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THE THEORY RELATING THE WOEHLER EQUATION TO CUMULATIVE DAMAGE IN THE DISTRIBUTION FATIGUE LIFE

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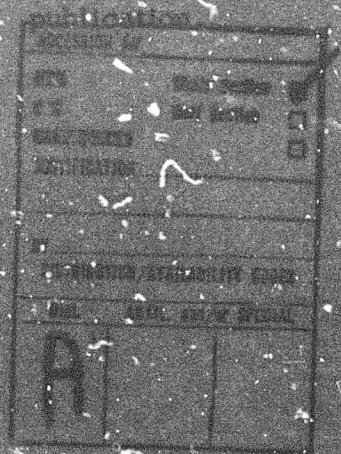

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A theoretical model which is able to encompass the observed fatigue growth phenomena of (1) crack acceleration or crack deceleration, (2) crack arrest, (3) crack jump at a change of load level, is proposed for the expected incremental crack growth rate under programmed loads. The parameters are calculated by techniques which match the crack growth rate using earlier data on striation counts obtained by electron fractography for an aluminum alloy. The implications of this model for determining an appropriate distribution of fatigue life are noted.

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PREFACE

The research reported herein was conducted under Air Force Contract F33615-73-C-4016 at Washington State University. The work was initiated under project 7071. The technical monitor of the contract was Dr. H. Leon Harter, Aerospace Research Laboratories, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio. The principal investigator was Prof. Sam C. Saunders with a research assistant, Verlin Drinen. The author acknowledges helpful conversations on the problems of fractographic interpretation of fatigue crack striations with J. Corey McMillan of the Boeing Commercial Airplane Company. The author is also grateful for permission to reproduce, in Figures 5 through 11, data which were taken from the joint work of J. Corey McMillan and Regis M. N. Pelloux.

This interim report covers work conducted during the period from September 15, 1972 until September 15, 1973. The manuscript was submitted for publication in final form in January, 1974.

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SECTION I

INTRODUCTION

The earliest model proposed for the determination of cumulative fatigue damage gave a result which is now called Miner's rule (see [1]) despite the presentation of this formula for life prediction 20 years earlier, in [2], by A. Palmgren. It is a deterministic model based on a linear damage theory which, strictly interpreted, has been known to be false for a long time. Its acceptance as an engineering tool in design has been hard to overcome because of its ease of application and the many situations in which the predictions are reasonably good. In fact, it was the best qualified of all those that were compared at one time; see references in [3].

The generalization from that simple deterministic model has proceeded mainly on two separate fronts. In one, it has been replaced by a probabilistic framework in which fatigue life is regarded as being stochastic in nature. In these models the statistical problems of determining the family of distributions and the regression of mean stress on life, along with the estimation of parameters, have come to the fore. Only recently, from such a model for fatigue crack growth, has Miner's rule been shown to be the expectation of fatigue life (which is stochastic) under repeated load spectra.

Under more realistic conditions than those assumed by Miner, namely that cumulative damage is dependent on load order, a modified Miner's formula was obtained which was theoretically capable of taking into account stress history. This result yields a formula for the expected number of spectra which can be repeated until failure. It contains as a special case the original Miner's rule with interactions neglected. A series of publications which

give this development are [4], [5], [3]. The article [3] is of an expository nature, providing a broader scope with some historical perspective.

On the other front there have been a great number of investigators studying the physical mechanisms of crack growth using various mechanisms involving fracture and dislocation theory of metals. Some of these which explore in detail the consequences of load order interactions are [6], [7], [8], [9].

One of the latest is Impellizzeri [7]. That paper gives, albeit in a deterministic form, a cumulative damage rule which can be used to predict the number of spectra which can be repeated until failure, taking into account the load interaction factors. It is a modification of the Miner's rule sum and agrees, except for notational differences, with the general results in [4], [5] for the expected value of a stochastic crack growth model.

Most of the present investigators assessing cumulative fatigue damage emphasize only one or the other of 1) identification of the important deterministic factors in crack initiation or 2) statistical problems associated with the pooling of the variation within material, manufacturing, loading, and environment into the several unknown parameters of a statistical law. Neither of these by itself, we feel, will ultimately be adequate for a complete theory. What is now needed is a more comprehensive interdisciplinary approach. The statistical theorist must utilize the knowledge gained by the investigators who have concerned themselves with quantifying the effects of such factors as stress intensity, residual stress and the interaction of varying peak stress levels in a loading sequence; see [7] and the reference there. Likewise the fracture mechanist must also utilize the knowledge which has been gained from the mathematical and statistical methodology of fitting parameters to life distributions.

That this problem is well recognized can be seen from the comments of J. Schijve which were made during the second F. J. Plantema Memorial Lecture, given at the 11th conference of the International Committee on Aeronautical Fatigue held in Stockholm in May 1969 (see [10]):

"Damage theories so far have been unable to include interactions between fatigue loads of different magnitudes (sequence effect) whereas laboratory tests have clearly indicated that such interactions do occur and can be qualitatively understood."

SECTION II

THE MECHANISMS OF FATIGUE CRACKING

Before an accurate theoretical model of the fatigue mechanism is constructed it is well to summarize some of the empirical results which have been obtained by earlier investigators. Thus we hope to make certain that the abstract model agrees with particular facts. In order to present a complete picture of the state of knowledge we paraphrase from the appropriate sections in the more inclusive report of McMillan and Pelloux in [11], rather than [8]:

The influence of maximum stress, stress range, and sequence of load application on the rate and mechanism of fatigue crack propagation in 2024-T3 aluminum alloy was carefully studied by means of electron fractography. Variable amplitude loading programs were designed to provide tests under the following conditions:

1) Constant maximum stress with three different levels of stress range,
2) Constant stress range with three and four levels of maximum stress,
3) Pseudorandom load application achieved by random distribution of the load spectra defined in items 1) and 2),
4) Uniform maximum stress with peak overloads and underloads.

The macroscopic growth rates were determined on center-notched crack growth panels and the fracture surfaces were examined by electron fractography. The analyses of the influence of program loads on the rate and mechanism of fatigue crack growth were accomplished by:

1) Comparing plots of crack length versus measured rates of crack propagation for the different programs. The measured rates were also compared with rates calculated by a computer program.

2) Relating the count and spacing of the fatigue striations observed on the fracture surfaces to the applied load program, by means of electron fractography.

These carefully controlled fatigue crack propagation tests, made under program and random loads, were followed by a study of striation counts and spacing measurements using electron microscopy, all of which help to provide a better understanding of the mechanisms of fatigue cracking. The main findings with respect to aluminum alloy 2024-T3 are as follows:

1) The advance of the fatigue crack front takes place only during the stress rise portion of a cycle.

2) The profile or sides of a striation are related to the loading and unloading sequence of a load cycle.

3) Microscopic crack growth rates were measured as a function of load amplitudes, ΔS , and maximum loads, S max, and an empirical crack growth rate equation was obtained:

$$\text{crack growth rate} = \text{constant } (\Delta S)^n (S \text{ max})^m .$$

This equation relates the relative growth rates at a given crack length under program loading.

4) In random load programs with constant maximum stress, the growth rate is the same as the programmed spectra rates. No sequencing effect is apparent in changes of stress variation alone.

5) In random load programs with variable maximum stress, the crack growth rates were higher than for the equivalent programmed load programs. The importance of the crack tip radius in controlling crack rates for random loads was pointed out.

6) The measured crack growth rate of program and random load crack propagation tests were compared with rates calculated by a computer program. The computed rates are not conservative but they do not differ from the measured rates by a factor of more than two. The application of Miner's rule resulted in $\sum \frac{n}{N}$ values of the order of .650 for the constant maximum load programs and of .850 for the variable maximum load programs.

It is the utilization of the facts just summarized, which were obtained in the investigation of McMillan and Pelloux loc. cit., without accepting their conclusions, that we undertake in the following pages.

SECTION III

THE PROGRAMMED LOAD IN THEORY

We shall introduce and utilize the same notation given in [5]. Let $\mathfrak{L}$ be a subset of the real-valued piecewise linear functions on the real line, each element of which is identically zero except for some interval $(0,m)$, where m is a positive integer. We call $\lambda \in \mathfrak{L}$ a <u>load spectrum</u> whenever the salient points of λ occur only at the positive integers, the slope of adjacent straight line segments alternates in sign, i.e., if we let primes denote differentiation,

$$-\text{sgn } \lambda'(x) = \text{sgn } \lambda'(x+1) \qquad \text{for all non-integral } x > 0 ,$$

and the function is continuous except possibly at the end points of its interval of support.

The least positive integer m such that $x \geq m$ implies $\lambda(x) = 0$ is called the <u>length</u> of the spectrum. Each unit interval across which $\lambda'(x) \neq 0$ is called a load <u>fluctuation</u>. The length is merely the number of load fluctuations. Two successive fluctuations, on one of which the load is increasing, is an <u>oscillation</u>. Note that we have taken the load oscillations to be of the same duration, i.e., the frequency of such oscillations is the same. In actual fact this may not be the case in practice, but we shall make this assumption because of the observed invariance of fatigue life to frequency in many cases of fatigue cycling. That is to say, we shall not consider cycles of such high frequency that thermal effects are encountered or of such low frequency that environmental changes and age affect the fatigue life. A justification of this assumption has been made before by Schijve.

In order to make some of the following precise we require the definition of continuation of a load function. Consider $\lambda_1, \lambda_2 \in \mathcal{L}$ with m_1 the length of λ_1. We define the binary operation $\lambda_1 * \lambda_2$ on $\mathcal{L}$ by

$$\lambda_1 * \lambda_2(x) = \lambda_1(x) + \lambda_2(x - m_1) \quad \text{whenever} \quad \lambda_1(m_1) = \lambda_2(0) .$$

One checks that $\lambda_1 * \lambda_2$ satisfies all the restrictions. In particular

$$\lambda * \lambda = \lambda^{2*} \quad \text{if} \quad \lambda(0) = \lambda(m) .$$

We define a programmed load as the repetition of a spectrum through its indefinite continuation. It is thus a periodic continuous real-valued piecewise linear function on the real line of infinite length. Let $\lambda \in \mathcal{L}$ be given with length $m > 0$; then $\underline{\lambda}$ is the programmed load with spectrum λ whenever $\underline{\lambda} = \lambda_1 * \lambda_2 * \cdots$, where

$$\lambda_1 = \lambda \quad \text{and} \quad \lambda_{j+1}(x) = \lambda_j(x - m) \quad \text{for} \quad j = 1,2,\ldots .$$

Thus λ_j represents the jth cycle of the programmed spectrum λ. Let λ^i_{j+1} be the partial repetition (i.e., the load history) up through the ith load fluctuation of the $(j+1)^{st}$ cycle; that is, λ^i_{j+1} is the restriction of λ_{j+1} in domain to the interval $(jm, jm+i)$ for $j = 0,1,\ldots$.

We now make the same probabilistic assumptions as in [5], namely,

1) For given $\lambda \in \mathcal{L}$ the ith incremental crack extension during the last fluctuation of the loading history λ^i is a nonnegative random variable $Z^i(\lambda^i)$ depending only upon λ^i. The random variables $Z^i(\lambda^i)$ for $i \geq 1$ are statistically independent.

This assumption implies the statistical independence of the crack extensions in each fluctuation not only from each other but from the total crack length as well. Of course we do not preclude functional dependence between successive distributions, given the programmed load. In this manner the dependence upon the order of loads is retained.

We assume as in [5] that

2) the incremental growth random variable $Z(\lambda)$ for any loading history λ has a survival distribution $R(\cdot:\lambda)$ which satisfies the inequality, for $x > 0$,

$$R(x:\lambda) \int_0^\infty R(t:\lambda)dt \geq \int_0^\infty R(t+x:\lambda)dt \ .$$

For a more nearly complete discussion and intuitive interpretation of this see [5].

We now make the critical assumption, which replaces the original interpretation of Miner's rule now considered to be incorrect, namely, that the load oscillations may be permuted in their order in any cycle without altering the resulting fatigue damage; see [5]. This assumption allows the resolution of a load history into an equivalent oscillation in terms of fatigue damage, i.e.,

3) There exists a finite set of loading oscillations, say

$$\Omega = \{\omega_0, \ldots, \omega_\kappa\}$$

such that for any programmed loading history λ^i there exists an equivalent $\omega_j \in \Omega$ for which, in distribution,

$$Z(\lambda^i) = Z(\omega_j^{i*}) \ .$$

The notation will be $\lambda^i \simeq \omega_j^{i*}$.

Consider a given $\lambda \in \mathcal{L}$ of length m . Then

$$Y_j(\lambda) = \sum_{i=1}^{m} Z_j^i(\lambda^i)$$

is the random crack extension under the jth cyclic repetition of λ, where different affixes i and j on Z_j^i indicate independent replications of the corresponding random variables.

Now we define the total crack length at the end of n cyclic repetitions of the load spectrum λ as

$$S_n(\lambda) = \sum_{j=1}^{n} Y_j(\lambda)$$

and

$$EY_j(\lambda) = \mu(\lambda) , \quad \text{var}[Y_j(\lambda)] = \sigma^2(\lambda) \quad \text{for} \quad j = 1,2,\ldots .$$

It follows from the independence that

$$\mu_n(\lambda) = ES_n(\lambda) = n\mu(\lambda) , \quad \sigma_n^2(\lambda) = \text{Var}[Y_j(\lambda)] = n^2\sigma^2(\lambda)$$

are the expected length and the variance of the crack at the nth cyclic repetition of the load spectrum λ.

Let W be the critical crack length, the occurrence of which defines failure, and assume

4) the random variable W is statistically independent of the crack length $S_n(\lambda)$ for each $n = 1,2,\ldots$.

From these assumptions let us define the random number $N(\lambda)$ of repetitions of the spectrum λ which occur before failure by

$$[N(\lambda) = n] = [S_{n-1}(\lambda) < W , S_n(\lambda) \geq W] .$$

Then, denoting by

$$EZ_1(\omega_j) = \mu(\omega_j) \ , \ \nu_j = EN(\omega_j) \ \text{ for } \ j = 1,\cdots,\varkappa \ ,$$

the information obtained from the Wöhler diagram of the distinguished set Ω of loading oscillations, we obtain, by the arguments in [5] and [12], the results which are a generalization of Miner's rule, namely,

$$EN(\lambda) = 1/ \sum_{j=1}^{\varkappa} \frac{n_j(\lambda)}{\nu_j}$$

where

$$n_j(\lambda) = \sum_{i \geq 1}^{\cdot\cdot} \{\lambda^i \simeq \omega_j^{i*}\}$$

and $\{\pi\}$ is the indicator of any relation π, being one if true and zero otherwise. In words $n_j(\lambda)$ is the number of cycles in the spectrum λ occurring in their respective order, which are equivalent in damage (or crack extension) with each cycle of the jth load in continuous repetition.

As a matter of historical comparison we recall the following.

The Miner-Palmgren rule said that if $\varkappa$ different load cycles are considered and ν_i equals the number of cycles to failure under repetition of the ith load cycle, then a loading spectrum which contains n_i applications of the ith cycle for $i = 1,\ldots,\varkappa$, can be repeated ν times until failure where

$$\nu = \frac{1}{\sum_{i=1}^{\varkappa} \frac{n_i}{\nu_i}} \ .$$

The original interpretation was that this was deterministic.

A main purpose of the subsequent investigation is to obtain information so that $n_j(\lambda)$ can be obtained in closed form. The relationship of these concepts to the distribution obtained in [12] and [13] is readily apparent but its study will not be undertaken in this paper.

SECTION IV

THE THEORETICAL BEHAVIOR OF FATIGUE CRACK GROWTH

Let us consider the behavior of a fixed load spectrum λ of length m, i.e., let there exist m oscillations λ_i for $i = 1,\cdots,m$ such that $\lambda = \lambda_1 * \lambda_2 * \cdots * \lambda_m$ and the resulting programmed load $\underline{\lambda}$ which has been defined in a preceding section. After sustaining k oscillations of the load ($k = 1,2,\cdots$) we let u_k be the expected crack length and $\Delta u_k = u_k - u_{k-1}$ for $k \geq 1$ with $u_0 = 0$. We set, for all integers k,

$$a_k = \lambda(k+\tfrac{1}{2}) , \qquad b_k = \lambda(k) .$$

We graph one such typical spectrum of m oscillations.

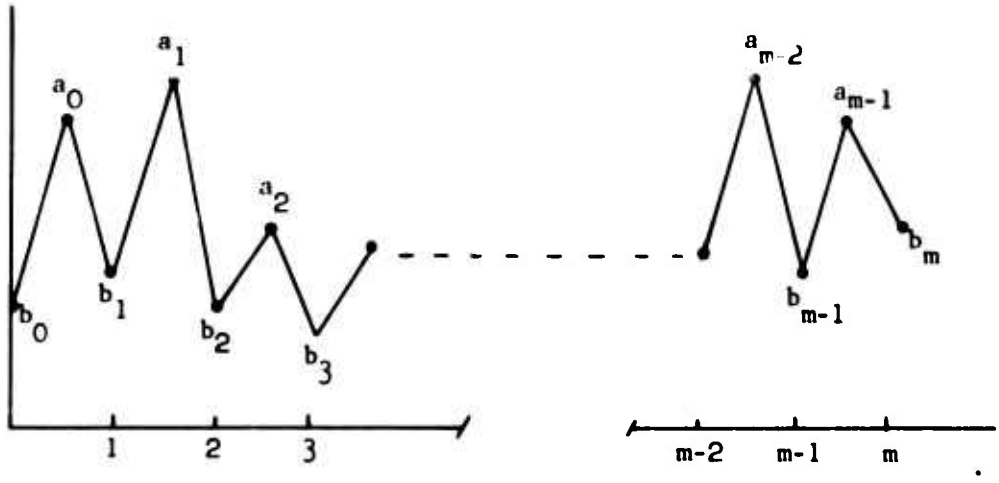

Figure 1

Thus the a_k's represent the maximum loads and the b_k's represent the minimum loads during the kth oscillation.

We now make the assumption: there exist two nonnegative functions f_1 and f_2, each a function of two variables, and a decreasing sequence of non-negative numbers $\{w_i\}_{i=1}^{\infty}$ which can be combined to give

$$\Delta u_k = [f_1(a_k,b_k) - \sum_{j=1}^{k} w_j f_2(a_{k-j}, b_{k-j})]^+ \qquad (1)$$

where $x^+ = \max(x,0)$ for any real x, and without loss of generality we take

$$\sum_{j=1}^{\infty} w_j = 1 \quad . \qquad (2)$$

This formulation is analogous to the metallurgical pronouncement that fatigue is a race between work hardening of the metal and the progressive relaxation of the yield stress. The first term $f_1(a_k,b_k)$ represents the propensity for propagation imposed by the kth oscillation, which we assume can be expressed as a function of the maximum and minimum stresses of that oscillation. The second term represents the cumulative effects of work hardening or resistance to crack propagation done in the preceding cycles. The question is whether such a simple mathematical model as presented in Eq(1) can explain all of the various phenomena encountered in the load order reactions of crack growth which have been discovered empirically; see [8] and [9].

The conditions under which crack growth was studied in [8] were steady-state conditions; namely, for any integer k

$$\Delta u_k = [f_1(\lambda_k) - \sum_{j=1}^{\infty} w_j f_2(\lambda_{k-j})]^+ \quad . \qquad (3)$$

Let us now turn to the mathematical formulation of the four empirical phenomena discussed in [8], to wit, crack deceleration, crack arrest, crack

acceleration and crack jump. To do this we consider the behavior of crack growth which has been experimentally determined in a specified aluminum alloy (namely, 2024-T3) under certain programmed loads. We now make

Remark 1: If we set $w_i = 0$ for $i \leq 0$ and define

$$\xi_i = \sum_{j=0}^{\infty} w_{jm+i} \quad \text{for any integer } i$$

then, if $i \leq 0$ and r is the least positive integer such that $rm + i \geq 0$, it follows that $\xi_i = \xi_{rm+i}$ and

$$\xi_1 \geq \xi_2 \geq \cdots \geq \xi_m \quad \text{and} \quad \sum_{i=1}^{m} \xi_{k-i} = 1 \quad \text{for any } k = 1,\cdots,m \ . \quad (4)$$

We now state

Lemma 1: The equation for incremental crack extension under steady-state conditions, given in Eq(3), may be rewritten as

$$\Delta u_k = [f_1(\lambda_k) - \sum_{j=0}^{m-1} \xi_{k-j} f_2(\lambda_j)]^+ \quad \text{for } k = 1,2,\cdots,m \ . \quad (5)$$

Proof: From Eq(3) we note that

$$\sum_{j=1}^{\infty} w_j f_2(\lambda_{k-j}) = \sum_{j=0}^{\infty} \sum_{i=1}^{m} w_{jm+i} f_2(\lambda_{k-jm-i}) \ .$$

Now using the periodicity of $\underline{\lambda}$, i.e., $\lambda_{jm+i} = \lambda_i$ for any integer j, we have the right hand side above

$$= \sum_{i=1}^{m} \sum_{j=0}^{\infty} w_{jm+i} f_2(\lambda_{m+k-i}) = \sum_{i=1}^{m} \xi_i f_2(\lambda_{m+k-i})$$

which by change of index is equal to Eq(5). ||

We examine first the programmed loads of constant amplitude which are elements of Ω . Fix $\omega \in \Omega$; then

$$\omega_0 = \omega_k \quad \text{for} \quad k = \pm 1, \pm 2, \cdots$$

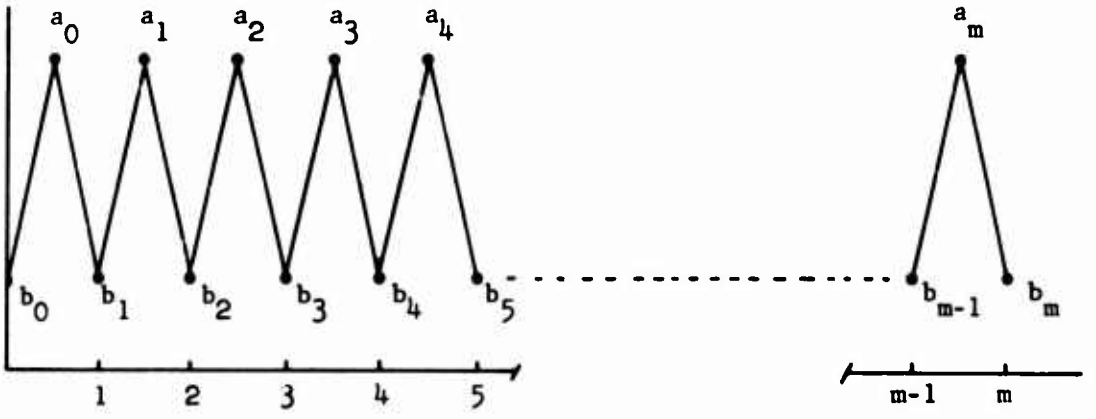

Figure 2

Here $a_0 = a_k$, $b_0 = b_k$ for $k = \pm 1, \pm 2, \cdots$.

From Remark 1 and Lemma 1 we have directly

Theorem 1: Fatigue crack growth will occur under a constant amplitude load with programmed oscillation ω_0 if and only if

$$1 < \frac{f_1(\omega_0)}{f_2(\omega_0)} \quad , \tag{6}$$

and if this obtains the crack growth rate is constant and equal to

$$\Delta u_k = f_1(\omega_0) - f_2(\omega_0) \quad \text{for} \quad k = 1, \cdots, m \quad . \tag{7}$$

Following our notational convention $f_1(\omega_0) = f_1(a_0, b_0)$ where a_0 is the maximum load and b_0 is the minimum load during load fluctuations, i.e.,

$$\omega_0(0) = b_0 \, , \quad \omega_0\left(\tfrac{1}{2}\right) = a_0 \, ,$$

we assume the following monotone behavior of the functions f_1 and f_2 for fixed $a_0, b_0 > 0$

$f_1(x,b_0)$ and $f_2(x,b_0)$ are monotone increasing for $x > b_0$,

while

$f_1(a_0,y)$ and $f_2(a_0,y)$ are monotone decreasing for $0<y<a_0$.

In order to avoid difficulties we shall presume that the maximum load a_0 is well under the yield stress of the material. In addition, if $a_0 = b_0$, there will be no load fluctuations (and hence no fatigue), and as a consequence we must have

$$1 \geq \frac{f_1(a,a)}{f_2(a,a)} \quad \text{for all admissible } a .$$

We next consider the programmed load which contains one peak overload.

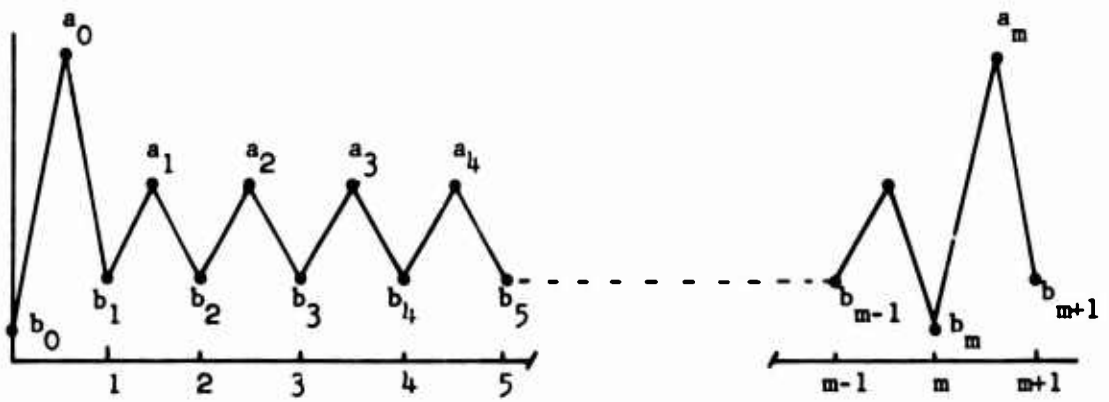

Figure 3

Here $\lambda_i = \lambda_1$ for $i = 2,\cdots,m-1$ but λ_0 is different, namely,

$$a_0 \geq a_1 , \qquad b_0 \leq b_1 .$$

where at least one of the inequalities must be strict; otherwise, $\lambda_0 = \lambda_1$.

It is known from [8] that the phenomena of crack deceleration and/or

crack arrest can occur following such a peak load. We now determine the behavior of our model under a programmed load of the type given in Figure 3.

Theorem 2: Following a peak overload on the last oscillation of a cycle of length m, the crack will arrest for $k \leq m-1$ oscillations if and only if

$$\frac{f_1(\lambda_1) - f_2(\lambda_1)}{f_2(\lambda_0) - f_2(\lambda_1)} \leq \xi_i \qquad \text{for } i = 1,\cdots,k \tag{8}$$

and a necessary and sufficient condition that the crack will advance subsequently in the cycle is that the reverse inequality holds for $i = k+1,\cdots,m-1$.

Proof: For any $j = 1,\cdots,m$ we know

$$\Delta u_j = [f_1(\lambda_j) - \sum_{i=0}^{m-1} \xi_{j-i} f_2(\lambda_i)]^+ \quad .$$

Since $\lambda_i = \lambda_1$ for $i = 1,\cdots,m-1$, we have

$$\Delta u_j = [f_1(\lambda_1) - \sum_{i=1}^{m-1} \xi_{j-i} f_2(\lambda_1) - \xi_j f_2(\lambda_j)]^+ \quad .$$

By adding and subtracting $\xi_j f_2(\lambda_1)$ and using Eq(4) we obtain

$$\Delta u_j = [f_1(\lambda_1) - f_2(\lambda_1) - \xi_j (f_2(\lambda_0) - f_2(\lambda_1))]^+ \quad . \tag{9}$$

Again using Eq(4) we let k be the largest value of j such that Eq(8) holds, and the result follows by the monotonicity of ξ_j . ||

We comment that from the above result and the postulated monotonicity of $f_2(\cdot, b_0)$ there will be a range of values for which crack arrest will occur and another range of values where the crack advance will only be reduced. We have the immediate

Corollary 1: Under the conditions of Theorem 2 the total crack advance during each cycle of the spectrum is

$$u_m - u_0 = (m-k)[f_1(\lambda_1) - f_2(\lambda_1)] - [f_2(\lambda_0) - f_2(\lambda_1)] \sum_{j=k+1}^{m} \xi_j$$

where k was defined in Eq(8).

Proof: Since $u_m - u_0 = \sum_{j=1}^{m} \Delta u_j$ the result follows from Eq(9) and the definition of k given in Eq(8). ||

We now consider the phenomenon of crack acceleration which may occur in the programmed load of the following type:

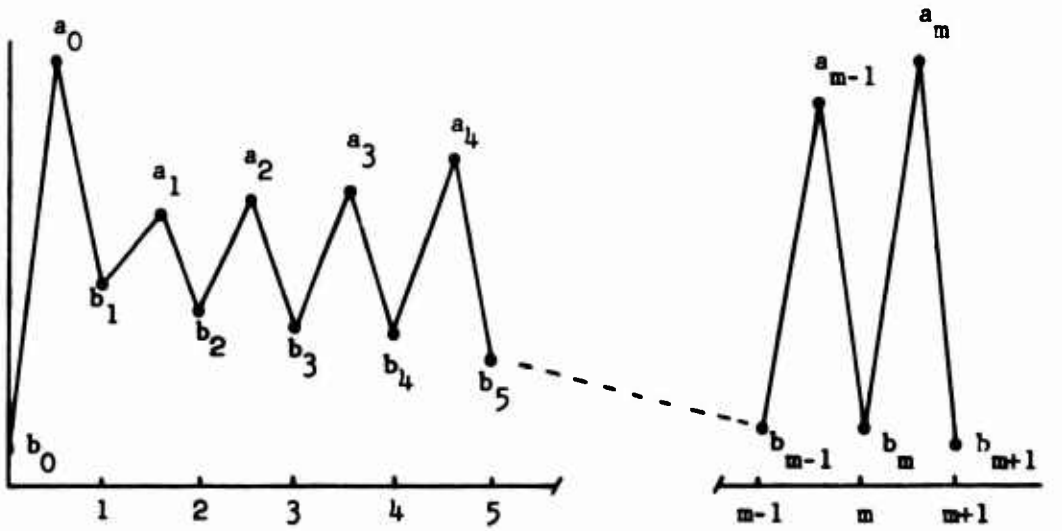

Figure 4

If we set

$$\rho_j = \xi_j - \xi_{j+1} \quad \text{for} \quad j = 1,\cdots,m-1\ , \quad \rho_m = 1 + \xi_m - \xi_1\ , \qquad (10)$$

then from Eq(4) we know $\rho_i \geq 0$ for $i = 1,\cdots,m$ and $\sum_{1}^{m} \rho_j = 1$ and $\rho_{-j} = \rho_{m-j}$.

Under a programmed load of the type seen in Figure 4 we can state a result about crack acceleration in

Theorem 3: If the maximum stress and stress variation in each oscillation of a cycle are such that for some $k = 1,\cdots,m-1$ we have $\Delta u_k > 0$ and

$$\sum_{j=1}^{m} f_2(\lambda_{k-j})\rho_j > f_2(\lambda_k) \tag{11}$$

where ρ_j for $j = 1,\cdots,m$ are defined in Eq(10), then the crack growth accelerates, i.e.,

$$\Delta u_{k+1} > \Delta u_k \quad . \tag{12}$$

Proof: Consider the condition under which $\Delta u_{k+1} > \Delta u_k$, i.e.,

$$f_1(\lambda_{k+1}) - \sum_{j=0}^{m-1} \xi_{k+1-j} f_2(\lambda_j) > f_1(\lambda_k) - \sum_{j=0}^{m-1} \xi_{k-j} f_2(\lambda_j) \quad .$$

Clearly $f_1(\lambda_k)$ is increasing for $k = 1,\cdots,m$. Now we see that

$$\sum_{j=0}^{m-1} \xi_{k-j} f_2(\lambda_j) - \sum_{j=0}^{m-1} \xi_{k+1-j} f_2(\lambda_j) \geq 0$$

if and only if

$$\sum_{j=0}^{m-1} \rho_{k-j} f_2(\lambda_j) \geq f_2(\lambda_k)$$

by adding $f_2(\lambda_k)$ to both sides and checking the definition in Eq(10). Thus Eq(11) is a sufficient condition for Eq(12). ||

One notes that Eq(11) would be satisfied if the maximum stress and the stress variation increased sufficiently in each oscillation.

SECTION V

A PARAMETRIC MODEL FOR CRACK GROWTH

Let us now postulate the form of the two functions which govern fatigue crack growth; viz., for $k = 1, 2$,

$$f_k(a,b) = A_k a^{\alpha_k}(a - b)^{\beta_k} \quad \text{for} \quad a \geq b \geq 0 \tag{13}$$

where A_i, α_i, β_i for $i = 1,2$ are unknown parameters, assumed to be positive, which must yet be determined. It is clear that these functions both satisfy the monotonicity requirements specified in the preceding section.

From Theorem 1 we know that for a programmed load of constant maximum stress a and minimum stress b the satisfaction of the condition

$$1 < \frac{f_1(a,b)}{f_2(a,b)} = \frac{A_1}{A_2} a^{\alpha_1-\alpha_2}(a - b)^{\beta_1-\beta_2}$$

is a necessary and sufficient condition that the crack advance. We thus assume that

$$0 < A_1 < A_2 , \quad \alpha_1 \geq \alpha_2 > 0 , \quad \beta_1 \geq \beta_2 > 0 \quad . \tag{14}$$

We note that as a consequence of the formulation of Eq(13) the "endurance limit", i.e., a value of the maximum stress below which fatigue will never occur (we say "never" because in this model we are not considering any stochastic variation in the damage caused by each oscillation of the load), will be altered if one takes different values of the minimum stress unless we assume $\beta_1 = \beta_2$. One of the consequences of the basic stochastic nature of the fatigue process is that such questions as whether or not some parameter is zero or just very small cannot be resolved by physical measurement.

Subsequently we propose to make the simplest mathematical assumptions possible which are consistent with the facts.

The crack growth rate per cycle, under constant amplitude loading, assuming it takes place, is

$$\Delta u = f_1(a,b) - f_2(a,b) = A_1 a^{\alpha_1}(a-b)^{\beta_1} - A_2 a^{\alpha_2}(a-b)^{\beta_2} .$$

The claim of reference [11] is that for a fixed maximum stress the logarithm of the crack growth rate per cycle is nearly a linear function of the logarithm of the stress variation. However, the evidence also shows that this slope increases as the variation in stress decreases (i.e., the slope is a decreasing function of the variation in stress); see p. 33 loc. cit.

Letting $x = \ln(a-b)$ with a fixed, we find as a function of x

$$\Delta u = A_1' e^{\beta_1 x}[1 - Ke^{-\epsilon x}] \tag{15}$$

for some constants

$$A_1' = Aa^{\alpha_1} , \quad K = \frac{A_2}{A_1} a^{\alpha_2 - \alpha_1} , \quad \epsilon = \beta_1 - \beta_2 \geq 0 . \tag{16}$$

We set

$$\frac{\partial \ln \Delta u}{\partial x} = \beta_1 + \frac{\epsilon K}{e^{\epsilon x} - K} \equiv p(x)$$

and we check that $p'(x) < 0$.

We have verified that the model Eq(13) behaves in this regard as does the actual fatigue crack itself as recorded in [11]. Now we verify another experimentally determined condition concerning how the growth rate behaves as a function of the maximum stress when the stress variation is held constant.

It is stated (p. 33 loc. cit.) that the logarithm of the crack growth rate is again approximately a linear function of the maximum stress but that the slope is a decreasing function of this maximum stress.

Letting $y = \ln a$ with fixed difference $a - b$, we have

$$\Delta u = A_1'' e^{\alpha_1 y} [1 - K' e^{-\delta y}] \tag{17}$$

where

$$A_1'' = A_1 (a-b)^{\beta_1} > 0 , \quad K' = \frac{A_2}{A_1} (a-b)^{\beta_2 - \beta_1} > 0 ,$$

and $\delta = \alpha_1 - \alpha_2 \geq 0$ are constants. By comparison with the preceding results we see that the requisite behavior of the growth rate follows also in this case.

Where we disagree with the conclusion of McMillan and Pelloux, p. 57 #3 in [11] is in taking the crack growth Δu to be of the same parameteric form as f_1 and f_2. Not only does this form seem unsubstantiated, but it does not allow for crack retardation.

The similarity of these two cases, Eq(15) and Eq(16), prompts us to propose the same method for the determination of the unknown parameter for the case of constant amplitude loading.

Theorem 2: A necessary and sufficient condition that there exist parameters $\epsilon > 0$, $K > 0$ such that the convex function q defined by

$$q(x) = \frac{K\epsilon}{e^{\epsilon x} - K} \qquad \text{for} \quad c \leq x \leq d \tag{18}$$

satisfies the conditions $q(c) = C > q(d) = D$ is that

$$d - c = \frac{1}{D} - \frac{1}{C} . \tag{19}$$

In that case ϵ is the solution of the equation $g(y) = 0$, where

$$g(y) = [1 + \frac{y}{C}]De^{y(d-c)} - D - y \ , \qquad (20)$$

and in terms of ϵ

$$K = \frac{Ce^{\epsilon c}}{C + \epsilon} \ . \qquad (21)$$

Proof: The two conditions $q(c) = C$, $q(d) = D$ yield the two equations

$$C(e^{\epsilon c} - K) = K\epsilon \ , \quad D(e^{\epsilon d} - K) = K\epsilon \ .$$

Solving for K in the first equation yields Eq(21). Using this to eliminate K in the second yields $g(\epsilon) = 0$ with g defined in Eq(20). Now g is a convex function such that $g(0) = 0$, $g(\infty) = \infty$. For $\epsilon \geq 0$ we must have $g'(0) < 0$, which is the condition of Eq(19). ‖

Lemma 2: If ϵ_0 is given, more accurate approximations can be found by Newton-Raphson iteration for $n = 0,1,\cdots$:

$$\epsilon_{n+1} = \frac{\epsilon_n(d - c)[1 + \epsilon_n/C] - 1 + e^{-\epsilon_n(d-c)}}{(d - c)[1 + \epsilon_n/C] + 1/c - \frac{1}{D}e^{-\epsilon n(d-c)}} \qquad (22)$$

and $\epsilon_n \to \epsilon$ as $n \to \infty$.

Proof: This follows directly from $\epsilon_{n+1} = \epsilon_n - \frac{g(\epsilon_n)}{g'(\epsilon_n)}$ and the nature of g . ‖

From the extant data we now seek to establish values of the parameters which determine both f_1 and f_2 . We first attempt to analyse the results of programmed spectra with constant maximum loads and variable amplitude. Unfortunately measurements of incremental crack advance for programmed spectra of only constant amplitude loading are not available in [11] and comparable experimental results of sufficient accuracy are not readily available elsewhere. Moreover, in [11] the crack growth rates for constant maximum loads with variable amplitude but with load order interchanged were not separated, and the data on the

slope of the logarithm of the crack advance rate were put together. This confounding of the data, we believe, does not interfere in our determination of the parameters. This is one of the more interesting, or fortuitous, results of this investigation, especially since the inference from the model agrees with the empirical conclusions of other investigators.

SECTION VI

THE DETERMINATION OF PARAMETERS USING SPECTRA OF CONSTANT MAXIMUM STRESS BUT VARIABLE AMPLITUDE

We first quote a statement of McMillan and Pelloux [11] p. 33, l. 8 et seq. concerning spectra of the type under discussion. "There was a certain scatter in the data but it showed that for constant S_{max} (maximum stress) the following relative crack rate equation could be written

$$\text{rate} = \text{constant} \quad (\Delta S)^n \tag{23}$$

(where ΔS is the stress variation) with $n = 1$ for ΔS between eight and twelve thousand psi and increasing from two to four for ΔS lower than eight thousand psi."

By comparing Eq(15) with Eq(23) we see, in our terminology, we wish to choose the free parameters so that, letting x denote the ℓn-stress range, we have

$$p(x) \geq 1 \quad \text{for} \quad \ell n(8 \times 10^3) \leq x \leq \ell n(12 \times 10^3)$$

but moreover so that $p(x)$ increases to between 2 and 4 whenever $x \leq \ell n(8 \times 10^3)$. In order to utilize Lemma 2 we set

$$\begin{aligned} d &= \ell n(12 \times 10^3) & c &= \ell n(6 \times 10^3) \\ p(d) &= 3/2 & p(c) &= 3 \\ \therefore \quad D &= 3/2 - \beta_1 & C &= 3 - \beta_1 \ . \end{aligned} \tag{24}$$

Unfortunately the scatter in the data will prohibit an easy and exact determination of the disposable parameters. (This is indicated in the paragraph following the one quoted.) Nevertheless, so that we retain the monotonicity which $p(\cdot)$ should exhibit over its range, our latitude in the

selection of β_1 must be restricted in order for its behavior to remain consistent with the facts.

We first must check that Eq(19) is satisfied, namely,

$$(3 - \beta_1)(\tfrac{3}{2} - \beta_1) \leq \frac{3}{2\ln 2} = 2.16404 \ .$$

Analysis reveals that this inequality is satisfied for $0.618 < \beta_1 < 3.88$. In order that $D \geq 0$ we must keep $0.618 < \beta_1 < 1.5$. It follows, since $\epsilon = \beta_1 - \beta_2$ and $\beta_2 \geq 0$, that $\epsilon \leq \beta_1$. Since g is convex, a necessary and sufficient condition for this is that $g(\beta_1) \geq 0$. Substituting Eq(24) into Eq(20), we see that

$$g(\beta_1) \geq 0 \quad \text{if and only if} \quad e^{\beta_1 \ln 2} \geq \frac{3 - \beta_1}{3 - 2\beta_1} \ .$$

One sees that this is true for $0 \leq \beta_1 \leq 1$. Thus the permissible range of β_1 is only

$$0.62 \leq \beta_1 \leq 1 \ . \tag{25}$$

We pick

$$\beta_1 = 1 \quad \therefore \ D = \frac{1}{2} \ , \quad C = 2 \ .$$

Utilizing Lemma 2 to calculate ϵ , we first guess $\epsilon_0 = 1$ and, to our surprise, calculate

$$\epsilon_1 = \frac{3(\ln 2) - 2 + 2e^{-\ln 2}}{3(\ln 2) + 1 - 4e^{-\ln 2}} = 1 \ .$$

Therefore we have

$$\epsilon = 1 \ , \quad \therefore \ \beta_2 = 0 \ . \tag{26}$$

From the determination just made we have

Lemma 3: For maximum stress a and minimum stress b the functions f_1 and f_2 extending and resisting crack growth are

$$f_1(a,b) = A_1 a^{\alpha_1}(a - b) , \quad f_2(a,b) = A_2 a^{\alpha_2}$$

where α_1 , α_2 are yet unknown parameters.

Thus we have the following conclusion on the behavior of crack growth which we state in

Theorem 3: Under a spectrum λ of constant maximum stress and variable stress amplitude, i.e.,

$$\lambda(k+\tfrac{1}{2}) = a \qquad k = 1,\cdots,m$$
$$\lambda(k) = b_k$$

the crack growth for the kth oscillation is independent of $\xi_1,\cdots,\xi_m$ and is given by

$$\Delta u_k = [A_1 a^{\alpha_1}(a - b_k) - A_2 a^{\alpha_2}]^+ \qquad k = 1,\cdots,m .$$

Proof: From the general result on steady state crack advance in Lemma 1 we have

$$\Delta u_k = [f_1(\lambda_k) - f_2(\lambda_j) \sum_{j=0}^{m-1} \xi_{k-j}]^+$$

but by Lemma 3 we know f_2 depends only upon the maximum stress and not upon the stress range. But for all $k = 1,\cdots,m$ we know $\sum_{j=0}^{m-1} \xi_{k-j} = 1$ and we have the result. ||

A conclusion of this theorem is that any spectrum with a constant maximum stress at each oscillation but variable minimum stress would yield

the same total crack advance per spectrum independently of the order in which the loads are applied.

This mathematical result coincides with the empirical conclusion of p. 30, l. 14 et seq. [11]:

"This shows that in the tests at constant maximum load the sequence of load application did not measurably influence the overall crack growth rates. This was confirmed by the fracture analysis with the electron microscope:

1. There was no marked crack front advance at the change of load amplitude between different spectra.

2. After a change of load amplitude, the spacing of the striations for the next load amplitude sequence reached a stable and uniform value on the first cycle of the new load amplitude sequence."

We conclude this section with the following calculation: Since $c = \ln(6 \times 10^3)$ for this series of experimental loading programs, P1, P2, P3, P4, we substitute the known parameters into Eq(21) to obtain

$$K = \frac{Ce^{\epsilon c}}{c + \epsilon} = \frac{2}{3} e^{\ln(6 \times 10^3)} = 4 \times 10^3 \ . \qquad (27)$$

From the definition of Eq(16) and the fact that the constant value of the maximum load was $a = 12 \times 10^3$, we have a determination of the ratio

$$\frac{A_2}{A_1} = K(12 \times 10^3)^{\delta} \ . \qquad (28)$$

Once a value of δ is determined this ratio can be calculated.

SECTION VII

THE DETERMINATION OF PARAMETERS USING SPECTRA OF CONSTANT STRESS VARIATION BUT VARIABLE MAXIMUM STRESS

We quote a conclusion of McMillan and Pelloux [11] p. 39, l. 8 et seq. with regard to programs P6, P7, P8, P9.

"The average microscopic rates of crack growth per load cycle for Spectra A and B of Programs P6, P7, P8, and P9 were measured within each program and compared with the larger growth rate (taken as unity) corresponding to S max = 14,000 psi. The effect of the maximum load amplitude on the relative crack growth rate per cycle is shown in Figure 6. The scatter was quite large, but a relationship of the type:

$$\text{rate} = \text{constant } (S \text{ max})^m$$

could be written to represent relative crack growth rate. At a given crack length and for a constant maximum load, m varied from two to three at large S max to three to five at smaller S max. In the same figure the relative microscopic rates confirm the fact that P9 had a larger growth rate than P8. This can be explained by assuming a crack growth retardation in Spectrum B of P8 because Spectrum B followed the higher load level of Spectrum A. The crack growth retardation could be due to a work softening of the plastic zone by Spectrum B and a consequent lower crack growth rate."

We now attempt to utilize the same procedure employed in the preceding case to determine the unknown parameters. Unfortunately, there are no data on crack growth rates for spectra each of the same constant stress variation but with different and constant maximum stress. The fractographic difficulty with analyzing such a spectrum would be that without some measurable stress variation in each program the appropriate section of the fracture face could

not be made to correspond with the imposed load. However, since only sections of spectra of constant stress variation could be identified with the crack growth measured, there is certainly an influence due to the preceding blocks of higher or equal maximum stress.

Consider

$$\Delta u_k = \left[f_1(\lambda_k) - \sum_{j=0}^{m-1} \xi_{k-j} f_2(\lambda_j) \right]^+ .$$

Since, by definition, $a_k - b_k$ is constant for $k = 1,\cdots,m$, we set

$$A_1'' = A_1(a_k - b_k) , \qquad K' = A_2/A_1'' ;$$

then utilizing the results of Theorem 3 we have

$$\Delta u_k = A_1'' a_k^{\alpha_1} \left[1 - \sum_{j=0}^{m-1} \xi_{k-j} K' a_j^{\alpha_2 - \alpha_1} \right]^+ .$$

Let a denote the constant value of the maximum stress within the block in the spectrum which is of interest. By examining the programmed loads P6, P7, P8, P9 we see

$$\sum_{j=0}^{m-1} \xi_{k-j} a_j^{-\delta} \geq a^{-\delta}$$

where

$$\delta = \alpha_1 - \alpha_2 .$$

It follows that

$$\Delta u_k \leq A_1'' a^{\alpha_1} [1 - K' a^{-\delta}] .$$

Thus the true crack growth rate within the block would be less than that in a corresponding constant amplitude test with the same maximum stress. To

proceed in the same manner as before we write .

$$\Delta u = A_1'' a^{\alpha_1}[1 - K' a^{-\delta}] .$$

and letting $y = \ell n\ a$ we have

$$\frac{\partial \ell n\ \Delta u}{\partial y} = \alpha_1 + \frac{K'\delta}{e^{\delta y} - K'} \equiv p^*(y) .$$

In order to utilize Lemma 2, we attempt to compensate for the reduction in the rate as demonstrated above. Accordingly we interpret the data to adjust by setting

$$c = \ell n(10^4) \qquad d = \ell n(1.4 \times 10^4)$$

$$p^*(c) = 5 \qquad p^*(d) = 3$$

$$\therefore \quad C = 5 - \alpha_1 \qquad D = 3 - \alpha_1 .$$

We now check that Eq(19) is satisfied, namely,

$$\ell n(1.4) < \frac{1}{3 - \alpha_1} - \frac{1}{5 - \alpha_1} .$$

Analysis shows this inequality satisfied for $1.355 < \alpha_1 < 6.644$ but since we must also have $D \geq 0$, we require $\alpha_1 \leq 3$. As before it follows, since $\delta = \alpha_1 - \alpha_2$ and $\alpha_2 \geq 0$, that $\delta \leq \alpha_1$. A necessary and sufficient condition for this is that $g(\alpha_1) \geq 0$ with the proper parametric values substituted into g . Using Eq(20) we see

$$g(\alpha_1) \geq 0 \quad \text{if and only if} \quad e^{\alpha_1 \ell n 1.4} \geq \frac{3}{5}\left(\frac{5 - \alpha_1}{3 - \alpha_1}\right) .$$

Numerical analysis shows this to be true for $0 \leq \alpha_1 < 2.2-$. Thus the combined restriction on α_1 is

$$1.35+ < \alpha_1 < 2.2- . \qquad (29)$$

We pick

$$\alpha_1 = 2 \quad \therefore \; C = 3 \; , \; D = 1 \; . \tag{30}$$

We now check, using Lemma 2, that, if we select a value $\delta_0 = 1.5$, we find $\delta_1 = 1.4966$. Thus we take

$$\delta = 3/2 \quad \therefore \; \alpha_2 = \alpha_1 - \delta = 1/2 \; . \tag{31}$$

We recall that $c = \ln(10^4)$ and hence

$$K' = \frac{Ce^{\delta c}}{C+\delta} = \frac{2}{3} \times 10^6 \; .$$

Since the stress variation was $a_k - b_k = 7 \times 10^3$ psi, we have

$$K' = \frac{A_2}{A_1}(7 \times 10^3)^{-1}$$

and thus

$$\frac{A_2}{A_1} = K'(7 \times 10^3) = 4.666 \times 10^9 \; .$$

However, we have precisely calculated, in Eq(27), that

$$\frac{A_2}{A_1} = K(12 \times 10^3)^{\delta} = 5.259 \times 10^9$$

the last equality obtained by using $\delta = \frac{3}{2}$ and the value of K from Eq(26). There is a discrepancy of only about 13% between these two determinations. (The author does not pretend that these virtually coincidental values were obtained on the first iteration.)

The average value of the two determinations of A_2/A_1 is 4.962×10^9, and thus for our future calculations we take the ratio

$$\frac{A_1}{A_2} = 2 \times 10^{-10} \; . \tag{32}$$

The disparity between these two determinations could be reduced by ad hoc adjustment of the initial values and/or at the expense of the rational exponents within the model; however the appeal of such values comes not only from their simplicity but also from the possibility that the basic mechanism of fatigue can be explained in simple mechanical terms, whatever the material.

From our determination of the exponents we obtain

Lemma 4: For a maximum stress a and a minimum stress b within a load oscillation we have, for the functions extending and resisting crack growth,

$$f_1(a,b) = a^2(a-b) \ , \quad f_2(a,b) = \sqrt{a} \ .$$

We now state

Theorem 4: Under a spectrum λ of length m with

$$\lambda(k+\tfrac{1}{2}) = a_k \ , \quad \lambda(k) = b_k \quad \text{for} \quad k = 1,\cdots,m$$

the crack growth resulting from the kth oscillation is given by

$$\Delta u_k = A_2\left[\frac{A_1}{A_2}\, a_k^2(a_k - b_k) - \sum_{j=0}^{m-1} \xi_{k-j} a_k^{\frac{1}{2}}\right]^+ \ . \tag{33}$$

Note that the appropriate values of ξ_i have yet to be determined.

Turning our attention to the actual dimensions of the variables involved in the calculations, we find the measurements of Δu, the crack growth rate, are expressed in microns which range in value from 0.1 to 1 whenever a, the maximum stress, is measured in ksi (ksi equals a thousand psi) varying from 5 to 15, and $(a-b)$, the stress variation, is also expressed in ksi, varying from 2 to 12. This mixture of metric and English systems results from the calibration of the test instruments used. However, consistent dimensioning can be assured by proper choice of units to express the constants A_1 and A_2.

We have thus so far determined that under the conditions extant, namely, 2024-T3 aluminum alloy cycled in laboratory air at room temperature with stress being expressed in tens of ksi and a crack length between 0.6 and 1.7 inches, the incremental crack advance is given by

$$\Delta u_k = A_0\left[2a_k^2(a_k - b_k) - \sum_{j=0}^{m-1} \xi_{k-j}\sqrt{a_j}\right]^+ \qquad (34)$$

for steady state advance at the kth oscillation.

Note that the constant of proportionality A_0 and the "memory" constants ξ_i $(i = 1,\cdots,m)$ must be determined by other measurements using different data.

A comment is now in order. The mechanism of resistance to crack growth results from the plastic yield zone which precedes the crack tip. The size and shape of this zone are determined by the loads of the spectrum which have been previously applied, as well as the geometry of the specimen. It is admittedly a simplification to suppose that the crack growth resistance can be expressed as a one parameter function of the number and magnitude of the loads preceding. We feel that this simple supposition yields a more realistic description of the phenomenon than any model proposed heretofore.

We let the weighting factors be, for some $p < p < 1$,

$$w_i = (1 - p)p^{i-1} \qquad i = 1,2,\cdots .$$

Hence for a spectrum of length m we have

$$\xi_i = \frac{(1 - p)p^{i-1}}{1 - p^m} \qquad i = 1,\cdots,m \qquad (35)$$

and clearly $\sum_1^m \xi_i = 1$.

SECTION VIII

CONCLUSIONS

In this study we have concentrated on the problem of determining a formula for the effect of load order interactions on the incremental fatigue crack growth resulting from the load oscillations in a loading spectrum. We have utilized only data on aluminum 2024-T3 alloy.

We have argued that the incremental growth, if of the form

$$\Delta u_k = \left[A_1 a_k^{\alpha_1} (a_k - b_k)^{\beta_1} - A_2 \sum_{j=0}^{m-1} \xi_{k-j} a_j^{\alpha_2} (a_j - b_j)^{\beta_2} \right]^+ ,$$

where a_k is the maximum stress and b_k is the minimum stress for the kth load oscillation with $\xi_1, \cdots, \xi_n$ as memory constants, will satisfy all the observed phenomena of crack growth, namely, crack jump, crack arrest, and crack acceleration and deceleration.

Since there is no load order interaction in the crack growth data obtained when the maximum stress is held constant, the determination of the parameter values $\beta_1 = 1$ and $\beta_2 = 0$ by the method chosen would seem to have more assurance than the determination made subsequently of α_1 and α_2 from the data obtained when the stress variation was held constant.

This is because we must guess at the reduction caused through load order interactions, i.e., the net effect of the reduction factor $\sum_{j=0}^{m-1} \xi_{k-j} a_j^{-\delta}$ with $\delta = \alpha_1 - \alpha_2$, in choosing the values of the slope of the ℓn growth to be somewhat higher than the central values indicated by the data. This has resulted in the values $\alpha_1 = 2$, $\alpha_2 = 1/2$. The necessity of this extrapolation is a consequence of the type of spectrum which was

studied. It is clear that those experiments in [11] were not run with the idea of estimating the parameters of such a model as we have proposed but merely of strengthening the hypothesis that crack structures could be directly related to the load oscillations within each programmed load.

It is conceivable that subsequent data, more suitable for estimation, will reveal that these chosen values were in error. However, since there are additional parameters which are adjusted to fit the actual growth data, such a discrepancy could only be shown by prediction from our model to other situations.

We have determined the following formula for incremental crack growth for the kth load oscillation, based on the somewhat ambiguous data (from our point of view) of [11]:

$$\Delta u_k = A\alpha\left[2a_k^2(a_k - b_k) - \sum_{j=0}^{m-1} \xi_{k-j}\sqrt{a_j}\right]^+ \qquad k = 1,\cdots,m$$

where a and b are both expressed in tens of ksi.

We note that, if we had $\alpha_1 = 3$ (instead of 2), then for sinusoidal loading the incremental crack growth would be approximately of the form $\Delta u_k = \text{constant}\ a_k^4$, which would be consistent with the so called fourth power rule given by Paris [14].

On the basis of the above formula, if $\omega_0 \cdots \omega_\kappa$ is the set of constant amplitude spectra of different maximum stress for which the mean number of cycles until failure is known, then for a given programmed spectrum λ with history until the kth oscillation, namely,

$$\lambda^k = (\lambda_1,\cdots,\lambda_k) ,$$

we have the equivalence relation

$$\lambda^k \simeq \omega_j^k \quad \text{if and only if} \quad \Delta u_k(\lambda) = \Delta u_k(\omega_j) \ .$$

Thus the formula

$$n_j(\lambda) = \sum_{k=1}^{m} \{\Delta u_k(\lambda) = \Delta u_k(\omega_j)\}$$

can be used to determine the expected life under repetition of the spectrum λ.

To show the overall agreement of the fitted crack growth model given in Eq(34) with the observed behavior, we exhibit in each of Figures 7, 8, 9, 10 and 11 two calculations (made with two different values of the memory constant p as defined in Eq(35)) of the incremental crack growth resulting from the programmed loads displayed. We also show the actual fracture surface of the aluminun alloy upon which the load was imposed. The photographs have been reproduced from reference [11] with the authors' permission.

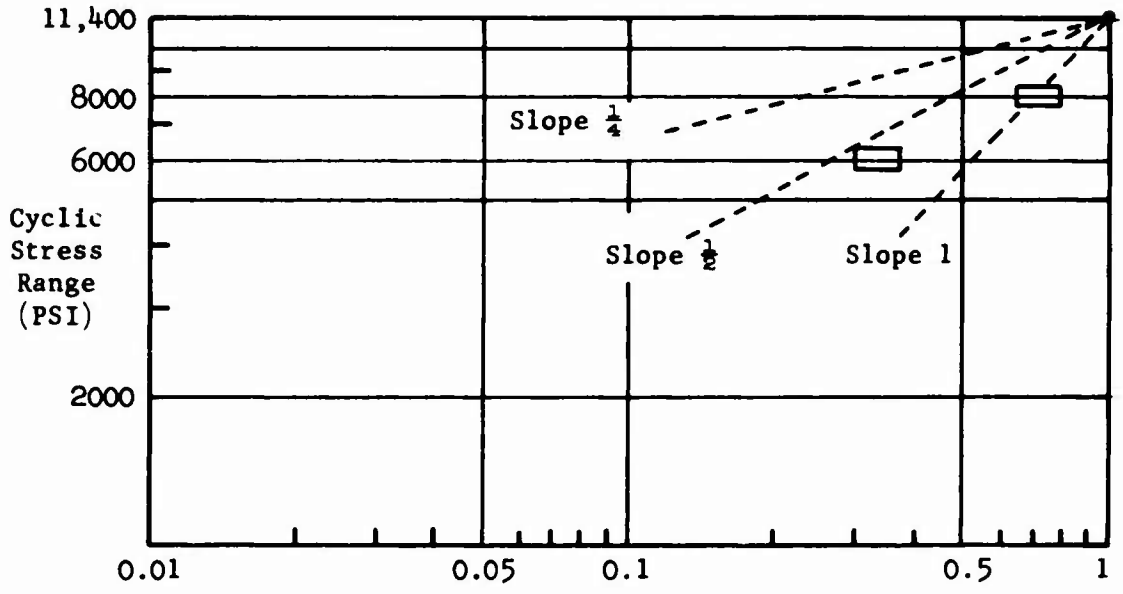

Figure 5

Scatter of Relative Crack Growth Rates at Constant Maximum Loads for Programs P1 , P2 , P3 , P4

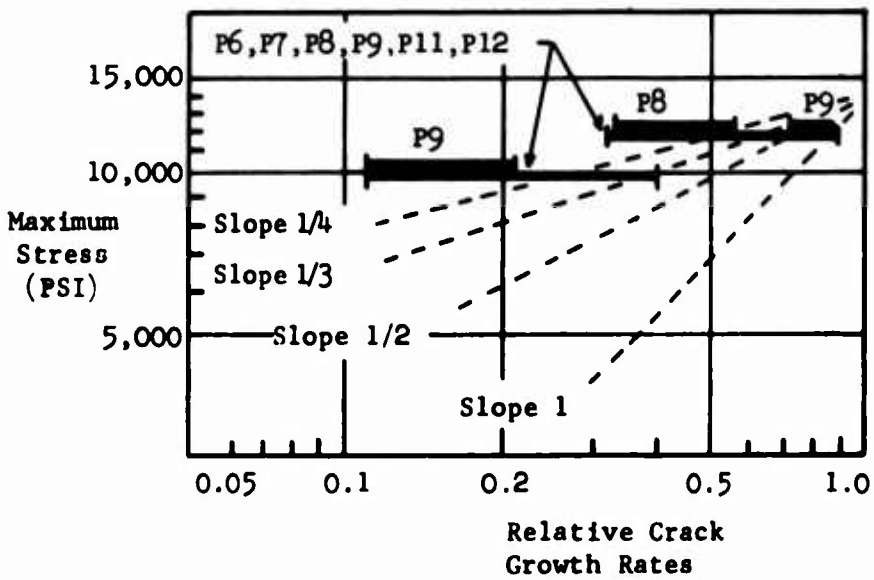

Figure 6

Scatter of Relative Crack Growth Rates at Constant Load Amplitude but Variable Maximum Load

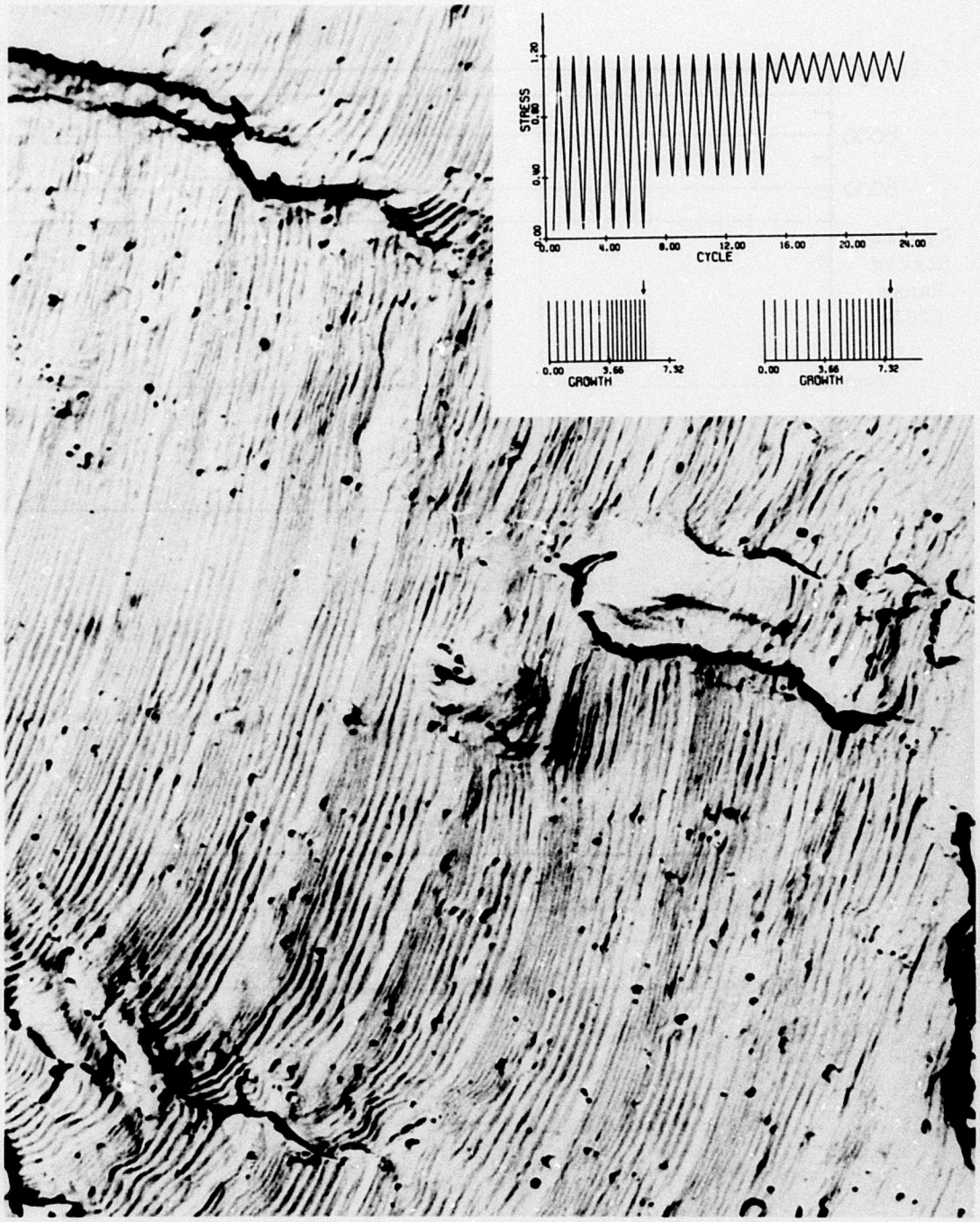

Figure 7

Typical fracture surface due to Program P 2, showing the uniformity of striation spacing during each of the first two groups of oscillations. Note the mathematical model predicts crack arrest during the third group of oscillations.

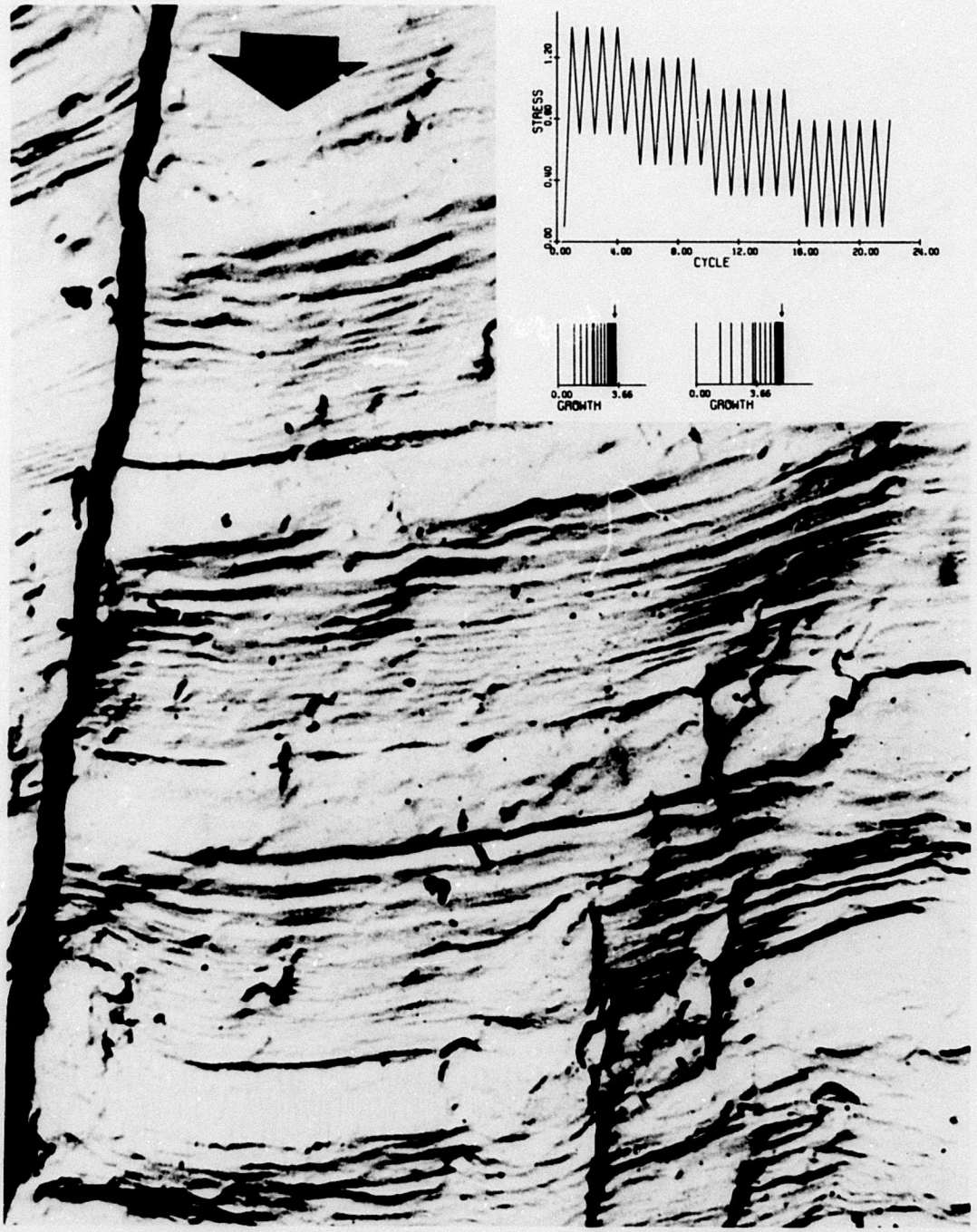

Figure 8

Typical fracture surface due to Program P 6. Note the large crack jump during the first oscillation, which the mathematical model predicts.

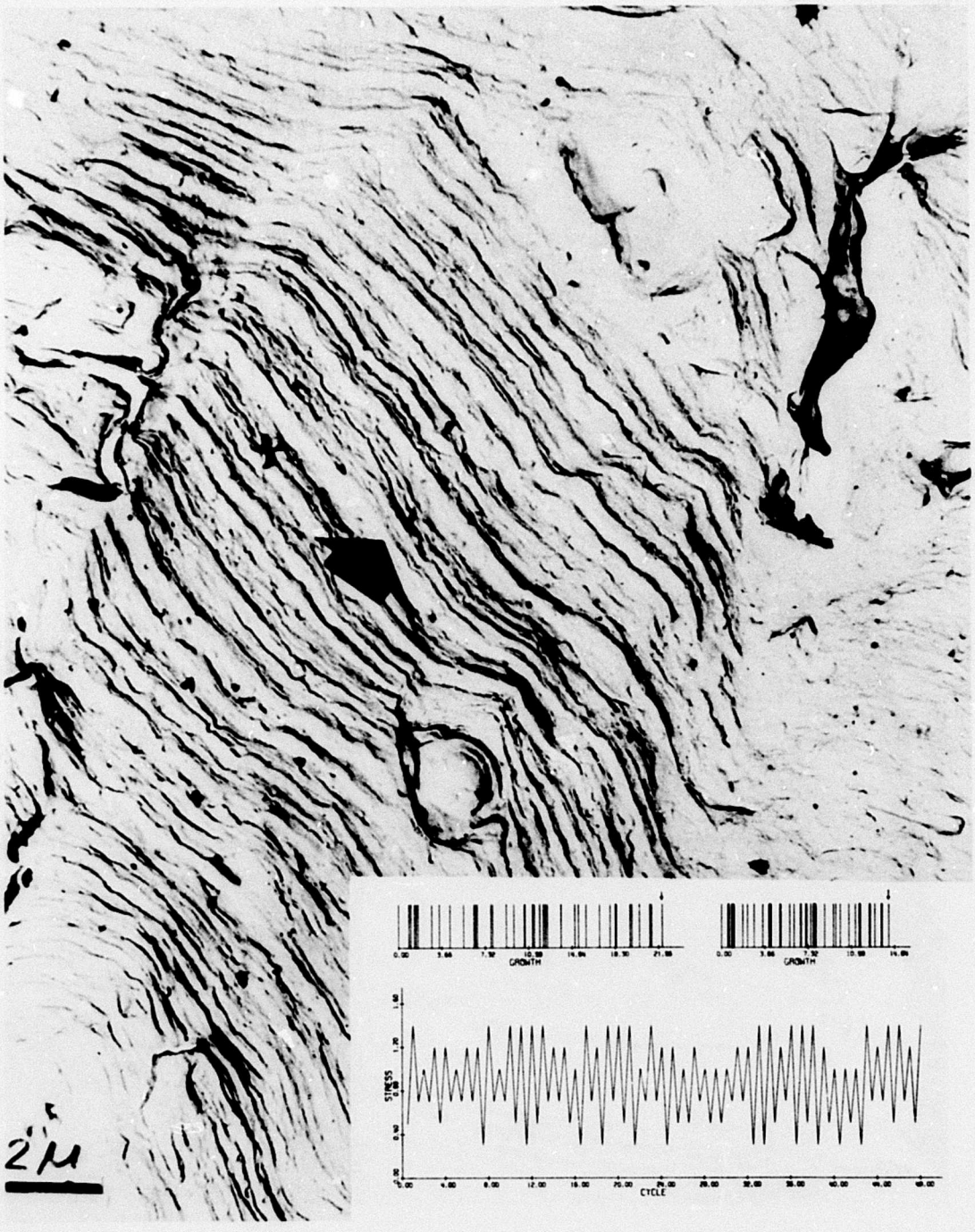

Figure 9

Fracture surface typography due to Program P 10. Note the sharpness of striation ridges indicating a sharp crack tip. The difficulty of identifying striations with actual loads is apparent of random loads.

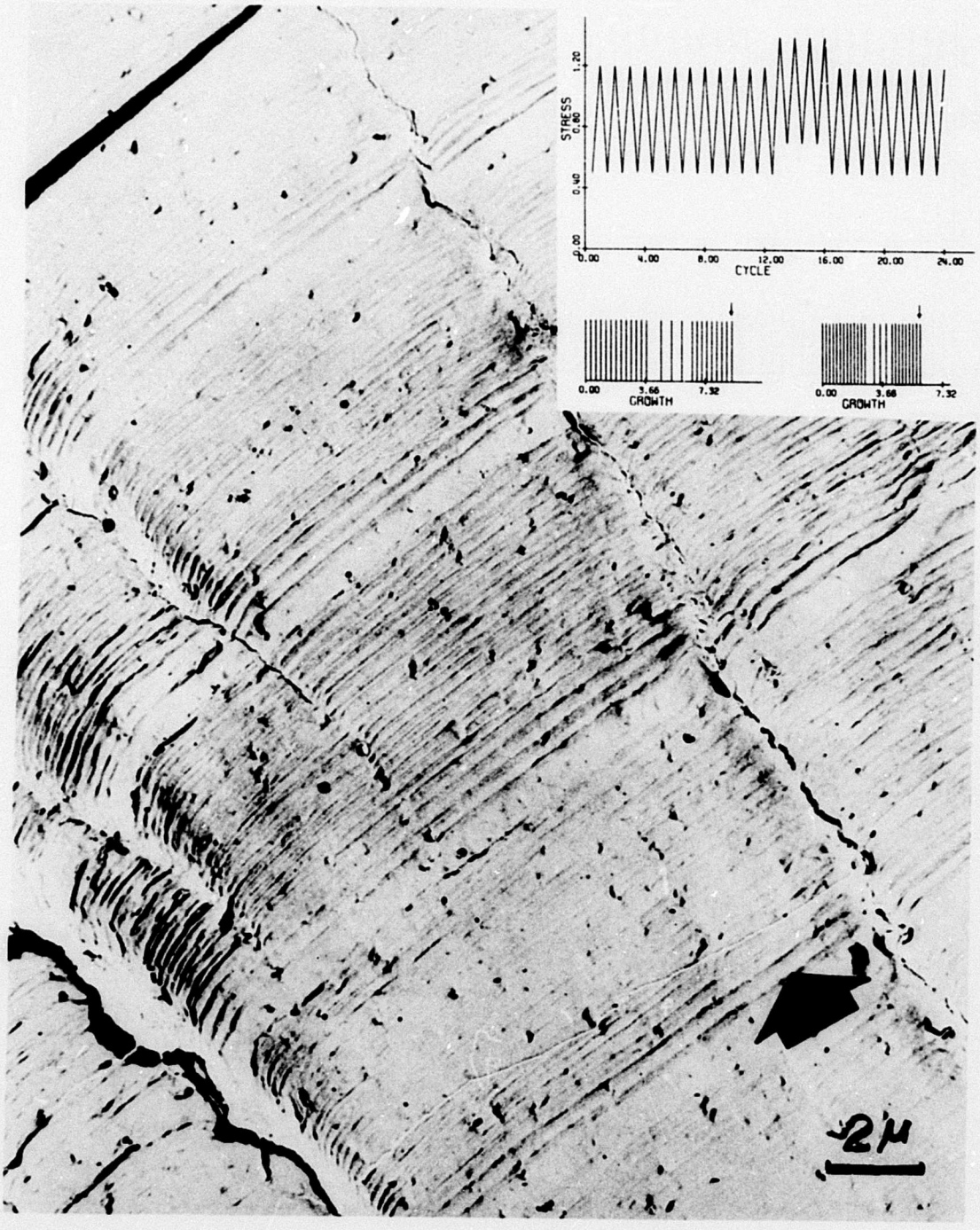

Figure 10

Typical fracture surface resulting from Program P 11. Note in the photograph the large striation spacing due to the load amplitude in the 13th oscillation, followed by three large striations corresponding to the next three oscillations. We see the mathematical model agrees with such behavior.

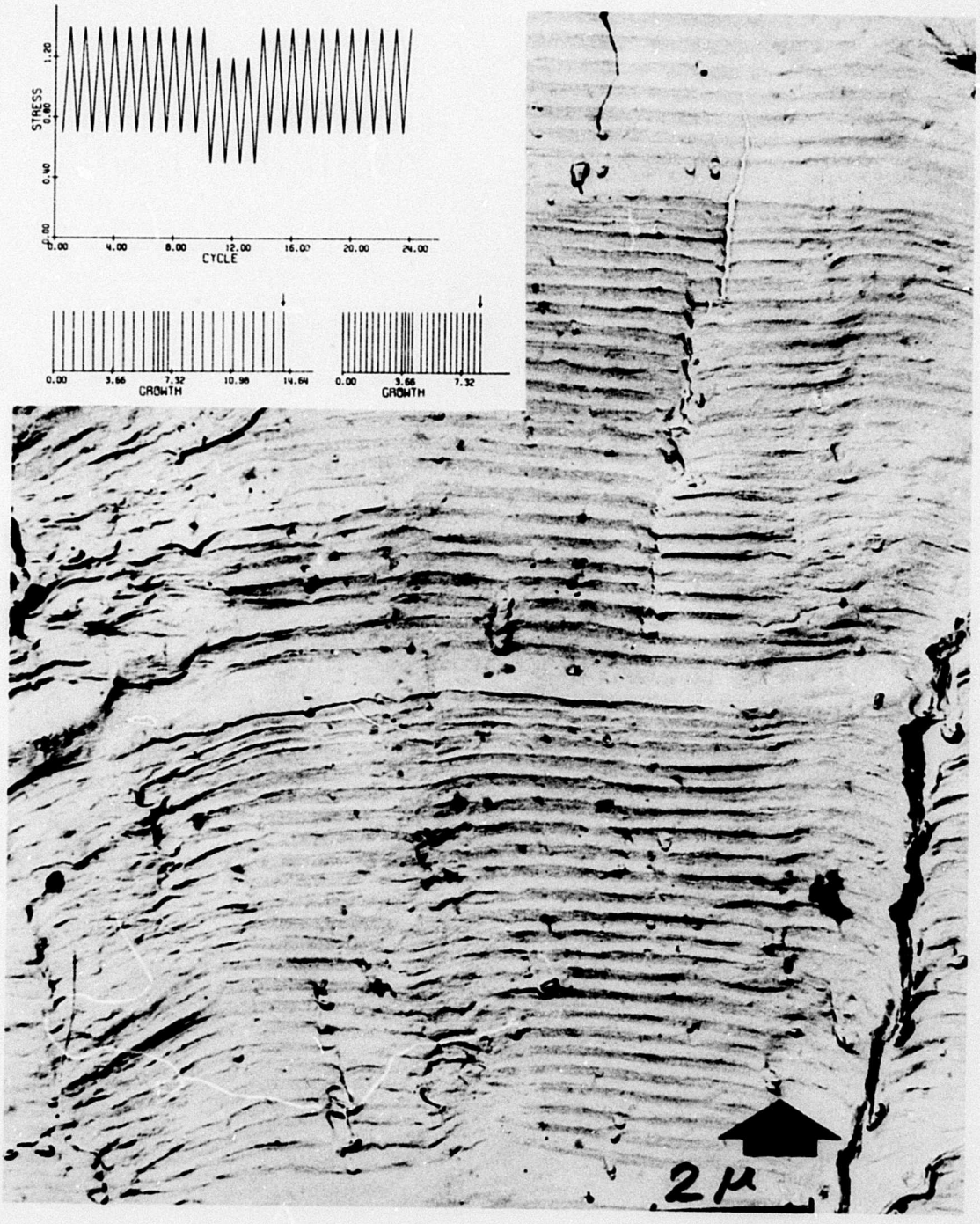

Figure 11

Typical fracture surface resulting from Program P 12. Note the large striation spacing due to the load oscillation in 14th cycle, preceeded by three smaller striations. The mathematical model agrees with this behavior.

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LIST OF SYMBOLS

$\mathcal{L}$	a set of functions of time, the values of which give the deflection of a metallic specimen being fatigued
λ_j	the jth load oscillation
$\lambda = \lambda_1 * \cdots * \lambda_m$	a load spectrum consisting of m load oscillations
$\underline{\lambda}$	a programmed load (or spectrum) consisting of an indefinite number of repetitions of λ
$Z(\lambda)$	the (random) incremental crack growth following the last fluctuation in λ
$\Omega = \{\omega_0, \cdots, \omega_\kappa\}$	the finite set of reference loading oscillations
$Y_j(\lambda)$	the total crack extension resulting from the jth repetition of spectrum λ
$S_n(\lambda)$	the total crack extension at the end of n cyclic repetitions of spectrum λ
$EY_j(\lambda) = \mu(\lambda)$	the expected (mean) crack extension per spectrum
$Var[Y_j(\lambda)] = \sigma^2(\lambda)$	the variance in crack extension per spectrum
$N(\lambda)$	the (random) number of repetitions of spectrum λ which can be repeated until fatigue failure occurs
a_k , b_k	the maximum and minimum values, respectively, of the stress during the kth load oscillation in the spectrum
$f_1(\lambda_k)$	the forcing function for crack extension during the kth fluctuation
$f_2(\lambda_k)$	the resisting function for crack extension during the kth fluctuation
w_j	the weighting factor which represents the "work hardening" at the crack tip resulting from the jth preceding load
$\Delta u_k(\lambda)$	the mean incremental crack extension during the kth fluctuation in the spectrum λ
ξ_i	the cumulative weighting factor for the ith load in the spectrum during steady state growth
$f_i(a,b)$ for $i = 1,2$	the resistance or forcing function expressed in terms of the maximum stress a and minimum stress b

$y = \ln a$	the natural logarithm of the maximum stress
$x = \ln(a - b)$	the natural logarithm of the stress variation
$p(x)$	the slope of the $\ln$-incremental crack growth as a function of x
α_i for $i = 1,2$	the exponent of maximum stress in f_i
β_i for $i = 1,2$	the exponent of stress variation in f_i
$\delta = \alpha_1 - \alpha_2$	
$\epsilon = \beta_1 - \beta_2$	
A	with or without affixes represents a constant associated with incremental crack growth behavior
$q(x) = p(x) - \beta_1$	
$g(y) = 0$	an equation the solution of which gives the value of ϵ satisfying specified boundary conditions
c , d , C , D	constants which represent the boundary conditions

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