)w_2] \\ \gamma &= x_3(0) = \omega(t_0) \\ \delta &= \angle^w + \pi\end{aligned}\tag{25}$$

In view of the analytic properties of the function $f(x, \tau, \epsilon)$, a Taylor series expansion of the solution $x(t, \epsilon)$ to (22) and (23) about $\epsilon = 0$ will exist for sufficiently small ϵ . Thus

$$x(\tau, \epsilon) = \sum_{n=0}^{\infty} \frac{\partial^n x(\tau, \epsilon)}{\partial \epsilon^n} \bigg|_{\epsilon=0} \frac{\epsilon^n}{n!}\tag{26}$$

and an approximate solution to the problem is represented by retaining a finite number of terms in the series (26). The corresponding approximation to the optimal control u^* given by (15) and (17) is obtained from

$$\hat{u}(\tau, \epsilon) = - \frac{\epsilon}{T-t_0} \sum_{n=0}^N \frac{\partial^n x_4(\tau, \epsilon)}{\partial \epsilon^n} \bigg|_{\epsilon=0} \frac{\epsilon^n}{n!}\tag{27}$$

The sub-optimal feedback control law follows by choosing $\tau = 0$ in (27) and letting the initial time t_0 in (25) be replaced by running time t .

Before proceeding with the calculations of the coefficients in (26), it should be pointed out that $\epsilon = 0$ corresponds to no control effort ($u = 0$) expended on the interval $t_0 \leq t \leq T$. This is evident since $\epsilon = 0$ implies $Q_2 = \infty$ for which $u = 0$ is the optimal solution in minimizing (12). In other words, the nominal solution in this approximation is the uncontrolled descent of the parachute.

In order to simplify the notation, let the components of the vector coefficients in (26) be represented by (a_n, b_n, c_n, d_n) according to

$$\begin{aligned} a_n(\tau) &= \left. \frac{\partial^n x_1(\tau, \epsilon)}{\partial \epsilon^n} \right|_{\epsilon=0}, & b_n(\tau) &= \left. \frac{\partial^n x_2(\tau, \epsilon)}{\partial \epsilon^n} \right|_{\epsilon=0} \\ c_n(\tau) &= \left. \frac{\partial^n x_3(\tau, \epsilon)}{\partial \epsilon^n} \right|_{\epsilon=0}, & d_n(\tau) &= \left. \frac{\partial^n x_4(\tau, \epsilon)}{\partial \epsilon^n} \right|_{\epsilon=0} \end{aligned}$$

The zeroeth order ($n = 0$) term in (26) is then obtained by solving

$$\begin{aligned} a'_0 &= \cos c_0 \\ b'_0 &= \sin c_0 \\ c'_0 &= 0 \\ d'_0 &= a_0(1) \sin c_0 - b_0(1) \cos c_0 \end{aligned} \tag{28}$$

subject to the boundary conditions

$$\begin{aligned} a_0(0) &= \alpha, & b_0(0) &= \beta, & c_0(0) &= \gamma \\ d_0(1) &= Q_1 [c_0(1) - \delta] \end{aligned} \tag{29}$$

The solution to (28) and (29) is readily obtained as

$$\begin{aligned} a_0(\tau) &= \alpha + \tau C \\ b_0(\tau) &= \beta + \tau S \\ c_0(\tau) &= \gamma \\ d_0(\tau) &= (1 - \tau)D + Q_1(\gamma - \delta) \end{aligned} \tag{30}$$

where $C = \cos \gamma$, $S = \sin \gamma$ and

$$D = \beta C - \alpha S \tag{31}$$

have been used for shorthand notation.

The first order ($n = 1$) term in (26) is obtained by differentiating both sides of (22) with respect to ϵ and then putting $\epsilon = 0$. This yields the system of equations for (a_1, b_1, c_1, d_1) :

$$\begin{aligned} a_1' &= -Sc_1 \\ b_1' &= Cc_1 \\ c_1' &= -d_0 \\ d_1' &= a_1(1)S - b_1(1)C + [a_0(1)C + b_0(1)S]c_1 \end{aligned} \quad (32)$$

The boundary conditions for (32), as for all subsequent terms in the series (26), are given by

$$A \left. \frac{\partial^n x(0, \epsilon)}{\partial \epsilon^n} \right|_{\epsilon=0} + B \left. \frac{\partial^n x(1, \epsilon)}{\partial \epsilon^n} \right|_{\epsilon=0} = 0 \quad n \geq 1. \quad (33)$$

This follows from (23) and the fact that (A, B, ζ) do not depend on ϵ . Thus (33) yields

$$\begin{aligned} a_n(0) &= 0, & b_n(0) &= 0, & c_n(0) &= 0 \\ d_n(1) &= Q_1 c_n(1), & n &\geq 1 \end{aligned} \quad (34)$$

The integration of (32) subject to (33) with a_0, b_0 and d_0 given by (30) yields

$$\begin{aligned} a_1(r) &= -S \left[\frac{D}{6} r^3 + (Q_1(\gamma - \delta) - D) \frac{r^2}{2} \right] \\ b_1(r) &= C \left[\frac{D}{6} r^3 + (Q_1(\gamma - \delta) - D) \frac{r^2}{2} \right] \\ c_1(r) &= (Q_1(\gamma - \delta) - D) r + \frac{D}{2} r^2 \\ d_1(r) &= d_1(0) - \left[\frac{1}{2} Q_1(\gamma - \delta) - \frac{1}{3} D \right] \\ &\quad + (1 + \alpha C + \beta S) \left[(Q_1(\gamma - \delta) - D) \frac{r^2}{2} + D \frac{r^3}{6} \right] \end{aligned} \quad (35)$$

where the initial condition on d_1 is given by

$$d_1(0) = 1/3(\alpha C + \beta S) D + Q_1 [(\gamma - \delta - 1/2 D)Q_1 - 1/2(\alpha C + \beta S)].$$

The first two terms of the sub-optimal feedback control law (27) with $r = 0$ and $N = 1$ have been obtained up to this point with

$$\begin{aligned} \hat{u}(0, \epsilon) &= -\frac{\epsilon}{T-t_0} [d_0(0) + d_1(0)\epsilon] \\ &= -\frac{\epsilon}{T-t_0} \left[1 + \frac{\epsilon}{3} (\alpha C + \beta S) \right] D + \\ &\quad -\frac{\epsilon}{T-t_0} [\gamma - \delta + \epsilon(\gamma - \delta - 1/2 D)Q_1 - \frac{\epsilon}{2} (\alpha C + \beta S)Q_1] \end{aligned} \quad (36)$$

The higher order terms in the series (26) become increasingly more tedious to obtain. For $n = 2$, the differential equations to be solved (after some simplification) are

$$\begin{aligned} a_2' &= -C[c_1]^2 - S c_2 \\ b_2' &= -S[c_1]^2 + C c_2 \\ c_2' &= -2d_1 \\ d_2' &= S a_2(1) - C b_2(1) + D[c_1]^2 + [1 + \alpha C + \beta S] c_2 \end{aligned}$$

where $c_1(\tau)$, $c_2(\tau)$, $d_1(\tau)$ are given by (35) and the boundary conditions by (34). These equations were integrated under the special case $Q_1 = 0$ with the result that the desired initial condition on d_2 is

$$\begin{aligned} Q_1 = 0: d_2(0) &= D \left[-\frac{1}{18} - \frac{2}{15} D^2 + \frac{2}{45} (\alpha C + \beta S) \right. \\ &\quad \left. + \frac{4}{15} (\alpha C + \beta S)^2 \right] \end{aligned}$$

The interesting feature about the case $Q_1 = 0$ is the observation that the sub-optimal control law contains D as a factor — at least as far as the first three terms in (27) have been computed. Thus it would appear as though the quantity D in (31),

$$D = x_2(0) \cos x_3(0) - x_1(0) \sin x_3(0) \quad (37)$$

plays a major role in the optimal feedback control law.

It is instructive to refer the above obtained sub-optimal control law back to the original coordinate system. Assuming $Q_1 = 0$ in (36) and referring to (37), (25), (21), (9) and (7), this results in

$$\hat{u} = \frac{(T-t)^2}{q_2} y' E v \left[1 + \frac{(T-t)^2}{3q_2} y' v + \dots \right] \quad (38)$$

In the above, $y = y(t)$ is the moving-origin coordinate system defined in (6), i.e.

$$y = p + (T - t) w,$$

and E is the 2×2 matrix

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which effectively generates the D term in (37), i.e.

$$y' E v = y_1 v_2 - y_2 v_1.$$

IV. Concluding Remarks

Given an estimate of the wind vector w at some initial time to intermediate between launch and touch down, and given the initial position and velocity vectors (relative to air) of the parachute, the optimal control of the parachute has been considered which strives to minimize the terminal distance of the parachute from the target at the time of touch down. A sub-optimal feedback control law has been obtained which can be used in a closed-loop manner based on continuous measurements of the position and velocity vectors of the parachute. The first order approximation to this control law is strikingly simple and takes the form

$$\hat{u} = \frac{(T-t)^2}{q_2} (y_1 v_2 - y_2 v_1) \quad (39)$$

where $y_1 = p_1 + (T - t)w_1$ and $y_2 = p_2 + (T - t)w_2$ are the horizontal components of the position vector $p = p(t)$ of the parachute at time t with the origin translated along $(T - t)w$ for each t in the interval $0 \leq t \leq T$, and $v = v(t)$ is the parachute velocity vector relative to air in the horizontal plane. In order for this control law to be effective, the initial launch position must be in the vicinity of the free descent path of the parachute which would hit the target at touchdown in the presence of a constant known wind. This is a reasonable assumption in view of the limited degree of control available in the gliding parachute system.

The above control law can be easily updated as new estimates of the wind become available. However, it is the D.C. or mean value of the wind, rather than the instantaneous value, which should be used as the estimate since the control law is optimal only for a constant wind vector w on the ensuing interval $T > t > t_0$. Although (39) is only a first order approximation, the term in brackets, $y_1 v_2 - y_2 v_1$, appears to be a homogeneous factor in each of the terms of the Taylor's series expansion (27). This was only validated for the first three terms of (27) and applies to the special optimization problem of minimizing the performance index (5) with $q_1 = 0$ and $q_2 > 0$.

The reciprocal of the weighting parameter q_2 in (39) represents the gain of the feedback controller which must be adjusted to achieve a small terminal distance error at touchdown without incurring excessive bank angles during the parachute maneuvering over the interval $0 \leq t \leq T$. The bank angle ϕ upon commands from the controller electronics must be designed to achieve the desired relationship

$$\phi = \tan^{-1} \frac{a(T-t)^2}{g q_2} [y_1 v_2 - y_2 v_1] \quad (40)$$

without significant delays.

Future work should include the following: (i) A computer simulation of the suboptimal control law and a comparison of the results with optimal control trajectories obtained by solving the two point boundary value problem for various initial conditions. (ii) Derivation of the appropriate filtering equations for estimating the wind based on available measurements. Computer simulation of the controller-filter combination for random wind profiles. (iii) A formulation of the problem with a random wind which seeks to minimize the expected value of the terminal distance from the target. In this vein, random terminal times might be appropriately studied to account for inaccurate altitude data. (iv) A formulation of the problem with the magnitude of the parachute velocity vector a control variable in addition to the direction.

V. References

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Appendix I

Reduction of (18) to Elliptic Integrals

Successive divisions of the equations in (18) by x_3' yields (with ϵ defined by (21):

$$\frac{dx_1}{dx_3} = -\frac{\cos x_3}{\epsilon x_4}, \quad \frac{dx_2}{dx_3} = -\frac{\sin x_3}{\epsilon x_4} \quad (41)$$

$$\frac{dx_4}{dx_3} = \frac{x_2(1) \cos x_3 - x_1(1) \sin x_3}{\epsilon x_4}$$

The third of these equations can be integrated directly:

$$x_4^2 = C_1 + \frac{2}{\epsilon} [x_1(1) \cos x_3 - x_2(1) \sin x_3] \quad (42)$$

where the constant of integration C_1 is given by

$$C_1 = [x_4(1)]^2 - \frac{2}{\epsilon} [x_1(1) \cos x_3(1) - x_2(1) \sin x_3(1)]. \quad (43)$$

Using the relation $\int \frac{dx_3}{x_4} = -\epsilon \int d\tau' = -\epsilon \tau$ for $\tau = 1$, the first integral is obtained as

$$\frac{\epsilon}{2} \sqrt{C_1 + \frac{2d}{\epsilon}} = |F(k, \phi_1) - F(k, \phi_0)| \quad (44)$$

where

$$d = \sqrt{|x_1(1)|^2 + |x_2(1)|^2}, \quad (45)$$

$$k^2 = \frac{4d}{C_1 \epsilon + 2d}$$

$$\phi_0 = 1/2 [x_3(0) + \tan^{-1} \frac{x_2(1)}{x_1(1)}] \quad (46)$$

$$\phi_1 = 1/2 [x_3(1) + \tan^{-1} \frac{x_2(1)}{x_1(1)}]$$

and $F(k, \phi)$ is an Elliptic integral of the first kind.

Substituting (42) into the first of the equations in (41) leads to the expression

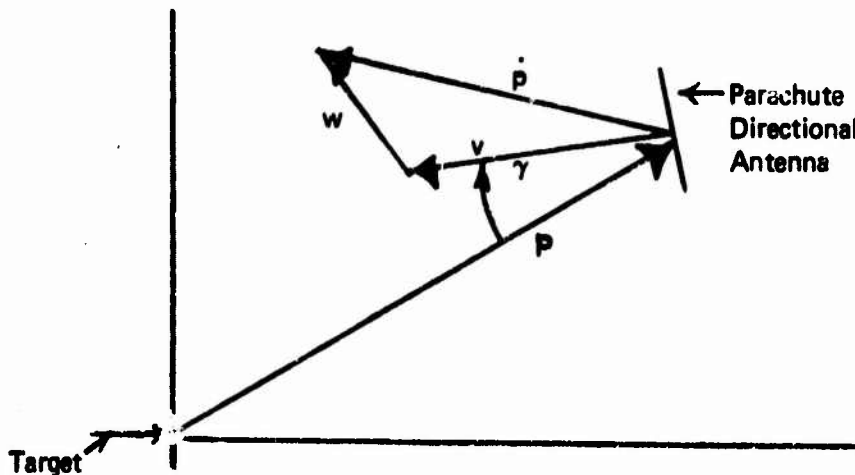
$$x_1(1) - x_1(0) = \pm \frac{1}{\epsilon} \left\{ \frac{\epsilon \sin \beta}{d} \left[\sqrt{C_1 + \frac{2d}{\epsilon} \cos(x_3(0) + \beta)} - \sqrt{C_1 + \frac{2d}{\epsilon} \cos(x_3(1) + \beta)} \right] + \sqrt{\frac{2 \cos \beta}{C_1 + \frac{2d}{\epsilon}}} \left[\left(1 - \frac{2}{k^2}\right) (F(k, \phi_1) - F(k, \phi_0)) + \frac{2}{k^2} (E(\phi_1 | \alpha) - E(\phi_0 | \alpha)) \right] \right\} \quad (47)$$

where ϕ_0 , ϕ_1 , d , C_1 and k are as defined above,

$$\beta = \tan^{-1} \frac{x_2(1)}{x_1(1)}, \quad \sin^2 \alpha = k^2$$

and $E(\phi | \alpha)$ is an Elliptic integral of the second kind.

Substituting (42) into the second equation of (41) will lead to a similar relation as (47) involving Elliptic integrals of the first and second kind. This equation in conjunction with (44) and (47) comprises three algebraic equations in the three unknowns $x_1(1)$, $x_2(1)$ and $x_3(0)$, given the initial conditions $x_1(0)$, $x_2(0)$ and $x_3(0)$. However, the arguments of the Elliptic integrals depend on the boundary data in a mixed way.



Vector Relations for Parachute Dynamics in the Horizontal Plane

Fig. 1