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SYMMETRY CODES AND THEIR INVARIANT  
SUBCODES

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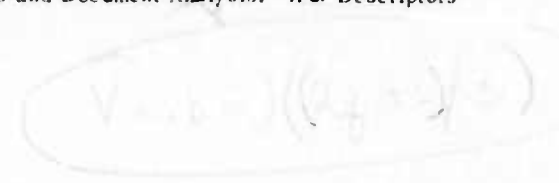
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SYMMETRY CODES AND THEIR INVARIANT SUBCODES

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## Symmetry Codes and Their Invariant Subcodes

### Abstract

We define and study the invariant subcodes of the symmetry codes in order to be able to determine the algebraic properties of these codes. An infinite family of self-orthogonal rate  $1/2$  codes over  $GF(3)$ , called symmetry codes, were constructed in [3]. A  $(2q + 2, q + 1)$  symmetry code, denoted by  $C(q)$ , exists whenever  $q$  is an odd prime power  $\equiv -1, (\text{mod } 3)$ . The group of monomial transformations leaving a symmetry code invariant is denoted by  $G(q)$ . In this paper we construct two subcodes of  $C(q)$  denoted by  $R_{\sigma}(q)$  and  $R_{\mu}(q)$ . Every vector in  $R_{\sigma}(q)$  is invariant under a monomial transformation  $\tau$  in  $G(q)$  of odd order  $s$  where  $s$  divides  $(q + 1)$ . Also  $R_{\mu}(q)$  is invariant under  $\tau$  but not vector-wise. The dimensions of  $R_{\sigma}(q)$  and  $R_{\mu}(q)$  are determined and relations between these subcodes are given. An isomorphism is constructed between  $R_{\sigma}(q)$  and a subspace of  $W = V_3^{\frac{2q+2}{s}}$ . It is shown that the image of  $R_{\sigma}(q)$  is a self-orthogonal subspace of  $W$ . The isomorphic images of  $R_{\sigma}(17)$  (under an order 3 monomial) and  $R_{\sigma}(29)$  (under an order 5 monomial) are both demonstrated to be equivalent to the  $(12, 6)$  Golay code.

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## Symmetry Codes and Their Invariant Subcodes

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## I. Introduction.

This paper defines and studies the invariant subcodes of the symmetry codes which were originally defined in [3]. The purpose of this study is the illucidation of properties of these subcodes in such a manner that these properties can be applied in determining characteristics of the symmetry code itself. For example, maximum length vectors in  $C(17)$  and  $C(29)$  can be determined from known maximum length vectors in the Golay code  $C(5)$ . The minimum weights are known for the first five symmetry codes. Estimates of the minimum weights of the larger symmetry codes have been obtained by locating a vector of weight 21 in  $R_C(41)$  (under an order 7 monomial) and a vector of weight 27 in  $R_C(53)$ , (under an order 3 monomial). An  $(n, k)$  error correcting code over  $GF(3)$  is a  $k$ -dimensional subspace of  $V_3^n = V$ . The weight of a vector  $x$ , denoted by  $w(x)$ , is the number of non-zero components it has. Symmetry codes are an infinite family of  $(2q + 2, q + 1)$  codes over  $GF(3)$  where  $q$  is an odd prime power  $\equiv -1 \pmod{3}$ . Each code is given in terms of a basis  $[I, S_q]$  where  $I$  is the  $q \times q$  identity matrix and  $S_q$  is the matrix described below.

We consider the elements of  $GF(q)$  to be ordered in some fixed way, and with this ordering we label the first  $q + 1$  coordinates with the elements of  $GF(q) \cup \{\infty\}$  with  $\infty$  taken as the first coordinate. We label the second  $q + 1$  coordinates by the same sequence of elements of

$GF(q) \cup \{\infty\}$  with dashes on them to distinguish them from the first  $q + 1$  coordinate labels. When  $q = p$  is a prime, for convenience we use the ordering  $\infty, 0, 1, \dots, p-1$  (and hence also  $\infty', 0', 1', \dots, (p-1)'$  for the right side). By definition,  $S_q$  is the  $(q + 1) \times (q + 1)$  matrix  $(s_{i', j'})$ ,  $i, j$  in  $GF(q) \cup \{\infty\}$ , such that  $s_{\infty', \infty'} = 0$  and for  $i', j' \neq \infty'$ ,  $s_{i', \infty'} = \chi(-1)$ ,  $s_{\infty', i'} = 1$ , and  $s_{i', j'} = \chi(j-i)$  where  $\chi(0) = 0$ ,  $\chi$  (a quadratic residue) = 1,  $\chi$  (a non-residue) = -1. We refer to the code generated by  $[I, S_q]$  as  $C(q)$ .

As a concrete example we write the basis for  $C(5)$  below.

| $\infty$ | 0 | 1 | 2 | 3 | 4 | $\infty'$ | 0' | 1' | 2' | 3' | 4' |
|----------|---|---|---|---|---|-----------|----|----|----|----|----|
| 1        | 0 | 0 | 0 | 0 | 0 | 0         | 1  | 1  | 1  | 1  | 1  |
| 0        | 1 | 0 | 0 | 0 | 0 | 1         | 0  | 1  | -1 | -1 | 1  |
| 0        | 0 | 1 | 0 | 0 | 0 | 1         | 1  | 0  | 1  | -1 | -1 |
| 0        | 0 | 0 | 1 | 0 | 0 | 1         | -1 | 1  | 0  | 1  | -1 |
| 0        | 0 | 0 | 0 | 1 | 0 | 1         | -1 | -1 | 1  | 0  | 1  |
| 0        | 0 | 0 | 0 | 0 | 1 | 1         | 1  | -1 | -1 | 1  | 0  |

$C(5)$  is a  $(12, 6)$  code and it is equivalent to the Golay code [2].

In [4] it was shown that each symmetry code is self orthogonal. The transformations on  $V$  which preserve the weights of all vectors are the monomial transformations. A monomial transformation can be viewed as a permutation of the coordinate indices of the vectors in  $V$  (the same permutation for each vector) coupled with multiplying some (or none) of the coordinates by minus one. The set of monomial transformations which send all the vectors in  $C(q)$  onto vectors in  $C(q)$  form a group denoted by  $G(q)$ . In [4] it was shown that  $G(q)$  contains  $PGL_2(q)$ .

In section II of this paper we construct two subcodes of  $C(q)$  denoted by  $R_{\sigma}(q)$  and  $R_{\mu}(q)$ . Every vector in  $R_{\sigma}(q)$  is invariant under a monomial transformation  $\tau$  in  $G(q)$  of odd order  $s$  where  $s$  divides  $q + 1$ . Also  $R_{\mu}(q)$  is invariant under  $\tau$  but not vector-wise invariant. The dimensions of  $R_{\sigma}(q)$  and  $R_{\mu}(q)$  are determined and relations between these subcodes are given. In section III an isomorphism is constructed between  $R_{\sigma}(q)$  and a subspace of  $W = V_3^{\frac{2q+2}{s}}$ . It is shown that the image of  $R_{\sigma}(q)$  is a self-orthogonal subspace of  $W$ . In section IV the isomorphic images of  $R_{\sigma}(17)$  ( $o(\tau) = 3$ ) and  $R_{\sigma}(29)$  ( $o(\tau) = 5$ ), are both demonstrated to be equivalent to the (12, 6) Golay code.

II. In this section we construct two subcodes of  $C(q)$ ,  $R_{\sigma}(q)$  and  $R_{\mu}(q)$  with the following properties. Every vector in  $R_{\sigma}(q)$  is invariant under a monomial transformation  $\tau$  in  $G(q)$  where the order of  $\tau$  is an odd number  $s$  dividing  $q + 1$ . Further,  $R_{\mu}(q)$  is also invariant under  $\tau$  but not vector-wise invariant. The dimensions of  $R_{\sigma}$  and  $R_{\mu}$  are determined, and relations between them are given.

In [4] it was shown that the mapping sending a monomial transformation  $\tau$  in  $G(q)$  onto the permutation  $\bar{\tau}$  it induces on the coordinate indices is a homomorphism of a subgroup of  $G(q)$  onto  $PGL_2(q)$  whose kernel has order 2. For the rest of this paper  $\tau$  denotes a monomial transformation in  $G(q)$  of odd order  $s$  where  $s$  divides  $(q + 1)$  such that  $\bar{\tau}$  is in  $PGL_2(q)$  and the order of  $\tau$  equals the order of  $\bar{\tau}$ .

Lemma 1. If  $s$  is an odd number dividing  $(q + 1)$ , then there exists a transformation  $\bar{\tau}$  in  $G(q)$  of order  $s$ . Further  $\bar{\tau}$  is in  $PGL_2(q)$ .

Proof: By [1] it is known that  $PGL_2(q)$  contains a cyclic subgroup of order  $\frac{(q+1)}{2}$ . Hence this subgroup contains an element  $\bar{\tau}$  of order  $s$  when  $s$  is any odd number dividing  $(q+1)$ . The monomial  $\tau$  in  $G(q)$  which maps into  $\bar{\tau}$  by the homomorphism described above is either of order  $s$  or  $2s$ . If it is of order  $s$  we are finished. If  $\tau$  is of order  $2s$  then  $\tau^2$  is of order  $s$ ,  $\bar{\tau}^2$  is also of order  $s$  (since  $s$  is odd),  $\bar{\tau}^2$  is in  $PGL_q(q)$  and the lemma is demonstrated.

The subcodes  $R_\sigma(q)$  and  $R_\mu(q)$  are the ranges of two linear transformations  $\sigma$  and  $\mu$  defined for  $x$  in  $C(q)$  as follows.

$$x\sigma = x + x\tau + \dots + x\tau^{s-1}$$

$$x\mu = x - x\tau$$

Even though  $\sigma$  and  $\mu$  are linear transformations, they are not monomial transformations; they are useful in obtaining information about  $\tau$ . Let  $K_\sigma(q)$  denote the kernel of  $\sigma$  and  $K_\mu(q)$  the kernel of  $\mu$ .

Theorem 1.  $R_\sigma(q)$ ,  $R_\mu(q)$ ,  $K_\sigma(q)$ ,  $K_\mu(q)$  are subcodes of  $C(q)$  such that

- 1)  $R_\sigma(q)$  is contained in  $K_\mu(q)$  and  $R_\mu(q)$  is contained in  $K_\sigma(q)$ , and
- 2)  $\tau$  leaves  $R_\mu(q)$  invariant and  $\tau$  leaves every vector in  $R_\sigma(q)$  invariant.

Proof: It is clear that  $R_\sigma(q)$ ,  $R_\mu(q)$ ,  $K_\sigma(q)$  are subcodes since they are vector subspaces contained in  $C(q)$ . If  $x\sigma$  is in  $R_\sigma(q)$  then  $(x\sigma)\mu = (x + x\tau + \dots + x\tau^{s-1})_\mu = (x + x\tau + \dots + x\tau^{s-1}) - (x\tau + x\tau^2 + \dots + x\tau^{s-1} + x) = 0$  so that  $R_\sigma(q)$  is contained in  $K_\mu(q)$ . Similarly  $R_\mu(q)$  is contained in  $K_\sigma(q)$ . If  $x\sigma$  is in  $R_\sigma(q)$ , then  $(x\sigma)\tau = (x + x\tau + \dots + x\tau^{s-1})_\tau = x\tau + x\tau^2 + \dots + x\tau^{s-1} + x = x\sigma$  and we see that  $\tau$  leaves every vector in  $R_\sigma(q)$  invariant. Since  $(x\mu)\tau = x\tau - x\tau^2$ ,  $\tau$  leaves  $R_\mu(q)$  invariant and the theorem is proved.

Remark: When  $s$  is divisible by 3,  $R_O(q)$  is contained in  $K_O(q)$ .

Proof: If  $y$  is in  $R_O(q)$ ,  $y = x\sigma = x + x\tau + \dots + x\tau^{s-1}$ . Hence  $y\sigma = (x + x\tau + \dots + x\tau^{s-1})\sigma = sy \equiv 0 \pmod{3}$ .

Lemma 2.  $\bar{\tau}$  is a product of disjoint cycles of length  $s$ . Further, if  $(i_1, \dots, i_s)$  is such an  $s$ -cycle for the left coordinate indices of  $V$ , then  $(i_1', \dots, i_s')$  is such an  $s$ -cycle for the right coordinate indices of  $V$ .

Proof: By their construction [4] the transformations in  $PGL_2(q)$  act on the left coordinate indices (and simultaneously on the right coordinate indices) as transformations on the projective line. Since  $s$  is an odd number which divides  $q + 1$ ,  $\bar{\tau}$  is either completely a product of disjoint cycles of length  $s$  or a product of disjoint cycles of length  $s$  with  $ks$  fixed points. But a projective transformation with three fixed points is the identity. Hence  $\bar{\tau}$  can have at most two fixed points on each side of coordinate indices. Since  $s$  divides  $q + 1$ , the number of left coordinate indices (and the number of right coordinate indices), this is only possible for  $k = 1$  and  $s = 2$ . The lemma follows from the fact that  $s$  is an odd number.

We let  $J$  be a set of left coordinate indices with the property that  $J$  contains exactly one index from each of these  $s$  cycles. Note that

$$|J| = \frac{(q+1)}{s}.$$

In order to determine the dimension of  $R_O(q)$  and  $R_\mu(q)$  we introduce the following terminology. We let the vectors in the basis  $[I, S_q]$  be denoted by  $(e_i, c(e_i))$  where  $e_i$  is the  $i^{\text{th}}$  row of  $I$  and  $c(e_i)$  is the  $i^{\text{th}}$  row of  $S_q$ .

Theorem 2.  $\dim R_{\sigma}(q) = \frac{(q+1)}{s}$  and  $\dim R_{\mu}(q) = \frac{(q+1)(s-1)}{s}$ .

Proof: Consider the set of  $\frac{(q+1)}{s}$  vectors  $\{(e_j + e_j\tau + \dots + e_j\tau^{s-1}, c(e_j) + c(e_j)\tau + \dots + c(e_j)\tau^{s-1})\}$  for  $j \in J$ . Since the order of  $\tau$  equals the order of  $\bar{\tau}$ ,  $e_j \neq \frac{1}{s} e_j \tau^i$ ,  $1 \leq i \leq s-1$ , so that  $(e_j + e_j\tau + \dots + e_j\tau^{s-1}) \neq 0$  for each  $j \in J$ . Hence by the definition of  $J$ , these vectors are linearly independent. Clearly they span  $R_{\sigma}(q)$ , and it thus follows that  $\dim R_{\sigma}(q) = |J| = \frac{q+1}{s}$ . Similarly  $\{(e_j\tau^k - e_j\tau^{k+1}), (c(e_j)\tau^k - c(e_j)\tau^{k+1})\}$  for  $j \in J$ ,  $k = 0, \dots, s-2$  is a basis of  $R_{\mu}(q)$ . Hence  $\dim R_{\mu}(q) = \frac{(q+1)(s-1)}{s}$ .

Remark: When  $\tau$  has even order ( $\neq 2$ ) which divides  $\frac{(q+1)}{2}$ , all the results of this paper hold when the order of  $\tau$  equals the order of  $\bar{\tau}$ . When the order of  $\tau$  equals twice the order of  $\bar{\tau}$ , then it is possible that Theorem 2 does not hold since the basis vectors described above can be zero.

Corollary 1.  $R_{\sigma}(q) = K_{\mu}(q)$  and  $R_{\mu}(q) = K_{\sigma}(q)$ .

Proof: By Theorem 1,  $R_{\mu}(q)$  is contained in  $K_{\sigma}(q)$  and  $R_{\sigma}(q)$  is contained in  $K_{\mu}(q)$ . In general,  $\dim R_{\mu}(q) + \dim K_{\mu}(q) = q+1 = \dim K_{\sigma}(q) + \dim R_{\sigma}(q)$ . By Theorem 2,  $\dim R_{\sigma}(q) = \frac{(q+1)}{s}$  and  $\dim R_{\mu}(q) = \frac{(q+1)(s-1)}{s}$ . Hence  $\dim R_{\mu}(q) = \dim K_{\sigma}(q)$  and  $\dim R_{\sigma}(q) = \dim K_{\mu}(q)$  and the corollary is demonstrated.

Note that since 3 divides  $(q+1)$  for every  $q \equiv -1 \pmod{3}$ , every symmetry code has a monomial transformation of order 3 leaving it invariant.

### III. The isomorphic image of $R_{\sigma}$ .

In this section we construct a linear transformation  $\phi$  from  $V$  onto  $W = V_3 \frac{2q+2}{s}$  where  $s$  is again an odd number dividing  $q+1$  with the following

properties. The dimension of  $\varphi(R_\sigma)$  equals the dimension of  $R_\sigma$ , the weight of  $\varphi(x)$  for  $x$  in  $R_\sigma$  is the weight of  $x$  divided by  $s$ , and  $\varphi(R_\sigma)$  is a self-orthogonal subspace of  $W$ .

In order to do this we let  $J$  be as in section II, and let  $J'$  be the elements in  $J$  with dashes on them. Note that  $J \cup J'$  contains  $\frac{2(q+1)}{s}$  elements. We consider the elements in  $J$  to have the same ordering they had in  $GF(q) \cup \{\infty\}$ . With this ordering we label the left half of the coordinate indices in  $W$  with the elements from  $J$ , and the right half with the elements from  $J'$ . We denote the unit vectors in  $W$  by  $\bar{e}_j$ ,  $j$  in  $J$  and  $\bar{e}_{j'}$ ,  $j'$  in  $J'$ .

Lemma 3. If  $x\tau = x$ , then the components of  $x$  on a cycle of  $\bar{\tau}$  are either all zero or all non-zero. Further, if  $x\tau = x$  and  $y\tau = y$ , then on the cycles of  $\bar{\tau}$  on which the components of both  $x$  and  $y$  are non-zero, the components of  $x$  equal plus or minus the components of  $y$ .

Proof: Let  $(i_1, \dots, i_s)$  be the coordinate indices of a cycle of  $\bar{\tau}$ . Let  $x_{i_j}$  be the  $i_j$ <sup>th</sup> component of  $x$ . If  $x\tau = x$ , then all the components of  $x$  on this cycle are determined by  $x_{i_1}$  and  $\tau$ . If  $y\tau = y$  also, then the components of  $x$  on this cycle equal the components of  $y$  on this cycle of  $x_{i_1} = y_{i_1}$ . If  $x_{i_1} = -y_{i_1}$  the components of  $x$  on this cycle are the negatives of the components of  $y$ . Since these are the only possibilities, the lemma is proved.

Theorem 3. There is a linear transformation  $\varphi$  from  $V$  onto  $W = V_3$   $\frac{2q+2}{s}$

such that 1)  $\dim \varphi(R_\sigma(q)) = \dim R_\sigma(q) = \frac{(q+1)}{s}$ , and

$$2) \quad w(\varphi(x)) = \frac{w(x)}{s}.$$

Proof: We let  $e_i$  and  $e_i'$ ,  $i \in GF(q) \cup \{\infty\}$  denote the unit vectors in  $V$ .

We define  $\varphi$  on these unit vectors as follows.

$$\begin{aligned} \text{If } j \in J, \quad \varphi(e_j) &= \bar{e}_j. & \text{If } i \notin J, \quad \varphi(e_i) &= 0. \\ \text{If } j' \in J', \quad \varphi(e_{j'}) &= \bar{e}_{j'}. & \text{If } i' \notin J', \quad \varphi(e_{i'}) &= 0. \end{aligned}$$

Define  $\varphi$  on the rest of  $V$  linearly. Clearly  $\varphi$  is a linear transformation from  $V$  onto  $W$ .

Recall that  $\{(e_j + e_j\tau + \dots + e_j\tau^{s-1}, c(e_j) + c(e_j)\tau + \dots + c(e_j)\tau^{s-1})\}$ ,  $j \in J$  is a basis of  $R_\sigma(q)$ . Since  $\varphi$  maps these vectors onto linearly independent vectors,  $\dim \varphi(R_\sigma(q)) = \dim R_\sigma(q) = \frac{(q+1)}{s}$  by Theorem 2.

Theorem 1 tells us that  $x\tau = x$  for all  $x$  in  $R_\sigma(q)$ . By Lemma 3 we know that the components of  $x$  on a cycle of  $\bar{\tau}$  are either all zero or all non-zero. Since  $\varphi$  projects on precisely one component from each  $s$ -cycle of  $\bar{\tau}$ ,  $w(\varphi(x)) = \frac{w(x)}{s}$ .

It was proven in [4] that  $C(q)$  is a self-orthogonal subspace of  $V$  so that  $R_\sigma(q)$  is certainly a self-orthogonal subspace of  $V$ . Even though  $\varphi$  does not preserve the property of self-orthogonality, we can prove that  $\varphi(R_\sigma(q))$  is a self-orthogonal subspace of  $W$ .

Theorem 4.  $\varphi(R_\sigma(q))$  is a self-orthogonal subspace of  $W$ .

Proof: Let  $x$  and  $y$  be vectors in  $W$  such that  $x = (\alpha_1, \dots, \alpha_{\frac{2q+2}{s}})$  and

$y = (\beta_1, \dots, \beta_{\frac{2q+2}{s}})$ . Then the inner product of  $x$  and  $y$ , denoted by  $(x, y)$ , is

$$\left( \sum_{i=1}^{\frac{2q+2}{s}} \alpha_i \beta_i \right) \pmod{3}. \quad \text{As is usual, } x \text{ and } y \text{ are orthogonal to each other}$$

if  $(x,y) = 0$ . In order to prove Theorem 4 we need to show that  $(x,y) = 0$  for all  $x,y$  in  $\varphi(R_\sigma(q))$  ( $x$  can also equal  $y$ ). In order to prove this, we introduce the inner product of  $x$  and  $y$  over the integers, denoted by

$[x,y]$ , where  $[x,y]$  equals  $\sum_{i=1}^s \alpha_i \beta_i$  by definition. We define  $[x,y]$  in

a similar fashion for  $x$  and  $y$  in  $V$ .

The proof of Theorem 4 is divided into two cases. The first case is 3 does not divide  $s$ . If  $x$  and  $y$  are in  $R_\sigma(q)$ , then  $x = x_1 + x_1\tau + \dots + x_1\tau^{s-1}$  and  $y = y_1 + y_1\tau + \dots + y_1\tau^{s-1}$  for some  $x_1$  and  $y_1$  in  $C(q)$ . By Lemma 3, all the elements in  $R_\sigma(q)$  which are not zero on a particular cycle of  $\tau$  have the same or opposite components on that cycle. Hence  $[x,y] = rs$  where  $r$  is the number of  $s$ -cycles of  $\tau$  (in both the left and right coordinates) in which both  $x$  and  $y$  have non-zero components. Since  $(x,y) = 0$ , 3 divides  $rs$ , but by assumption 3 does not divide  $s$  so that 3 divides  $r$ . By the definition of  $\varphi$ ,  $[\varphi(x), \varphi(y)] = r$  so that  $(\varphi(x), \varphi(y)) = 0$  for all  $x,y$  in  $R_\sigma(q)$ . Hence  $\varphi(R_\sigma(q))$  is self-orthogonal in this situation. We now consider the case that  $s = 3j$ , i.e.,  $\tau^{3j} = 1$ . We let  $x$  and  $y$  be in  $R_\sigma(q)$ , and we have  $x = x_1 + x_1\tau + \dots + x_1\tau^{3j-1}$ ,  $y = y_1 + y_1\tau + \dots + y_1\tau^{3j-1}$  for  $x_1, y_1$  in  $C(q)$ . Then

$$\begin{aligned} [x,y] &= \sum_{i=0}^{3j-1} [x_1, y_1\tau^i] + \sum_{i=0}^{3j-1} [x_1\tau, y_1\tau^i] + \dots + \sum_{i=0}^{3j-1} [x_1\tau^{3j-1}, y_1\tau^i] \\ &= \sum_{i=0}^{3j-1} [x_1\tau^i, y_1\tau^i] + \sum_{i=0}^{3j-1} [x_1\tau^i, y_1\tau^{i+1}] + \dots + \sum_{i=0}^{3j-1} [x_1\tau^i, y_1\tau^{i+3j-1}] \end{aligned}$$

by rearranging terms. Now  $[u,v] = [u\tau^i, v\tau^i]$  for all  $u$  and  $v$  in  $V$

since  $\tau^i$  is a monomial transformation over  $GF(3)$ . Hence  $[x, y] = 3j[x_1, y_1] + 3j[x_1, y_1\tau] + \dots + 3j[x_1, y_1\tau^{3j-1}]$ . Since  $x_1$  and  $y_1\tau^i$  ( $i=0, \dots, 3j-1$ ) are all in  $C(q)$  which is self-orthogonal, each  $[x_1, y_1\tau^i]$  is divisible by 3 so that  $[x, y] = 9r$  for some  $r$ . Each cycle of  $\tau$  is a  $3j$ -cycle, and by the definition of  $\varphi$ ,  $\varphi$  projects onto one coordinate from each  $3j$ -cycle so that  $[\varphi(x), \varphi(y)] = 3r$ . Hence  $(\varphi(x), \varphi(y)) = 0$ , and  $\varphi(R_O(q))$  is a self-orthogonal subspace of  $W$  for this case also.

IV. Invariant subcodes of  $C(17)$  and  $C(29)$  are isomorphic to the Golay code.

In this section we apply these ideas to  $C(17)$  and  $C(29)$ . The  $\tau$  for  $C(17)$  has order 3 and the  $\tau$  for  $C(29)$  has order 5. We describe these two monomial transformations explicitly, and exhibit bases for  $R_C(17)$  and  $\varphi(R_O(17))$ .

In order to exhibit these monomial transformations we introduce the following convention. We let  $\overline{\chi(i)}$  times a column index mean that we multiply the column by  $\chi(i)$  where  $\chi(i) = 1$  for  $i$  a quadratic residue, and  $\chi(i) = -1$  for  $i$  a non-residue. This convention is used in order to avoid confusion with negatives in  $GF(17)$ .

We can represent  $\tau$  as a monomial transformation on the columns of  $V$  as follows.

$$\tau(\infty) = 0, \quad \tau(16) = \infty; \quad \tau(i) = \overline{\chi(i+1)} \left( \frac{16}{i+1} \right), \quad i \neq \infty, 16;$$

$$\tau(\infty') = 0', \quad \tau(16') = \infty'; \quad \tau(i') = \overline{\chi(i'+1)} \left( \frac{16}{i'+1} \right), \quad i' \neq \infty', 16'.$$

The generators of the subgroup of  $G(17)$  which is isomorphic to  $PGL_2(17)$  are given in [4, p. 131]. It is easy to verify that  $\tau$  is a product of two

of these generators so that  $\tau$  is in  $G(17)$ . A straightforward check shows that  $\tau$  has order 3. If we rearrange the columns of  $V$  to correspond to the cycles of  $\bar{\tau}$ , the following is a basis of  $R_{\sigma}(17)$ .

| $\infty$ | 0 | 16 | 1 | 8 | 15 | 2 | 11 | 7 | 3 | 4 | 10 | 5 | 14 | 9  | 6 | 12 | 13 | $\infty'$ | 0' | 16' | 1' | 8' | 15' | 2' | 11' | 7' | 3' | 4' | 10' | 5' | 14' | 9' | 6' | 12' | 13' |
|----------|---|----|---|---|----|---|----|---|---|---|----|---|----|----|---|----|----|-----------|----|-----|----|----|-----|----|-----|----|----|----|-----|----|-----|----|----|-----|-----|
| 1        | 1 | 1  |   |   |    |   |    |   |   |   |    |   |    |    |   |    |    | -1        | -1 | -1  |    |    |     | 1  | -1  | 1  | 1  | 1  | -1  | -1 | 1   | 1  | 1  | -1  | 1   |
|          |   |    | 1 | 1 | 1  |   |    |   |   |   |    |   |    |    |   |    |    |           |    |     |    | 1  | 1   | 1  | 1   | -1 | 1  | -1 | -1  | 1  | -1  | 1  | 1  | -1  | -1  |
|          |   |    |   |   |    | 1 | -1 | 1 |   |   |    |   |    |    |   |    |    | -1        | 1  | 1   |    | 1  | 1   | 1  | 1   | -1 | 1  | -1 | -1  | 1  | -1  | 1  | 1  | -1  | -1  |
|          |   |    |   |   |    |   |    |   | 1 | 1 | -1 |   |    |    |   |    |    | -1        | 1  | 1   |    | 1  | 1   | 1  | 1   | -1 | 1  | -1 | -1  | 1  | -1  | 1  | 1  | -1  | -1  |
|          |   |    |   |   |    |   |    |   |   |   |    | 1 | -1 | -1 |   |    |    | -1        | -1 | -1  |    | -1 | -1  | -1 | 1   | -1 | 1  | -1 | -1  | 1  | -1  | 1  | 1  | -1  | -1  |
|          |   |    |   |   |    |   |    |   |   |   |    |   |    |    | 1 | -1 | -1 | -1        | -1 | -1  |    | -1 | -1  | -1 | 1   | -1 | 1  | -1 | -1  | 1  | -1  | 1  | 1  | -1  | -1  |
|          |   |    |   |   |    |   |    |   |   |   |    |   |    |    |   |    | 1  | -1        | -1 | -1  |    | -1 | -1  | -1 | 1   | -1 | 1  | -1 | -1  | 1  | -1  | 1  | 1  | -1  | -1  |

From this we get the following basis for  $\varphi(R_{\sigma}(17))$  by choosing  $J = \{\infty, 1, 2, 3, 5, 6\}$ .

| $\infty$ | 1 | 2 | 3 | 5 | 6 | $\infty'$ | 1' | 2' | 3' | 5' | 6' |
|----------|---|---|---|---|---|-----------|----|----|----|----|----|
| 1        |   |   |   |   |   | -1        | 1  | 1  | -1 | -1 |    |
|          | 1 |   |   |   |   |           | 1  | 1  | -1 | -1 | 1  |
|          |   | 1 |   |   |   |           | 1  | 1  | 1  | 1  |    |
|          |   |   | 1 |   |   |           | 1  | -1 | 1  | -1 | -1 |
|          |   |   |   | 1 |   | -1        | -1 | 1  |    | 1  | 1  |
|          |   |   |   |   | 1 | -1        | 1  |    | -1 | 1  | -1 |

It is known [4] that the minimum weight of  $C(17)$  is 18, so that the minimum weight of  $\varphi(R_{\sigma}(17))$  is 6. It follows from the theorem in [2] that  $\varphi(R_{\sigma}(17))$  is equivalent to the Golay (12, 6) code over  $GF(3)$ .

A monomial transformation  $\tau$  of order 5 in  $G(29)$  is given by the following.

$$\tau(\infty) = 0, \tau(24) = \infty; \tau(i) = \overline{\chi(i+5)} \left( \frac{28}{i+5} \right), i \neq \infty, 24,$$

$$\tau(\infty') = 0', \tau(24') = \infty'; \tau(i') = \overline{\chi(i'+5)} \left( \frac{28}{i'+5} \right), i' \neq \infty', 24'.$$

As in the previous case it can be verified that  $\tau$  is a product of

generators of the subgroup of  $G(29)$  which is isomorphic to  $PGL_2(29)$ . Given  $\tau$ , a basis of  $R_\sigma(29)$  can be computed similar to the basis of  $R_\sigma(17)$ . The minimum weight in  $C(29)$  is 18 and since the weight of every vector in  $R_\sigma(29)$  is divisible by 5, the minimum weight of  $R_\sigma(29)$  must be at least 30. It is exactly 30 since the basis vectors have weight 30. Hence the minimum weight of  $\varphi(R_\sigma(29))$  is 6. It then follows as above that  $\varphi(R_\sigma(29))$  is equivalent to the Golay Code.

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