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July 1973

Technical Report

for NWRC-TR-7

MARKOV MODELS AND DYNAMIC ANALYSIS OF ASW EFFECTIVENESS

By: W. H. FRYE and A. J. KORSAK

Prepared for:

NAVAL ANALYSIS PROGRAMS (CODE 431)
OFFICE OF NAVAL RESEARCH
ARLINGTON, VIRGINIA 22217

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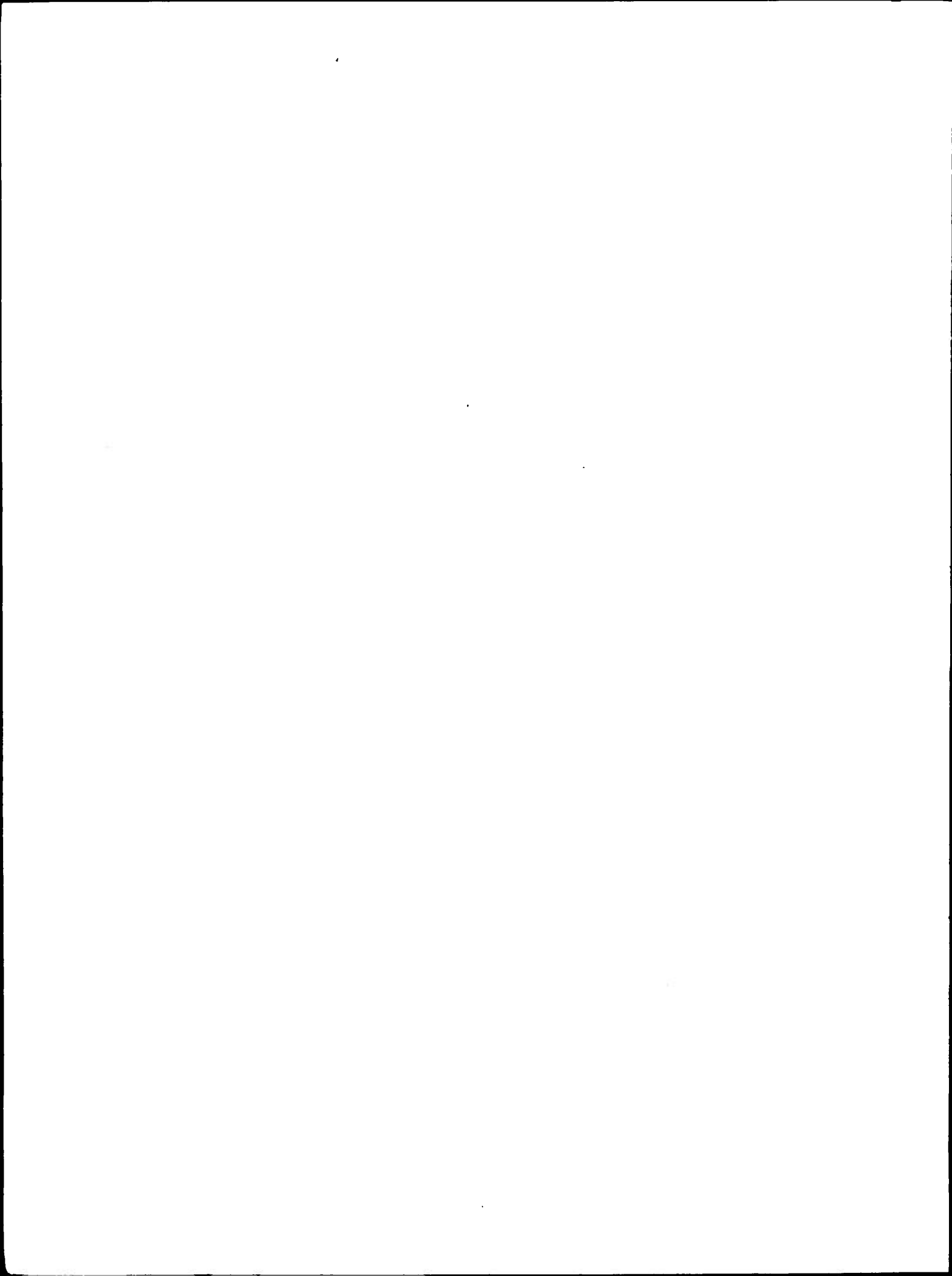
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13. ABSTRACT This research report presents methodology for evaluating ASW system effectiveness in dynamic operational contexts. It shows that conditional probability models may be generalized by using Markov modeling methods and suggests three models that include false contacts. Parameter estimation methods are given for Markov models and the results of experimentation with several methods are presented. Some of the methods use operational data not currently used in effectiveness modeling; hence, it is suggested that improved effectiveness modeling may be possible with Markov methods because more data can be used to estimate model parameters. State space structuring is investigated and results of numerical experimentation are given for different sets of states applied to the same set of data. (Data were generated by hypothetical non-Markov dynamic processes and a Monte Carlo simulation model.) The principal effectiveness measures are identified and ASW applications of them discussed. The improvability of the measures (in the variance reduction sense) is investigated and the probability of mission success measure is shown to be essentially unimprovable by the use of Markov models. Three theorems are given that give insight into parameter estimation for Markov chain models; these algebraic identities are useful for estimating the potential improvability of the effectiveness measures.			

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CONTENTS

LIST OF ILLUSTRATIONS	xi
LIST OF TABLES.	xv
PREFACE	xvii
 I INTRODUCTION AND SUMMARY	 1
A. Research Objectives	1
B. Research Findings	4
C. Research Activities	8
D. Possible Operational Applications	13
 II SUMMARY OF QUEUEING MODELS OF ASW UNITS.	 15
A. Summary of Work During the First Phase.	15
1. ASW Unit Model	17
2. ASW Force Effectiveness Model.	21
B. Further Computations with the Single Unit Model . .	25
1. Special Numerical Integration Method for Solving Large Flowgraphs	25
2. Results of a Simulation Approach	26
C. Proposed Unit Model Extensions to Represent Classification.	32
 III CONDITIONAL PROBABILITY MODELS	 39
A. Introduction.	39
B. A Conditional Probability Model as a Markov Model .	40
C. Limitations of Conditional Probability Models . . .	48
 IV MARKOV AND SEMI-MARKOV MODELS IN ASW EFFECTIVENESS ANALYSES	 51
A. Introduction.	51
B. Useful Properties of Markov Models.	51

C.	Measures of Effectiveness	52
D.	Formulating a Markov Model.	55
1.	Defining a Set of States	55
2.	Example of a Set of States	56
3.	Estimation of Markov Model Parameters.	57
V	BASIC MARKOV CHAIN THEORY.	59
A.	States, Transitions, Paths, and Loops	59
B.	Transition Probabilities.	61
C.	Partitioned Form of the Transition Matrix	61
D.	Multistep Transition Matrices and Absorption Probabilities	62
E.	z Transforms.	65
F.	Inversion of the Matrix $(I - Pz)$ and Applications	66
G.	Probability Generating Functions.	69
H.	Summary of Effectiveness Measures Derived	73
I.	A Four-State Example.	74
VI	PARAMETER ESTIMATION FOR MARKOV PROCESSES.	79
A.	Introduction.	79
B.	Factoring the Estimation Problem.	79
C.	Estimating Transition Probabilities	80
1.	Using Transition Data (n_{ij}).	80
2.	Using Mean Time-in-State Data (t_{ij}).	84
3.	Using Mean First-Passage Times (m_{ij})	89
4.	Using the First and Second Moments of First-Passage Times.	94
5.	Using $P(\text{State})$ as a Function of Time	95
D.	Estimating the Density Function of the Transition Times	98
1.	Introduction	98
2.	The Holding Time Parameter (p_{ii})	99
3.	Adding Auxiliary Delay States.	100
4.	Adding Auxiliary States and Transitions.	102
E.	Discrete Time Semi-Markov Processes	109
F.	Testing for the Markov Property	110

VII IMPROVABILITY OF THE MEASURES OF EFFECTIVENESS	113
A. Introduction.	113
B. Approximating an Arbitrary Model by a Markov Model.	114
C. Applying Theorem 1.	114
D. Implications of the Theorem 1	116
VIII PROBABILITY STATE TIME DIAGRAM.	119
IX NUMERICAL EXPERIMENTATION: EXAMPLE ONE.	131
A. An Artificial Process	131
B. The Dynamics of the Process	134
C. Six Markov Approximations to the Actual Process	136
1. Matching Average Time Spent in a State	136
2. Matching Average Times in a State, Conditioned on the Next State.	138
3. Introducing Delay States	139
4. Using Actual $P(\text{State})$ as a Function of Time and Least Squares.	139
5. Semi-Markov Approximation.	143
6. Incorporating Time into the State Definition	145
D. Summary	146
X MARKOV MODELS THAT INCLUDE FALSE CONTACTS.	149
A. Introduction.	149
B. A Markov Chain Model for an ASW Unit in the Presence of False Contacts.	150
C. A Mixed Markov Model for an ASW Unit in the Presence of False Contacts.	158
D. A False Contact Model Formed from a "Product"	164
XI NUMERICAL EXPERIMENTATION: EXAMPLE TWO (FALSE CONTACTS).	169
A. Introduction.	169
B. Monte Carlo Experiments for Estimation of Transition Probabilities.	173

C.	Approximating the Actual Process by Markov and Semi-Markov Processes	180
1.	Markov Chain Model with Matching Average Waiting Time	182
2.	The First Semi-Markov Approximation: Geometrically Distributed T_{ij}	182
3.	The Second Semi-Markov Approximation: Uniform Distribution	184
4.	The Third Semi-Markov Approximation: Averaged h_{ij}	185
D.	Approximations Using Auxiliary States	185
1.	The Fourth Semi-Markov Approximation: One-State Memory	188
2.	A Loopless Version of the False Contact Model.	190
3.	A Semi-Markov Model with Complete Memory	193
E.	Testing the Markov Property	200
XII	MARKOV MODELS AND MONTE CARLO SIMULATION	207
A.	Introduction.	207
B.	The Simulation Model.	208
C.	Selection of States for the Markov Model.	209
D.	Estimating the Transition Probabilities from Transition Data	215
E.	Estimating the Parameters from Average Time-in-State Data.	220
F.	Average Time Until Detection of the High Value Unit.	224
XIII	COMBINING PARAMETER ESTIMATES.	229
A.	Introduction.	229
B.	Combining Independent, Unbiased Estimates	229
C.	Application to Estimation of Transition Probabilities	232
D.	The Variance of the Probability of Success.	235
E.	Optimal Allocation of Sampling in a Simple Markov Chain	237

XIV POSSIBLE OPERATIONAL APPLICATIONS.	245
A. Effectiveness Applications.	245
1. False Contacts	245
2. Countermeasure Environments.	245
B. Other Applications.	248
1. Distance Between a Submarine and a High Value Target	248
2. Radiation from an Aircraft Carrier	249
3. Submarine Classification Process	250
4. Submarine Exposure	250
C. Recommendations	250
APPENDICES	
A RELATED ASW MODELING.	253
B SELECTION OF PARAMETER (1).	267
C MASON'S RULE.	273
D LEAST SQUARES AND QUADRATIC PROGRAMMING	279
E THEOREMS ON PARAMETER ESTIMATION.	285
F DETERMINING A SET OF STATES	299
REFERENCES.	305
DISTRIBUTION LIST	309

ILLUSTRATIONS

2.1	Prototype Queuing Representation of an ASW System .	18
2.2	Alternative Representation for Figure 2.1	19
2.3	Representative ASW Force for Model Demonstration. .	23
2.4	Illustrative Force Effectiveness Model Results. . .	24
2.5	Cumulative Probability Distribution for Contact-Time in System.	27
2.6(a)	Probability Distribution for Unsaturated Time . . .	28
2.6(b)	Probability Distribution for Saturated Time	29
2.6(c)	Probability Distributions for Waiting Time.	30
2.7	Steady-State Probability Densities.	31
2.8	Structure of Single-Unit Model Including Classification.	35
2.9	A Markov Model of the Classification Process. . . .	36
3.1	States in the Weapon System Effectiveness Model . .	41
3.2	Transition Diagram for the Weapon System Effectiveness Model	41
3.3	Data for the Weapon System Effectiveness Model. . .	43
3.4	Success Probabilities with Loops.	45
5.1	Transition Diagram for Four-State Markov Chain. . .	60
5.2	Transition Diagram with Transition Probabilities. .	74
5.3	Imbedded Chain for Figure 5.2	77
6.1	Transition Diagram with Auxiliary States.	103
6.2	Density Function of Waiting Time in State i	105
6.3	Auxiliary States for Transitions out of 1	109
8.1	Illustrative Probability State Time Diagram	121
8.2	Ratios of Line Segments Determine Conditional Probabilities	122

8.3	Markov Process with Six States.	123
8.4	Probability State Time Diagram for First State of Process in Figure 8.3.	124
8.5	Detect Line Quantized; Classify Exponentials Added in Each Segment	125
8.6	Histogram for a Probability State Time Diagram from a Finite Set of Data	129
9.1	Probability State Time Diagram for an Artificial Stochastic (Non-Markov) Process with States O, D, C, A, and F.	132
9.2	Transition Diagram for the Imbedded Process with Conditional Probabilities from Figure 9.1	133
9.3	Comparison of P(State) as a Function of Time Curves for the Actual Process and Markov Approximation	135
9.4	Comparison of P(State) as a Function of Time Curves for the Actual Process and Auxiliary Delay States Approximation	140
9.5	Comparison of P(State) as a Function of Time Curves for the Actual Process and Least Squares Approximation	142
10.1	Transition Diagram for False Contact Models 1 and 2	151
11.1	Probability State Time Diagram for False Contact Model	172
11.2	Transition Diagram and Probabilities for Imbedded Process of False Contact Model.	174
11.3	Comparison of P(State) as a Function of Time Curves for Actual Process and Semi-Markov, Geometric h_{ij} Approximation (Five States)	183
11.4	Comparison of P(State) as a Function of Time Curves for Actual Process and Semi-Markov, Uniform h_{ij} Approximation (Five States)	186
11.5	Comparison of P(State) as a Function of Time Curves for Actual Process and Semi-Markov, Averaged h_{ij} Approximation (Five States)	187
11.6	Transition Diagram and Probabilities for One-State Memory Model.	189

11.7	Comparison of $P(\text{State})$ as a Function of Time Curves for Actual Process and Semi-Markov, Averaged h_{ij} with One-State Memory	191
11.8	Transition Diagram for Loopless Model	194
11.9	New State Definitions for Semi-Markov Model with Complete Memory	196
11.10	Transition Diagram with New States and Transition Probabilities	197
11.11	Comparison of $P(\text{State})$ as a Function of Time Curves for Actual Process and 47 State Semi-Markov Process Approximation	199
12.1	Imbedded Markov Chain for Simulation Model.	209
12.2	Unwinding the Imbedded Markov Chain	211
12.3	Mean Times of First Entry into State i ($\bar{T}_i$)	212
12.4	Transition Diagram for Imbedded Chain of Simulation Model.	216
12.5	Transition Diagram for Markov Chain Approximation	217
12.6	Comparison of $P(\text{No Detection by Time } t)$ for Simulation Model and Markov Chain with Parameters Estimated from 1000 Independent Trials.	218
12.7	Comparison of $P(\text{No Detection by Time } t)$ for Transition Probabilities Estimated from Transition Data and Average Time-in-State Data	223
12.8	Semi-Markov Model with All Transitions at Their Average Times	225
A.1	Basic Structure of Wagner ASW Model	258
A.2	Interaction Within the "Inner Sanctum" of the Sensor Teams.	259
B.1	Root Loci for Various W Values.	270
C.1	Transition Diagram for a Typical Markov Chain	276
E.1	Number of Transitions for a Four-State Markov Chain	293

TABLES

2.1	Significant Categories of Contact	33
2.2	Effects of "Certain" Submarine Contacts	36
6.1	Transition Counts n_{ij} for a Single Path	83
6.2	Transition Counts n_{ij} for Three Paths	83
9.1	Transition Probabilities Derived from Figure 9.1. .	133
9.2	Transition Probability Matrix	137
9.3	Estimates of $p_{ij}^{(5)}$ Using $P(\text{State})$ as a Function of Time	143
9.4	Results from Semi-Markov Approximation--28 States .	144
10.1	Values of Degradation Factor Caused by False Contacts.	162
10.2	Degradation Factor d Caused by the Presence of False Contacts.	164
11.1	Transition Probability Matrix (p_{ij}) for Imbedded Chain	175
11.2	Probability of Being in a State as a Function of Time $\times 1000$	176
11.3	Mean Transition Times ($\bar{T}_{ij}$)	177
11.4	Standard Deviations of Transition Times (σ_{Tij}). . .	177
11.5	Estimates for p_{ij} --25 Replications for Each Case. .	178
11.6	Estimates for p_{ij} --100 Replications for Each Case .	179
11.7	Numbers of Samples Within the One-Sigma Interval. .	181
11.8	Measures of Dynamic Fit	192
11.9	Comparison of Actual and Markov $P(\text{State})$ as a Function for Imbedded Chain	198
11.10	Comparison of Path Probabilities.	203
12.1	Computer Simulation Outputs	213
12.2	Comparing Means and Variances	214

12.3	Simulation Output When Using Average Time-in-State Data.	221
13.1	Typical Values of Optimal N_1^*, N_2^*	243

PREFACE

This report concludes the Queueing Methodology study that has been directed toward the development of dynamic models for evaluating anti-submarine warfare (ASW) system effectiveness. The project was sponsored by the Director, Naval Analysis Programs, of the Office of Naval Research (ONR). Mr. R. H. Dickman was the ONR Project Scientific Officer. The research effort was performed jointly by Stanford Research Institute's (SRI) Naval Warfare Research Center (NWRC), Mr. L. J. Low, Director, and the Information Sciences Laboratory, Mr. D. R. Brown, Director. Mr. W. H. Frye of NWRC was the project leader for that part of the study leading to this report.

The first report on the project, "Dynamic Analysis of ASW Effectiveness--A Queueing Approach," was published in March 1972.^{1*} This report now concludes the project with the presentation of the research results that complement the results from the first report.

The following SRI personnel contributed to this final report:

William H. Frye (NWRC, project leader)

Andrew J. Korsak (Information Sciences Laboratory).

In addition, computer programming assistance was provided by D. L. Alderman and D. G. Ayers of NWRC.

* References are listed at the end of this report.

I INTRODUCTION AND SUMMARY

A. Research Objectives

The objective of the Queueing Methodology study was to develop new methodology for evaluating ASW system effectiveness in dynamic contexts. It should be specifically noted that the study was not directed toward the solution of any particular real-world ASW problem. Rather, the problem addressed is how to adequately analyze and measure ASW system effectiveness in complex, dynamic, tactical situations.

To understand the approach taken in this research effort, some background in the nature of naval effectiveness modeling is needed. Models, modeling efforts, and effectiveness analysis can be divided into two categories: analytical or operational. Roughly speaking, analytical models are usually constructed on the assumption that the model structure and parameters are known or will somehow be estimated by the time the model is used on a real problem. Parameter estimation, and the structuring of the model so that its parameters can be estimated, is usually of secondary concern in such modeling efforts. In contrast, operational models focus on the data available (or to become available), and the parameters are estimated from these data.

Most ASW effectiveness modeling efforts are analytical, and in many respects analytical models are more highly developed than operational models. War games and Monte Carlo simulation, which are common forms of analytical models, have advanced considerably in many technical respects over the past decade or so. In particular, complex dynamic tactical situations are now routinely modeled by analytical means. Whatever the advantages of these advanced analytical models, they suffer from many

inherent weaknesses, particularly the difficulties in estimating parameters or more generally in demonstrating model validity.

On the other hand, operational models--which are nearly always some form of conditional probability model--have their own weaknesses. Although parameter estimation (and therefore model validity) is not a major problem (by virtue of its very construction), the operational models seem to have limited usefulness and flexibility.

Both analytical and operational models were studied during the research effort. The first part of the study was devoted to analytical queueing models that are presented in detail in an interim report.^{1*} Throughout this part of the study, the model parameters were assumed to be known and queueing models were formulated for making ASW effectiveness calculations. As the study progressed, however, it became evident that these calculations could be handled more conveniently by Markov methods than by queueing methods. Moreover, the interim report also clearly shows that Markov methods could conveniently be used to calculate such dynamic quantities as the:

- Expected number of contacts lost while in the ASW system (over a given period of time).
- Probability of missing a true contact (failure to detect) for a given level of contact activity (false and true contacts).
- Expected time to detection.
- Expected time from detection to kill.

Several limitations of the queueing models emerged as the study progressed. On the one hand, parameter estimation proved to be as formidable a problem for the queueing models as it is for dynamic Monte Carlo simulation models. On the other hand, a major difficulty arose from computational considerations: The more complex queueing models required so much

* References are listed at the end of the report.

computer time that direct simulation became more attractive for acquiring numerical results. The conclusion that the queueing models chosen for study were at a dead end therefore became inescapable, and a reevaluation of research objectives was accordingly made. The particular queueing models that had been studied in this phase proved to be valueless for the next phase. Nevertheless, the experience gained with Markov methodology (including the formulation of ASW problems in terms of states and transitions, as well as formulas and numerical techniques for obtaining answers) proved to be valuable during the remainder of the study. ✓

Research effort during the second part of the study was concentrated on a complementary aspect of effectiveness modeling; namely, model structuring and parameter estimation. This led to consideration of dynamic operational models instead of dynamic analytical models. Examination of the classical conditional probability models such as the weapon system effectiveness (WSE) models was considered as the starting point for this part of the research effort. Thus, the emphasis in the second part of the study was on the development of methods for structuring dynamic models and estimating their parameters rather than methods for calculating measures of effectiveness when the model parameters are known. More specifically, the objectives for this part of the study were to:

- Analyze conditional probability effectiveness models and determine their limitations.
- Generalize conditional probability effectiveness models (using Markov methods if they proved appropriate) to:
 - Remove some of the limitations on conditional probability models; in particular, determine ways to make them dynamic by incorporating false contacts.
 - Include, when appropriate, other dynamic aspects of the tactical problem being modeled.
- Develop methods from simulated data and Fleet operational data for estimating parameters in the dynamic operational models.

- Define and investigate ways to formulate new state spaces, and investigate potential ASW effectiveness applications if operational Markov models proved appropriate.
- Investigate methods for reducing the variance of effectiveness measure estimates and for using variance estimates in exercise design.

B. Research Findings

This section summarizes research findings, and Section C summarizes research activities for the second part of the study. Every attempt has been made to separate what was found from what was done. In many respects, however, the findings are rather general and may often be difficult to relate to research activities without a detailed study of the body of the report.

During the final phase of the Queueing Methodology study, research findings ascertained:

- ✓ • Queueing theory itself has no direct, unique applicability to the analysis of ASW system effectiveness. However, some of the conceptual and computational methods--particularly Markov methods--often used in the analysis of queueing models are well-suited to analyzing effectiveness in today's complex tactical environments.
- Conditional probability models, the most commonly used operational models, have many limitations.
 - They are static models, not dynamic. One consequence of this is that they are defined for only a limited number of tactical situations, usually one-on-one engagements.
 - They have limited utility, flexibility, and predictive ability in new situations.
 - Meaningful sensitivity analyses cannot be performed.
 - They do not explicitly include factors such as false contacts that degrade ASW system capability.
 - They use only a limited portion of the data available for parameter estimation.

- They have few--if any--environmental or operational variables, and causative variables may be difficult to isolate.
- Markov (and semi-Markov) methods are not only be used increasingly in analytical ASW effectiveness modeling but also appear to have considerable potential for operational effectiveness modeling in dynamic ASW environments. ✓
- A Markov chain model can be regarded as the natural generalization of a conditional probability model. This generalization can remove many of the limitations on traditional, conditional probability models.
- Operational Markov models have both advantages and disadvantages. The principal advantage is that operationally meaningful measures of effectiveness can be defined for a large variety of dynamic tactical contexts. The four principal overall effectiveness measures that Markov models can define are: ✓
 - The probability of mission success.
 - The average time until mission failure (or success).
 - The probability that mission failure does not occur by a specified time.
 - The probability of occupying a designated state at a random time.

Many measures of effectiveness used in naval analysis studies are equivalent to one of these four measures. A wide variety of missions and tactical situations can be treated by suitably defining success and failure states.

The principal disadvantage is that some properties of operational Markov models were found to be rather "sterile." That is, the output effectiveness measure sometimes too perfectly reflects the data inputs used to estimate parameters. The four effectiveness measures differ markedly in this respect; specifically, when data from "complete" exercises are used to estimate the transition probabilities in a Markov chain model, it was found that: ✓

- The estimated probability of success obtained from Markov calculations is identical to the ratio of the number of successes to the number of exercises.

- The estimated conditional mean time until failure obtained from Markov calculations is identical to the ratio

$$\frac{\sum (\text{times until failure})}{\text{total number of failures}}$$

provided that only the path data leading to failure are used to estimate the Markov parameters.

These assertions can be translated into implications concerning the statistical improvability of the measures:

- The variance of the estimate of Prob(success) is not reducible by using a Markov model.
- The variance of the Markov estimate of the average time to fail (given that failure ultimately occurs) is reducible by at most a factor roughly equal to the probability that failure ultimately occurs. The estimate will in general be biased when all the path data are used in a non-Markov context.
- The most difficult step in defining a useful operational Markov model is expected to be determining a satisfactory set of states ("state space"). A set of reasonable requirements for the state space are:
 - States should be mutually exclusive and exhaustive.
 - Each state should be operationally observable.
 - Each state should be of operational significance.
 - States should be of reasonable generality, though not necessarily so general as the traditional state sequence of Opportunity + Detect + Classify + Attack + Kill.
 - States should be defined so that the effectiveness measure(s) can be calculated by probability statements about state occupancy or by mean time-to-occupy statements. This often can be accomplished by defining success and failure states as absorbing states.
 - The statistical requirement that knowledge only of the current state should determine the transition probabilities for the next step should be reasonably well met.
- Effectiveness (and performance) measures can be defined at two levels in a Markov model. Lower-level measures often

relate to the performance of individual subsystems or ASW functions such as detection and classification. These measures, which are quantities such as Prob(Classify/Detect) and the mean time to classify, can be associated with the microstructure of the model; that is, with transition probabilities. Overall effectiveness measures that depend on the entire set of model parameters are at a different level.

- Degrading factors such as false contacts can be explicitly included to permit systematic study of the relationship between the degradation parameters and the effectiveness measures.
- Many calculations of interest can be routinely made; in particular, the dynamic response can be calculated when appropriate and sensitivity analysis can be routinely performed.
- Many kinds of Fleet operational data, both static and dynamic, can in theory be used to systematically estimate the parameters of a Markov model. One type of useful static data (transition data) is now used to estimate the the conditional probability parameters in conditional probability models. Other types of potentially useful data are rarely if ever used in effectiveness modeling; an example is average time-in-state data that may be considered dynamic data.
- Standard methods are available for many parameter estimation problems in Markov modeling. Other methods were investigated and evaluated during the current research. Many parameter estimation problems lead to least squares or quadratic programming problems. When a satisfactory set of states has been found for a Markov model, parameter estimation is not expected to be a significant problem.
- Operational Markov models may be used to obtain improved estimates of certain of the traditional effectiveness measures.
- Selecting a suitable set of states for an operational Markov model in ASW will often require the aggregation of states and/or the addition of auxiliary states to make the Markovian assumption valid. It was found during the research that auxiliary states can often improve the fit, and specific methods are given for introducing them. Aggregation of states by systematic methods is more difficult; one method

sometimes useful is least squares using Prob(state) as a function of time as input. As a general method, least squares appear to offer several uses in the structuring of the state space and in estimating parameters. The general use seems to result from the linearity of many of the sets of equations associated with Markov theory. Quadratic programs are needed when the transition probabilities estimated from least squares are infeasible.

- Markov models may be useful in comparing dissimilar models and in validating simulation models. The comparison (or validation) can be accomplished by determining a suitable common set of states for the two models, estimating the probability of occupying each state for all relevant instants of time for each model separately, and comparing the two sets of Prob(state) as a function of time curves.
- Current analysis reports from operational exercises contain inadequate data for Markov modeling. Only minor changes in analysis procedures should be required to collect data for a Markov model with a moderate number of states, however.
- A Markov model may be useful in resolving some experimental design problems encountered in Fleet exercise planning. One simple situation in the study was analyzed by using an optimization method that incorporated the cost of experiments. The optimum number of samples of each available type were determined analytically on the assumption that the objective would minimize the variance of the Prob(success) estimator subject to the cost constraints.

C. Research Activities

The findings in Section B were arrived at by considering the combined research activities from both phases of the project. An overview of the individual sections of this report conveniently summarizes these activities.

A summary of the work done on queueing models during the first phase of the project is given in Section II. This section also introduces some results from the analytical queueing models that obtained after the publication of the interim report.¹

In Section III, conditional probability models are defined, discussed, and shown to be a special form of Markov model. Parameter estimation for this model is briefly discussed, and some intuitive attempts are made to introduce false contacts by modifying the estimation formula. The modified formulas prove to be incorrect, however, and the correct formula (derived from Markov theory) is given. Although the limitations of conditional probability models are enumerated, several arguments are given to justify their use. Finally, the suggestion is given that many of the limitations on conditional probability models can be removed by generalizing to a Markov model.

Section IV opens with a brief general discussion of useful properties of Markov models in ASW effectiveness analysis. Four measures of effectiveness of rather general use, but readily definable and calculable in a Markov model, are then given. Also discussed is model formulation that includes defining a set of states so that the measures may be conveniently calculated. State space definition, asserted to be the most difficult step in model formulation, and rules of thumb for formulating a set of states are given. Parameter estimation in a Markov model applied to an ASW situation is briefly discussed. Finally, the possibility is mentioned of using operational data not currently used for modeling.

Basic Markov chain theory and formulas are given in Section V. This material is needed for either analytical or operational modeling; for the latter, it is needed both for calculating effectiveness measures and as a basis for several parameter estimation methods. Algebraic formulas for all the measures are included. Common-sense interpretations of the absorption probability formulas and their elements are given where appropriate to aid the understanding.

In Section VI, parameter estimation for a Markov model is discussed at length. We point out the usefulness of factoring the estimation problem

into static and dynamic parts. Parameters for the static part, which correspond to a conditional probability model (or the imbedded chain in Markov theory), may be estimated from transition data only. The dynamic parameters can then be separately estimated from mean time-in-state operational data.

Methods for estimating transition probabilities are then given for various assumptions about the data available. The input data are assumed to be transition data, mean time-in-state data, mean first-passage time data, data on the first and second moments of first passage times, or probability of state as a function of time data. The need for least squares and quadratic programming are shown in several instances, but these are discussed in more detail later in the section.

Holding (or waiting) times in a state, which are a function of the self-transition probabilities (p_{ii}) in a Markov chain, are the subject of the remainder of the section. The p_{ii} estimator is first shown to be a simple function of the observed mean time in state i in a Markov chain. Then, various ways to better approximate the holding time behavior by enlarging the chain through the addition of auxiliary states are suggested and examples are given. The section concludes with some chi-squared goodness-of-fit tests that may be used to test the reasonableness of the Markov assumption.

Section VII deals with the statistical problem of reducing the variance of the four effectiveness measures by using a Markov model. Improvement is measured relative to the estimates obtainable directly from the data. Certain mathematical identities (proved as theorems in an appendix) are the key tool for discussing this important problem area. Several implications of the theorems conclude the section.

Anticipating that most real-world ASW problems will not have a Markov structure (at least when a natural set of states is used), we need to know

how well Markov methods work on non-Markov problems. Therefore, in Section VIII, a dynamic non-Markov model--conveniently described by a Probability-State-Time (PST) diagram--is introduced, motivated, and related to a Markov model. A PST diagram is used in conjunction with a dynamic (Markov) version of the WSE model in Section IX. Numerical experimentation uses data from the PST-defined model, and the resulting dynamic behavior given is compared with the dynamics of the actual process.

Various methods are used to estimate parameters, and auxiliary states are sometimes added along the lines of the suggestions in Section VI. Although the addition of auxiliary states improved the dynamic fit, all Markov chain models fit rather poorly. Therefore, two more sophisticated Markov models were tried. The first, a semi-Markov model, resulted in a substantial improvement. The last attempt used time itself in the state definitions: The result was an essentially perfect dynamic fit at the cost of a greatly increased number of states.

In Section X, three false-contact models were defined by adding states for the prosecution of false contacts to the traditional states--search, opportunity, detect, classify, and kill. Markov formulas are given for the Prob(Success) effectiveness measure; the form is a product of conditional probabilities divided by a false contact correction factor. For the second of these models, the influence of false contacts on effectiveness takes the form of a multiplicative correction factor to apply to the traditional product of conditional probabilities. The third and most complex false contact model was defined by a cartesian product method; it has 27 states and includes false attacks and earlier stages of false-contact prosecution.

Further numerical expectation, with a PST diagram for the first false contact model in Section X--is performed in Section XI. Five methods were used to estimate parameters with the original state space. Three of these

methods were semi-Markov and the other two were Markov chains; none worked well. Monte Carlo experiments were also performed to study sampling problems.

After the earlier parameter estimation attempts failed with the original state space, some methods were tried with auxiliary states. Some improvement resulted from the most elaborate of these models, but the fit remained unsatisfactory. By the end of the experimentation, the reasons for the many failures became apparent.

A new topic, Markov models and Monte Carlo simulation, is the subject of Section XII. A single Monte Carlo model including a submarine, an aircraft carrier, and an acoustic decoy was selected for study with the use of the mean survival time measure. The idea was to use the simulation to generate data for estimating the parameters of a Markov chain model. A simple, natural set of states proved to be satisfactory for this simulation. Parameters were determined in two different ways and led to almost identical estimates. Probability of state as a function of time comparisons were made between the approximating Markov chain and the simulation model's direct output for assessing the validity of the approximation. Finally, some basic sensitivity analyses were performed by using the fitted parameters, and an estimate was made of the increase in survival time as a result of an increase in the primary decoy parameter (called "capture time").

Section XIII deals with two questions: how to combine different estimates of the same parameter in an optimal way, and how to use a Markov model in exercise design. Both of these questions, which are treated in a very limited manner, entail the use of estimates of variance. For question one, formulas are given for combining two or three independent estimates of an unknown constant, and the formulas are applied to the transition probabilities in a Markov chain. To answer the second question,

an example of a simple three-state Markov chain model is assumed; a further assumption is that the number of samples at each of the two transient states can be freely chosen. Formulas are then derived for the optimum number of samples under a cost constraint, and a table is provided to illustrate the optimal sample sizes for various costs.

D. Possible Operational Applications

Possible operational applications are discussed in Section XIV, the final section. Many forms of false contact models are possible using Markov concepts. The choice of a state depends on the application and the problem features the analyst wishes to distinguish. Markov models also may be helpful in studying the difficult subject of countermeasure effectiveness because states can be defined in terms of information. States and Markov models are then discussed in relation to the UPTIDE series of exercises, and suggestions are made for analysis using some of the concepts from this report. Other possible applications include modeling: range from a submarine and an HVU, electromagnetic radiations from an aircraft carrier, submarine exposure, and the submarine's classification process.

Two recommendations conclude the body of the report:

- (1) A research project in an area of current interest to the Navy should be undertaken to apply the dynamic Markov modeling methods developed during the study. The model should be developed for class of tactical problems where wholly satisfactory measures of effectiveness have not yet been developed because of the dynamic nature of the problem, the complexity of the problem, or the difficulty of relating lower level performance measures to the overall effectiveness measures. The Naval Analysis Programs Office of the Office of Naval Research could sponsor such research as a continuation of the Dynamic Analysis study.
- (2) The technique of using a Markov model to compare dissimilar models or to assist in validating a simulation model should

be suggested to Navy analysts concerned with simulation validation and effectiveness modeling.

Six appendices follow the body of the report. Appendix A discusses ASW modeling which is conceptually related to the report, and compares the other approaches with those in the interim report and the present report. Appendix B treats a special topic which relates to the analytical Queueing Models. Appendix C gives Mason's rule for determining transmissions in a flowgraph; the rule was applied to obtain the measures of effectiveness for the two of the false contact models. Appendix D discusses least squares and quadratic programming that were needed for parameter estimation. Appendix E has proofs of the three important identities relating to parameter estimation. Finally, Appendix F amplifies the earlier discussion of state selection.

II SUMMARY OF QUEUEING MODELS OF ASW UNITS

A. Summary of Work During the First Phase

During the first year of the contract, the emphasis was on modeling ASW operations in terms of queueing theory.¹ The objective was improved representation of ASW dynamics that would lead to more realistic measures of effectiveness than those currently available with traditional formulas consisting of products of success probabilities (the WSE model, for example). Because "congestion" from many simultaneous contacts causes some degree of delay in ASW system processing, the suggestion to attempt queueing models in ASW analysis arose.

After the literature on queueing theory and its applications was reviewed, the conclusion was that a direct application of readily available results on single queue/multiple server systems is infeasible. ASW situations simply do not exhibit properties of a single "waiting line" in any sense. What was needed to relate to ASW appeared to a degree in articles on networks of queues, such as in models of time-sharing and other computing systems. Again, however, no direct parallel was found because contacts could not be justifiably regarded as passing through a series of ASW process stages without proper consideration of strong interaction between the stages. Therefore, the decision was made to construct a new set of congestion models that would exhibit queueing-like characteristics, be amenable to analysis by techniques in queueing theory, and relate better to the ASW situations considered. This resulted in queueing analysis techniques found for solution of the models created; the key technique was the method of imbedded Markov chains. Additional techniques found of great value in computations with the imbedded Markov models were:

- Laplace transform solutions.
- Flowgraph techniques to organize computations efficiently.
- Matrix flowgraphs to aid in model formulation.
- Successive over relaxation (SOR) method for solving large systems of linear differential equations associated with Markov processes.
- Padé approximants to e^{tA} for speeding up numerical integration with Markov models.

Other methods useful in queueing analysis were not adaptable to ASW models considered on this project.² These methods include

- Contour integration of Laplace transforms for waiting time distribution in tandem queues.
- Integral equations for waiting time distributions.
- Integro-differential equations for transient waiting time distributions.

Although the queueing literature reports very powerful results with the above techniques, these methods are mainly for a single queue; no clear way appears as feasible as the Markov techniques for generalizing their applicability to more complex congestion models.

The work on queueing models was divided into two areas: at the single ASW unit level and at the force level. This two-pronged approach decision was made on the basis that congestion in ASW operations probably originates in processing delays of contact information on individual ASW screen or search units, but that the overall effect is on the force level with regard to decisions to prosecute, maneuver for improved contact data, or evade for safety. Consequently, work was started on the following two separate but closely related models.

- A model for analysis of congestion effects in an ASW unit providing distributions for its periods of saturated and unsaturated detection capability.

- A model for computing the probability of detecting a target by a typical ASW force configuration of units with known saturated and unsaturated period distributions and detection versus range characteristics. A summary discussion of these models is presented in the following two subsections.

1. ASW Unit Model

For the unit ASW model it was assumed that as a contact passes through the system, it occupies a unit of sonar service while in the first "server"--i.e., while undergoing detection--and continues to occupy that unit of sonar service even when in any of the three queues following the first server. Similarly, the Combat Information Center (CIC) resource is assumed to be occupied at the rate of one unit per contact from the moment of entering "localization" and up to leaving the system and so on.

The unit model assumed that each ASW function serves a limited number of multiple contacts simultaneously. The unit model representation weighs all contacts uniformly for the load each places on a server, or equivalently, for the difficulty of service.

The command and control and classification functions were omitted from the initial model.¹ Similarly, certain other features of an ASW engagement have not been modeled. Some of these are: hand-off of functions from other ASW units, lost contact processes, and unsuccessful kill attempts and reattack process.

Based on the above considerations, the initial multiple queueing situation of Figure 2.1 was selected for study. The meaning of Figure 2.1 follows.

Acoustic stimuli, if they are locally generated as they become available for detection, await the availability of the detection server D in a fictitious queue Q_D . This queue is created to measure the statistics of missed and delayed detection caused by preoccupation of the

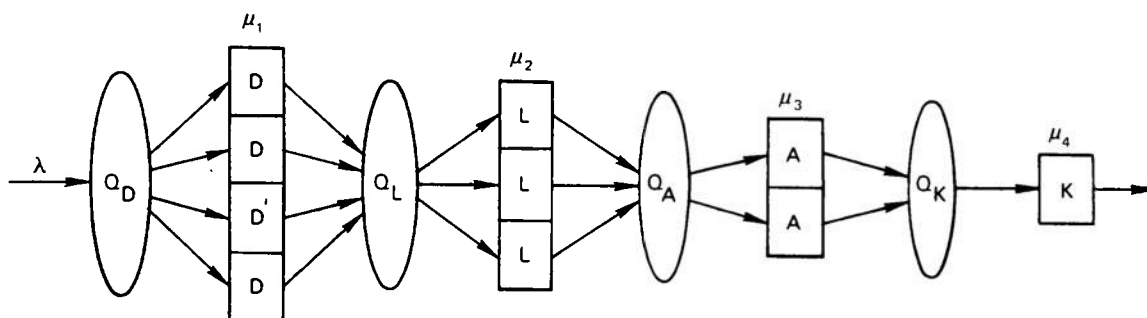


FIGURE 2.1 PROTOTYPE QUEUING REPRESENTATION OF AN ASW SYSTEM

detection process. As soon as an element of the detection server is available (four degrees of detection server occupancy are used here for specificity), the stimulus progresses "through" the detection phase according to prescribed detection time statistics. From this point the stimulus is a contact. Contacts not in process of localization are represented as "waiting" in a queue Q_L in the same manner as the contacts progressed from a queue awaiting detection to a queue awaiting localization. So, the contacts progress from the latter queue, through the localization process, through a queue awaiting the attack function, and so on, through service by the kill function.

The number of parallel service units at each stage was selected for development purposes to be just enough so that queueing is possible at the following stage. That is, in view of service at each stage being "tied up" until a contact leaves the system, at least one less unit of service is necessary at each successive stage. In other words, the server capacities were chosen to be simple enough for a first study, but complicated enough to allow queues to form in all possible stages.

An alternative schematic representation of the situation is presented in Figure 2.2. This figure shows the idea of the presence of a contact being felt in several places at once, in that a contact

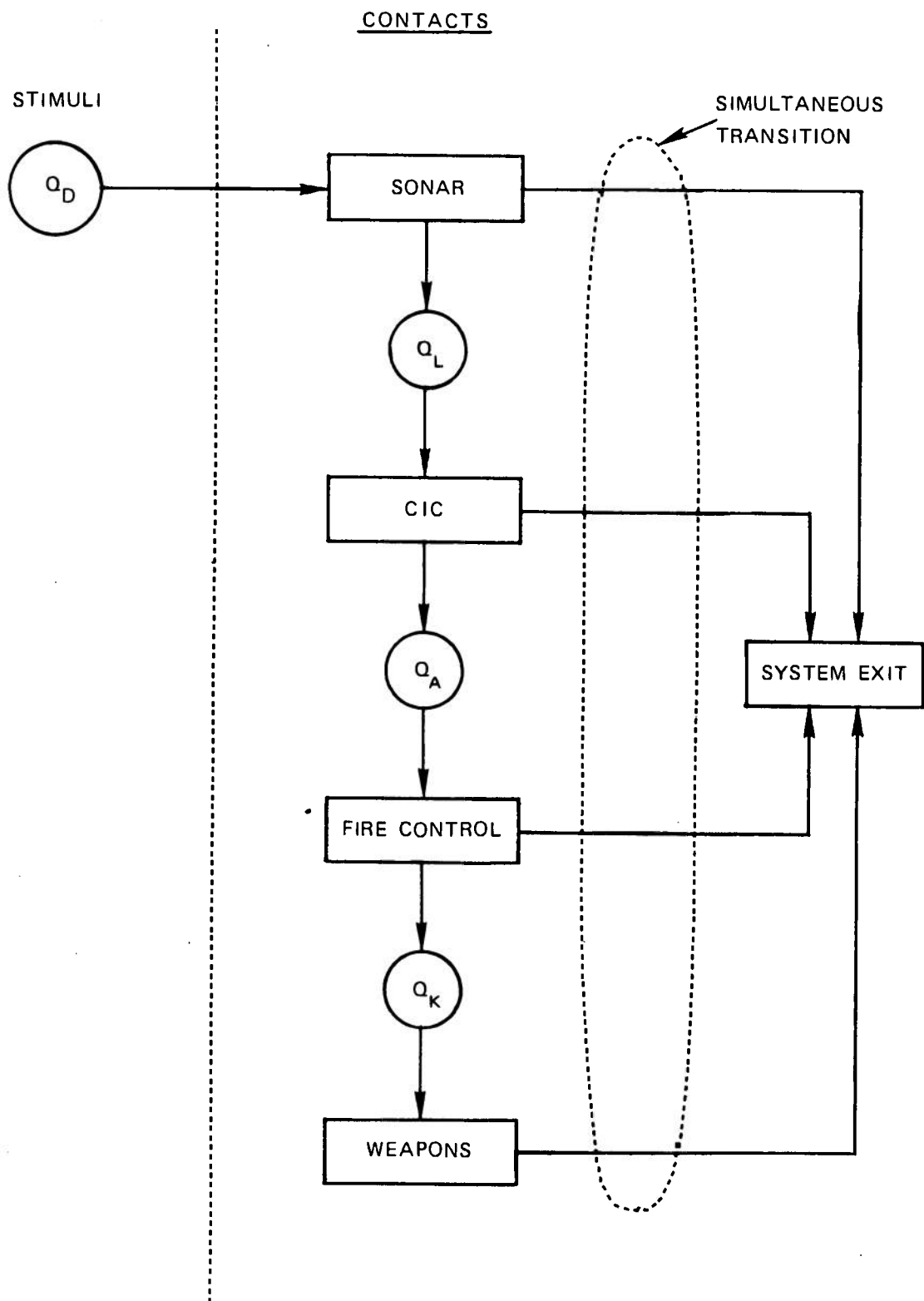


FIGURE 2.2 ALTERNATIVE REPRESENTATION FOR FIGURE 2.1

progressively uses the resource of sonar (detection), then sonar, CIC (localization), and so on, until it finally exits from the system.

As a start it was assumed that all parallel service units at each stage have the same exponential service distribution, with parameters as in Figure 2.1. It must be remembered that a completion of a service does not make a given service unit available for other contacts; such a completion merely tells when a transition of a contact to the next state occurs (either into the next server or the queue ahead of it). The input distribution for all true and false contacts is also assumed to be exponential with parameter λ . (Both the input and service distributions can be made more general by using a network of parallel and series exponential holding times.)

Various attributes of factors associated with queueing systems in general were considered in structuring a suitable model for evaluating ASW system performance. The primary statistical distributions were judged to be:

- Unsaturated time distribution (for entire system).
- Saturated time distribution (for entire system).
- Waiting time distribution (in first queue, i.e., ahead of detection).
- Distribution of time spent in entire system.

Other significant distributions are:

- Queue lengths at the various stages.
- The number of contacts exiting any stage in a given time.

The primary statistical distributions are needed in the force effectiveness model.¹ The others may be useful for evaluating the effects of physical storage capacity or other limitations at various stages of the ASW system and for determining the nature of the input distributions to

other ASW platform queueing models coupled to a given one through a command and control hand-off system.

The above statistical distributions can be used to calculate the following possible measures of effectiveness of an ASW unit:

- Probability that a contact is lost.
- Expected number of contacts lost while in the system, over a given period of time.
- Probability of missing a true contact (failure to detect) for a given level of contact activity (false and true).
- Expected time to detect.
- Expected time from detection to kill.

If the classification function were added to the model, the following additional measures would be obtainable:

- Probability of losing a contact before classifying.
- Expected time to classify.
- Probability that a false contact is pursued all the way through to the kill function.

Computation was aimed only at numerical calculation of the statistical distributions as these are the most difficult to obtain. The above measures of effectiveness were not considered in the current effort because fleet-operational data were unavailable to render them meaningful. Graphs comparing analytical solutions and simulation results for some of the computations that were performed are presented in Section II-B.

2. ASW Force Effectiveness Model

The force level queueing model developed during the first phase of this study considered an ASW screening situation in which ASW units

maneuver within a protected area surrounding a task force. The individual ASW units were assumed to experience congestion from the abundance of contacts according to the unit queueing model previously described. In the numerical example worked out, only exponential approximations to the distributions of the unit model were used.

The force model was based on the idea that units could be combined in a single detection device whose statistics for "windows" of possible target capture result from joint probability of detection by any one of the ASW units within range of a target. Kinematics of both targets and ASW units were included along with a simple range-law for probability of detection given opportunity. The example used for computations is described by Figure 2.3 that shows initial positions and velocity vectors of the ASW units participating. A lengthy series of calculations was performed to obtain probabilities of a target ever being detected prior to its penetrating the protected zone, given that the target approaches the force from all possible (quantized) positions of initial appearance, as indicated in Figure 2.4 (all probabilities have been multiplied by 100 in the figure).

The assumption in combining ASW units into a force was that each unit individually experiences alternating periods of being saturated and not saturated by contacts it is processing, as determined by the unit model. In addition, a waiting time distribution accounted for the first period of the series of busy and idle periods of any ASW unit--with regard to the time arrival of a specific contact--because the given contact could arrive during either a busy or idle time. The probability that a target is ever detected in time for effective counteraction was expressed in terms of probabilities of detection by each of the sensors during each of their unsaturated periods up until the time that the target penetrates the protected zone.

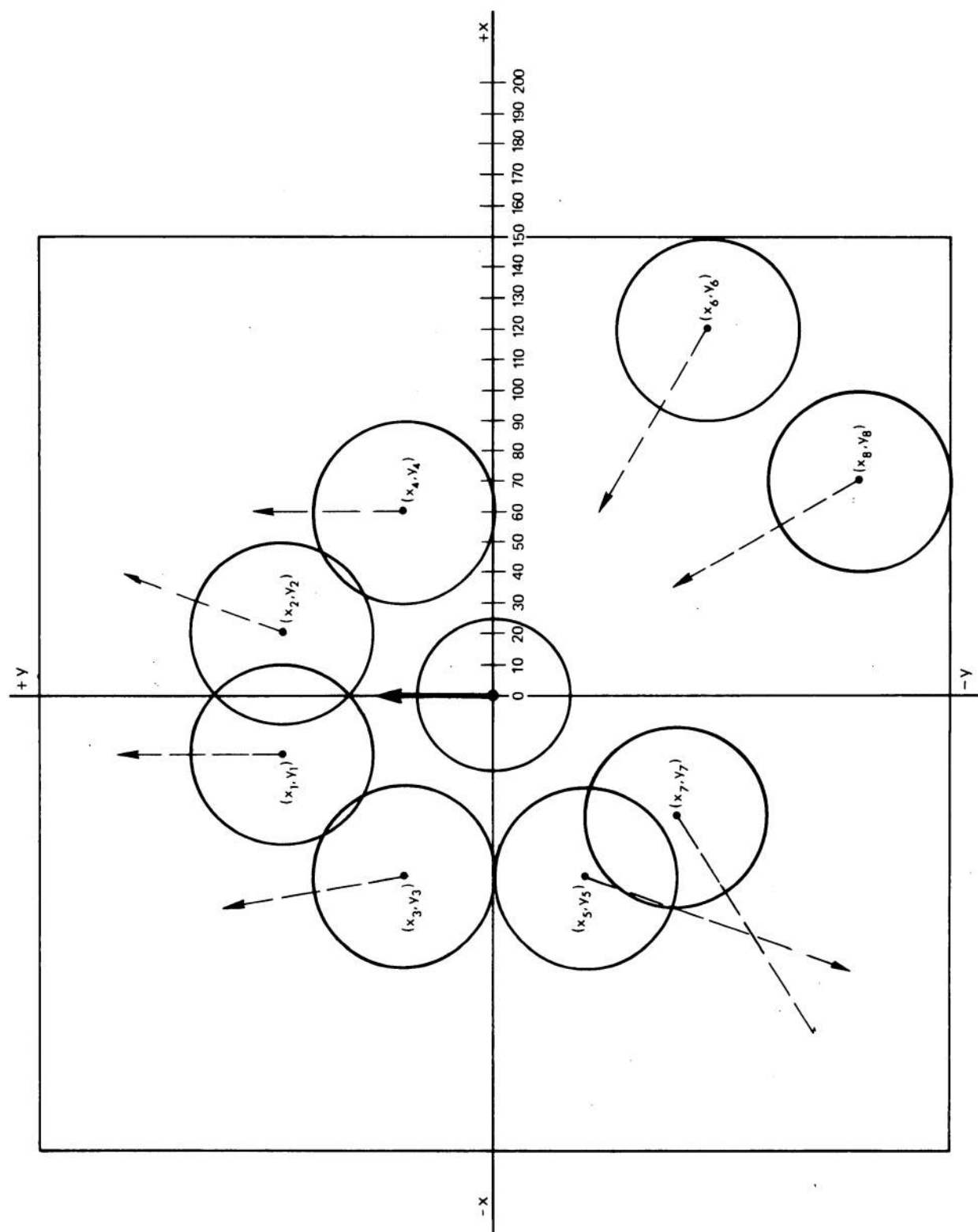
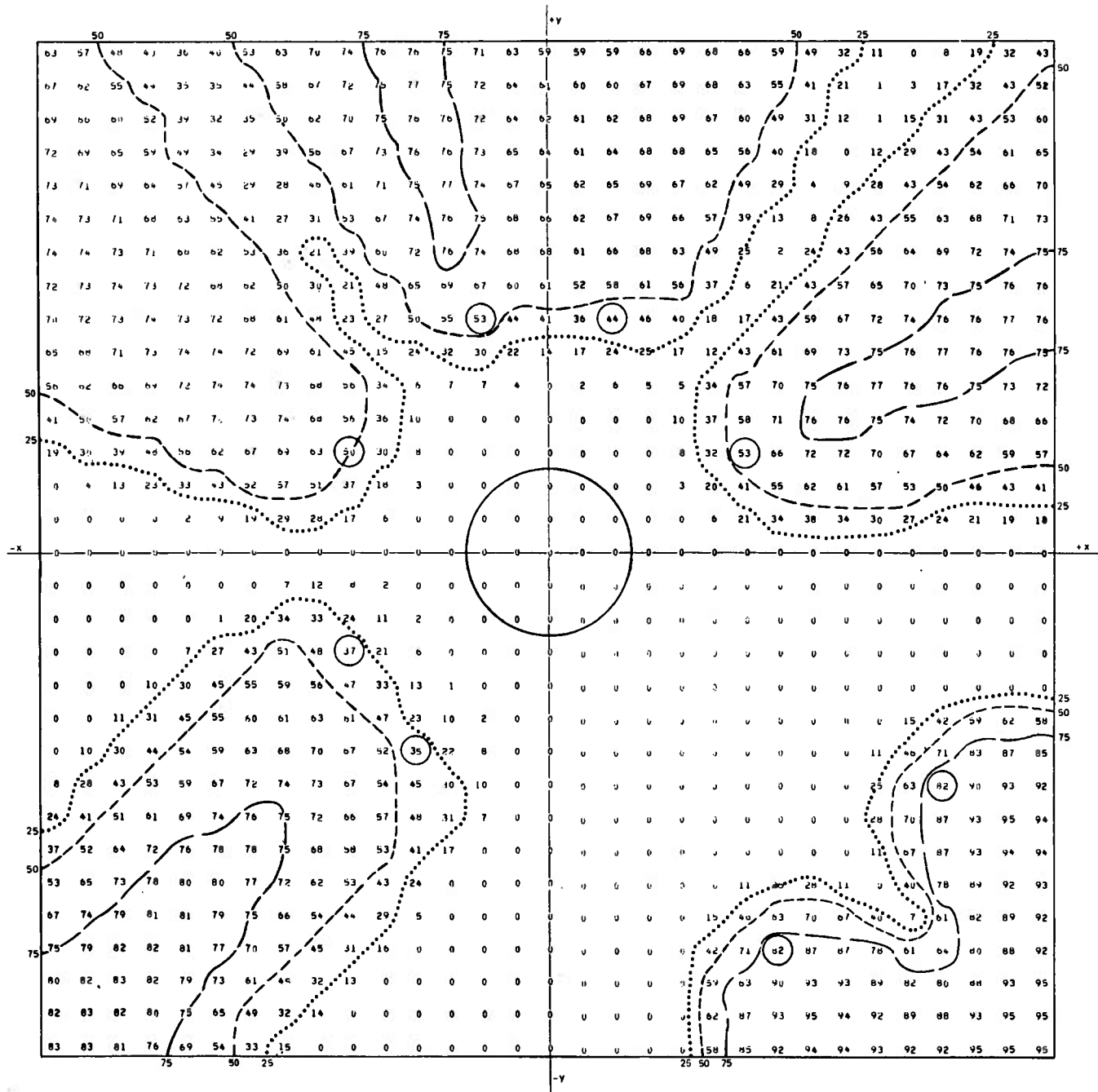


FIGURE 2.3 REPRESENTATIVE ASW FORCE FOR MODEL DEMONSTRATION



LEGEND:

- 0.25 PROBABILITY
- 0.50 PROBABILITY
- 0.75 PROBABILITY

NOTE:

INDICATED VALUES ARE PROBABILITIES:
EXAMPLES: 0.75, 0.21, and 0.08.

FIGURE 2.4 ILLUSTRATIVE FORCE EFFECTIVENESS MODEL RESULTS

B. Further Computations with the Single Unit Model

The most complex of all flowgraphs introduced in the previous year's work on a queueing model of a single ASW unit was for "time in the system," i.e., the time from the moment a contact arrives at the detection periphery to the time that it is finally killed. Since only true targets were considered, all contacts were assumed to be eventually localized, attacked, and then killed. The same methodology would apply equally well, however, to a more general semi-Markov model with classification, false contacts, and possible loss of contacts.

Because of the large number of states in the time-in-system flowgraph (see Ref. 1, p. III-39 and Figures III-19 to III-22), actual computations of distributions for this case were not completed during the first year's work. Still more efficient numerical techniques had to be set up early in the current year before results were obtainable. In addition to the continued analytical work, a Monte Carlo simulation model was prepared and run in parallel with the analytical model. Both approaches led to almost identical plots for all distributions, i.e., contact waiting time, ASW system saturated and unsaturated time, and time to process a specified contact from opportunity to kill. The results for these distributions and the simulation data will be described further below.

1. Special Numerical Integration Method for Solving Large Flowgraphs

A special integration technique, SOR, was used to handle the large size of the Markov model for contact time in the system.* The advantage of using this method and an explanation of it are laid out in

* We are indebted to Dr. Samuel Schechter who recommended the methods used here.

Appendix B of Ref. 1. Steps were taken to reduce integration time of this large system of equations to as low a value as possible per integration step. Some convergence difficulties were encountered for the over-relaxation iterations, as the method is quite critical with regard to choice of the over-relaxation parameter ω . A detailed discussion of the problems associated with the selection of ω is given in Appendix B of this report.

The analytically computed probability distribution for contact time in the system is given in Figure 2.5 along with the simulation data.* Almost identical results were obtained by the two methods up to $t = 0.6$ hours, but the analytical integration was not carried out any further because of the relatively high computing cost compared to simulation cost.

2. Results of a Simulation Approach

Although analytical modeling was the primary concern of the first phase of the current project, it was decided to compare results obtained so far with data collected from a direct simulation of the single ASW unit model. Use was made of the General Activity Simulation Program (GASP). The results obtained from simulation were essentially identical to the analytical results. Figures 2.6(a), (b), and (c) are copies of the analytical results (Figure III-26 of Ref. 1) with the simulation results superimposed. Similarly, Figure 2.7 and Figure 2.2 demonstrate the comparison of steady-state and time-in-system results.

* Each integration "step" here is not the usual "infinitesimal" one, however, as we are using Padé approximants of $e^{A\Delta t}$ accurate up to $\|A\Delta t\| \approx 1$ or 2 as opposed to 0.001, or something of that order, in common integration. Hence, only a few "steps" are used in total--by our method--to achieve enough data points to plot curves for the desired distributions.

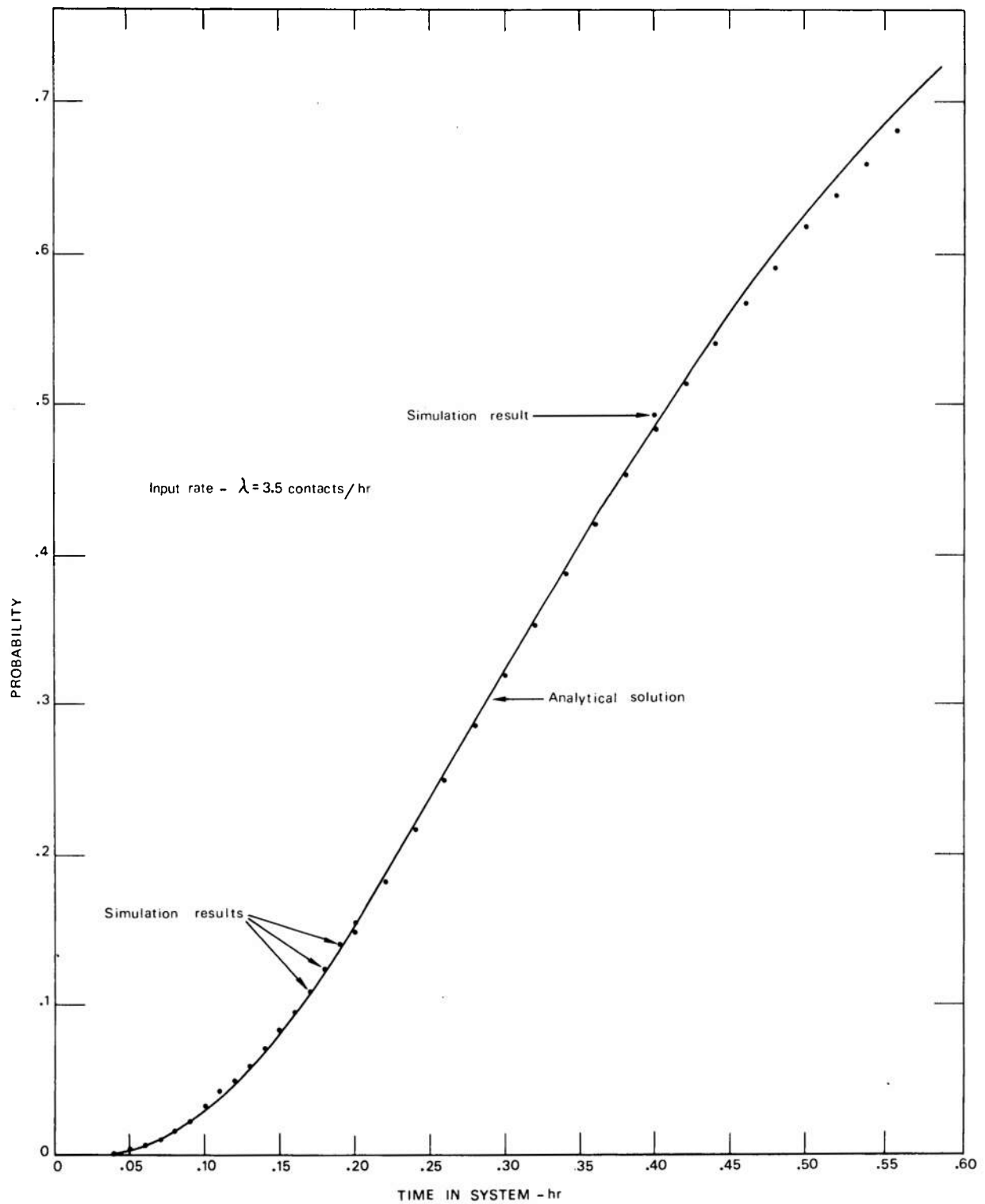


FIGURE 2.5 CUMULATIVE PROBABILITY DISTRIBUTION FOR CONTACT-TIME IN SYSTEM

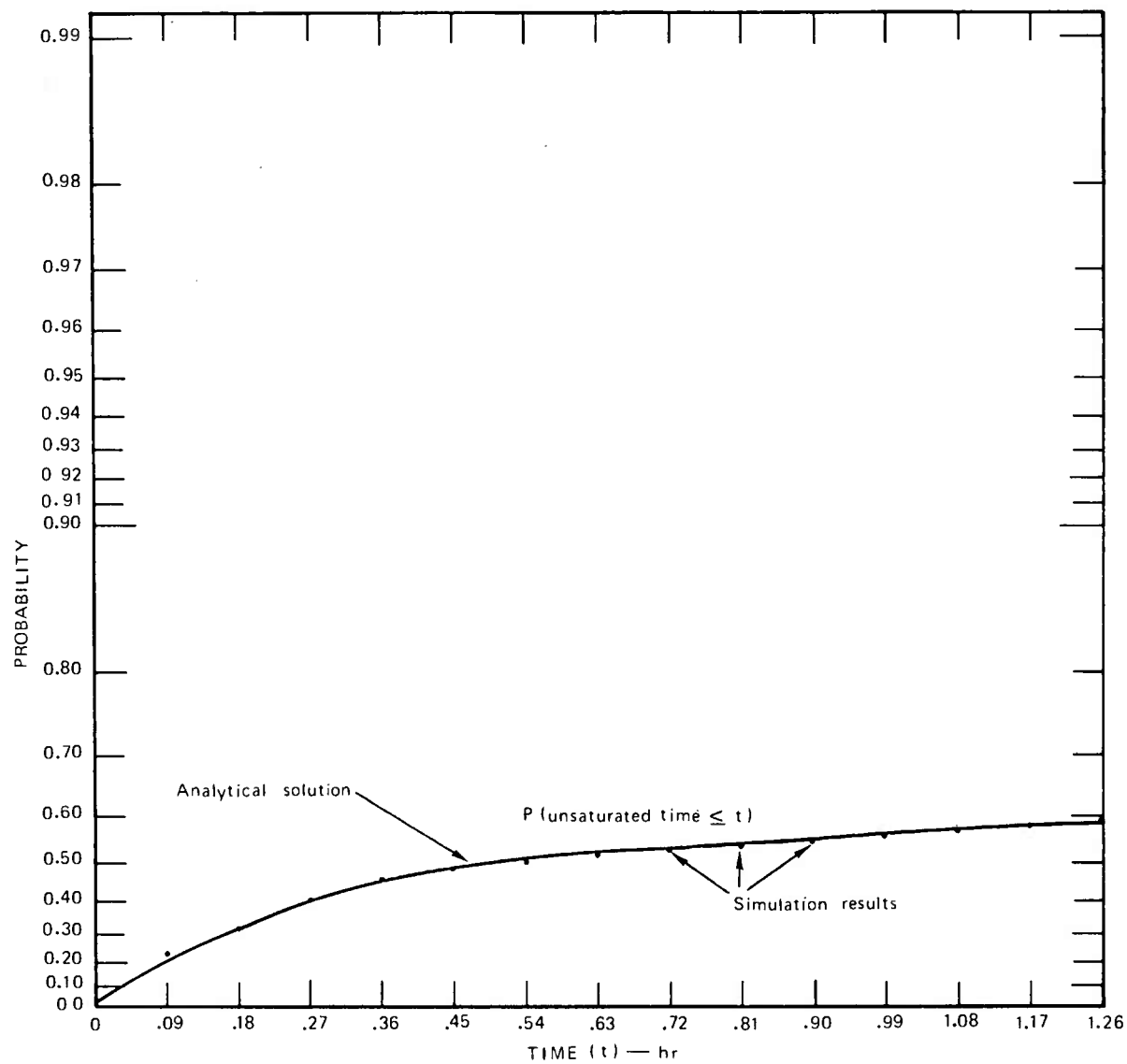


FIGURE 2.6(a) PROBABILITY DISTRIBUTION FOR UNSATURATED TIME

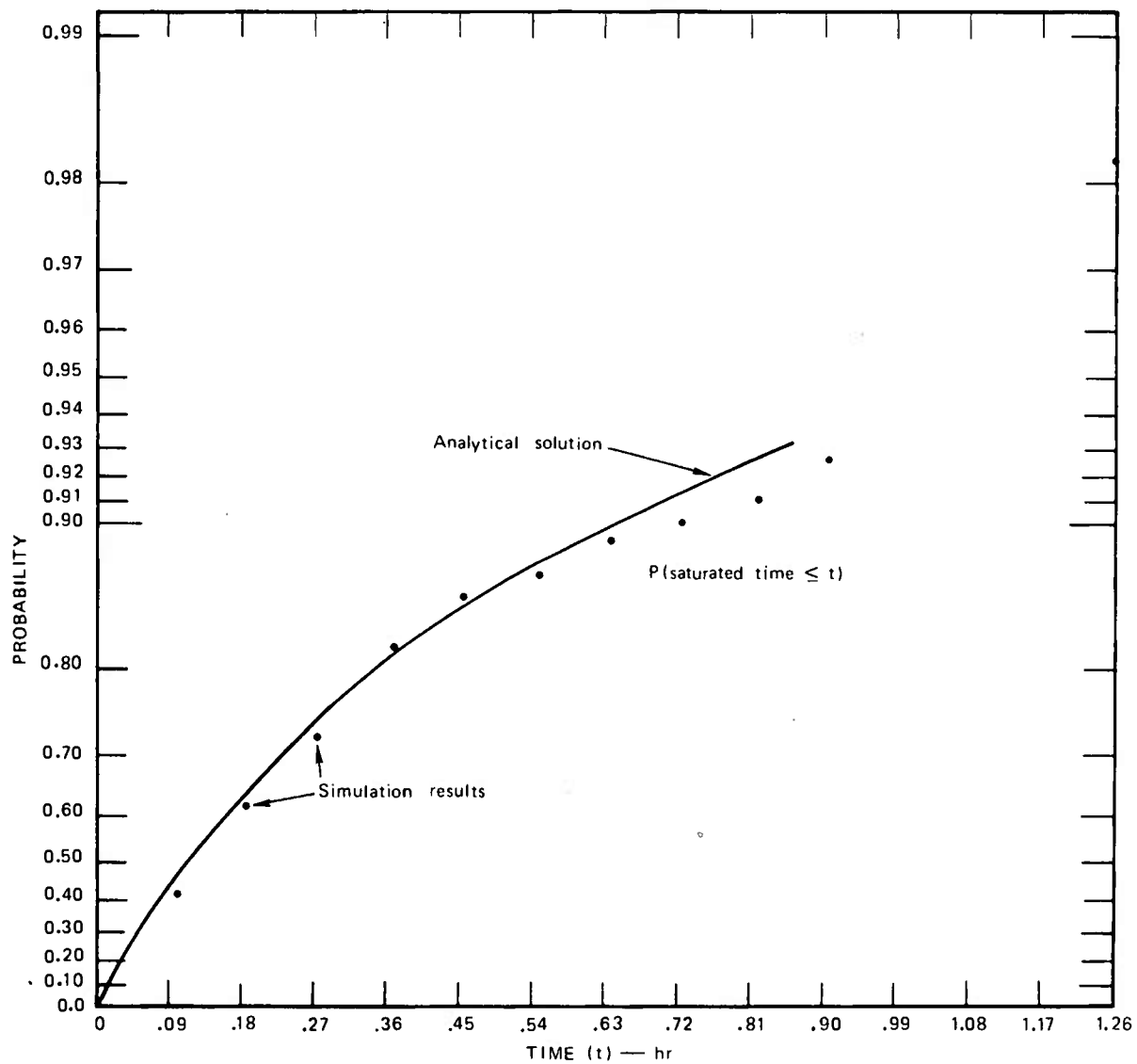


FIGURE 2.6(b) PROBABILITY DISTRIBUTION FOR SATURATED TIME

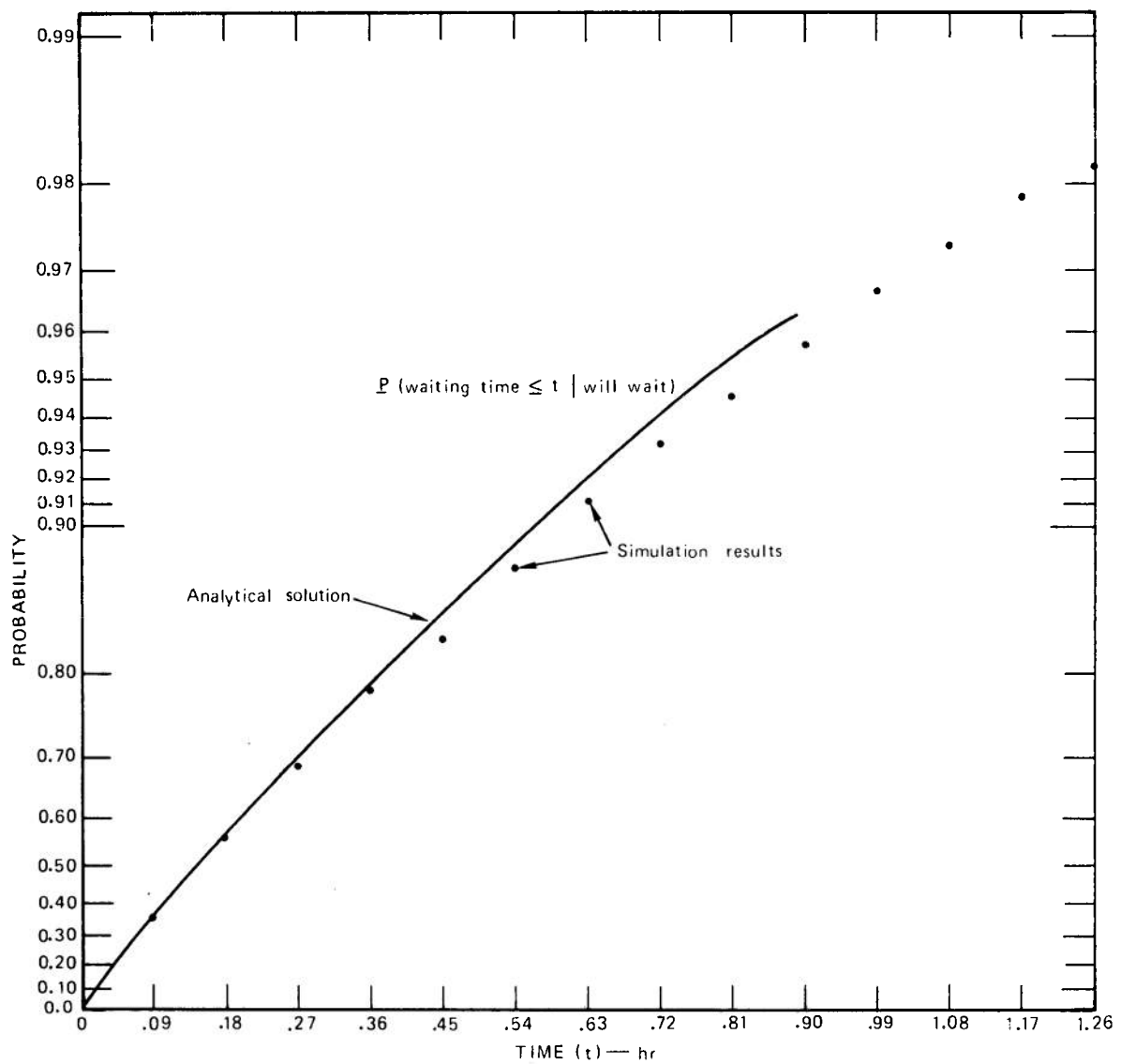


FIGURE 2.6(c) PROBABILITY DISTRIBUTIONS FOR WAITING TIME

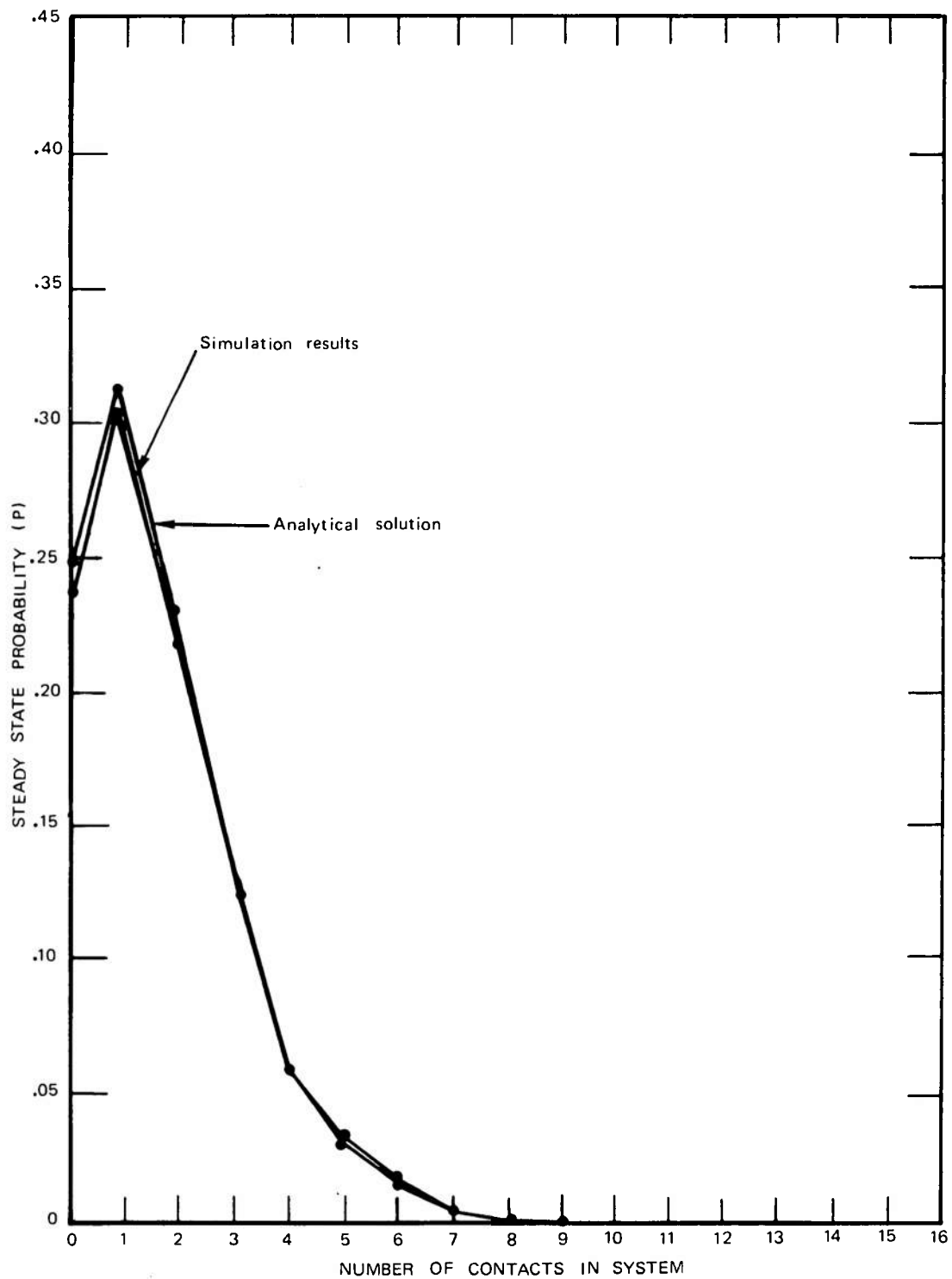


FIGURE 2.7 STEADY-STATE PROBABILITY DENSITIES

The fact that simulation verifies the results of the analytical work is rewarding because rather complex analytical modeling and computation were performed. The slight differences between some of the simulation and analytical curves probably result from statistical fluctuations. (The simulation was run for an equivalent of up to 10,000 hr of model time, thus processing a total of about 35,000 contacts for an arrival rate of 3.5/hr.)

A disappointing feature of the above comparison (in terms of the desirability of an analytical approach) is that the simulation approach took a great deal less computer effort to obtain some of the results, even with highly dimensional Markov models. Thus, further analytical work in the direction of more realistic models--including classification, nonexponential distributions for contact arrivals, and the like--during the current year had to be restricted to Markov and semi-Markov models of modest dimensions, having in the order of 3 to 28 states.

C. Proposed Unit Model Extensions to Represent Classification

The process of classifying contacts was omitted from the queueing models of the first year of the current project so that complexity of initial models could be reduced and analytical approaches could be tested on more tractable computational problems. This section describes how classification may be included within the same analytical framework.

To include classification in the queueing models, the dimensionality of an associated Markov process imbedded in the queueing process has to be increased. Classification is basically a decision process whose function is to separate true from false contacts; therefore, at least these two categories of contacts must be considered. In practice, several categories are used, but model construction was begun with the categories

possible sub, nonsub, and certain sub. Furthermore, classification is not always perfect; hence, there will be actually six significant categories of contacts as shown in Table 2.1. The false nonsub category acts

Table 2.1

SIGNIFICANT CATEGORIES OF CONTACT

Contacts	Classified		
True	Possible sub	Certain sub	Nonsub
False	Possible sub	Certain sub	Nonsub

as a "sink"; that is, any contacts falling into it are assumed not to be picked up again as possible subs. All contacts start out as possible subs, by the definition of contact. It is presumed that true and false contacts might obtain different transition rates among the three classification categories. Also, all transition rates would probably vary with the contact's position in the queueing system and with the degree of congestion of the system.

Two main proposed approaches to specific formulation of a single ASW unit queueing model have evolved during the current project.

Model 1--Classification is assumed to be performed "instantly" just before any previously modeled transition of the Markov model (except for an arrival), i.e., at the completion of a detection service, a localization service, and the like. This approach is relatively easy to implement because it merely modifies the manner in which transitions take place in the Markov models developed to date and extends them to include the extra categories of contacts mentioned above. Thus, just at the completion of detection of a particular contact if that contact moved in the previous model to the localization server, it would now move to

it as still a possible sub with a certain probability, or as a certain sub with another probability, or it may exit into the false contact bin with the remaining probability. Similarly, at the completion of a localization service, contacts would split off into different categories. Presumably, the probability of classification--i.e., passing from the possible sub to either the certain or nonsub states--would be modeled with a higher and higher probability as a contact progresses further and further into the queueing system. This model is relatively easy to implement by simulation, and initial analytical effort has been made to set up the appropriate Markov model.

Model 2--An alternative approach is to assume that classification is a process in which the decisions are made at separate times from the times of a contact's transition from one queueing service to another in the system. One way to handle this is to regard classification as being performed by a separate server whose completion of service competes with others in the Markov model for determination of the next transition. In practice, classification is really accomplished jointly by the personnel at the various ASW function locations, and presumably a final decision rests on the commanding officer who is not associated expressly with any one of the servers of our Markov model and receives information from all of them. Thus, it is logical to consider an additional "competing process" that contributes to the Markov model's transition decisions and includes classification (Appendix A of Ref. 1).

Because the second model is basically no more difficult to simulate than the first one, it was chosen for extending the currently available simulation model. A detailed set of flow charts to implement this approach were prepared but not programmed because of the change of emphasis toward models that are more easily related to operational parameters.

Although the single ASW unit model with classification added was not carried through to a computational analysis, recording its conceptual attributes is worthwhile. The structure of either Model 1 or Model 2 above is depicted in Figure 2.8. The "Black Box" modeling the classification process will be explained further below. The dotted lines represent the direction of effect of any classification decision. The assumption

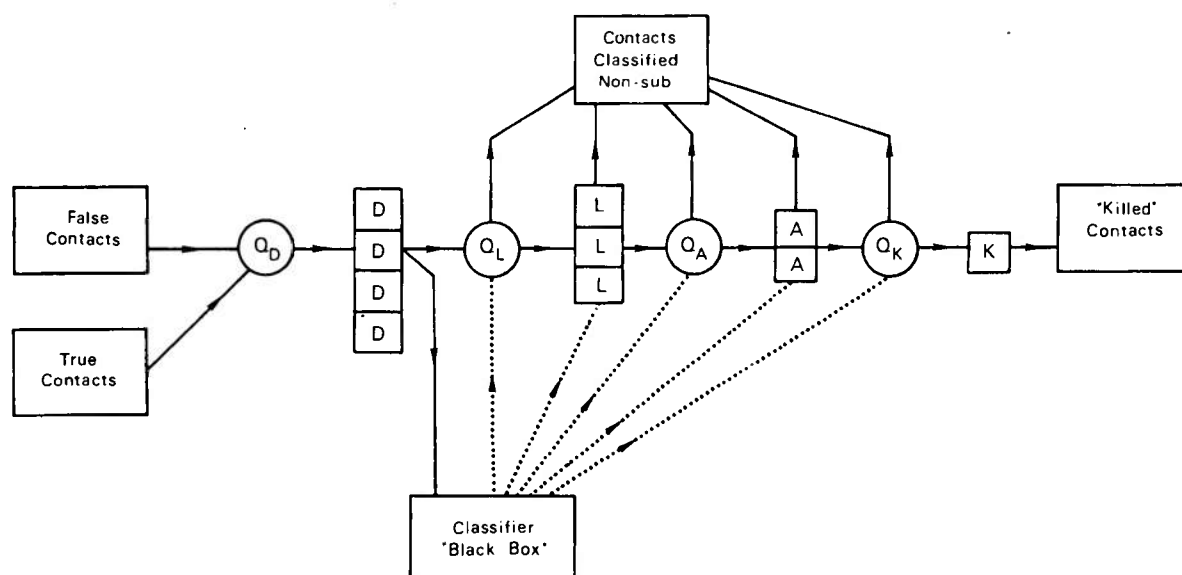


FIGURE 2.8 STRUCTURE OF SINGLE-UNIT MODEL INCLUDING CLASSIFICATION

was made that a decision to classify a contact nonsub immediately rejects the contact out of the system so that other contacts can move up in their queues. A decision that some contact is certain sub has been tentatively assumed to have the possible effects shown in Table 2.2.

The features shown in the table model possible effects of a classification decision on the operation of the basic single-unit model. However, considering the reverse effect of system state transitions on the classification process itself may also be important. First of all, a model of the structure of the "Black Box" in Figure 2.8 must be chosen.

Table 2.2

EFFECTS OF "CERTAIN" SUBMARINE CONTACTS

Case	Result
A new certain sub contact in a queue is held up by a not-yet-classified contact in next service.	The not-yet-classified contact is displaced by the new certain sub contact, and the former goes back into queue, with all service to date invalidated--i.e., it will start from scratch the next time it enters that service.
New certain sub contact is in a service.	The service rate may possibly be raised, thus indicating more concerted effort by personnel.

In the investigations by Operations Research Incorporated (ORI) in a parallel effort related to this project,³ the classification process was basically represented by a "growth of information" about a contact versus statistical sampling of that information to arrive at a classification decision. A simplified view of this approach was adopted by representing growth of information as shown in Figure 2.9. The growth transition rate λ_c would naturally be higher than the regression rate μ_c in Figure 2.9, and would thereby ensure that a contact would eventually be classified (if it stayed in the system long enough). The behavior of the model in Figure 2.9 would be approximately like that of an "exploding" queue with intensity $\rho = \lambda/\mu > 1$, in which the occurrence of a classification would

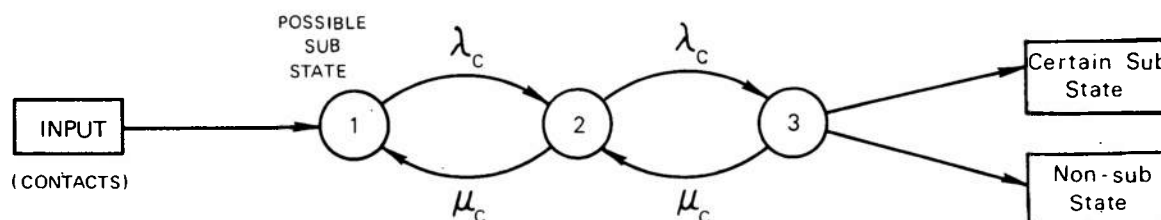


FIGURE 2.9 A MARKOV MODEL OF THE CLASSIFICATION PROCESS

be represented by an overflow from the truncated queue. This model might even be adaptable to including queueing features separate from those of the rest of the model.

Additional statistical measures expected to be derivable from the unit ASW model with classification added are:

- The distribution of time a true contact spends in the system for a given false contact ratio at the detector input.
- The relative degrees of congestion at various parts of the system caused by presence of false contacts.

III CONDITIONAL PROBABILITY MODELS

A. Introduction

Basically, two types of effectiveness models are used in ASW studies. The first type of model--often called an analytical model--is used to estimate (or predict) effectiveness when basic parameters of the problem such as detection ranges, sweep widths, environmental variables, and so on, are known. Equations (mathematical or mathematical and logical) are given from which system effectiveness may be calculated when these input quantities are known or are assumed to be known. The ASW operations modeled may never have actually been performed (nonexistent equipment may be included) so that data are unavailable for estimating many of the model parameters. The usual procedure is to use the available data to estimate the input variables of the model--which often characterize equipment or systems--and to use these data also to structure the model wherever possible, and to hypothesize the remainder of the structure. The effectiveness measure for this kind of model is often a probability of success, such as the probability of a destroyer detecting a submarine, or a submarine detecting, classifying, closing, and successfully attacking another submarine.

The second type of model--often called an operational model--also requires inputs, but these are directly observed quantities from real ASW operations or from ASW exercises. Instead of assuming values of input variables and computing intermediate quantities such as conditional probabilities, an operational model estimates them directly from observed outcomes.

A common form of model used both as an analytical and operational model is the conditional probability model. The basic idea is to choose a series of requirements necessary and sufficient for mission success. These requirements often are directly related to equipment functions or system or subsystem functions; often, an interface of a subsystem with another subsystem is where achievement of a requirement is to be observed. Because the achievement of a requirement is necessarily conditional on an opportunity for the portion of the system to meet the requirement and it is often assumed to be a probability, this form of model is known as a conditional probability model.

B. A Conditional Probability Model as a Markov Model

Conditional probability models can be conveniently described by a simple diagram made up of branches and nodes. These branches correspond to requirements and the nodes correspond to cumulative requirements met.

An example of a typical conditional probability model of the operational type is a simplified version of the WSE model developed and used in the Big Daddy series of sub-on-sub barrier exercises performed several years ago. The diagram for this model is shown in Figure 3.1. The labels within the nodes, called states, are oversimplified descriptions of possible stages in the sequence of events leading to kill. More complete definitions are as follows.

Opportunity	A detection opportunity is presented.
Detect	To detect the transmitting submarine given a detection opportunity.
Classify	Correctly classify a submarine as a submarine, given that a submarine has been detected.
Attack	Satisfy attack conditions on the submarine, given that the submarine was correctly classified.
Kill	Submarine is destroyed, or placed out of action, given that an attack was made.



FIGURE 3.1 STATES IN THE WEAPON SYSTEM EFFECTIVENESS MODEL

The arc (or branch) from detect to classify in a sense represents the requirements for classification because a transition from detect to classify is made only when the classification requirements are met. Other branches may be similarly interpreted by consideration of the nodes (or states) they connect.

The diagram of Figure 3.1 can be modified by adding a node (state) labeled fail, for failure to kill the transiting submarine. Since failures can occur by failing to detect given opportunity, to classify given detection, and so on in sequence, branches can be added to show transitions from individual states to Fail as shown in Figure 3.2.

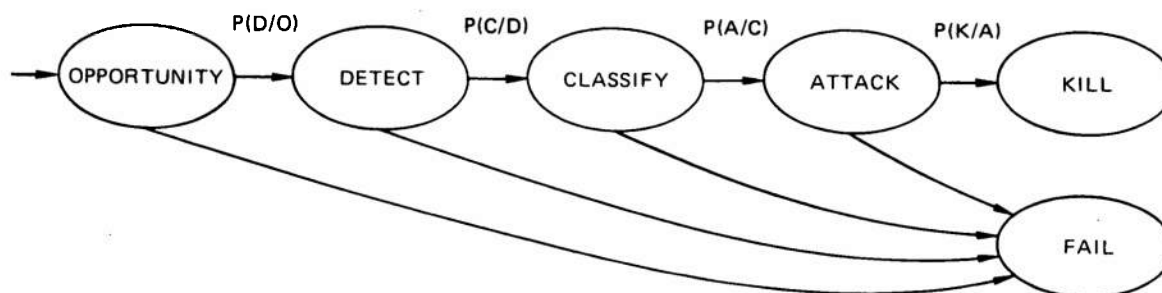


FIGURE 3.2 TRANSITION DIAGRAM FOR THE WEAPON SYSTEM EFFECTIVENESS MODEL

In terms of the diagram in Figure 3.2, a given barrier exercise resulting in a detection opportunity can be summarized by specifying the path followed from the opportunity state to one of the terminal states kill or fail. If states are represented by their first letters, then a string of symbols representing the path can be determined from each exercise. For example, the path ODCF means the transiting submarine presented a detection opportunity, it was then detected and classified by the

barrier submarine, but attack was not accomplished; hence, the transiting submarine failed in its mission. The desired path for the barrier submarine is the unique path from opportunity to kill represented by ODCAK.

The measure of effectiveness for the barrier submarine mission (WSE) is now defined as the probability of kill of a transiting submarine, given that a detection opportunity is presented.

By the use of standard formulas for conditional probability, the probability of kill given opportunity can be written:

$$P(K|O) = P(D|O) \times P(C|O \text{ and } D) \times P(A|O \text{ and } D \text{ and } C) \\ \times P(K|O \text{ and } D \text{ and } C \text{ and } A) \quad . \quad (3.1)$$

By assumption, however, detect can only occur following opportunity, and classify can only occur following detect. Therefore, the only remaining conditions in each conditional probability are the latest ones, so that the formula for WSE becomes:

$$WSE = P(K|O) = P(D|O) \times P(C|D) \times P(A|C) \times P(K|A) \quad . \quad (3.2)$$

In terms of the state diagram of Figure 3.2, WSE is the probability that a path terminates in kill, given that it started in opportunity. If the conditional probabilities of formula (3.2) are recorded on the branches of Figure 3.2 as shown, WSE is seen to be the product of the probabilities along the path from opportunity to kill. The value of WSE is estimated from exercise data, where each conditional probability is estimated by a ratio--number of transitions to next state: number of

opportunities for this transition. For example, the estimate of $P(C|D)$ is:

$$\frac{\text{number of classifications}}{\text{number of detections}} = P(C|D) \quad . \quad (3.3)$$

Now assume that data from a number of transit exercises are available, and there are:

N transits presenting detection opportunity

n_1 transits resulting in detection

n_2 transits resulting in classification

n_3 transits resulting in attack

n_4 transits resulting in kill.

The diagram of Figure 3.2 can be used to tabulate these numbers of transitions, and the result is shown in Figure 3.3. The values at the nodes are the numbers of paths entering the given node.

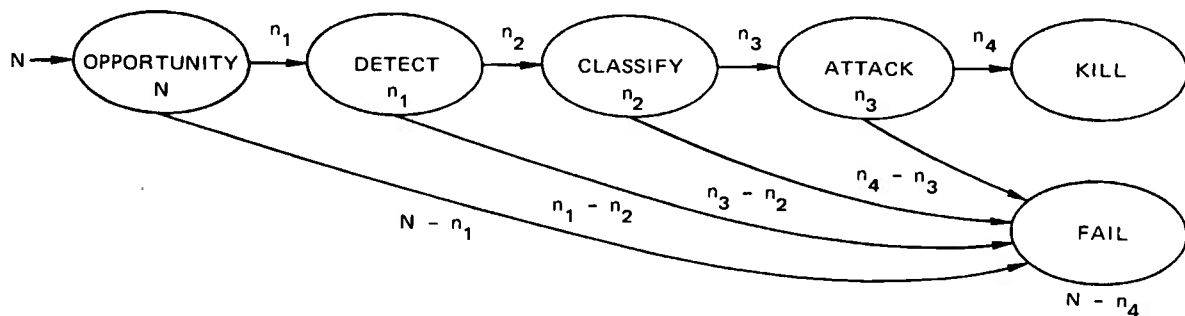


FIGURE 3.3 DATA FOR THE WEAPON SYSTEM EFFECTIVENESS MODEL

Replacing each conditional probability in the definition of WSE by its estimate results in the following estimate of WSE.

$$WSE = \frac{n_1}{N} \cdot \frac{n_2}{n_1} \cdot \frac{n_3}{n_2} \cdot \frac{n_4}{n_3} \quad . \quad (3.4)$$

Because the n_1 , n_2 , and n_3 terms all cancel, the end result is

$$WSE = \frac{n_4}{N} = \frac{\text{number of kills}}{\text{number of transits presenting opportunity}} \quad (3.5)$$

If a transit presenting an opportunity is considered a trial, and a kill is a successful trial, then

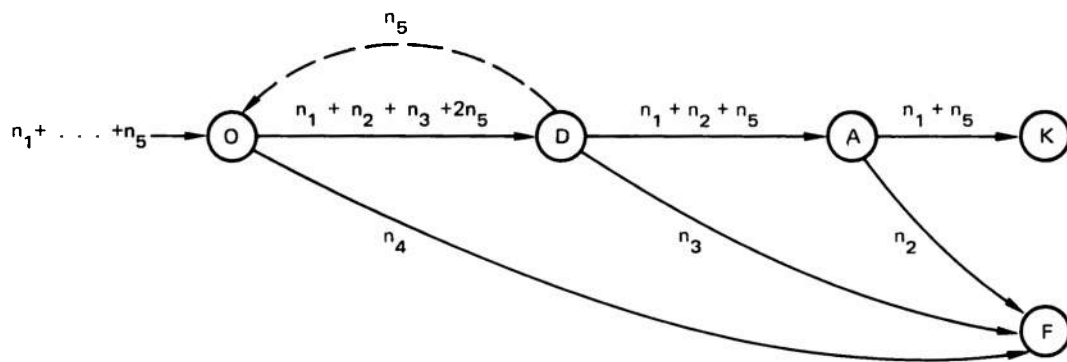
$$WSE = \frac{\text{number of successes}}{\text{number of trials}} \quad (3.6)$$

which is the usual estimator for the probability of success for Bernoulli trials. If the assumption can be accepted that a set of transit exercises can be considered a set of independent Bernoulli trials, then simply counting the number of successes would provide this estimate of WSE directly.

The intermediate states (detect, classify, attack) did not matter because the transition counting entailed in passing through these states cancelled out. This cancellation will always occur in this kind of model. Further discussion of this point will be deferred to Section VII. At this point, it suffices to note that the cancellation occurs when each transit begins with opportunity and ends with kill or failure, provided that no repeated entries occur in any state.

The assumption was made above that each exercise could be translated into a path from the starting node to either of the terminal nodes without returning to any state after leaving it. In terms of the diagram, this means that no path formed a "loop." If one attempts to calculate success probabilities in a problem with loops by the usual conditional methods, the results will be in error.

An example of an erroneous attempt is shown in Figure 3.4, where the detect and classify states from Figure 3.2 have been merged for simplicity.



PATH NUMBER	OBSERVED PATH	NUMBER OF TIMES OBSERVED
(1)	O D A K	n_1
(2)	O D A F	n_2
(3)	O D F	n_3
(4)	O F	n_4
(5)	O D O D A K	n_5

FIGURE 3.4 SUCCESS PROBABILITIES WITH LOOPS (BASED ON FIGURE 3.3)

Assume that the observed paths and the numbers of times each path was observed are recorded as shown on the bottom of Figure 3.4. For example, n_1 paths are designated O D A K, meaning that there were n_1 occurrences of the sequence Opportunity $\rightarrow$ Detect $\rightarrow$ Attack $\rightarrow$ Kill. A return to an earlier state (a loop) occurs in the last path since opportunity/detect occurs more than once in this path. Because paths 1 and 5 terminate in kill, the number of successes is $(n_1 + n_5)$ and the Bernoulli estimator of probability of kill $P_b(K|O)$ is

$$P_b(K|O) = \frac{n_1 + n_5}{\sum_{i=1}^5 n_i} \quad (3.7)$$

In the path of type 5, the opportunity state is entered twice. That is, following an initial opportunity is a first detection, then the contact is lost, and a second opportunity is presented. This then presents two opportunities.

The number of possible transitions between all pairs of states can be conveniently recorded on the branches between states as shown in Figure 3.4. To obtain these values, each path through from opportunity to kill or fail can be traced and added to the counter on each branch as the transition is made, or the number of transitions from the encoded path definitions can simply be counted.

Consider next the usual conditional probability estimator for probability of Kill.

$$P(K|O) = P(D|O) \times P(A|D) \times P(K|A)$$

$$= \frac{\text{number of detections}}{\text{number of opportunities}} \times \frac{\text{number of attacks}}{\text{number of detections}} \times \frac{\text{number of kills}}{\text{number of attacks}} \quad (3.8)$$

Using transition data from Figure 3.4 and estimating each fraction as

(number of transitions out of a state to designated state)

results in (total number of transitions out of the state)

$$\frac{\text{number of detections}}{\text{number of opportunities}} = \frac{n_1 + n_2 + n_3 + 2 n_5}{n_1 + n_2 + n_3 + n_4 + 2 n_5}$$

$$\frac{\text{number of attacks}}{\text{number of detections}} = \frac{n_1 + n_2 + n_5}{n_1 + n_2 + n_3 + 2 n_5}$$

$$\frac{\text{number of kills}}{\text{number of attacks}} = \frac{n_1 + n_5}{n_1 + n_2 + n_5}$$

Putting these ratios into Eq. (3.8) shows that the number of detections and attacks again cancel, resulting in

$$P(K|O) = \frac{\text{number of kills}}{\text{number of opportunities}} = \frac{n_1 + n_5}{n_1 + n_2 + n_3 + n_4 + 2 n_5} \quad (3.9)$$

The numerator of Eq. (3.9) agrees with that in Eq. (3.7), but the denominator of Eq. (3.9) is larger than that in Eq. (3.7) by n_5 , which is the number of second opportunities. Because n_5 is nonnegative, the Bernoulli estimator of Eq. (3.7) cannot be less than the usual estimator of Eq. (3.8); that is,

$$P_b(K|O) \geq P(K|O) \quad . \quad (3.10)$$

Therefore, the usual estimator of Eq. (3.8) is incorrect; it estimates a different quantity from $P(K|O)$. The Bernoulli estimator is the correct estimator to use in this situation, and its formula in terms of estimates of conditional probabilities will be derived by Markov methods in Appendix C on Mason's rule. The formula, given here without justification, is

$$P(K|O) = \frac{P(D|O) P(A|D) P(K|A)}{1 - P(D|O) P(O|D)} \quad . \quad (3.11)$$

The new conditional probability in the denominator $P(O|D)$ is the conditional probability associated with the detect-to-opportunity branch. That is, it is the probability of another opportunity following detection (as contrasted with attack or fail), conditioned on detection occurring.

Still other loopless estimators may be attempted with conditional probability methods. For example, the return loop from detect to opportunity may be simply ignored. The ratios estimating $P(D|O)$ and $P(K|A)$ remain the same, but $P(A|D)$ changes to

$$\tilde{P}(A|D) = \frac{n_1 + n_2 + n_5}{(n_1 + n_2 + n_5) + n_3}$$

because the n_5 returns to opportunity are ignored.

Substitution of this value in Eq. (3.8) gives a new estimator

$$\tilde{P}(K|0) = \frac{(n_1 + n_2 + n_3 + 2 n_5)(n_1 + n_5)}{(n_1 + n_2 + n_3 + n_4 + 2 n_5)[(n_1 + n_2 + n_5) + n_3]}$$

which is again incorrect because it differs from the Bernoulli estimator.

C. Limitations of Conditional Probability Models

It was just observed that the probability of success estimator arrived at by successive conditioning to form a conditional probability model is identical to the estimator for Bernoulli trials when each transit exercise started from the same state (opportunity) and ended in success (kill) or fail. (Exercise data of this type will be called "complete" exercise data henceforth.) Further limitations of conditional probability models are:

- Only a limited number of situations are defined--usually one-on-one engagements. Applicability is limited to tactical situations in which a single set of serial requirements for mission success can be identified.
- They are static, and therefore problems in which time is an essential element may not be capable of being treated. Dynamic operational data (such as average time to prosecute a false contact) cannot be directly used in estimating the parameters of a conditional probability model.
- They have few if any controllable variables. Hence, they are of limited use in improving equipment or operations. They cannot be used to improve tactics.
- Degrading factors such as false contacts do not appear explicitly. Hence, the quantitative influence of these degrading factors cannot be assessed.
- Only a limited portion of the available data is used for parameter estimation because only static transition data can be used to estimate their parameters.
- They have limited predictive power.
- Confidence limits are difficult to obtain.

In view of these limitations, the following justify the use of a conditional probability model.

- Conditional probability models incorporate all the pertinent operational and environmental factors directly and therefore require a minimal number of hypothesized relationships.
- Data are often available from sources other than complete exercises. In particular, data are often available from truncated exercises (those beginning in an intermediate state/ending in an intermediate state).
- The value of the probability of success is not the only useful information. For example, breaking down an operation into phases and observing transition information as is done in estimating transition probabilities often reveals points where system operation may be improved.
- Some transitions may be unobservable, and complete exercises may therefore be impossible. For example, live torpedoes cannot be fired at exercise submarines so that the attack-to-kill transitions cannot be directly observed. Data from other sources, such as simulation, may have to be used in estimating some transition probabilities.
- The individual conditional probability estimates may be studied, analyzed, or estimated as a function of parameters of the mission/environment.

IV MARKOV AND SEMI-MARKOV MODELS IN ASW EFFECTIVENESS ANALYSIS

A. Introduction

In the previous section, conditional probability models were discussed and analyzed, and it was pointed out that a conditional probability model can be considered a special case of a Markov model. Moreover, some of the limitations on conditional probability models can be removed by considering more general Markov models.

In this section, some additional general reasons for considering Markov (and semi-Markov) models for ASW effectiveness analysis are first given in outline form. Following this is a discussion of the measures of effectiveness that a Markov model may estimate. Finally, the steps entailed in formulating and using a Markov model and in incorporating one or more of these measures of effectiveness are briefly discussed. Later sections in the report deal more fully with model formulation and parameter estimation.

B. Useful Properties of Markov Models

In addition to removing some of the limitations on conditional probability models, the reasons for considering Markov models in ASW effectiveness analysis are:

- Markov models are general and flexible in the following ways.
 - Many, perhaps most, measures of effectiveness used in naval warfare studies and analyses can be formulated in terms of an appropriate Markov model. (Conditional probability models are a special case of a Markov model.) New measures of effectiveness can also be formulated that may be useful where effectiveness has remained elusive.

- Markov models can be used at various levels, ranging from one-on-one engagements/encounters to the task force level.
- Markov models offer a way to combine data from diverse sources in a single model. Measures of effectiveness at various levels can all be introduced into a single model.
- Markov models are fundamentally dynamic in nature, but the dynamic and static aspects conveniently factor to allow the dynamic aspects to be analyzed separately. Shifting from static to dynamic modeling presents new opportunities to exploit operational data that are currently not used in effectiveness modeling.
- For certain of the measures of effectiveness, Markov models show a remarkable insensitivity to the actual nature of the underlying random process. That is, they are very "robust" in the statistical sense.
- Markov models are computationally convenient.
 - Analytical (closed-form) solutions can be found for small models.
 - Systematic computational procedures exist for determining all of the reasonable measures of effectiveness commonly used for Markov models of any reasonable size.
 - Many dynamic properties of the model can be routinely calculated.
- Many systematic methods exist for estimating Markov model parameters from observed data. Several of these methods employ data not ordinarily used in ASW effectiveness modeling.
- Markov models may be useful for comparing dissimilar models and for validating simulation models.

C. Measures of Effectiveness

Four principal measures of ASW mission effectiveness will be considered in this report. All of these measures can be defined in terms of a Markov process. Many, or perhaps most, measures of effectiveness commonly used in naval warfare analysis studies are one of these measures.

Probability of Mission Success--In this first measure of effectiveness, success must be suitably defined for the mission being considered. Classical conditional probability models used in ASW analysis are of this type, where success often means kill of the enemy submarine. In other contexts, success may be simply detection of the enemy submarine, or detection and classification of the enemy submarine. This measure is fundamentally a static measure.

To incorporate this measure into a Markov model, a state labeled Success must be introduced and made an absorbing (trapping) state. The probability of success will then be the probability of the process being absorbed in the success state. This probability depends on the static structure of the Markov process rather than on its dynamics.

Mean Time to Success (or Mean Time to Fail)--In this second measure of effectiveness, for example, an aircraft carrier in the objective area may be considered to fail in its mission when an enemy submarine detects it. The mean time to detection of the carrier is an appropriate measure for many studies.

In Markov process terms, the success (or fail) state is required as above, and success/fail are again made absorbing states. Methods for calculating the mean time to absorption in a designated trapping state are available from Markov theory. The dynamics of the Markov model are entailed in the calculation of this measure.

Probability of Success (or Fail) at or by a Specified Time--In this third measure of effectiveness, if an aircraft carrier can be considered to succeed in its mission after no detection for (say) ten days, then the chosen measure of effectiveness may be

$$P(\text{carrier is not detected in ten days}) \quad .$$

Again the Markov formulation requires appropriately defined success/fail states that are absorbing states. The measure becomes

$$P(\text{process is not absorbed in the fail state by time } T)$$

where T is the specified duration. This measure again depends on the dynamics of the Markov model.

Probability of Occupying a Designated State at a Randomly Chosen Instant of Time--This fourth measure of effectiveness is appropriate in situations where mission success cannot be identified with an absorbing state, as in a trailing mission.

In contrast to the Markov formulations of the preceding three measures, this measure is defined in terms of the steady-state (as opposed to transient) properties of the Markov process. The measure again depends on the dynamics of the process.

The first of these measures is a static measure and the other three are dynamic. However, the static measure is imbedded within the dynamic Markov model so that all are computable from a single dynamic model. In many contexts, the first three measures may all be incorporated within a single Markov model.

When a Markov model is applied to a real naval operational problem, the assumptions required for a Markov model will seldom, if ever, be exactly fulfilled. The four measures vary markedly in their sensitivity to the underlying assumptions, however. The probability of success measure is singularly insensitive to the actual nature of the process being modeled. The second measure, the mean time to success (or fail), is also very "robust." The reasons for this insensitivity (robustness) are discussed in detail in Section VII. The last two measures, the probability of not failing by time T and the probability of occupying a

designated state at a randomly chosen point in time, depend much more heavily on the assumptions required for a Markov process. Both measures, it should be noted, are special cases of the probability of occupying a state at an arbitrary time t . For this reason, most of the experimental work discussed later in the report has been devoted to finding methods for restructuring the state space and estimating the model parameters to obtain satisfactory estimates of $P(\text{state})$ as a function of time t .

D. Formulating a Markov Model

1. Defining a Set of States

A Markov model basically consists of a set of states and rules for determining how transitions will occur between states. In applications to ASW effectiveness analysis, the most basic and probably the most difficult step will be defining the set of states. How the states are defined will generally depend on which of the four measures of effectiveness is being considered. A goal would be to find a single set of states that is adequate for all four measures.

In Markov theory, a state essentially describes a possible condition of the process being modeled. All possible conditions must be anticipated and states defined accordingly; at any instant of time, the Markov process must be in some state but not be in more than one state. In operational terms, the states must be defined so that the state can be determined for each point in time when what actually transpired during the ASW operation (or exercise) is known. Thus, from the definition of the set of states and suitable information about the exercise (say from the analysis report), a sequence of states can be determined that represents that exercise. This sequence, together with the times at which transitions between states were made, will often comprise the basic data for estimating the parameters of a Markov model.

A special feature of many of the Markov models considered in this report is that the process starts in a single state that can be designated the starting state. For the first three measures of effectiveness, there must also be a set of terminal (absorbing) states; often, the states of success and fail will suffice. In ASW operations, a natural starting state is often either search or detection opportunity.

2. Example of a Set of States

The conditional probability model (WSE model) discussed in the previous section had states denoted opportunity, detect, classify, attack, kill, and fail. The process makes a transition from opportunity to detect when detection occurs, and the state remains detect until classification (i.e., the classify state is entered) or failure occurs. The terminology can sometimes be misleading: note that to be in the detect state means that detection has occurred but neither classification nor failure has yet occurred.

The critical part of the definition that is impossible to satisfy in practice is the requirement that knowledge of the state alone determines how the transition to the next state shall be made. More specifically, the requirement is that all transitions shall be made at random by consulting a set of transition probabilities associated with the state being occupied. Emphatically, because transitions out of a designated state will sometimes be made to one state and sometimes to another in no way implies that the transitions can be adequately described by a set of probabilities associated with the occupied state. That is, not all transition behavior, even apparently random behavior, can be described by a Markov model.

Later sections of this report deal with the problem of formulating a Markov model for a stochastic process whose states do not

satisfy the Markov property. Various attempts are made to restructure the state space by adding auxiliary states so that the new set of states more nearly satisfies the Markov requirements.

3. Estimation of Markov Model Parameters

When a satisfactory set of states has been found, with or without the aid of chi-squared "goodness of fit" tests, the next step is to estimate the model parameters from the available data. Since all the measures of effectiveness can be calculated in terms of the model parameters, the model construction is then complete. Parameter estimation is covered in more detail in Section VI, and the calculation of the measures of effectiveness from estimated parameters are detailed in Section V.

A major part of this report concerns the methods for estimating the parameters in a Markov model defined to determine one or more of the measures of effectiveness listed above. The methods must assume observable data from fleet operations and exercises as their basic inputs. The observable data need not be limited to that conventionally used in static models such as conditional probability models, however. Indeed, one of the principal benefits of a change from static to dynamic modeling may be that potentially more data are available from operations and exercises with which to construct and validate effectiveness models.

The parameters of a conditional probability model are usually estimated from transition data; that is, from counts of the observed numbers of transitions between the states of the model. Additional potential data available for estimating the parameters of a dynamic Markov model include such quantities as the:

- Average time each state is occupied.
- Average time of the first entry into each state.

- Average length of time spent in each state i before its transit to another state j .
- Mean time to absorption in designated absorbing states.
- Probability of occupying each state as a function of time.

In more sophisticated applications, the variances of these various times may be used as well as the mean values. The equations relating many of these quantities to basic parameters of the process are linear and thus imply that suitably modified least squares methods can be used to estimate the basic parameters when the equations are inverted. Since three of the four principal measures entail the estimation of the probability of occupying a state as a function of time, special emphasis has been placed on methods for estimating parameters to make $P(\text{state})$ as a function of time adequately fit the observed data.

The complementary side of the shift from static to dynamic modeling is also important. A dynamic model with its parameters can be used to predict properties of the system or operation being modeled. These predictions can then be compared with observed data to determine the adequacy of the model and to suggest changes in model structure or parameters. Static models are more limited in this respect; with less structure in the model, fewer predictions can be made.

V BASIC MARKOV CHAIN THEORY

In this section, the basic concepts of Markov chains are summarized and formulas are derived for computing several measures of effectiveness when the basic parameters (the transition probabilities) of the Markov chain are known. The development here is algebraic and uses matrix notation and operations for brevity and compactness. Other derivations, often carried out using flowgraph methods, are given as needed in later sections. A single example of a Markov chain with two transient states and two absorbing states follows the theoretical development.

A. States, Transitions, Paths, and Loops

The principal elements of a Markov chain (the simplest Markov process) are a set of states and a set of transitions between states. A state is basically the description of a possible condition of a system and the selection of a set of states is a crucial and often difficult task in applications. In this section, the states are simply designated by integers 1, 2, ..., N.

States must be defined so that the system being modeled is in one and only one of the states at any given time. For a Markov chain, the transitions between states (changes of state) are defined at integer times $n = 1, 2, 3, \dots$. By convention, the process starts in some state (usually generically designated as state i) at time $n = 0$ and makes its first transition to a state j at time $n = 1$.

The Markov chains of interest here have one or more absorbing states which are states that are never left once they are entered. Entry into an absorbing state is associated in applications with the end of the

process since nothing of interest happens afterward. Absorbing states in this report are generally defined by mission success (the success state)/mission failure (the fail state). Other states are transient states by definition. Eventually, if the process continues long enough, some absorbing state will be entered to end the process.

A transition diagram may be constructed for any Markov chain. It consists of nodes representing the states and branches (or arcs) that represent possible transitions. Figure 5.1 shows an example with four states--two are transient and two are absorbing. For concreteness, states 1 and 2--the transient states--may be identified with a surface

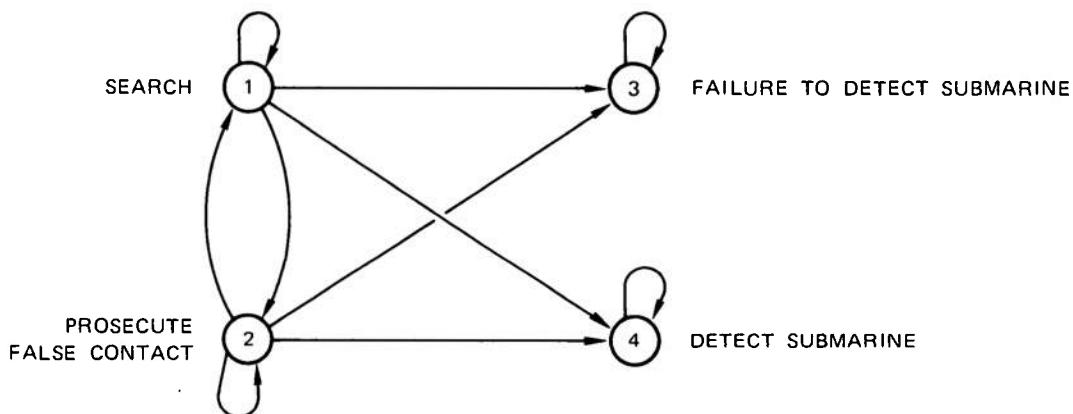


FIGURE 5.1 TRANSITION DIAGRAM FOR FOUR-STATE MARKOV CHAIN

ship searching for a submarine and prosecuting a false contact. The success state (state 4) may represent detection of a submarine; correspondingly, the failure state (state 3) may be associated with failure to detect. As shown by the diagram, transitions take place between states 1 and 2 as the ship first searches, then prosecutes a false contact, then returns to search and so on. Transitions from a state to itself (self-transitions) represent waiting times or holding times in a state; the branches on the diagram showing self-transitions are called self-loops. A path from one state to another consists of a sequence of

branches from one state to another and may be of various lengths. A path from a state back to itself without visiting any state more than once is called a loop. Thus the path from state 1 to state 2 and back to state 1 is a loop.

B. Transition Probabilities

Transitions are made from state to state in accordance with a set of transition probabilities that are the only numerical parameters of a Markov chain. One probability is defined for each branch of the transition diagram because the lack of a branch implicitly means that the corresponding transition probability is zero. These probabilities may be arranged in matrix form, with the rows being associated with the transitions out of a state and the columns being associated with transitions into a state. In a stationary Markov chain, the transition probabilities do not depend on time and are denoted by p_{ij} . The formal definition is:

$$p_{ij} = P(\text{state at time } n \text{ is } j \mid \text{state at time } n - 1 \text{ is } i) \text{ for all states } i, j \text{ and all times } n = 0, 1, 2, \dots$$

C. Partitioned Form of the Transition Matrix

The transition matrix for the Markov chain in Figure 5.1 is denoted by P , with elements p_{ij} . It can be partitioned according to transient and absorbing states and the resulting submatrices denoted by Q , R , and the identity matrix I . That is,

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix} \quad (5.1)$$

where the submatrices Q and R are defined by

$$Q = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \quad R = \begin{bmatrix} p_{13} & p_{14} \\ p_{23} & p_{24} \end{bmatrix} \quad (5.2)$$

and I is a 2×2 identity matrix.

In general, Q is a square matrix of dimension r, where r is the number of transient states. Matrix R has r rows and a columns (i.e., it is dimensioned $r \times a$), where a is the number of absorbing states. The identity is dimensioned $a \times a$. It should be noted that rows of P are associated with the state occupied before a transition, and columns of P with the state to which the transition is made. Thus, the submatrix Q is associated with transitions from transient states to transient states, and the submatrix R with transitions from transient states to absorbing states. The identity matrix I is associated with the transitions from absorbing states to absorbing states. Since absorbing states are defined as states that are never left once they are entered, the probabilities are unity on the diagonal and zero off the diagonal.

D. Multistep Transition Matrices and Absorption Probabilities

The transition matrix P defines transition probabilities for single transitions. If P is multiplied by itself, the transition matrix for two consecutive transitions is obtained with elements denoted by a superscript "2" to denote the number of consecutive transitions.

$$P^2 = \left(p_{ij}^{(2)} \right) \quad (5.3)$$

If the matrix multiplication is carried out in using the partitioned form on the right side of Eq. (5.1), the result is expressible in terms

of the submatrices of P which have the same dimensions as the original partitioning. Equation (5.4) gives the result:

$$P^2 = \begin{bmatrix} Q^2 & (I + Q)R \\ 0 & I \end{bmatrix} . \quad (5.4)$$

In the example, both identity matrices I are of dimension 2; in general the dimensions differ.

Successive multiplication of P by itself yields transition matrices for larger numbers of consecutive transitions. The n-step transition matrix P^n is readily shown to have the partitioned structure shown in Eq. (5.5).

$$P^n = \begin{bmatrix} Q^n & (I + Q + Q^2 + \dots + Q^{n-1})R \\ 0 & I \end{bmatrix} = \begin{pmatrix} p_{ij}^{(n)} \end{pmatrix} . \quad (5.5)$$

This form reveals the process dynamics. The submatrix Q^n represents those series of n transitions that remain in transient states for all n transitions. In cases of interest in this report, the process will always be absorbed in an absorbing state so that all elements of Q^n will approach zero as the number of transitions n becomes infinite.

The other submatrix associated with starting the process in transient states is $(I + Q + Q^2 + \dots + Q^{n-1})R$. Multiplying in the R and expanding into its component terms gives Eq. (5.6).

$$R + QR + Q^2R + Q^3R + \dots + Q^{n-1}R . \quad (5.6)$$

The first term, R, represents those cases where absorption occurs on the first transition. The second term, QR, represents transitions that

remained in the transient states for one transition and were absorbed on the second transition. Later terms represent absorption after some number of transitions from the starting transient state to other transient states, followed by absorptions. In general, $Q^k R$ corresponds to absorption on the $(k + 1)^{st}$ transition--the first k transitions were between transient states and the $(k + 1)^{st}$ transition to an absorbing state.

By means entirely analogous to those used to sum a geometric series, the limit of the sum in Eq. (5.6) can be shown to be $(I - Q)^{-1}R$. [The necessary auxiliary formula is shown in Eq. (5.7):

$$I + Q + Q^2 + \dots + Q^n + \dots = \sum_{n=0}^{\infty} Q^n = (I - Q)^{-1} \quad .] \quad (5.7)$$

Therefore, the "infinite step" transition matrix, which is the limit of Eq. (5.5) as n becomes infinite, has the form shown in Eq. (5.8).

$$\lim_{n \rightarrow \infty} P^n = P^\infty = \begin{bmatrix} 0 & (I - Q)^{-1}R \\ 0 & I \end{bmatrix} \quad . \quad (5.8)$$

Only the last a columns associated with absorbing states have nonzero entries. In particular, the submatrix $(I - Q)^{-1}R$ gives the absorption probabilities for all starting states. That is, the $(i, k)^{th}$ element of $(I - Q)^{-1}R$ is the probability of being absorbed in the k^{th} absorbing state, given that the process started in state i at time zero.

This result is usually obtained by setting up a system of simultaneous linear equations in the unknown absorption probabilities.⁴ It will be convenient to denote the matrix $(I - Q)^{-1}R$ by A , with elements a_{ik} .

E. z Transforms

Many quantities besides the absorption probabilities a_{ik} just found by matrix multiplication can be obtained by using z transforms that are also called geometric transforms. The z transform is analogous to the Laplace transform but is easier to grasp intuitively because the variable z is associated with the number of transitions made between states.

Some additional notation is needed at this point. Let the matrix $\Phi(n)$ be the matrix whose elements $\phi_{ij}(n)$ are the probabilities of occupying state j after n transitions (i.e., at time n) when the process started in state i at time n = 0. Let S(n) be the state at time n. Equation (5.9) defines $\Phi(n)$, and Eq. (5.10) relates $\Phi(n)$ to the transition matrix P.

$$\Phi(n) = P[S(n) = j | S(0) = i] = [\phi_{ij}(n)] \quad (5.9)$$

$$\Phi(n) = P^n = \left(p_{ij}^{(n)} \right) \quad (5.10)$$

The z transform of a sequence of quantities $[f(n)]$, which may be scalars or matrices, is defined to be the infinite sum

$$f(z) = \sum_{n=0}^{\infty} f(n) z^n \quad (5.11)$$

Applying this to the sequence of matrices $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, ... and using Eq. (5.11) results in Eq. (5.12).

$$\Phi(z) = \sum_{n=0}^{\infty} \Phi(n) z^n = \sum_{n=0}^{\infty} P^n z^n \quad (5.12)$$

Using the general relationship in Eq. (5.7) after replacing Q by Pz results in Eq. (5.13).

$$\Phi(z) = (I - Pz)^{-1} \quad (5.13)$$

Each element $\phi_{ij}(z)$ of the matrix in Eq. (5.13) is a function of z called the transmission from state i to state j. From Eq. (5.7), the $(i,j)^{th}$ element has the form shown in Eq. (5.14) when states i and j are distinct.

$$\phi_{ij}(z) = p_{ij}z + p_{ij}^{(2)}z^2 + p_{ij}^{(3)}z^3 + \dots \quad (5.14)$$

F. Inversion of the Matrix (I - Pz) and Applications

By the use of the partitioning shown in Eq. (5.1), the matrix I - Pz can be written in the form of Eq. (5.15).

$$I - Pz = \begin{bmatrix} I - Qz & -Rz \\ 0 & I(1 - z) \end{bmatrix} \quad (5.15)$$

The inverse of I - Pz is easily verified by direct matrix multiplication to be the matrix shown in Eq. (5.16).

$$(I - Pz)^{-1} = \begin{bmatrix} (I - Qz)^{-1} & \frac{(I - Qz)^{-1}Rz}{1 - z} \\ 0 & \frac{I}{1 - z} \end{bmatrix} \quad (5.16)$$

Each of the matrices in the top row is useful for deriving formulas for calculating effectiveness measures. It should be noted that the

coefficient of z^n in the expansion of $(I - Qz)^{-1}$ is the matrix Q^n that occupies this same position in Eq. (5.5).

Similarly, the coefficient of z^n in the expansion of

$$\frac{(I - Qz)^{-1} R z}{I - z}$$

found in the upper right position is $(I + Q + Q^2 + \dots + Q^{n-1})R$, which occupied this same position in Eq. (5.5). The $(1 - z)$ term appearing in the denominator of Eq. (5.16) has the function of accumulating the probabilities in the absorption states. When this factor is ignored and z is set equal to unity, the upper right hand matrix reduces to $(I - Q)^{-1}R = A$, the matrix of absorption probabilities found earlier in Eq. (5.8).

The elements of the submatrix $(I - Q)^{-1}$ that keeps recurring are also meaningful quantities: the $(i, j)^{th}$ element is the mean number of entries into a transient state j before absorption occurs, given that the process is started in state i .

To show this, let $(I - Q)^{-1}$ be denoted by T , with elements t_{ij} . Define random variables $x_{ij}(n)$ and e_{ij} by

$$x_{ij}(1) = \begin{cases} 1 & \text{if } S(n) = j \\ 0 & \text{if } S(n) \neq j \end{cases} \quad (5.17)$$

$$e_{ij} = \sum_{n=0}^{\infty} x_{ij}(n) \quad (5.18)$$

where the subscript i again indicates that the process starts in state i at time $n = 0$. From Eq. (5.18), the mean value of e_{ij} is

$$\overline{e}_{ij} = \sum_{n=0}^{\infty} \overline{x}_{ij}(n) \quad . \quad (5.19)$$

Because $x_{ij}(n)$ has only two values (unity and zero), its mean is

$$\begin{aligned} \overline{x}_{ij}(n) &= 1 \cdot P[S(n) = j | S(0) = 1] + 0 \cdot P[S(n) \neq j | S(0) = 1] \\ &= \phi_{ij}(n) = p_{ij}^{(n)} \quad . \end{aligned} \quad (5.20)$$

Therefore, the mean number of entries into a transient state j before absorption ($\overline{e}_{ij}$) is

$$\overline{e}_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)} = t_{ij} \quad (5.21)$$

by Eq. (5.14) with $z = 1$ and the definition of t_{ij} .

If the process is truncated after the M^{th} step (instead of being allowed to continue until absorption occurs), the mean number of entries into j before absorption or truncation is

$$\overline{e}_{ij}^{(M)} = \sum_{n=0}^M \overline{x}_{ij}(n) = \sum_{n=0}^M p_{ij}^{(n)} = \sum_{n=0}^{\infty} p_{ij}^{(n)} - \sum_{n=M+1}^{\infty} p_{ij}^{(n)} \quad . \quad (5.22)$$

Thus, $\overline{e}_{ij}^{(M)}$ is seen to be the i, j^{th} element of the matrix

$$\begin{aligned} (I - Q^{M+1})(I - Q)^{-1} &= (I - Q)^{-1} - Q^{M+1}(I - Q)^{-1} \\ &= (I - Q)^{-1} - (Q^{M+1} + Q^{M+2} + \dots) \quad . \end{aligned} \quad (5.23)$$

Truncation at M is therefore accomplished by premultiplying the fundamental matrix $(I - Q)^{-1}$ by a "correction factor" $I - Q^{M+1}$.

In the untruncated case, the average number of transitions before absorption when the process starts in state i can be denoted by $\bar{e}_i$. Its value is obtained by summing over all r transient states

$$\bar{e}_i = \sum_{j=1}^r t_{ij} \quad \text{for } i = 1, 2, \dots, r \quad . \quad (5.24)$$

In applications later, the mean number of entries into state j will be associated with the mean time spent in state j before absorption, i.e., before the end of the exercise. One parameter estimation method will be developed by using a matrix $\hat{T}$ whose elements are the average total time spent in each of the transient states during the exercise. For calculation of a useful effectiveness measure, however, a different mean time is needed: the mean time until absorption, given that absorption occurs in state k . This quantity $\bar{e}(i,k)$ is also calculated from the elements in the partitioned form of $(I - Pz)^{-1}$ but is somewhat more difficult to derive.

G. Probability Generating Functions

Returning to the z transform of the one-dimensional sequence $f(0)$, $f(1)$, $f(2)$, ..., we can interpret $f(n)$ as the probability that an integer-valued random variable X assumes the value n . This transform, known as the probability generating function of the random variable X , is defined by

$$f(z) = \sum_{n=0}^{\infty} f(n)z^n = \sum_{n=0}^{\infty} P(X = n)z^n \quad (5.25)$$

and can then be differentiated with respect to z to obtain

$$f'(z) = \sum_{n=0}^{\infty} nf(n)z^{n-1} = \sum_{n=0}^{\infty} nP(X = n)z^{n-1} \quad . \quad (5.26)$$

When z is set equal to unity, the result is the expected (mean) value of X :

$$E(X) = f'(z = 1) = \sum_{n=0}^{\infty} nP(X = n) \quad . \quad (5.27)$$

It will be necessary to know later that the second moment of X and hence its variance ($\text{var } X$, or σ_X^2) can also be found by differentiating $f(z)$. Specifically, the second derivative evaluated at $z = 1$ equals $E(X^2) - E(X)$

$$\begin{aligned} f''(z = 1) &= \sum_{n=0}^{\infty} n(n-1)P(X = n) = \sum_{n=0}^{\infty} n^2 P(X = n) - \sum_{n=0}^{\infty} nP(X = n) \\ &= E(X^2) - E(X) \quad . \end{aligned} \quad (5.28)$$

Since the variance of X is defined as the mean of the square of X minus the square of the mean of X , we have

$$\begin{aligned} \text{var } X &= E(X^2) - E^2(X) = [E(X^2) - E(X)] + [E(X) - E^2(X)] \\ &= f''(z = 1) + f'(z = 1) - [f'(z = 1)]^2 \quad . \end{aligned} \quad (5.29)$$

In the matrix in Eq. (5.16), the individual elements of the sub-matrix $(I - Qz)^{-1}Rz/(1 - z)$ are the generating functions of the cumulative distribution of the number of transitions before absorption. Two changes must be made to use these elements:

- The $(1 - z)$ must be dropped because it accomplishes the accumulation; dropping the $(1 - z)$ leaves the conditional probability generating function of the density function of the number of transitions.
- The appropriate absorption probability a_{ik} must be used to divide what remains after the factor $(1 - z)$ is dropped so that the conditioning on absorption in state k is reflected.

Algebraically, the $(i,k)^{th}$ element of $(I - Qz)^{-1}Rz$ divided by a_{ik} -- the $(i,k)^{th}$ element of $(I - Q)^{-1}R$ --is the probability generating function of the number of transitions before absorption in k , given absorption in k and starting state i . The first derivative of this generating function, evaluated at $z = 1$, is the desired measure of effectiveness $\bar{e}(i,k)$.

Differentiation of the probability density function may be carried out in matrix terms by using the auxiliary formula

$$\frac{d}{dz} (I - Qz)^{-1} = (I - Qz)^{-1}Q(I - Qz)^{-1} \quad . \quad (5.30)$$

Differentiating $M(z) \equiv (I - Qz)^{-1}Rz$ as a product results in

$$\frac{d}{dz} [M(z)] = (I - Qz)^{-1}Q(I - Qz)^{-1}Rz + (I - Qz)^{-1}R \quad . \quad (5.31)$$

When evaluated at $z = 1$, this expression reduces in a few simple steps to

$$\frac{d}{dz} [(I - Qz)Rz] \Big|_{z=1} = (I - Q)^{-1}[(I - Q)^{-1}R] = TA \quad . \quad (5.31)$$

The matrix on the right is therefore equal to the product of the fundamental matrix $T = (I - Q)^{-1}$ and the absorption probability matrix $A = (I - Q)^{-1}R$, both of which are elements in the top row of the partitioned form of $(I - Pz)^{-1}$ in Eq. (5.16).

The $(i,k)^{th}$ element of Eq. (5.32), denoted by $\bar{e}(i,k)$, is the mean time to absorption state k , given absorption in state k and starting state i at $n = 0$

$$\begin{aligned} \bar{e}(i,k) &= \frac{(i,k)^{th} \text{ element of } (I - Q)^{-1}[(I - Q)^{-1}R]}{P(\text{absorption in state } k | \text{starting state } i)} \\ &= \sum_{j=1}^r t_{ij} a_{jk} / a_{ik} \quad . \end{aligned} \quad (5.32)$$

The equation $A = TR = (I - Q)^{-1}R$ can be made quite intuitive when t_{ij} is known to represent the average number of transitions into state j before absorption, given starting state i . The product $t_{ij}p_{jk}$ is the mean number of times that state k is entered from state j , given starting state i . The sum over all transient states j corresponds to the inner product of row i of $T = (I - Q)^{-1}$ and the column of R that represents state k . The value of the sum is the mean number of entries into state k , given starting state i . Because a single trial is implicit in the calculation of this mean, the absorption probability itself equals the mean number of entries:

$$E(\text{number of entries into } k | \text{starting state } i) =$$

$$1 \times P(\text{absorption in state } k | \text{start in state } i)$$

$$+ 0 \times P(\text{absorption in state other than } k | \text{start in state } i) \quad .$$

This result is found by a different method.^{5*}

Another measure useful in some contexts is the probability of not failing before a specified time. The $(i,k)^{\text{th}}$ element of $(I + Q + \dots + Q^{M-1})R$ is the probability of occupying absorption state k at time M (i.e., the probability of being absorbed in k at or before time M). Therefore, the probability of not failing by time M is

$$\overline{p}_{if}(M) \equiv 1 - P(\text{absorption into the fail state } f \text{ by } M)$$

$$= 1 - (i,f)^{\text{th}} \text{ element of } (I + Q + \dots + Q^{M-1})R \quad . \quad (5.33)$$

* Reference 5, Vol. I, Equation 4.1.117.

Operationally this quantity may have an interpretation such as the probability of surviving (or remaining undetected) for at least M time intervals.

The sum $I + Q + Q^2 + \dots + Q^{M-1}$ is easily shown to equal $(I - Q^M)(I - Q)^{-1}$, so that the measure $\overline{p}_{if}(M)$ also depends on the fundamental matrix $T = (I - Q)^{-1}$. Specifically

$$\overline{p}_{if}(M) = 1 - \left[(i,f)^{\text{th}} \text{ element of } (I - Q^M)(I - Q)^{-1}R \right] \quad . \quad (5.34)$$

The matrix $(I - Q^M)(I - Q)^{-1}$ has elements that represent the mean number of entries into state j in the first M - 1 transitions; that is, the mean number of entries matrix $(I - Q)^{-1}$ corrected for truncation at M.

H. Summary of Effectiveness Measures Derived

In summary, the submatrices $T = (I - Q)^{-1}$ and $A = (I - Q)^{-1}R$ found in the partitioned form of $(I - Pz)^{-1}$ [Eq. (5.16)] have the following uses and interpretation when the process starts in state i at time $n = 0$:

- The elements a_{ik} of $A = (I - Q)^{-1}R$ are the probabilities of absorption in state k, as measures of effectiveness, they are the probabilities of mission success. When premultiplied by $(I - Q^M)$, the elements are the probabilities of absorption by time M; the $(i,f)^{\text{th}}$ element of the product is $(1 - \text{probability of the mission not failing by time M})$.
- The elements t_{ij} of $T = (I - Q)^{-1}$ are the mean number of entries into a transient state j; they represent the mean time spent in state j during the exercise. The sum

$$\sum_{j=1}^r t_{ij}$$

is the mean time before absorption; operationally, this is the mean duration of the exercise.

- The $(i,k)^{th}$ element of the product $TA = \{(I - Q)^{-1}(I - Q)^{-1}R\}$ --when divided by the corresponding element a_{ik} of A --is the mean time to absorb in state k , given absorption in state k . Operationally, these times are the mean time until mission success (or failure) occurs, given that success (or failure) occurs.

I. A Four-State Example

Four states, the allowed transitions between these states, and the values of the transition probabilities p_{ij} are shown in Figure 5.2.

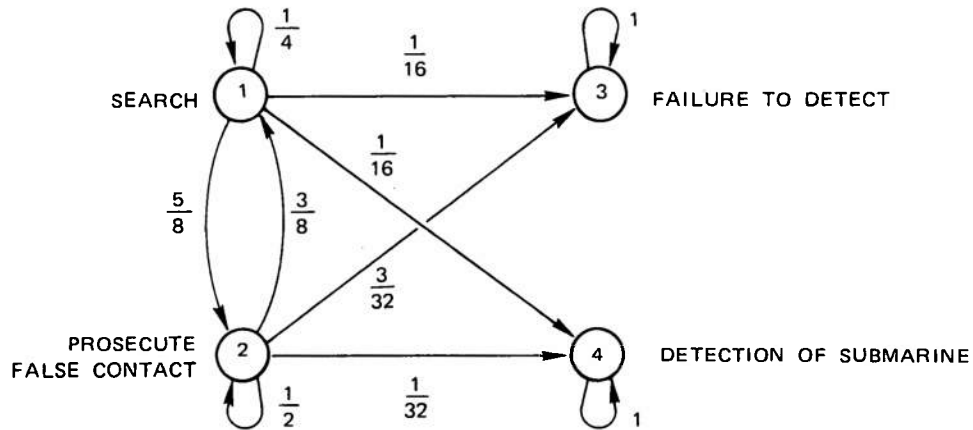


FIGURE 5.2 TRANSITION DIAGRAM WITH TRANSITION PROBABILITIES

The transition matrix is

$$P = \left(\begin{array}{cc|cc} \frac{1}{4} & \frac{5}{8} & \frac{1}{16} & \frac{1}{16} \\ \frac{3}{8} & \frac{1}{2} & \frac{3}{32} & \frac{1}{32} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}, \quad (5.37)$$

where the submatrices Q and R are

$$Q = \begin{pmatrix} \frac{1}{4} & \frac{5}{8} \\ \frac{3}{8} & \frac{1}{2} \end{pmatrix} \quad R = \begin{pmatrix} \frac{1}{16} & \frac{1}{16} \\ \frac{3}{32} & \frac{1}{32} \end{pmatrix} . \quad (5.38)$$

The fundamental matrix $(I - Q)^{-1} = T$ is found by first calculating $(I - Q)$.

$$(I - Q) = \begin{pmatrix} \frac{3}{4} & -\frac{5}{8} \\ -\frac{3}{8} & \frac{1}{2} \end{pmatrix} . \quad (5.39)$$

Inverting this matrix results in

$$(I - Q)^{-1} = \begin{pmatrix} \frac{32}{9} & \frac{40}{9} \\ \frac{24}{9} & \frac{48}{9} \end{pmatrix} . \quad (5.40)$$

Therefore, if the process starts in the search state the average number of entries into the search state (before absorption) is $32/9$ and the average number of entries into the prosecute false contact state is $40/9$. Since these are also the average lengths of time spent in the respective states, the process spends more time prosecuting false contacts than it does in searching. The sum of the elements in the first row ($32/9 + 40/9 = 72/9 = 8$) is the mean time until absorption, or the mean duration of the exercise given that the exercise started in the search state. The absorption probability matrix $(I - Q)^{-1}R = A$ is

$$A = \begin{pmatrix} \frac{32}{9} & \frac{40}{9} \\ \frac{24}{9} & \frac{48}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{16} & \frac{1}{16} \\ \frac{3}{32} & \frac{1}{32} \end{pmatrix} = \begin{pmatrix} \frac{23}{36} & \frac{13}{36} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} . \quad (5.41)$$

When starting in search, the probability of detection is 13/36 and the probability of failing to detect is 23/36 since the second column represents the absorption state 4 (detect), and the first row is associated with starting in state 1 (search).

Conditional mean times to fail and detect are obtained from computing $(I - Q)^{-1}(I - Q)^{-1}R = TA$ and normalizing by dividing by the appropriate absorption probability. The result is the matrix

$$[\bar{e}(i,k)] = \begin{pmatrix} \frac{1696 \times 36}{324 \times 23} & \frac{896 \times 36}{324 \times 13} \\ \frac{1704 \times 3}{324 \times 2} & \frac{888 \times 3}{324} \end{pmatrix} \cong \begin{pmatrix} 8.193 & 7.658 \\ 7.889 & 8.222 \end{pmatrix} \quad (5.42)$$

Reading from the first row to reflect starting in search (state 1) results in

$$\begin{aligned} \bar{e}(1,3) &= \text{mean time to fail given that failure} \\ &\quad \text{ultimately occurs} = 8.193 \end{aligned} \quad (5.43)$$

$$\begin{aligned} \bar{e}(1,4) &= \text{mean time to detect given that detection} \\ &\quad \text{ultimately occurs} = 7.658 \end{aligned}$$

As a check, one may verify that the weighted sum of these conditional mean times--where the weights are the corresponding absorption probabilities--equals the mean duration of the exercise.

The probabilities of fail and success at time M are more difficult to obtain numerically because the matrix Q must be raised to the M^{th} power. Choosing $M = 4$ arbitrarily and calculating Q^4 gives

$$Q^M = Q^4 \cong \begin{pmatrix} 0.220 & 0.366 \\ 0.220 & 0.366 \end{pmatrix} \quad (5.44)$$

The probability of detect by time $M = 4$ is therefore 0.160 and the probability of fail by that time is 0.254.

The imbedded chain corresponding to Figure 5.2 is shown in Figure 5.3 and was obtained from Figure 5.2 by removing the self-transitions at states 1 and 2 and normalizing the remaining transition probabilities to

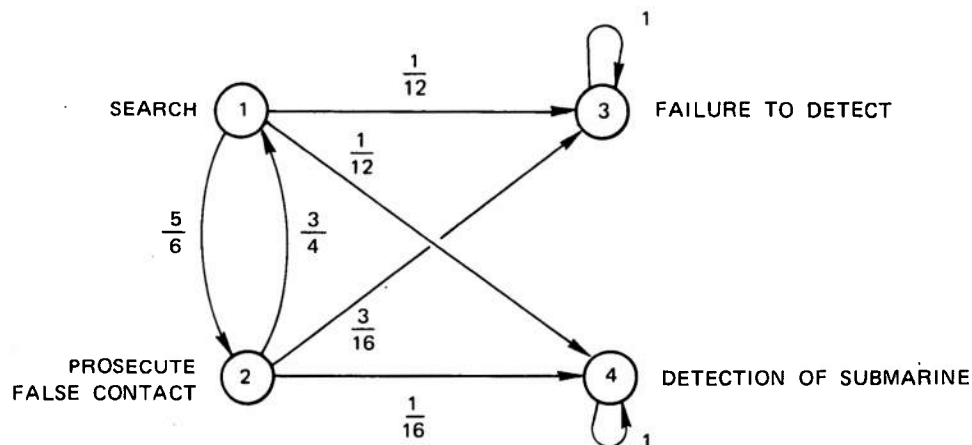


FIGURE 5.3 IMBEDDED CHAIN FOR FIGURE 5.2

other states. Transition probabilities p'_{ij} now have a different meaning. They are conditioned on leaving state i :

$$p'_{ij} = P[S(n+1) = j | S(n) = i \text{ and } S(n+1) \neq i]$$

for $i = 1, 2$, and all j .

The transition probability matrix for the imbedded chain P' is

$$P' = \left(\begin{array}{cccc} 0 & \frac{5}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{3}{4} & 0 & \frac{1}{16} & \frac{1}{16} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) = \left(\begin{array}{cc} Q' & R' \\ 0 & I \end{array} \right) . \quad (5.46)$$

$(I - Q')^{-1}$ works out to be

$$(I - Q')^{-1} = \left(\begin{array}{cc} \frac{24}{9} & \frac{20}{9} \\ 2 & \frac{24}{9} \end{array} \right) \quad (5.47)$$

so that the mean number of entries into states 1 and 2 are $24/9$ and $20/9$, respectively. In terms of the original chain, $24/9$ is the mean number of searches before absorption and the mean number of false contact prosecutions, respectively. Since the average time in the search state per entry into search is $1/(1 - 1/4) = 4/3$, the average time in search before absorption is

$$(\text{mean number of searches}) \times (\text{mean length of one search}) = \frac{24}{9} \times \frac{4}{3} = \frac{32}{9} ,$$

which agrees with the mean time in search calculation performed above in the original chain.

It can be easily verified that absorption probabilities are the same for the imbedded chain as for the original chain, i.e.,

$$A' = (I - Q')^{-1} R' = (I - Q)^{-1} R = A . \quad (5.48)$$

VI PARAMETER ESTIMATION FOR MARKOV PROCESSES

A. Introduction

Section V outlined Markov chain theory and derived matrix formulas for calculating three measures of effectiveness: $P(\text{success})$; mean time until mission failure, given that failure occurs; and $P(\text{fail does not occur by a specified time } M)$. All of these measures were calculated in terms of the transition probabilities p_{ij} .

In analytical models such as those in Ref. 6, the transition probabilities are found by analytical means. However, for reasons indicated in Section III-A, operational models are of more interest in this report and require statistical methods to estimate parameters from operational data. Several methods for estimating parameters are developed in this section, assuming that the state space is given. Least squares and quadratic programming methods are shown to be applicable in many situations. Some of the parameter estimation methods are applied to artificial problems and to simulation in Sections IX, XI, and XII.

Weighting methods for combining two or more estimates of a parameter are treated in Section XIII.

B. Factoring the Estimation Problem

In some applications, factoring the parameter estimation into two problems is worth consideration. The first problem is to estimate the parameters of the imbedded Markov chain, where this chain is found from a given chain by removing all self-transitions and normalizing. Algebraically, the p_{ii} are set to zero for all transient states i , and the other p_{ij} in the same row are divided by $1 - p_{ii}$ so that the sum of the

outgoing probabilities is unity:

$$\sum_{i \neq j} \frac{p_{ij}}{1 - p_{ii}} = \frac{1}{1 - p_{ii}} (1 - p_{ii}) = 1 \text{ for all transient states } i.$$

The imbedded chain for the four-state example of Figure 5.2 was shown in Figure 5.3 of Section V.

Note that transitions before absorption in the imbedded chain are always to a different state. In this chain, the amount of time spent in a transient state is not properly accounted for; hence, the correspondence between state and time is lost. In particular, $P(\text{state})$ as a function of time and such quantities as mean times till absorption are generally different from the corresponding quantities in the original chain. However, since transitions out of a state occur to other states in the same ratio as in the original process, the absorption probabilities are identical to those in the original process.

Inasmuch as the time relationship is destroyed in the imbedded chain, it is considered static in the same sense that a conditional probability model is static. Putting the self-transitions back in makes the model dynamic. Fortunately, the self-transitions can be added to the imbedded chain without disturbing the static (transition) structure, which implies that the static and dynamic aspects of the model can be studied and analyzed somewhat independently. This separation of estimation into two parts (imbedded chain and holding time) is used in later examples and is explained in Subsection D.

C. Estimating Transition Probabilities

1. Using Transition Data (n_{ij})

In a true Markov context, the most desirable data for estimating transition probabilities are counts of the number of transitions (n_{ij})

from state i to state j for all states i and j . Transition counts may be available for either the original or the imbedded Markov chain. In the original chain, at least some of the n_{ii} will be nonzero; but in the imbedded chain, all n_{ii} are zero by definition. When the n_{ij} for the imbedded chain are given, the parameters $\hat{p}_{ij}$ that are found will be for the imbedded chain; other data must be used to estimate the p_{ii} and to normalize the p_{ij} by multiplying each by the factor p_{ii} . Let

n_{ij} = observed number of transitions from state i to state j

$$n_i = \sum_{j=1}^N n_{ij} = \text{number of transitions out of state } i \quad .$$

Then the quantity $\hat{p}_{ij}$ is the optimal estimator for p_{ij} in both the maximum-likelihood and minimum chi-squared sense:

$$\hat{p}_{ij} = \begin{cases} n_{ij}/n_i & \text{for all } i,j \text{ when } n_i > 0 \\ 0 & \text{otherwise} \end{cases} \quad .$$

Obviously, each $\hat{p}_{ij}$ lies in the closed interval $(0,1)$. Also, by the definition of n_i , the sum over j for a fixed i is always unity.

This estimator is entirely analogous to that used to estimate the parameter p in a series of Bernoulli trials because it equals the ratio of the number of successes to the number of trials. Furthermore, $\hat{p}_{ij}$ is easily shown to be unbiased (its expected value is p_{ij}) and to be the minimum variance unbiased estimator.⁷ With these properties, it is unlikely that this estimator can be improved upon. The difficulty with this estimator is primarily that considerable effort may be required to determine the transition counts; currently, they are rarely given in ASW exercise reports.

Determination of the n_{ij} from an ASW exercise is simple in principle. From the beginning of the exercise, the times when state changes (transitions) occur are recorded with the corresponding transition times. Whenever it is known whether the process is Markov at the outset, it is desirable to first represent the data in path form; that is, first record the sequence of states entered from start until absorption, together with the times of transition. A sample path that starts in state 1 and is absorbed in state 5 may be represented by two rows of data--one row for the states entered and another for the transition times. An example is the data

state sequence: 1 2 1 2 3 1 2 5

transition times: 0 5 6 8 9 11 12 14 .

An equivalent path form that can be used when transitions can be assumed to occur at only integer times is the sequence

1 1 1 1 1 2 1 1 2 3 3 1 2 2 5 .

Reading from the left, the n^{th} element of this sequence is the observed state at time n , for $n = 0, 1, \dots, 14$. For theoretical purposes, the second representation is preferable because all the information is shown in transition form. In either case, the n_{ij} can readily be determined from the path data by simply counting transitions. The matrix of n_{ij} for the single path shown above is shown in Table 6.1. A single "one" is shown for the 5-5 transition; this results in the desired estimate of unity for state 5 that is assumed absorbing.

The transition counts for other paths may be accumulated in the same n_{ij} matrix, and one matrix for each path need not be recorded. For example, if the data consist of three paths shown on the left for a

Table 6.1

TRANSITION COUNTS n_{ij} FOR A SINGLE PATH

i \ j	1	2	3	4	5	n_i
1	5	3	0	0	0	8
2	1	1	1	0	1	4
3	1	0	1	0	0	2
4	0	0	0	0	0	(0)
5	0	0	0	0	1	(1)

five-state process with absorbing states 4 and 5, the matrix is that shown on the right in Table 6.2.

Table 6.2

TRANSITION COUNTS n_{ij} FOR THREE PATHS

Paths	States	i \ j	1	2	3	4	5	n_i
1	1 1 1 2 3 5	1	2	2	1	1	0	6
2	1 2 2 3 1 4	2	0	1	3	0	0	4
3	2 3 3 1 3 4	3	2	0	1	1	1	5
		4	0	0	0	1	0	(1)
		5	0	0	0	0	1	(1)

When experimentation is needed to find a satisfactory set of states, it may be desirable to proceed differently. First, a number of variables may be selected from which states are to be defined; these values are recorded continuously by going through the exercise or sequence of tactical events one time. For a tentative definition of state space, one can proceed through the list of these variables to determine when

changes of state occur and to record the transitions accordingly. When the situation is complex, having a computer program compute the n_{ij} from lists of the auxiliary variables may be desirable. Suggestions about variables to choose for defining states are made in Section IV-D-1.

2. Using Mean Time-in-State Data (t_{ij})

In some tactical situations in which state is defined, the total time spent in each state is tabulated in the exercise report. An example of this form of data is the tabulation of the total time the Orange submarines are in the defined states for the recent series of UPTIDE exercises. If the underlying process is Markovian (that is, if these states are actually states in the Markovian sense) these data may be used to estimate the transition probabilities in a Markov chain. Because less data collection and data analysis effort should be required to obtain the mean time-in-state data, developing a parameter estimation method for these data is worthwhile.

At least two theoretical difficulties must be faced, however. First, the theory requires the mean time data to be available for all possible starting (transient) states. Second, only the transition probabilities for the transient states can be determined so that additional data are required for the calculation of absorption probabilities and other quantities involving transitions into absorbing states.

The basis for the estimators is the theoretical expression given in Section V that relates the transient portion (Q) of the transition matrix (P) to the mean number of entries into each transient state before absorption. The matrix equation given there was

$$T = (I - Q)^{-1} \quad (6.1)$$

where the elements of T , denoted by t_{ij} , were the mean numbers of transitions into transient states j before absorption, conditioned on the process starting in state i .

The elements of Q are simply p_{ij} for transient states i and j . When estimates of all the t_{ij} are given, we may denote the resulting matrix by $\hat{T}$, with entries $\hat{t}_{ij}$. When the starting state is always state 1 (as it is assumed to be in most of this report), the second and later rows of $\hat{T}$ may possibly be filled in by using the same data used in the first row after all data were removed before the first entry into state i . (This method was used successfully on an example reported in Section XII which may be consulted for details.)

The straightforward approach to estimating the elements of Q in Eq. (6.1) would be simply to invert both sides and transpose elements and thereby obtain a matrix $\hat{Q}$ whose elements are estimates of the transition probabilities p_{ij} .

$$\hat{Q} = I - (\hat{T})^{-1} \quad (6.2)$$

When a satisfactory fit is obtained, this simple method may be adequate. However, possible difficulties include:

- $\hat{T}$ may not have an inverse, or may be very nearly singular.
- Some elements of $\hat{Q}$ may be negative, or greater than unity.
- Nonzero elements may result for some transitions known to be physically impossible.

A procedure that circumvents the first and last difficulty may be illustrated by an example. Suppose the T matrix is given for a process with three transient states (designated 1, 2, 3) and two absorbing states. On physical grounds, let us rule out transitions 1 - 1 and 2 - 3 so that

the transient part of the transition matrix is of the form

$$Q = \begin{pmatrix} 0 & p_{12} & p_{13} \\ p_{21} & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

(here the carets (^) are omitted for convenience). Thus, $I - Q$ becomes

$$I - Q = \begin{pmatrix} 1 & -p_{12} & -p_{13} \\ -p_{21} & 1 - p_{22} & 0 \\ -p_{31} & -p_{32} & 1 - p_{33} \end{pmatrix} . \quad (6.3)$$

Equation (6.1) was

$$T = (I - Q)^{-1} . \quad (6.4)$$

Premultiplying each side of Eq. (6.4) by $(I - Q)$ in Eq. (6.3) gives

$$(I - Q)T = (I - Q)(I - Q)^{-1} = I , \quad (6.5)$$

or by expansion and transposition

$$QT = T - I . \quad (6.6)$$

The matrix equation in Eq. (6.6) represents a set of nine simultaneous linear equations in the seven unknown transition probabilities for the allowed transitions. When $(I - Q)$ has an inverse, solving Eq. (6.6) is equivalent to solving this set of equations, which in expanded form are:

$$\begin{aligned}
t_{21}p_{12} + t_{31}p_{13} &= t_{11} - 1 \\
t_{22}p_{12} + t_{32}p_{13} &= t_{12} \\
t_{23}p_{12} + t_{33}p_{13} &= t_{13} \\
t_{11}p_{21} + t_{21}p_{22} &= t_{21} \\
t_{12}p_{21} + t_{22}p_{22} &= t_{22} - 1 \\
t_{13}p_{21} + t_{23}p_{22} &= t_{23} \\
t_{11}p_{31} + t_{21}p_{32} + t_{31}p_{33} &= t_{31} \\
t_{12}p_{31} + t_{22}p_{32} + t_{32}p_{33} &= t_{32} \\
t_{13}p_{31} + t_{23}p_{32} + t_{33}p_{33} &= t_{33} - 1
\end{aligned} \tag{6.7}$$

The equations in Eq. (6.7) may or may not have a solution in the algebraic sense. However, a least squares solution can always be found.

Equation (6.7) can be written in the form

$$Ap = b \tag{6.8}$$

where A is a 9×7 matrix, p is a 7×1 matrix (i.e., a column vector with 7 components), and b is a 9×1 matrix. Specifically, p may be chosen as the vector $(p_{12}, p_{13}, p_{21}, p_{22}, p_{31}, p_{32}, p_{33})^T$, where T denotes transpose, and b may be chosen as the column of values on the right side of Eq. (6.7). The first row of A is then $(t_{21}, t_{31}, 0, 0, 0, 0, 0)$, the zeroes reflecting the absence of all p_{ij} excepting p_{12} and p_{13} in the first equation of Eq. (6.7).

The modern approach to least squares proceeds directly from the matrix form in Eq. (6.8). The sum of the squared deviations is

$$\begin{aligned}
d^2 &= (Ap - b)^T (Ap - b) = (p^T A^T - b^T)(Ap - b) \\
&= p^T A^T Ap - p^T A^T b - b^T Ap + b^T b = p^T A^T Ap - 2b^T Ap + b^T b \quad . \quad (6.9)
\end{aligned}$$

The derivative of the scalar d^2 with respect to the vector p is then found and set to zero:

$$2A^T Ap - 2A^T b = 0 \quad . \quad (6.10)$$

Solving this by inverting yields the unconstrained least-squares solution

$$\hat{p} = (A^T A)^{-1} A^T b \quad . \quad (6.11)$$

Nothing in this formulation constrains the unknowns p_{ij} to satisfy the probability constraints

$$p_{ij} \geq 0 \quad , \text{ and } \sum_{j=1}^3 p_{ij} \leq 1 \quad . \quad (6.12)$$

If these constraints are badly violated, a quadratic programming formulation may be used. Appendix D explains quadratic programming in more detail, but here it need only be said that arbitrary linear inequality and equality constraints may be introduced; and in particular, the probability constraints Eq. (6.12) can be added. The general form of a quadratic program is: find a vector x to minimize

$$z = x^T D x \quad (6.12)$$

subject to constraints

$$\begin{aligned}
A_1 x &= b_1 \\
A_2 x &\leq b_2 \\
A_3 x &\geq b_3 \\
x &\geq 0
\end{aligned}
\tag{6.13}$$

where D , A_1 , A_2 , A_3 are given matrices of constants; the b_1 , b_2 , b_3 are given column vectors; and the scalar z (the sum of the squared deviations) is to be minimized.

3. Using Mean First-Passage Times (m_{ij})

When a set of states is defined for a class of tactical situations, it is possible to obtain transition probability estimates from data on the time between the entry into one state and the entry into another. To illustrate this, suppose that search, detect, and classify are states in a Markov chain model. Then such quantities as the average time from the beginning of search until detect (i.e., entry into the detect state), the average time from the beginning of search until the time of entry into classification, the average time from the entry of detection until the entry of classification can be estimated from operational data. These quantities may themselves be considered as dynamic measures of effectiveness since they depend upon the dynamics of the ASW system or unit being considered. For the current purpose, however, they are not considered as end-products but as parameters required for the estimation of the four measures of effectiveness considered earlier. All of these (end-product) measures of effectiveness are calculated from the transition probabilities in a Markov chain model so methods are necessary for estimating the transition probabilities from time-between-states information.

In technical terms, a set of quantities of interest from which transition probabilities may be estimated are called first-passage times. The first-passage time from state i to state j is defined as the number of transitions to arrive in state j , given that the state is i at time zero. First-passage times are defined only for recurrent chains; that is, for Markov chains all of whose states can be reached from any other state. When all states do not communicate in this manner, a modified definition entailing conditional first-passage times may be used; but this results in very complex expressions. Therefore, the formulation given here is for the case all states communicate; by definition, there are no absorbing states. The measure of effectiveness appropriate to these situations is the probability of occupying a designated state at a randomly chosen time; this is appropriate, for example, for the trailing mission.

Data requirements are a complete set of estimates of mean first passage times ($\bar{m}_{ij}$), where "complete" means all (i,j) combinations that are physically and logically possible. As in the previous section, using data from paths that all start in the same state may be possible by ignoring all transitions before the first entry into state i when data for row i are being calculated.

A possible objection to the use of mean first passage time data may be raised at this point. When the analysis required to obtain the mean first-passage times is considered, it seems that paths must be traced from each possible starting state (i) to the entry of an absorbing state (k), and that this tracing would yield path information that can be translated into transition counts n_{ij} . Since methods are available for estimating the transition probabilities p_{ij} from the n_{ij} , it then might be asked whether the mean first-passage times contribute anything new to the problem. The answer to this will depend on whether the process is actually

Markovian, the measure of effectiveness, and what the "goodness of fit" measure is for the model. In a true Markov context where only the absorption probabilities are desired, the transition information is sufficient. However, if the model is realistically at least somewhat non-Markovian and is also required to fit dynamically [in the $P(\text{state})$ as a function of time sense] then the transition probabilities derived from the mean first-passage times should give useful additional information.

Equations relating the mean first passage times and transition probabilities may be found in any standard text on Markov chains.⁸

Letting m_{ij} be the mean first passage time from i to j and p_{ij} be the usual transition probability from i to j , the equations are

$$m_{ij} = 1 + \sum_{\substack{k=1 \\ k \neq j}}^N p_{ik} m_{kj} \quad 1 \leq i, j \leq N \quad . \quad (6.14)$$

This is a set of N^2 linear equations in the N^2 unknown transition probabilities. In theory, with exact values of the m_{ij} known, there should be a unique solution to this linear system that yields the transition probabilities. The probabilities comprising the solution should automatically satisfy the nonnegativity and row sum conditions

$$p_{ij} \geq 0 \quad , \quad \sum_{j=1}^N p_{ij} = 1 \quad \text{for } i = 1, 2, \dots, N \quad . \quad (6.15)$$

In applications, however, there are two reasons why straightforward solution of the set of linear equations in Eq. (6.14) will seldom be possible. First, the model will rarely be exactly Markovian as required by the theory. Second, the amount of data is limited so that the estimators m_{ij} will have noise components even in the Markovian case. Three outcomes are possible when one attempts to solve the system in Eq. (6.14).

- (1) The system of equations in Eq. (6.14) may have a solution with nonnegative p_{ij} which also satisfies the row conditions in Eq. (6.15).
- (2) The system may have a solution with some p_{ij} outside the range zero to unity, or the row conditions in Eq. (6.15) may not be satisfied.
- (3) The system may be inconsistent and hence have no solution.

The last two outcomes are the result of the non-Markovian of the real processor noise components in the estimators. Only the first outcome gives an acceptable solution, but the second and third possibilities are often to be expected in working with a real problem.

a. Constrained Least Squares

A first method for obtaining an acceptable set of estimates from the set of equations of Eq. (6.14) will be through the application of constrained least squares. This model is ordinary least squares augmented by linear constraints on the unknown quantities. The constrained least squares formulation follows. The solution always exists, but the nonnegativity constraint may be violated.

First, the system in Eq. (6.14) is replaced by a single quantity to be minimized:

$$d^2 = \sum_{i=1}^N \sum_{j=1}^N \left(1 + \sum_{\substack{k=1 \\ k \neq j}}^N p_{ik} m_{kj} - m_{ij} \right)^2 \quad (6.16)$$

The quantity d^2 is obviously nonnegative and the lowest possible value is therefore zero in Eq. (6.16). The value zero is achieved as a minimum if--and only if--the set of equations in Eq. (6.14) is consistent. To

ensure acceptability of the minimizing set of p_{ij} , however, it is necessary to add the row constraints of Eq. (6.15). The constrained least-squares problem then takes the form of minimizing

$$d^2 \text{ by selecting } p_{ij} \quad (i, j = 1, 2, \dots, N) \quad (6.17)$$

to satisfy

$$\sum_{j=1}^N p_{ij} = 1 \quad \text{for } i = 1, 2, 3, \dots, N \quad (6.18)$$

The double summation defining d^2 can be rewritten in matrix form as

$$d^2 = (Ap - b)^T (Ap - b) \quad (6.19)$$

when b is an observable random vector, p_{ij} is the unknown vector of transition probabilities, and A is a matrix of constants determined from the m_{ij} . The row constraints of Eq. (6.19) also can be written in matrix form as

$$Bp = k \quad , \quad (6.20)$$

where k is a known vector and B a known matrix. The solution to Eq. (6.17) subject to Eq. (6.18) is then that given in Appendix D with the covariance matrix V equal to the identity matrix

$$\tilde{p} = \hat{p} + \left[\left(A^T A \right)^{-1} B^T \right] \left[B \left(A^T A \right)^{-1} B^T \right]^{-1} (k - B\hat{p}) \quad (6.21)$$

where $\hat{p}$ is the solution to the unconstrained problem

$$\hat{p} = \left(A^T A \right)^{-1} A^T b \quad . \quad (6.22)$$

b. Quadratic Programming Solution

The constrained least squares formulation minimized a sum of squares subject to linear constraints derived from the row constraints that the sum of the transition probabilities out of each state should sum to unity. Unfortunately, there is a possibility that some probabilities may lie outside the range zero to unity while satisfying the row conditions. When the nonnegativity constraints

$$p_{ij} \geq 0 \quad \text{for } i, j = 1, 2, 3, \dots, N \quad (6.28)$$

are also introduced, the result is a quadratic program to minimize

$$p^T D p + c^T p$$

subject to

$$p \geq 0 \quad (6.24)$$

$$A p = b$$

where

p is the unknown vector of transition probabilities

b and c are known constant vectors

D is a known matrix of constants which is symmetric and positive semidefinite.

4. Using the First and Second Moments of First-Passage Times

In addition to the mean first-passage times (the first moments), estimates of the second moments of first-passage time (s^2) may also be

available. For a Markov chain, the following relations hold when all states communicate.⁵

$$s_{ij}^2 = 2m_{ij} - 1 + \sum_{\substack{k=1 \\ k \neq j}}^N p_{ik} s_{ij}^2 \quad i, j = 1, 2, \dots, N \quad (6.25)$$

This set of equations has nearly the same form as Eq. (6.14) in the previous section, and the same remarks apply concerning its solution. In particular, a constrained least-squares approach may be taken to obtain the estimators p_{ij} when the estimates of the first and second moments $\bar{m}_{ij}$ and $\bar{s}_{ij}^2$ are available. Since second moments are used as well as first moments in the fitting process, the statistical properties of the estimators can be expected to be at least somewhat improved.

5. Using P(State) as a Function of Time

Other data that may be available in ASW applications are estimates of the probability of occupying each state as a function of time. For a defined set of states with time origin at the beginning of the engagement or exercise, the estimates of $P(\text{state})$ for a fixed t may be from the ratios

$$\hat{p}[S(t) = i] = y_i(t) = \frac{\begin{array}{c} \text{Number of times state } i \text{ was} \\ \text{occupied at time } t \end{array}}{\begin{array}{c} \text{Number of times state } i \text{ could have} \\ \text{been occupied at time } t \end{array}} \quad (6.26)$$

for $i = 1, 2, \dots, N$. That is, $y_i(t)$ is proportional to the observed number of times that state i was occupied at time t , the denominator being the normalizing constant that makes the sum of the $y_i(t)$ equal unity for each fixed t . Other ways of obtaining the estimates $y_i(t)$ may be found more practical in some applications. In economics applications, the $y_i(t)$ are often from such aggregated quantities as the relative proportions of total sales for each competing firm in a particular market.

Analogous quantities may be found in ASW, for example, by considering the relative proportions of time a submarine spends in various information states. Because deceptive tactics and decoys are used to force a submarine into lower information states, this may be a way to study the impact of particular tactics/decoys on transition probabilities and hence on measures of effectiveness derived from these transition probabilities.

To proceed with the mathematical development, let the N state process be observed from time $t = 0$ until time $t = T$. Let the given data be

$$y_j(t) = \hat{p}[S(t) = j] \quad \begin{matrix} t = 0, 1, \dots, T \\ j = 1, 2, \dots, N \end{matrix} \quad (6.27)$$

Then the relationship between the unknown transition probabilities p_{ij} and the observed $y_i(t)$ are

$$y_j(t) = \sum_{i=1}^N y_i(t-1)p_{ij} + u_j(t) \quad \begin{matrix} j = 1, 2, \dots, N \\ t = 1, 2, \dots, T \end{matrix} \quad (6.28)$$

If the $y_j(t)$ were the $P[S(t) = j]$ from a true Markov process with parameters p_{ij} , all the error terms $u_j(t)$ would be zero. The approach will again be to apply least squares; a set of p_{ij} is sought to minimize the sum of the squared errors.

Vector and matrix notation is needed for compactness. Accordingly, let

$$y_j = [y_j(1), y_j(2), \dots, y_j(T)]^T \quad (6.29)$$

$$p_j = (p_{1j}, p_{2j}, \dots, p_{Nj})^T \quad (6.30)$$

$$u_j = [u_j(1), u_j(2), \dots, u_j(T)]^T \quad (6.31)$$

$$A_j = \begin{bmatrix} y_1(0) & y_2(0) & \dots & y_N(0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(t-1) & y_2(t-1) & \dots & y_N(t-1) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(T-1) & y_2(T-1) & \dots & y_N(T-1) \end{bmatrix} \quad (6.32)$$

(note that A_j is actually independent of j). Then in partitioned (component) form, the Eqs. (6.28) become

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_N \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ \vdots \\ p_N \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_N \end{bmatrix} \quad (6.33)$$

Equations (6.33) can be written more compactly as

$$y = Ap + u \quad (6.34)$$

Conventional least squares can be applied: minimize the sum of the squared deviations $u^T u$ by minimizing

$$u^T u = (y - Ap)^T (y - Ap) \quad (6.35)$$

whose solution is the optimal estimator $\tilde{p}$

$$\tilde{p} = (A^T A)^{-1} A^T y, \quad (6.36)$$

provided that $A^T A$ is nonsingular (which it is when A_j is of full rank).

It can be shown that the row sum conditions ($\sum_j p_{ij} = 1$ for all i) are automatically satisfied by $\tilde{p}$, but the $\tilde{p}_{ij}$ are not guaranteed to lie between zero and unity.⁹

Estimates that restrict the transition probabilities to the range (0,1) may also be developed. One again minimizes the scalar quantity

$$d^2 = (y - Ap)^T (y - Ap) \quad , \quad (6.37)$$

subject to the row constraints and $p \geq 0$. This is again a quadratic program, soluble with a modified simplex algorithm.

The least-squares model is applied to the $y_i(t)$ calculated from an artificially constructed non-Markov process in Section XI.

D. Estimating the Density Function of the Transition Times

1. Introduction

In a Markov chain, the only parameters are the transition probabilities p_{ij} ; these determine the static (transition) and the dynamic behavior of the process. As mentioned in Subsection B, the diagonal terms (p_{ii}) determine the holding (or waiting) times in the states and hence determine the dynamic behavior. It is often convenient to estimate the p_{ii} after the off-diagonal p_{ij} is estimated by data from the imbedded chain--that is, from data that depended only on transitions made between different states.

This section begins by showing how to estimate the p_{ii} when the other p_{ij} and average-time-in-state data ($\bar{T}_i$) are available. The method can be applied when it can be assumed that the holding time in a state is geometrically distributed and is independent of the state to which the

transition is made. Since these conditions are often not satisfied in applications, methods are given that may be tried when the dynamic fit obtained by using a Markov chain is unsatisfactory. Techniques of this kind generally use auxiliary states to represent the transition time between selected pairs of states (i,j). When the density function of the transition times is chosen to be the functional form $h_{ij}(n)$, the model is called a semi-Markov model; some techniques for estimating the parameters of these densities conclude the section.

2. The Holding Time Parameter (p_{ii})

In the discrete time Markov process (Markov chain), the transition probabilities determine both the static (transition) and dynamic behavior of the process. In particular, the density function of the time spent in a state is geometrically distributed, independent of the destination state. Denoting the self-transition probability of state i by p_{ii} , the probability of making a transition to some other state at time n is $p_{ii}^{n-1}(1 - p_{ii})$, because (n - 1) self-transitions (returns to state i) must first occur, followed by a transition to some other state. Therefore, the average number of transitions required to leave state i is

$$\begin{aligned}\bar{T}_i &= \sum_{n=1}^{\infty} nP(\text{state } i \text{ is left on the } n^{\text{th}} \text{ transition after entry}) \\ &= \sum_{n=1}^{\infty} n p_{ii}^{n-1} (1 - p_{ii}) = (1 - p_{ii}) \cdot \frac{1}{(1 - p_{ii})^2} = \frac{1}{1 - p_{ii}} \quad . \quad (6.38)\end{aligned}$$

The simplest possible estimation procedure to fit Markov chain parameters to observed data on transitions in the imbedded Markov chain and to observed average transition time data exploits this formula. When the mean time ($\bar{T}_i$) spent in each state is known or is estimated from operational data, the optimal estimator $\hat{p}_{ii}$ is found by solving Eq. (6.39)

$$\bar{T}_i = 1/(1 - \hat{p}_{ii}) \quad (6.39)$$

for the estimator $\hat{p}_{ii}$. The result is

$$\hat{p}_{ii} = 1 - 1/\bar{T}_i \quad (6.40)$$

After these estimates of the self-transitions (the diagonal elements of the transition matrix P) have been determined, the estimates $\hat{p}'_{ij}$ for the other (off-diagonal) transition probabilities for the imbedded process can be normalized by multiplying each $\hat{p}'_{ij}$ by $(1 - \hat{p}_{ii})$. The resulting set of $\hat{p}_{ij}$ ($i, j = 1, 2, \dots, N$) are nonnegative, satisfy the row constraints for each i , and are therefore feasible estimates of the transition probabilities. This estimation procedure is used on sample problems in Sections IX and XI. Based on these examples, when the actual model is non-Markovian and this simple procedure is used without further modifications, a poor dynamic fit seems to be expected.

3. Adding Auxiliary Delay States

One possible way to improve the dynamic $[P(\text{state}) \text{ as a function of time}]$ fit entails the addition of auxiliary states. This addition does not influence the probabilities of transitions between the original states, but does influence the times from one state to another.

If it is reasonable to assume that the waiting time density is independent of the destination state, the natural approach is to increase the number of parameters that characterize the waiting time density. The obvious two parameters are the mean and variance of the waiting time. (A Markov chain waiting time density has only a single parameter (p_{ii}) that determines both the mean waiting time and the variance of the waiting time.) An approach that sometimes works is to add auxiliary delay states,

say D_i auxiliary states for state i , to give the resulting waiting time density function the same sample mean and variance as the estimated waiting time density. If m_i is the (sample) mean waiting time in state i and v_i is the corresponding variance, then the D_i should be chosen to satisfy

$$D_i + \frac{1}{1 - \hat{p}_{ii}} = m_i, \quad D_i = \text{nonnegative integer} \quad (6.41)$$

$$\frac{\hat{p}_{ii}}{(1 - \hat{p}_{ii})^2} = v_i. \quad (6.42)$$

Equation (6.41) was derived to fit the mean waiting time by assuming that D_i delay states (with a mean time of delay D_i) are followed by the original state i [with mean time $1/(1 - \hat{p}_{ii})$]. Equation (6.42), derived to match the variances, does not involve D_i since the delaying transitions always occur and do not involve the variance. Solving Eq. (6.42) for $\hat{p}_{ii}$, the estimator of p_{ii} , gives

$$\hat{p}_{ii} = \frac{(2 + v_i^{-1}) \pm \sqrt{4v_i^{-1} + v_i^{-2}}}{2} \quad (6.43)$$

for the appropriate choice of sign before the radical. Using this value of $\hat{p}_{ii}$ in Eq. (6.41) and solving for the number of delay states D_i ,

$$D_i = m_i - \frac{1}{1 - \hat{p}_{ii}}. \quad (6.44)$$

If the computed value of D_i is negative, this method will not work. In any case, rounding to an integer is generally necessary and results in some error in fit in the mean time. The fit to the observed mean time is usually more important than the fit to the observed variance. The rounded D_i may now be used to determine a new $\hat{p}_{ii}$ from Eq. (6.41). Then the error in variance may be calculated from Eq. (6.42); if the error is

not excessive, these modified estimates of $\hat{p}_{ii}$ and D_i may be used in the modified Markov chain model with D_i auxiliary states.

A numerical comparison between the original simple estimating method and the auxiliary delay states method can be found in Section IX. A considerable improvement resulted from the addition of delay states in this example.

4. Adding Auxiliary States and Transitions

The delay method described in the last subsection was systematic, simple, and permitted the matching of two parameters to an observed or theoretical holding time density function. The method does not always work, however, and in any case the auxiliary delay states are not used efficiently. This inefficiency results from the simple form of the transitions among the delay states, for each delay state merely "holds" the process in state i for one more transition. When the fullest possible set of transitions are allowed between the auxiliary states, we may expect to use the given number of states more efficiently by selecting parameters to approximate given holding time densities. This is the approach taken in this section.

An example is used to illustrate the approach. Figure 6.1 shows an example in which three auxiliary states (i' , i'' , i''') and the indicated transitions allowed between them have been added to represent a holding time density $h_i(n)$, where $h_i(n) = P(\text{process makes a transition out of state } i \text{ at time } n / \text{state } i \text{ was entered at time zero})$.

For definiteness, suppose that all the conditional transition probabilities c_{ij} (conditioned on a state change from i to j) are given. These c_{ij} are determined from the usual transition probabilities p_{ij} by normalizing over the transitions out of state i

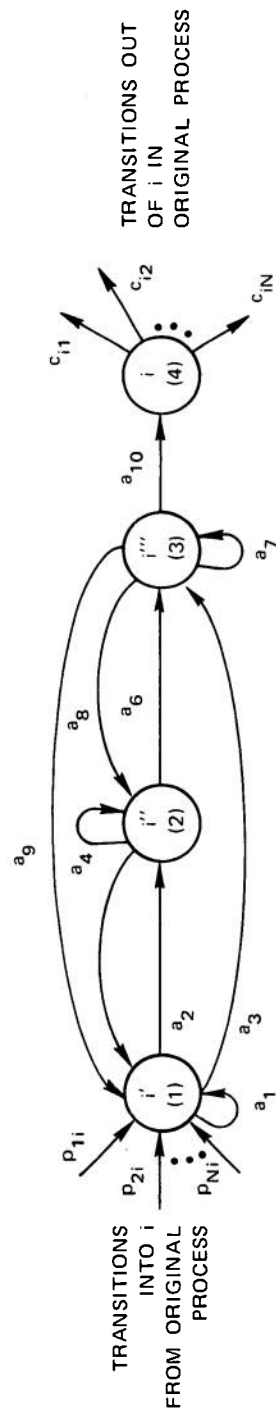


FIGURE 6.1 TRANSITION DIAGRAM WITH AUXILIARY STATES

$$c_{ij} = p_{ij} \sqrt{\sum_{\substack{k=1 \\ k \neq i}}^N p_{ik}} \quad \text{for } j = 1, 2, \dots, i-1, i+1, \dots, N \quad .$$

All inputs to state i (the original state) are sent to state i' , and states i'' and i''' are placed between i' and i as shown, with transition probabilities for all auxiliary branches being indicated by a_i in the diagram. The overall transition structure of the process is readily seen to be unchanged because the transitions from i to any other state j occur with the correct probability c_{ij} . The structure of the network of auxiliary states and transitions added to the original process determine the density of the waiting time.

Ten new parameters (a_1 through a_{10}) are shown on the diagram. However, the linear (row) constraints

$$\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ a_4 + a_5 + a_6 &= 1 \\ a_7 + a_8 + a_9 + a_{10} &= 1 \end{aligned} \tag{6.45}$$

reduce the number of free parameters to seven. By allowing these ten parameters to vary subject to the three row constraints and the usual nonnegativity conditions ($a_i \geq 0$), we may expect to realize a variety of shapes for the waiting time density. Several examples of densities are shown in Figure 6.2 for the parameter sets detailed on the figure.

Analytically, it is straightforward to determine the probability generating function $g_i(z)$ of the waiting time density. The transmission from state i' to state i is $g_i(z)$.

From Mason's formula in Appendix C, the graph determinant Δ of the auxiliary network is

$$\Delta = 1 + z(-a_1 - a_4 - a_7) + z^2(a_2a_5 + a_6a_8 + a_1a_4 + a_1a_7 + a_4a_7) + z^3(-a_2a_6a_9 - a_1a_6a_8 - a_2a_5a_7 - a_4a_7) \equiv 1 + Az + Bz^2 + Cz^3 \quad (6.46)$$

The numerator of the transmission is

$$a_3(1 - a_7 - a_8 - a_9)(1 - a_4)z + a_2a_6(1 - a_7 - a_8 - a_9)z^2.$$

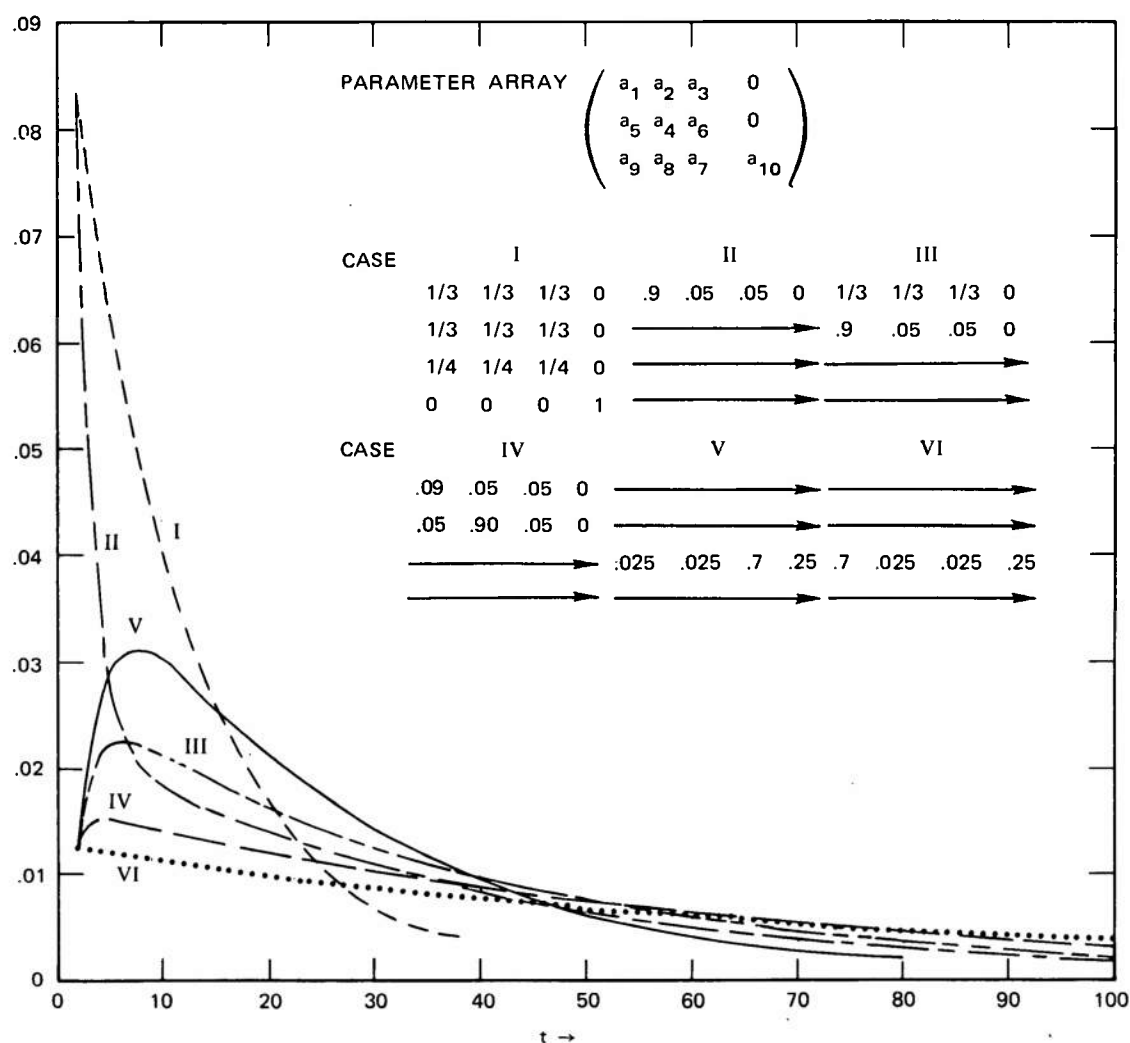


FIGURE 6.2 DENSITY FUNCTION OF WAITING TIME IN STATE i

Letting A, B, and C be the constants in the graph determinant as in Eq. (6.46), the transmission $g(z)$ is then the ratio

$$g_i(z) = \frac{[a_3(1 - a_4)z + a_2a_6z^2][1 - a_7 - a_8 - a_9]}{1 + Az + Bz^2 + Cz^3} \quad (6.47)$$

Expanding $g_i(z)$ in an infinite series in the variable z gives the density of the waiting time with the coefficient of z^n equalling the probability that the transition out of i occurs on the $(n+1)^{st}$ step. The expansion results in complex expressions for the coefficients and therefore does not offer a constructive method for finding the a_i parameters.

A straightforward method is available for choosing the a_i to approximate the holding time density $h_i(n)$. If the cumulative distribution of the waiting time is denoted by $H_i(n)$

$$H_i(n) = \sum_{m=0}^n h_i(m) = P(\text{holding time in } i \leq n) \quad , \quad (6.48)$$

then $H_i(n)$ is related to the probability of being absorbed in state 4(=state i) of the auxiliary network by

$$y_4(n) \equiv P(\text{absorption in state 4 by time } n) = H_i(n)$$

$$\text{for } n = 0, 1, 2, \dots \quad (6.49)$$

Therefore, the probability of occupying one of the states (the absorbing state 4) is given as a function of time. If the probabilities of occupying the other states as a function of time were also known, the least squares approach given earlier should suffice. To apply the methods of Section VI-C, estimates of the $P(\text{state})$ as a function of time relations $[y_i(n)]$ for states $i = 1, 2$, and 3 should be made such that $y_i(n) \geq 0$ for $i = 1, 2, 3$ and $\sum_{i=1}^4 y_i(n) = 1$ for all n . The form of these $y_i(n)$ should

result to zero, and solving for $a = \hat{a}$:

$$\hat{a} = \left(A^T W^{-1} A \right)^{-1} A^T W^{-1} y \quad . \quad (6.52)$$

This solution has the same potential difficulties as earlier least squares procedures; in particular, some a_i may be outside the interval (0,1). As before, Quadratic programming may be used to obtain feasible a_i .

The auxiliary state method may be used in many other situations to approximate holding time densities. The holding time density may be made to depend on the terminal state by adding auxiliary states between state i and state j . That is, auxiliary states and transitions are added for each branch of the original transition diagram instead of for each state.

Several destination states j may be handled simultaneously by the least squares approach above by introducing a single set of states to represent all the transitions out of state j . Given the transition probabilities for the imbedded chain, the $(N - 1)$ densities $h_{ij}(n)$ can be accumulated as in Eq. (6.48) to obtain the $y_j(n)$

$$y_j(n) = p_{ij} \cdot \sum_{m=0}^n h_{ij}(m) \quad j = 1, 2, \dots, N; j \neq i \quad . \quad (6.53)$$

The above least squares procedure can be used with these $y_j(n)$ ($N = 0, 1, 2, \dots, T$) as data. Varying numbers of auxiliary states may be tried until a satisfactory fit is obtained.

An example consisting of 3 states is sketched in Figure 6.3. States 1, 2, and 3 comprise the original process; states a and b are auxiliary states. All possible transitions between states 1, a , and b are allowed, but states 2 and 3 cannot be left once entered. Twelve

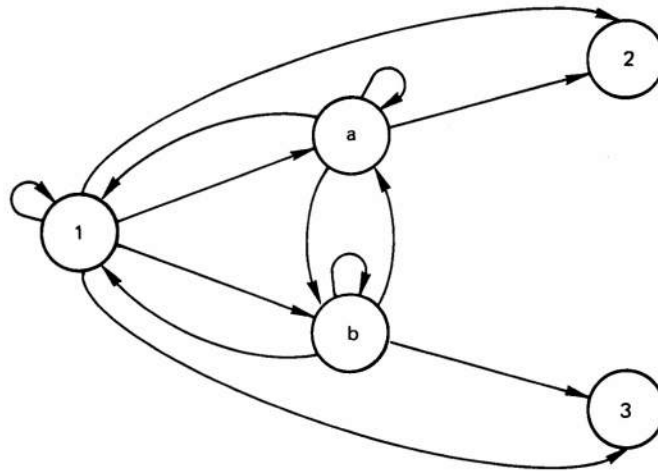


FIGURE 6.3 AUXILIARY STATES FOR TRANSITIONS OUT OF 1

transition probabilities and three row constraints are implied by the figure so that nine parameters are available to approximate the holding time behavior represented by the constants p_{12} , p_{13} , and the densities $h_{12}(n)$, $h_{13}(n)$.

E. Discrete Time Semi-Markov Processes

A generalization of a Markov chain is a discrete time semi-Markov process. In addition to the transition probabilities required to define a Markov chain, a semi-Markov process also requires the specification of a set of discrete conditional density functions $h_{ij}(n)$ to define the time of the transition from i to j , given that the transition from state i will be to state j . Thus, the transition out of i can be thought of as being made in two steps:

- The p_{ij} determine the next state (j) as for a Markov chain.
- The time of the transition (measured from the time state i is entered) is chosen randomly from the density function $h_{ij}(n)$.

It is important to note that each semi-Markov process also has implicit in it a Markov chain called the imbedded Markov chain. This

imbedded chain is determined by the transition probabilities p_{ij} alone. That is, the imbedded chain concerns the sequences of states entered by the process and ignores times of transition. The probabilities of absorption in Absorbing states are the same for a semi-Markov process and its imbedded Markov chain, but all dynamic quantities such as the density function of the time of absorption will in general be different.

Parameter estimation for the imbedded Markov chain of a semi-Markov process can be accomplished with Markov chain methods because the transition and the dynamic (time-dependent) behavior again factor as they did in a Markov chain. The first step in fitting a semi-Markov process to observed data is therefore to fit the imbedded chain; if this cannot be fit satisfactorily, the semi-Markov model is inappropriate. The next step is to fit the dynamic behavior; indeed, the approach of the previous subsection can be regarded as one systematic method to accomplish this.

Other methods may also be used for estimating the parameters of the $h_{ij}(n)$. Theoretical reasons may sometimes be found for assuming a functional form for the $h_{ij}(n)$; data may then be used to estimate such parameters as the mean and variance of the functional form. Adding auxiliary states to a Markov chain to approximate the semi-Markov process is equivalent to assuming a functional form. The advantage of the auxiliary state method is that the Markov chain formulas can continue to be used on the approximating chain.

Various other methods for estimating parameters in a semi-Markov model were investigated during the study. These results are available as internal memoranda and are therefore not included in this report.

F. Testing for the Markov Property

Statistical tests may be devised for testing the reasonableness of the Markov assumption with the classical chi-squared goodness of fit

test. One form of this test results from forming "cells" into which observations may fall and comparing the observed number of observations in each cell with the expected number of observations in each cell.⁷

The test statistic is

$$\chi^2_{i-1} = \sum_{i=1}^k \frac{(f_i - n\pi_i)^2}{n\pi_i}, \quad (6.54)$$

where

k = number of cells

f_i = observed number of observations in cell i

π_i = probability that an observation falls in cell i

n = number of observations.

The test statistic χ^2_{k-1} is asymptotically distributed as a chi-squared random variable with $(k - 1)$ degrees of freedom. When the fit is good, the deviations $(f_i - n\pi_i)$ are small; when the fit is poor, they are large. A confidence level (α) (such as 0.95) is arbitrarily set, and the hypothesis that the data are consistent with a Markov chain model is rejected if the value of χ^2_{k-1} from Eq. (6.54) exceeds the rejection limit found in chi-squared tables from the confidence level α and parameter k .

A modified version of this classical test can be applied to a Markov chain to obtain a test statistic

$$\chi^2_{(n-1)T} = \sum_{t=1}^T \sum_{i=1}^n N(t) [\bar{y}_i(t) - \hat{y}_i(t)]^2 / \hat{y}_i(t)$$

which is chi-squared with $(n - 1)T$ degrees of freedom when all $\hat{y}_i(t)$ are nonzero. Definitions of the quantities included are:

n = the number of states

$\hat{p}_{ij}$ = an estimate of p_{ij} ($i, j = 1, 2, \dots, n$)

$\hat{y}_j(t)$ = the predicted probability of being in state j at time t
given by

$$\hat{y}_j(t) = \sum_{i=1}^n y_i(t-1) \hat{p}_{ij} \quad (j = 1, 2, \dots, n)$$

$y_i(t)$ = observed proportion of times state i is occupied at time
 t ($t = 1, 2, \dots, T$)

$N(t)$ = number of observations at time t .

(For complete ASW exercises $N(t) \equiv N$, the number of exercises.)

VII IMPROVABILITY OF THE MEASURES OF EFFECTIVENESS

A. Introduction

The orientation to measures of effectiveness taken in this report is that of classical statistics: the measures are definite, fixed numbers exist in the abstract but are unknown. They can, however, be estimated from data--the analyst's task is to devise methods for obtaining estimators that are good in the statistical sense. A good estimator should have a small (preferably zero) bias, and its variance should be small. In most ASW applications, the estimators may be assumed to be unbiased (the mean or average value of the estimator is the true value) so that estimators may be judged by comparing their variances. For practical purposes the smaller the variance, the better the estimator.

This section addresses the question of improvability of the estimators of the four principal measures of effectiveness in terms of variance reduction when the model parameters must be estimated from fleet operational data alone. It will be shown that:

- (1) The probability of success estimator given by the ratio (successes/trials) is essentially unimprovable.
- (2) The mean time until failure, given that failure occurs, is improvable by a factor roughly equal to the probability of failure.

The manner in which these conclusions were reached must be discussed to properly qualify them. The first conclusion results from a theorem (proved in Appendix E) essentially stating that the "cancellation" noted in conditional probability models (see Section III) also occurs in any Markov model. The second conclusion results from a second theorem, also proved in Appendix E.

B. Approximating an Arbitrary Model by a Markov Model

In view of the material in the previous sections on parameter estimation, it seems reasonable to assume that a Markov chain model can be constructed to approximate a wide variety of other kinds of models. For example, a Monte Carlo simulation model may be regarded as a Markov model whose states are defined in terms of the model's variables. The description of the state space would be extremely difficult for large models owing to the very large number of states. There are so many states that a given state has almost no chance of being reentered on any given replication. Nevertheless, the basic structure is Markov and the number of states is finite.

The first step in the argument, then, is to assert that the arbitrary effectiveness model being considered can be satisfactorily approximated by a Markov chain model with a finite number of states.

C. Applying Theorem 1

When transition data are available to estimate the parameters of the approximating Markov chain, the only estimating method needed is the simple estimator

$$\hat{p}_{ij} = \begin{cases} n_{ij}/n_i & \text{for } n_i > 0 \\ 0 & \text{for } n_i = 0 \end{cases} \quad (7.1)$$

where the n_{ij} are the observed numbers of transitions from state i to state j and the n_i are the sums $\sum_j n_{ij}$. In the section where this estimator was introduced, it was stated that the quantities $\hat{p}_{ij}$ are optimal according to both the maximum likelihood and the minimum chi-squared criteria; furthermore, they are unbiased and of minimum variance among

the unbiased estimators. It is therefore unlikely that the $\hat{p}_{ij}$ can be improved in the statistical sense. Thus, it appears reasonable that the measures of effectiveness calculated from the approximating Markov model when the true parameters p_{ij} are replaced by their estimates $\hat{p}_{ij}$ [obtained from fleet operational data by using Eq. (7.1)] will be as good as or superior to other estimators in the statistical sense. This is the second step in the argument. [It is recognized that the second step lacks rigor: the $\hat{p}_{ij}$ are not necessarily maximum likelihood estimators for the overall (measures of effectiveness) parameters, and maximum likelihood estimators are not always minimum variance estimators.]

The third step of the argument leading to the conclusions (1) and (2) results from the respective application of the theorem and the conjecture. The theorem states that the absorption probabilities obtained from the Markov chain calculation $\hat{p}_s = [(I - Q)^{-1}R]_{1s}^*$ when the parameters of Q and R are the estimators $\hat{p}_{ij}$ from Eq. (7.1) are identical to the ratios

$$\bar{p}_k = \frac{\text{number of times absorbed in state } k}{\text{number of trials}},$$

provided that the process always starts in state 1 and ends in some absorbing state. In operational terms, all ASW exercises start in a designated starting state and continue until the outcome of the exercise is resolved, that is, until an event occurs to terminate the exercise according to rules agreed upon ahead of time.

The second theorem is similar in form to the theorem 1 but concerns the mean time to failure, given that failure occurs. However, a crucial

* The subscripts (1,s and 1,f) denote the element in the first row and the s^{th}, f^{th} columns respectively of the indicated matrix.

change is needed in the estimation procedure. Instead of all the data being used to estimate the p_{ij} , only the data for paths that terminate in failure are used. The conjecture is: the mean time until fail (given that failure ultimately occurs) calculated from the Markov formula^{*}

$$T_f = [(I - Q)^{-1}(I - Q)^{-1}R]_{1f}/a_{1f}$$

is identical to the ratio

$$\overline{T}_f = \frac{\text{sum of the numbers of transitions before Failure occurs}}{\text{number of trials on which Failure occurred}}$$

D. Implications of the Theorem 1

The implications of theorem 1 are numerous and can be viewed in both positive and negative senses. On the negative side, the theorem implies that to construct a Markov chain model to obtain an improved estimate of $P(\text{success})$ is entirely futile when only data obtained from complete (start to finish) exercises can be used because the result is identical to the simpler estimate (number of successes/number of trials). At the outset of the current research, this limitation was not recognized; the theorem came as a surprise and somewhat changed the direction of the research effort.

Viewed positively, some implications of the theorem are:

- The probability of success estimator can be improved (in the variance reduction sense) if other data are available to improve the estimates of the p_{ij} .
- The number of states in the Markov model and the manner in which they are defined are entirely arbitrary concerning the estimation of p_s . State definitions may therefore be made based on the need to obtain better estimates of the other measures of effectiveness.

- The Markov property--apparently required to derive the formula for p_s --is not at all required. That is, the estimator $(I - Q)^{-1}R$ is very "robust" in that it is completely insensitive to the actual nature of the random process generating the sequences of states.
- It also follows that postexercise analysis of states and their transitions is not a critical matter because state definitions are arbitrary. In particular, how the inevitable borderline cases are treated in analysis is unimportant so long as each trial traced a path from the starting node to the correct terminal node.
- Arguments sometimes used to show the need for "aggregating" states to obtain sufficient data for estimating transition probabilities are invalid for the probability of success measure--the number of states and their definitions are entirely arbitrary, and the variance of the $\hat{p}_s$ estimator is independent of the state space.

Many of the same implications hold for the second theorem regarding the conditional mean time until fail measure. Overall, the implications are more favorable because the mean time estimator uses only part of the data available (those paths leading to failure); the remainder of the data can be used to improve the estimators $\hat{p}_{ij}$ and hence the conditional mean time till failure $\hat{T}_f$ derived from the $\hat{p}_{ij}$. The extent to which the Markov property is required for improving the p_{ij} is currently unknown, although experimental work discussed in a later section appears to indicate that this property is not required. All that is apparently required is homogeneity in the sense that the transitions in the transient portion of the Markov chain model should be generated by a random process independent of where the process is finally absorbed.

The factor by which the variance of $\hat{T}$ can be reduced should be roughly p_f = probability of fail. This follows because p_f is the approximate fraction of the data that is used to estimate $\hat{T}_f$ and the variance of the sample mean $\bar{T}_f$ is σ_T^2/N , where σ_T^2 is the true variance of the time to fail (T_f) and N is the number of trials. Experimentation with numerical

examples in non-Markovian situations with the required homogeneity properties may shed some light on this variance reduction question.

VIII PROBABILITY STATE TIME DIAGRAM

This study largely concerns the development of methods for applying Markov, semi-Markov, or modified Markov methods to real operational problems. Because any real operational problems are unlikely to be exactly Markovian, it is necessary to demonstrate that Markov methods can be used in non-Markovian situations.

Three main possibilities to experiment with to obtain data from non-Markovian processes are:

- Operational data from ASW exercises
- Simulated data from digital computer simulation runs
- Artificially constructed problems.

Operational data of the type needed for this study are either unavailable in a convenient form or not available at all. Transition and dynamic data (such as mean times between states and transition counts n_{ij}) available from sources such as the UPTIDE analysis volumes are time-consuming to extract; therefore, the use of operational data was ruled out for the current study. Data from simulation are easier to obtain, but their use would make analysis of Markov methods application more difficult because the true probabilistic dynamic structure of the simulation model is invariably unknown. Thus the third alternative, the use of artificially constructed dynamic probabilistic problems, was adopted.* Although the problems constructed for study had their own special strongly non-Markovian structure, methods that did not exploit it were always sought.

* Some effort was also devoted to applying the methods to data from a simulation model (see Section XII).

That is, the study of artificial problems with special structure was hopefully expected to find methods of more general applicability.

To create artificial dynamic probabilistic problems sufficiently complex to test the adequacy of Markov approximation and sufficiently simple to make reasonably straightforward calculations of the true values of the quantities of interest is no simple matter. After some experimentation, a suitable method was found: a dynamic probabilistic model is defined by a Probability State Time (PST) diagram. From the PST diagram the many conditional probabilities, probabilities of state as a function of time, and dynamic quantities such as the mean and variance of the transition time between selected pairs of states can be readily determined in an easily understandable way.

The concept of defining a random process by a diagram is basically simple: assume that the process depends upon a single random variable u , which is assumed to be uniform on the interval $(0,1)$. This variable is to be associated with the vertical dimension on the diagram. The horizontal dimension is time (t), and the state changes are specified merely by drawing a line in the (u,t) plane. Thus, a random process including the states of opportunity, detection, classification, attack, and kill is defined in Figure 8.1. A sample function of the random process, which corresponds to an ASW tactical encounter or exercise, is now determined from the diagram by choosing a value of u at random, and drawing a horizontal line u units from the top of the rectangle. Whenever this line intersects one of the lines defining a state change, a transition into the designated state occurs and the time at which the transition occurs is the corresponding value of t . Thus, line ① represents the sample function for which detection occurs at time t_1 , classification occurs at time t_2 , attack occurs at time t_3 , and failure occurs at time t_4 . Note that kill does not occur because the K line representing a transition

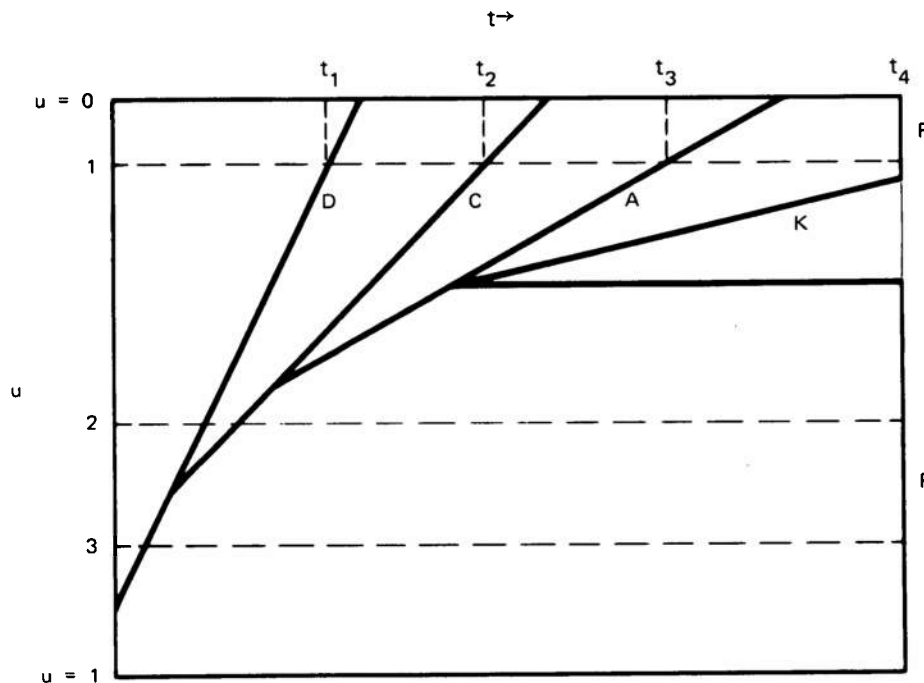
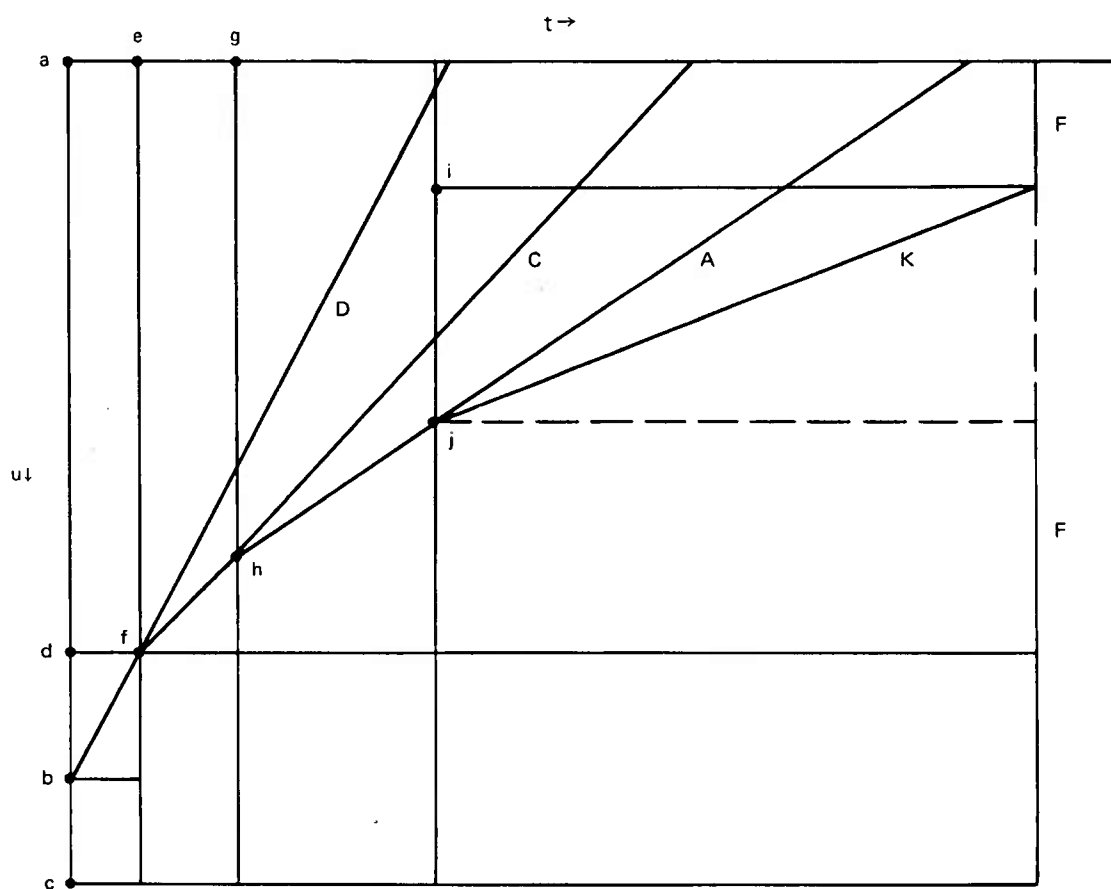


FIGURE 8.1 ILLUSTRATIVE PROBABILITY STATE TIME DIAGRAM

into the kill state is not intersected by line ①. From $t = 0$ to $t = t_1$, the state is opportunity; from t_1 to t_2 , the state is detect; in general, the most recent state change determines the state at time t , which is the first intersection to the left of the point on the PST diagram. The sample function represented by line ② detects and classifies but does not attack and therefore fails. The sample function for line ③ detects but does not classify and therefore fails. All fails in this example occur at some fixed time large enough to represent the maximum length of an exercise. (The instructions to the hypothetical exercise participants were to continue trying to achieve mission success by proceeding through the states in sequence until this maximum time, with failure resulting if kill has not occurred by the maximum time.)

The slanted horizontal lines representing state changes in Figure 8.1 represent conditional uniform distributions. That is, the time of detection is uniformly distributed given that detection occurs,

Conditional probabilities are easily read from the PST diagram by taking ratios of lengths of appropriate line segments. The probability of detection (given opportunity, represented by the vertical line at $t = 0$) is the ratio ab/ac as shown on Figure 8.2, while the other conditional



122

probabilities are

$$P(C/D) = ef/ab$$

$$P(A/C) = gh/ef$$

$$P(K/C) = ij/gh$$

$$P(K/O) = ij/ac.$$

Many other conditional probabilities, including those dependent on conditioning on knowledge of a state at an earlier time, can be found similarly.

A random process defined by a PST diagram has a special structure because the probability densities for the transition times are not independent. (The independence of these densities for Markov and semi-Markov processes is one of the most convenient properties of a Markov model.) Interdependence is expected to be one of the major difficulties to be faced in applications of Markov methods so that experimenting with Markov methods on problems with some form of interdependence was considered appropriate.

Even a Markov chain can be approximated by a PST diagram, however. Figure 8.3 is the transition diagram of a six-state Markov chain, with transition probabilities shown on the branches representing the allowed transitions. In Figure 8.4, a PST diagram shows the density function for

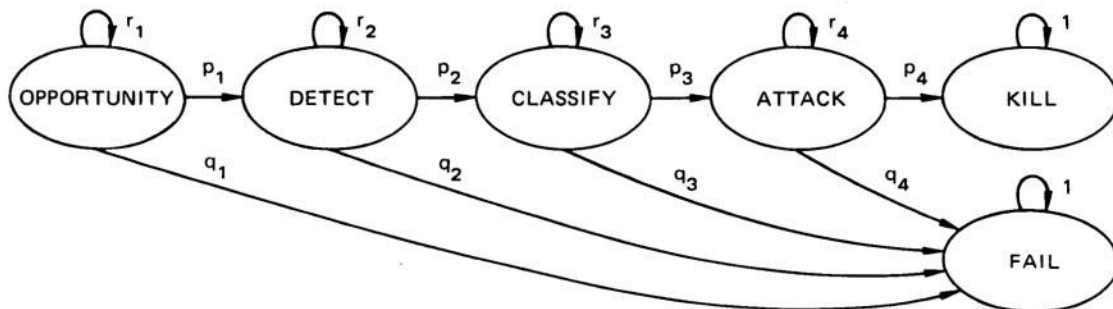


FIGURE 8.3 MARKOV PROCESS WITH SIX STATES

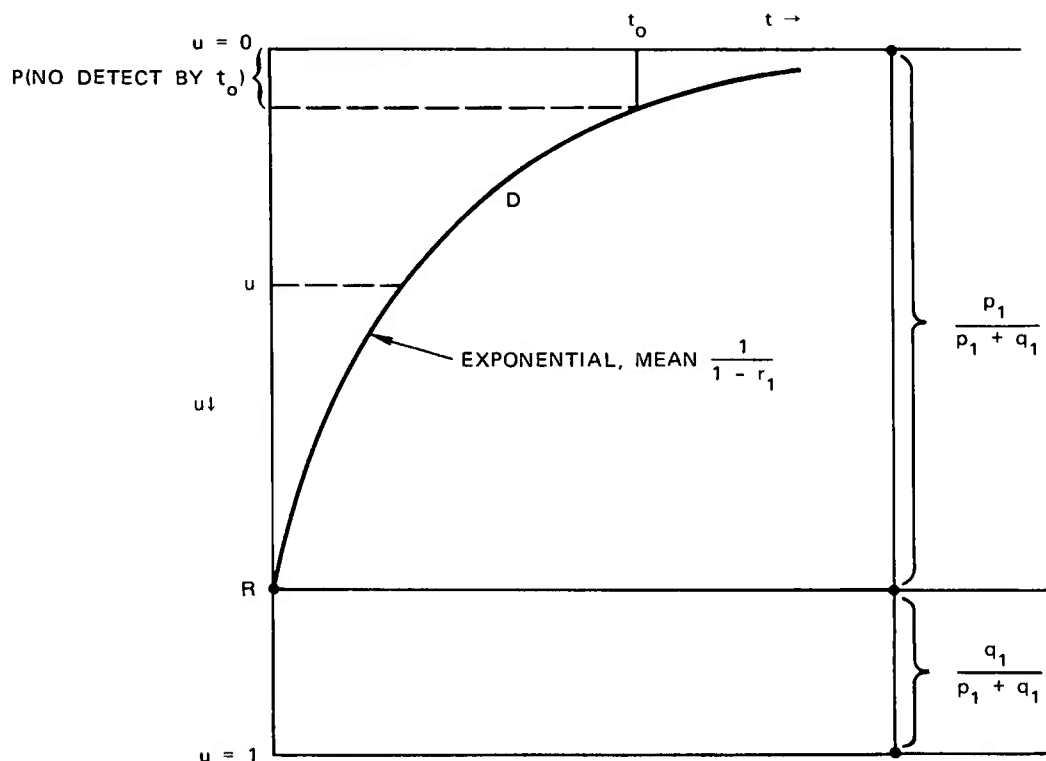


FIGURE 8.4 PROBABILITY STATE TIME DIAGRAM FOR FIRST STATE OF PROCESS IN FIGURE 8.3

the transition from Opportunity to Detect. (The line is drawn for a continuous exponential distribution instead of a discrete geometric distribution for convenience.)

Since the time from detect till classification in the Markov chain is geometrically distributed given detection (i.e., the time in the detect state is geometrically distributed), a curve shaped like that in Figure 8.4 must be repeated once for each value of u so that the density of time to classify--given detection--can be independent of the time to detect--given opportunity. This is impossible geometrically so an approximation must be used. The detection curve can be approximated by a number of vertical line segments and a classification curve can be drawn for each such vertical line segment as shown in Figure 8.5. The resulting set of lines can approximate the true transition lines as accurately as

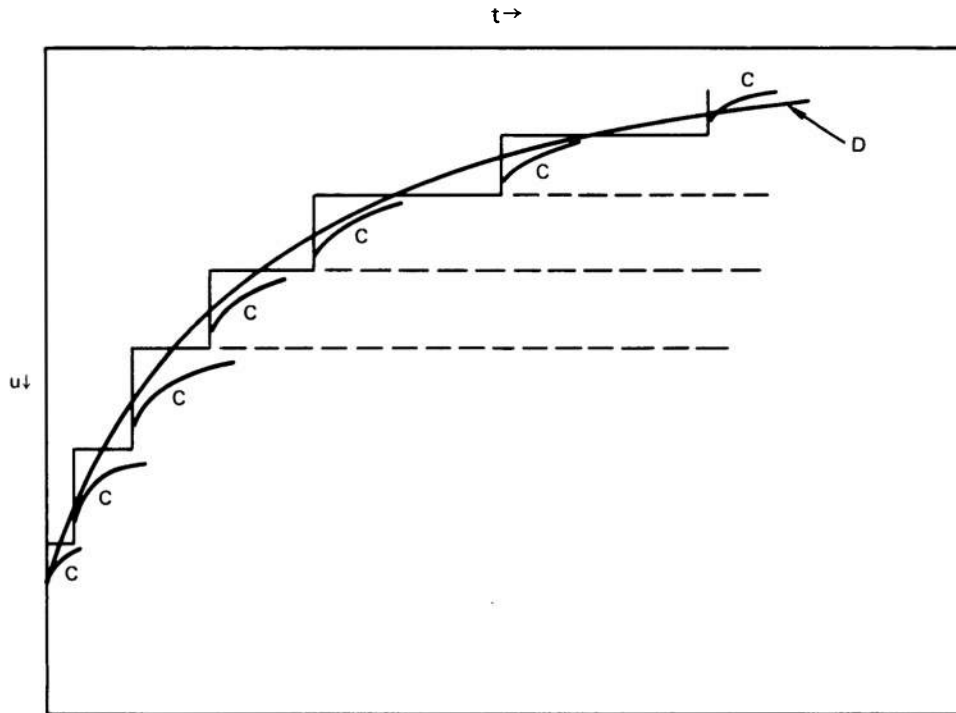


FIGURE 8.5 DETECT LINE QUANTIZED; CLASSIFY EXPONENTIALS ADDED IN EACH SEGMENT

desired by choosing intervals sufficiently small. Obviously, this procedure must be repeated again for later states, with the subdivisions getting finer and finer and the curves for later transitions becoming increasingly more wild. It is not proposed here that these approximations actually be carried out. Figure 8.5 is shown merely to demonstrate that a Markov chain can be approximated by a PST diagram.

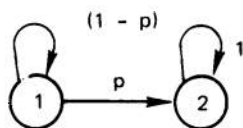
Just as each Markov process has a Markov chain imbedded within it, a PST diagram has a random process imbedded within it. In general, the imbedded PST process is not Markovian. (In some cases, as when the imbedded chain has no loops, it will be Markovian. The first example studied in this report has an imbedded chain that is Markovian, but the second example does not.)

It may be mentioned that in certain respects a random process defined by a PST diagram is more general than a Markov chain, and in other respects is less general. The correspondence between the two classes of processes is best seen by thinking of paths through the states as the basic elements, not the states and the transition probabilities.

A Markov chain containing loops has infinitely many paths, each with probability equal to the product of the probabilities along the path. The PST-defined process has finitely many paths, and each of these paths has a probability determined by the diagram.

If a PST diagram had infinitely many paths, it would be a Markov chain if the assigned probabilities for each path could be represented as products of probabilities along the respective paths. In general this would be impossible to do, implying that the class of PST-defined processes with infinitely many paths is richer than the class of Markov chains over the same set of states. On the other hand, if a Markov chain is limited to a finite number of paths on the grounds that all finite problems necessarily end in a finite period of time, then the PST-defined process is somewhat less general.

An example will clarify the comparison. Consider the two-state process shown in the sketch below. The path probabilities for three



selected short paths are:

Path	Probability	
	Markov	PST
11	$(1 - p)^2$	p_1
12	$(1 - p)p$	p_2
122	p	p_3

The probabilities of the three paths may be independently assigned for a PST definition; let their values be p_1 , p_2 , p_3 as shown. Then the processes can be the same only if the probabilities for these paths agree, that is,

$$(1 - p)^2 = p_1$$

$$(1 - p)p = p_2$$

$$p = p_3$$

If these equations have a solution, it must be $p_3 = p$, $p_1 = (1 - p_3)^2$, and $p_2 = (1 - p_3)p_3$ so that p_3 determines p_1 and p_2 . Hence p_1 , p_2 are not arbitrary and the Markov chain is seen to be more restricted than the PST-defined process.

There are further computational advantages of the PST diagram method for defining a random process. Measuring ratios of line segments has shown that certain conditional probabilities are easily determined, and other conditional probabilities will be found determinable by essentially the same method. The condition will usually define a subset of the probability space which is composed of the union of a number of disjoint line segments, and the event of interest will be yet another union of disjoint line segments. The conditional probability is then the ratio of the total length of the line segments.

The dynamic structure of the model, primarily represented by the set of probabilities of occupying a state as a function of time, is easy to determine from a PST diagram. To determine the probability of being in the detect state at time t , for example, requires summing the lengths of all vertical line segments at t which represent the detect state.

Since the total length (over all states) of the line segments is unity, the sum of lengths gives the desired probability directly.

Certain kinds of dependence can be incorporated into a problem defined by a PST diagram. When compensating factors that tend to make the sum of two successive transition times a constant, for example, the horizontal distance between the two state change lines is that constant. An operational example illustrating this sort of dependence might be the dependence of time to classify, given detection, for detections delayed by an inattentive sonar operator. If detection occurs late because of lack of attention to the sonar scope, the signal to noise ratio may be high at detection, implying a shorter time to classify than normal detections. Thus, the sum (time to classify, given detection) + (time to detect, given opportunity) may tend to be nearer a constant than an assumption of independence of stages would imply.

Although it is not being suggested here that actual ASW operational problems are of the type defined by PST diagrams, it is possible to take a (necessarily finite) set of real operational data (or simulated data) to form the analogue of a histogram in PST form. Data required are a set of paths, each consisting of a sequence of states in the order they were entered and the transition times for each path. Each path could then be represented by a series of dots on a horizontal line, as in Figure 8.6. The probability variable u has no value to associate with a path, which poses a problem for plotting the vertical dimension. If all paths are considered equally likely, the paths should be separated by the same distance, with the ordering remaining indeterminate. In seeking structure in the process, one can experiment with various ways of ordering the paths. (If structure can be found, the interesting problem of interpreting the probability dimension could then be undertaken. Intuition suggests that the probability variable is in some sense a generalized environmental

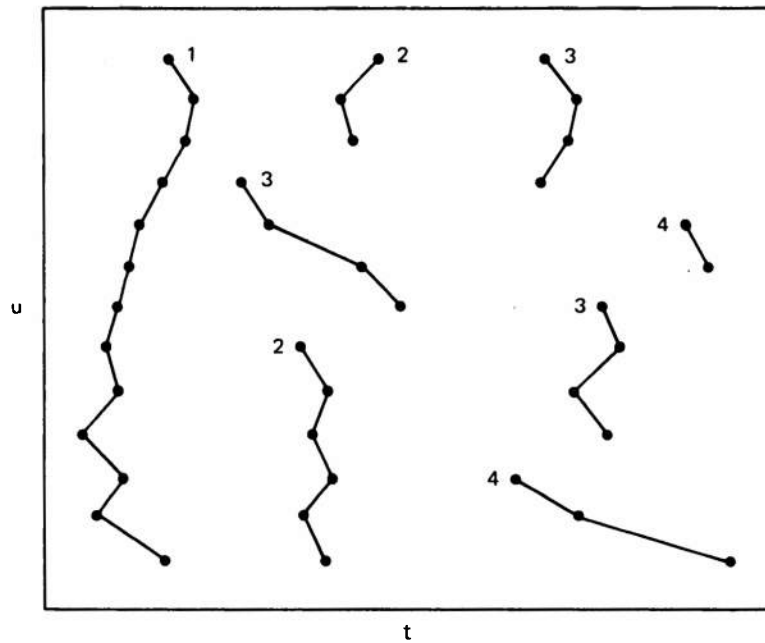


FIGURE 8.6 HISTOGRAM FOR A PROBABILITY STATE TIME DIAGRAM FROM A FINITE SET OF DATA

variable.) Figure 8.6 shows a hypothetical example consisting of 13 sample paths for a process with four states. Dots represent data points and connecting lines represent a possible form for the full problem.

IX NUMERICAL EXPERIMENTATION: EXAMPLE ONE

A. An Artificial Process

In this section, a six-state random process is defined by a PST diagram so that various methods of estimation may be tried and compared. The structure is quite simple (no loops are allowed); each transition must be either to the next state on the path to the kill state or to fail. Figure 9.1 shows the PST diagram. The vertical line at the left represents the opportunity state at $t = 0$. Transitions to intermediate states detect, classify, and attack are to the right, and the rightmost lines represent entry into one of the two trapping states kill and fail. Because of its choice of states and lack of loops, this example may be considered to represent the classical conditional probability model of ASW. The imbedded chain is a Markov chain, but the entire process is strongly non-Markovian by its very construction.

In this artificial example, no attempt was made to choose realistic values for the time required to change from state to state. As can be noted from the form of the diagram, all distributions representing times between states are conditionally uniform. For example, the time to detect (give that detection occurs) is uniform from time 0 to time 20 units. The time to classify, given that classification occurs, is also uniform from 0 to 20 time units. A maximum time of 20 units in any nontrapping state is allowed and thus implies that failure occurs if the next state is not reached in 20 time units.

Conditional probabilities implied by the diagram are readily found to be those in Table 9.1. This set of transition probabilities and the possible transitions are also shown in Figure 9.2.

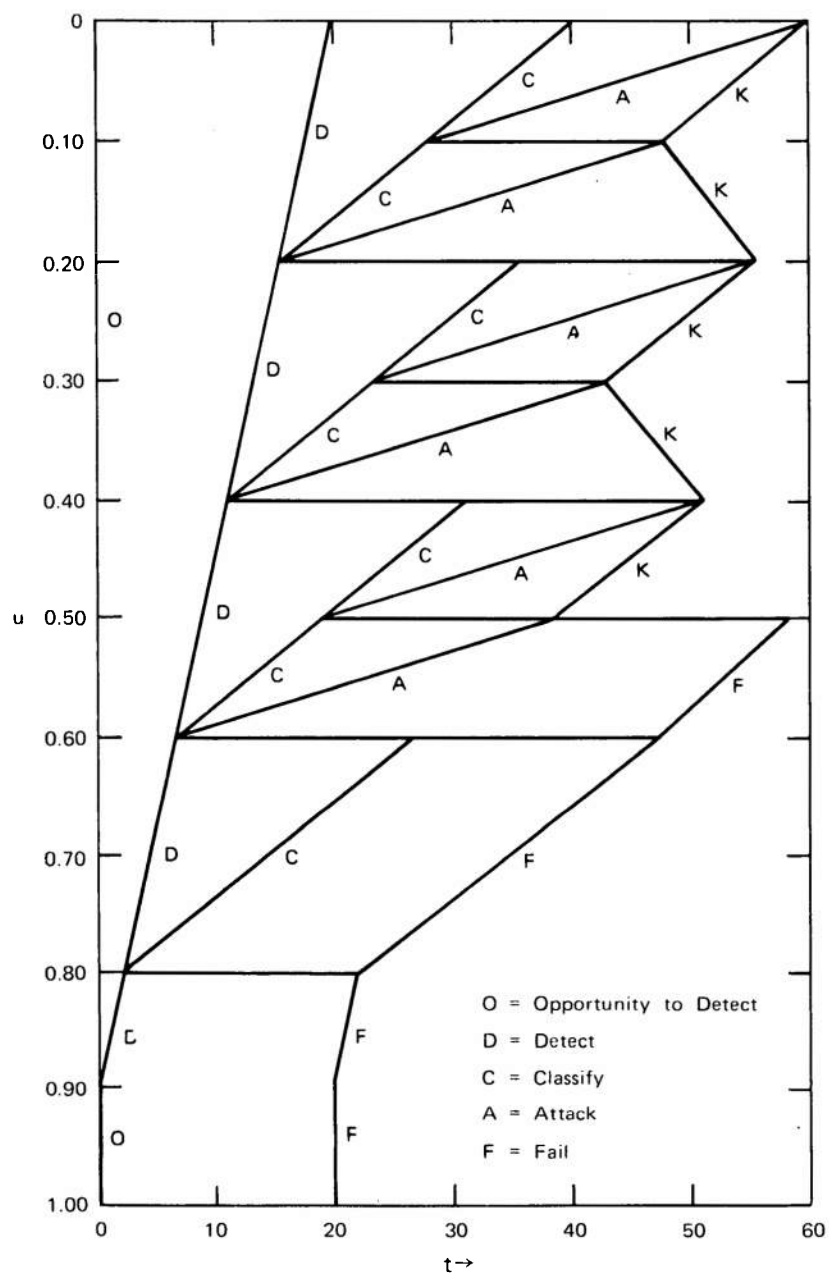


FIGURE 9.1 PROBABILITY STATE TIME DIAGRAM FOR AN ARTIFICIAL STOCHASTIC (NON-MARKOV) PROCESS WITH STATES O, D, C, A, AND F

Table 9.1

TRANSITION PROBABILITIES DERIVED FROM FIGURE 9.1

$P(D/O) = 9/10$ (Detection occurs for $0 \leq u \leq 0.9$)
 $P(C/D) = 8/9$ (Classification occurs for $0 \leq u \leq 0.8$)
 $P(A/C) = 6/8$ (Attack occurs for $0 \leq u \leq 0.6$)
 $P(K/C) = 5/6$ (Kill occurs for $0 \leq u \leq 0.5$)
 $P(F/O) = 1/10$ ($0.9 \leq u \leq 1.0$)
 $P(F/D) = 1/9$ ($0.8 \leq u \leq 0.9$ out of $0 \leq u \leq 0.9$)
 $P(F/C) = 2/8$ ($0.6 \leq u \leq 0.8$ out of $0 \leq u \leq 0.8$)
 $P(F/A) = 1/6$ ($0.5 \leq u \leq 0.6$ out of $0 \leq u \leq 0.6$)

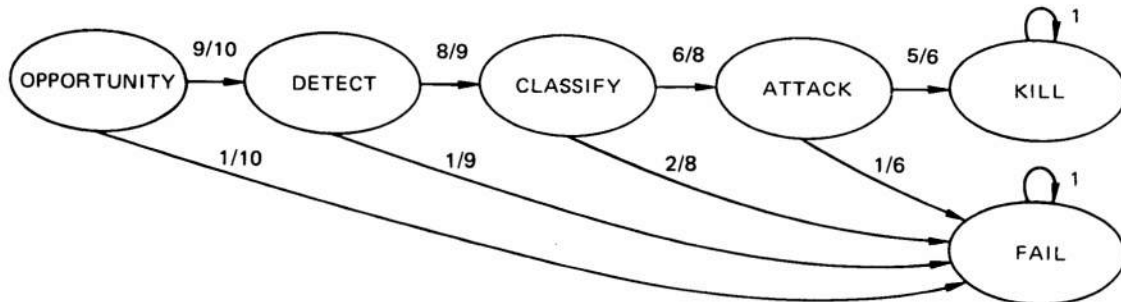


FIGURE 9.2 TRANSITION DIAGRAM FOR THE IMBEDDED PROCESS WITH CONDITIONAL PROBABILITIES FROM FIGURE 9.1

Any exercise starting in opportunity and ending in kill or fail can be thought of as a path through this Markov diagram; transitions from state to state can be tabulated for estimating conditional probabilities, and these estimates can be used as in the WSE model to estimate the probability of kill given opportunity. The probability of kill given opportunity is the product of the conditional probabilities along the single

path from opportunity to kill. When the true transition probabilities are used, the estimator for $\hat{p}_k$ gives

$$P(K/O) = \frac{9}{10} \times \frac{8}{9} \times \frac{6}{8} \times \frac{5}{6} = 5/10 = 0.5 \quad . \quad (9.1)$$

Examination of the PST diagram in Figure 9.1 shows that Kill occurs when $0 \leq u \leq 0.5$ and Failure occurs otherwise; hence, the conditional probability estimator in Eq. (9.1) is exact (when infinite data are available) since the distribution of u is assumed uniform on $(0,1)$. It appears, therefore, that a Markov model of this process has been successfully determined since all conditional probabilities and the measure of effectiveness $P(K/O)$ are correct.

B. The Dynamics of the Process

Thus far the behavior of the process in time has not been considered. From the PST diagram of Figure 9.1, the probabilities of occupying each state as a function of time can be determined by drawing a vertical line at time t and measuring the lengths of the line segments representing each state along this vertical line. The resultant probabilities of state as a function of time are shown in Figure 9.3 for all six states for all times from 0 to 60.

Several attempts have been made to fit a dynamic Markov model to this non-Markovian process by a variety of methods. The methods are to estimate by:

- (1) Matching average time-in-state, unconditionally.
- (2) Matching average time-in-state, conditioned on the next state.
- (3) Introducing additional states to represent delays.
- (4) Using actual $P(\text{state})$ as a function of t and least squares.

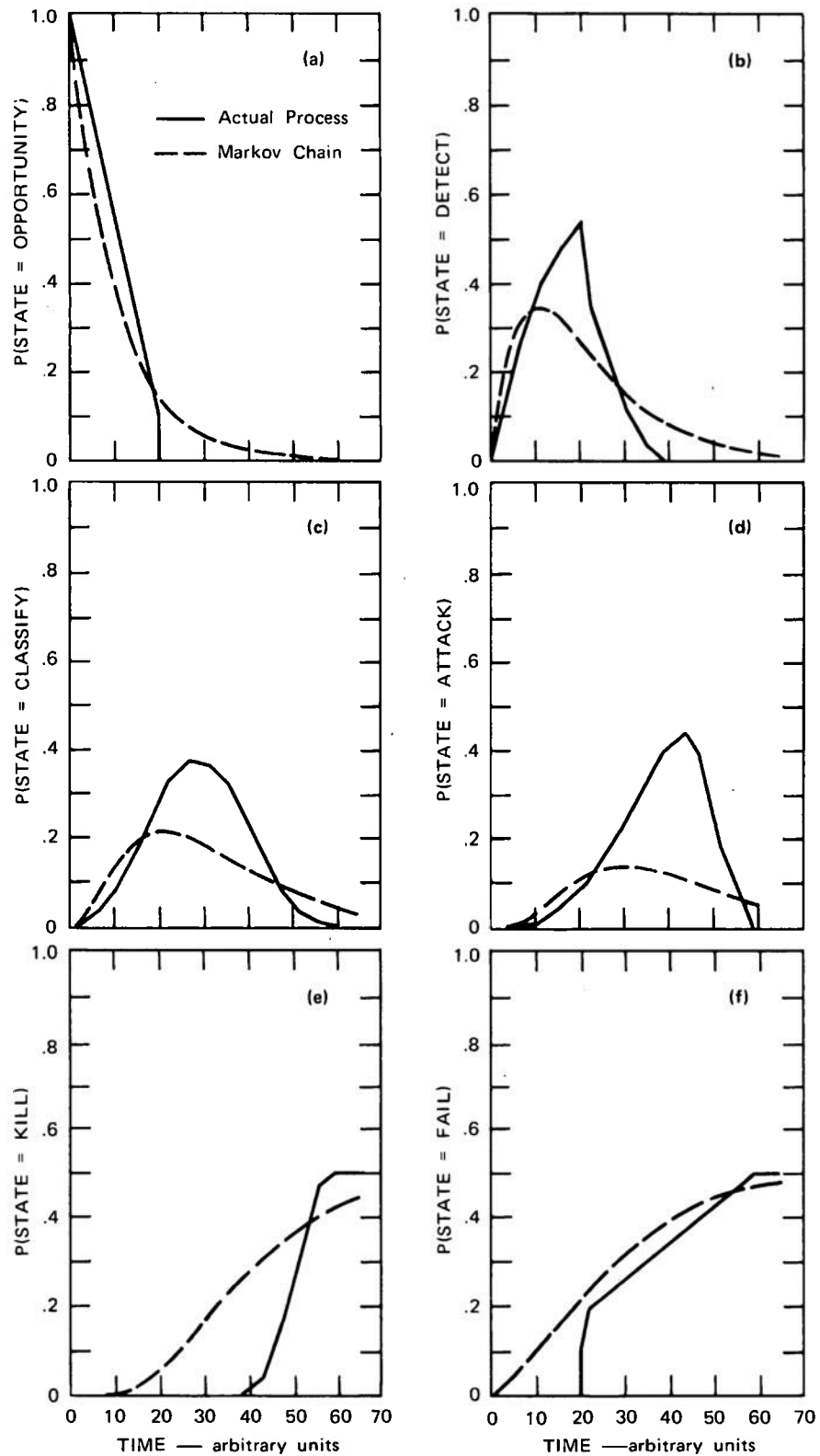


FIGURE 9.3 COMPARISON OF $P(\text{STATE})$ AS A FUNCTION OF TIME CURVES FOR THE ACTUAL PROCESS AND MARKOV APPROXIMATION

- (5) Fitting a semi-Markov process, after introducing additional states as needed.
- (6) Defining a new process whose states incorporate time in the state definition.

These methods will be discussed in turn. In all cases, infinite data are assumed to avoid dealing concurrently with sampling problems and structural problems.

C. Six Markov Approximations to the Actual Process

1. Matching Average Time Spent in a State

In Section VI, the average time spent in a state i with self-transition probability p_{ii} was shown to be the quantity $1/(1 - p_{ii})$, which when inverted results in $p_{ii} = 1 - 1/\bar{T}_i$ --where $\bar{T}_i$ is the average time spent in the state before exit. Transition probabilities for the imbedded process are then each multiplied by $(1 - p_{ii})$ to normalize them. Average times-in-state ($\bar{T}_i$) are readily calculated from the PST diagram, and the p_{ii} and normalization calculations give the transition matrix shown in Table 9.2 for the approximating Markov chain. With this set of parameters, then, the approximating Markov chain model has the following properties:

- The conditional probabilities are exactly the same as the actual process of Figure 9.1, and the probabilities of kill and fail are the same.
- The average time in each state (except fail and kill) is the same as the average time for the actual process.

To determine $P(\text{state})$ as a function of time, P^n can be calculated by using values from Table 9.2; the results are plotted on Figure 9.3. A comparison of the true values with the Markov chain values shows that merely matching average times-in-state is insufficient for a good dynamic fit to the actual non-Markovian process.

Table 9.2

TRANSITION PROBABILITY MATRIX

i/j	Opportunity	Detect	Classify	Attack	Kill	Fail
Opportunity	0.90909	0.081818	0	0	0	0.009091
Detect	0	0.91	0.08	0	0	0.01
Classify	0	0	0.9	0.075	0	0.025
Attack	0	0	0	0.933757	0.055202	0.011041
Kill	0	0	0	0	1	0
Fail	0	0	0	0	0	1

General features of the dynamic fit, which tended to persist for other Markov attempts also, are:

- The actual random process may have sharp cutoffs at points where the probabilities reach their final values, but the Markov approximations reach their final values asymptotically.
- The shapes of the approximating curves are reasonable but lack the sharp corners of the actual process.
- The approximating $P(\text{state})$ as a function of t curves usually leads the actual curves for small t , cross over the actual curves, then lag the actual curves thereafter.
- Appreciable probability remained in the transient states at $t = 60$ (the point at which all probability has been absorbed in the actual process).

2. Matching Average Times in a State, Conditioned on the Next State

The method given above has the disadvantage that the average waiting times in a state are computed by averaging over all states to which transitions are made. Thus, the average time in the opportunity state is

$$0.9 \times (\text{average waiting time in opportunity} | \text{detect occurs}) \\ + 0.1 \times (\text{average waiting time in opportunity} | \text{fail occurs}) \quad .$$

Since fail is an absorption state, it is obvious that if only the average waiting time in opportunity--given that detect occurs--is used as the waiting time, the new p_{ii} should better approximate the dynamics.

Carrying out the new average time calculations and repeating the process used for the first method resulted in a fit that was only

slightly better than that of Method (1), however. [The improvement was so slight that the dynamic response for Method (2) was not plotted.]

3. Introducing Delay States

In Section VI, it was suggested that the waiting time density of an approximating Markov chain could better approximate the true density if delays were introduced to permit fitting both the mean waiting time and its variance instead of just the mean. The methods of that section were used to determine numbers of delay states D_i ; the new transition matrix was raised to successive powers to determine $P(\text{state})$ as a function of time once again. Figure 9.4 shows the results and compares them with the values from the actual process. The opportunity curve is somewhat improved [over the curve of Method (1)], detection and classification are considerably improved but still lead the actual curves, attack is improved but leads the actual curve and the peak probability is far too small, and kill is somewhat improved. The final state (fail) has a somewhat poorer fit than that obtained by Method (1).

4. Using Actual $P(\text{State})$ as a Function of Time and Least Squares

A fourth method of estimating the p_{ij} was a modified version of the least squares model in Section VI-C-5. The modification consisted of using $P(\text{state})$ as a function of time data for $t = 0, 5, 10, \dots, 60$ so that the transition probabilities being estimated were the five-step probabilities $p_{ij}^{(5)}$. The difference equations used were:

$$y_1(n) = y_1(n-1)p_{11} \quad (9.2)$$

$$y_2(n) = y_1(n-1)p_{12} + y_2(n-1)p_{22} \quad (9.3)$$

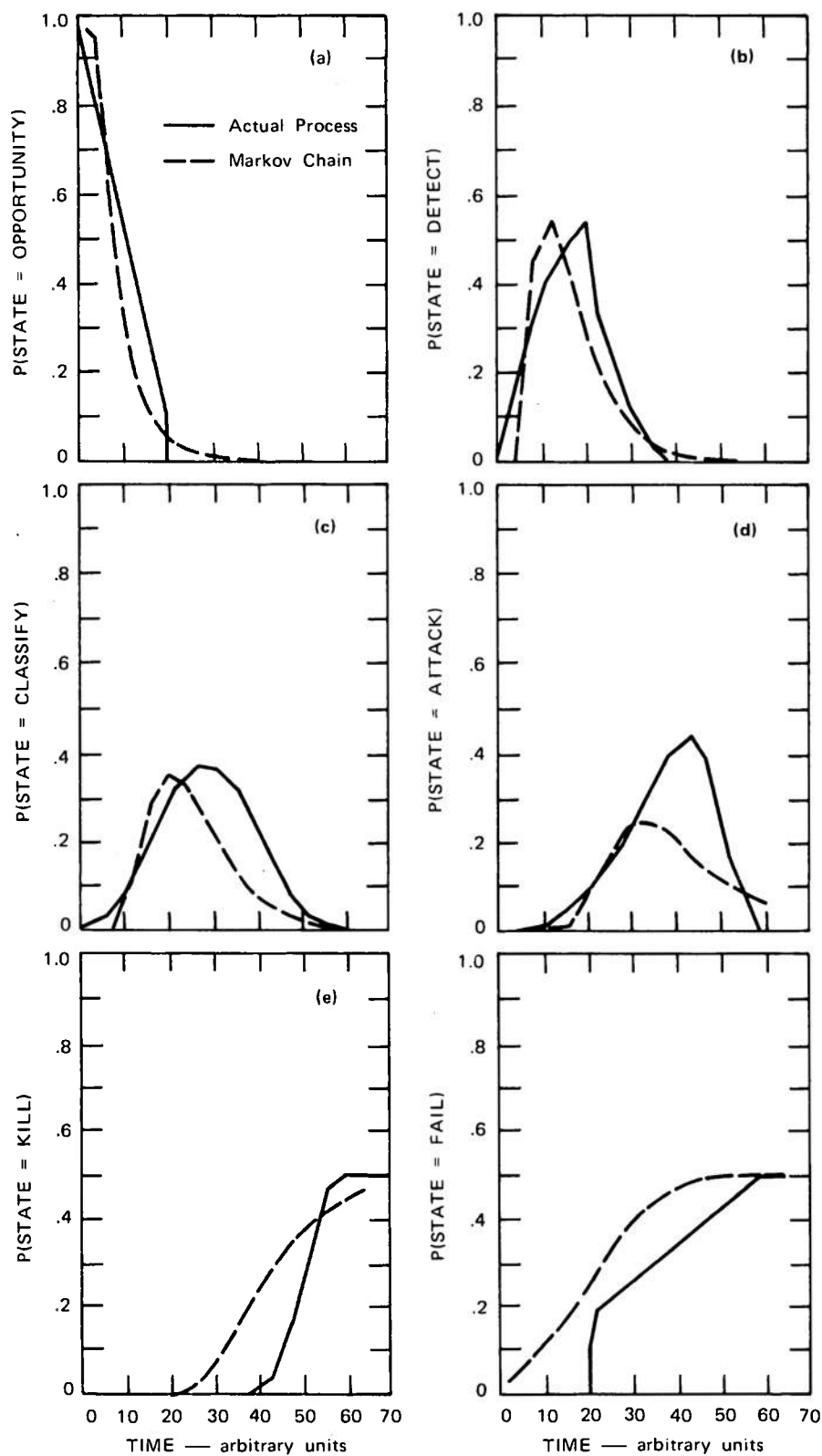


FIGURE 9.4 COMPARISON OF $P(\text{STATE})$ AS A FUNCTION OF TIME CURVES FOR THE ACTUAL PROCESS AND AUXILIARY DELAY STATES APPROXIMATION

$$y_3(n) = y_2(n-1)p_{23} + y_3(n-1)p_{33} \quad (9.4)$$

$$y_4(n) = y_3(n-1)p_{34} + y_4(n-1)p_{44} \quad (9.5)$$

$$y_5(n) = \sum_{i=1}^4 y_i(n-1)p_{i5} + y_5(n-1) \quad (9.6)$$

$$y_6(n) = y_4(n-1)p_{46} + y_6(n-1) \quad (9.7)$$

where

$$y_i(n) = P(\text{state} = i \text{ at time } 5n) \text{ from the actual process.}$$

To constrain the sum of probability out of each node to be unity, Eq. (9.6) was modified by replacing the p_{i5} by

$$p_{i5} = 1 - (p_{ii} + p_{i,i+1}) \quad \text{for } i = 1, 2, 3, 4 \quad (9.8)$$

This reduces the number of variables to eight and ensures that

$$\sum_{j=1}^6 p_{ij} = 1$$

for all states i .

The estimates of transition probabilities are shown in Table 9.3. Parentheses show values obtained by normalizing after setting the (3,5) element to zero (this element was negative in the least squares solution). This five-step transition matrix was raised to successive powers of n to determine the predicted $P(\text{state})$ as a function of time in the usual manner; the results are shown in Figure 9.5.

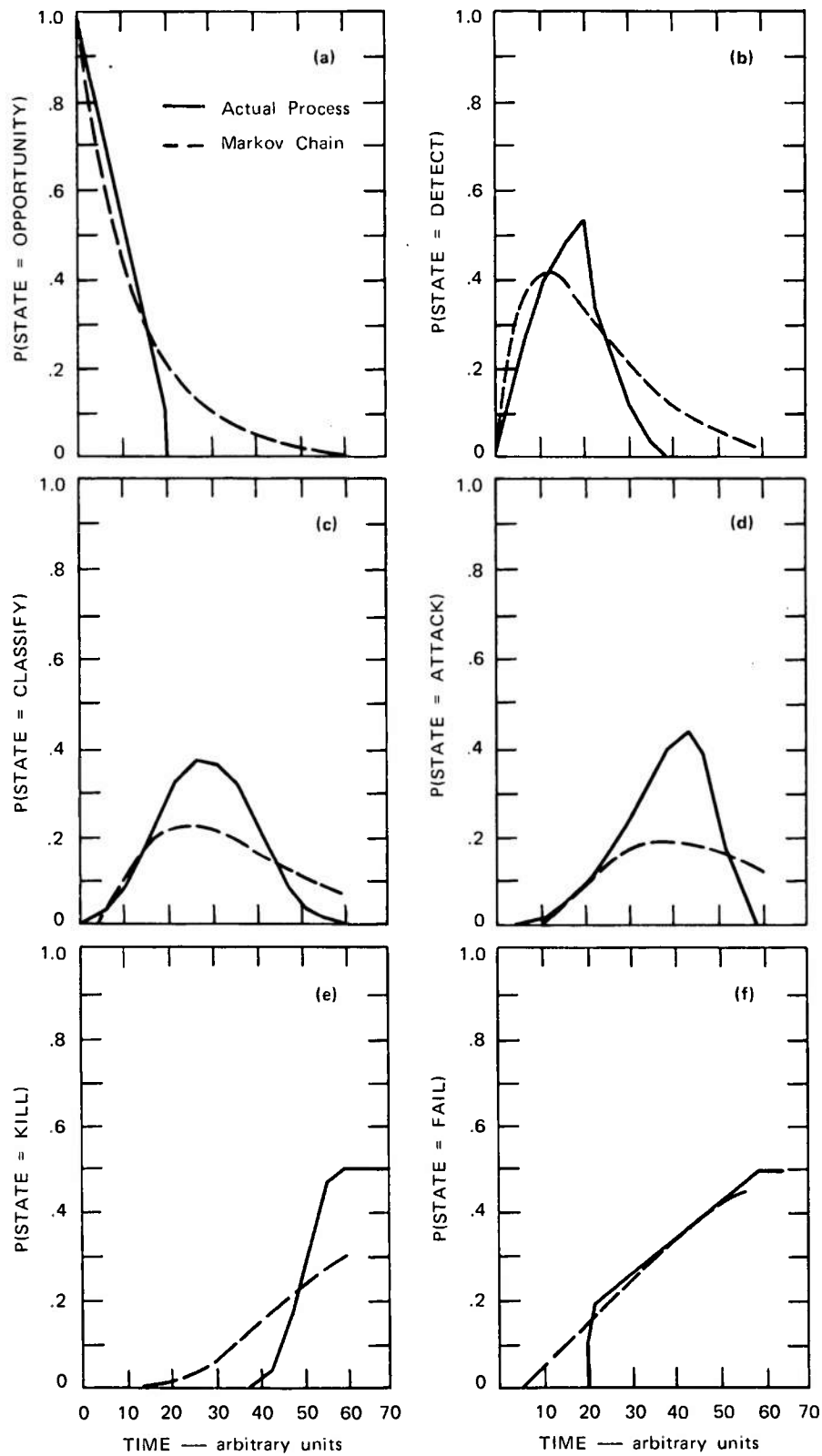


FIGURE 9.5 COMPARISON OF $P(\text{STATE})$ AS A FUNCTION OF TIME CURVES FOR THE ACTUAL PROCESS AND LEAST SQUARES APPROXIMATION

Table 9.3

ESTIMATES OF $p_{ij}^{(5)}$ USING P(STATE) AS A FUNCTION OF TIME

1/j	1	2	3	4	5	6
1	0.683	0.317	0	0	0	0
2	0	0.6039	0.2762	0	0.1119	0
3	0	0	(0.620)	(0.380)	(0)	0
4	0	0	0	0.6186	0.1421	0
5	0	0	0	0	1	0
6	0	0	0	0	0	1

Because the input data leading to these estimates contained no transition information, it is of interest to see what estimate of the probability of kill p_K is implied by these $\hat{p}_{ij}$. In contrast with the true value $p_K = 0.5$, the estimate is readily found to be $\tilde{p}_K = 0.432$. Therefore, it appears that estimators optimal for estimating the absorption probability p_K may not be optimal for predicting $P(\text{state})$ as a function of time when Markov methods are applied to non-Markov situations.

5. Semi-Markov Approximation

The first four methods used successively better estimates of the densities of waiting time in a state without incorporating some of the many dependent aspects of the true model into the estimators. Methods (5) and (6) used auxiliary states defined so as to remove some of the dependencies. An additional refinement in Method (5) was that the waiting times were not required to be geometrically distributed. That is, the fifth approximation was by a semi-Markov process, not by a Markov chain.

The fifth attempt to "Markovize" the actual process used a semi-Markov model with 28 states--the original 6 and 22 others. Density

functions of the transitions were allowed to vary according to the state being entered (semi-Markov property). The Z transform method was used for the calculations and was carried out by hand. Because of the labor entailed, only a few comparisons were made. These are given in Table 9.4.

Table 9.4

RESULTS FROM SEMI-MARKOV APPROXIMATION--28 STATES

Time	State	P(state)[true]	P(state)[semi-Markov]	Error
5	Detect	0.2	0.273	0.07
10	Detect	0.38	0.4	0.02
20	Detect	0.53	0.537	0.01
30	Detect	0.145	0.145	0
35	Detect	0.045	0.045	0
40	Detect	0	0	0
16	Classify	0.19	0.222	0.03
20	Classify	0.29	0.3164	0.04
40	Classify	0.23	0.2182	0.01
30	Attack	0.255	0.224	0.03
40	Attack	0.42	0.36	0.06
50	Attack	0.26	0.33	0.07

The dynamic behavior of the semi-Markov model approximates the actual behavior considerably better than any of the earlier Markov chain approximations used in Methods (1) through (4). It is encouraging to see that a semi-Markov model can fit such a complex dynamic probabilistic process as that defined by the PST diagram. However, the semi-Markov model used in Method (5) used knowledge of the actual process itself; this knowledge will not be available in applications to real problems. (All other approximating methods in this report use only observed data so that this fitting attempt was an exception to the general rule: use the artificial problems to find promising methods that can be transferred to real problems.)

6. Incorporating Time into the State Definition

The estimating methods given so far can all be viewed as systematic ways to add auxiliary states and transitions to improve the fit of the dynamic behavior represented by $P(\text{state})$ as a function of time. In the sixth estimating method tried on this problem, time itself was used to define auxiliary states. The method can be shown to be equivalent to a model with time-varying transition probabilities.

In the original process, there are six states and time varied from 0 to 60. In the new process, the states are the ordered pair (S, T) , where "S" denotes the state in the original process and T is an integer value of time. For example, the new state $(C, 12)$ or C12 represents the state "the original process is the classify state at time 12." The derived process has $6 \times 60 = 360$ states because times from 1 to 60 are used.

Any sample path from the original process, together with the transition times, now generates a path in the new process. For example, the value of $u = 0.15$ in the original process results in

Detection at time 17

Classification at time 22

Attack at time 31

Kill at time 52.

In the derived process, the path is $O1,^* O2, \dots, O16, D17, D18, \dots, D21, C22, C23, \dots, C30, A31, A32, \dots, A51, K52, K53, \dots, K60$.

A computer program was written to select a set of sample paths from the PST diagram and to perform the processing necessary to estimate

* The O = Opportunity.

P(state) as a function of time. When infinite data are assumed to be available, this model fits the original process essentially perfectly in the P(state) as a function of time sense.* This was first hypothesized and then verified numerically. (When u was varied systematically from zero to 1 in steps of 0.001--i.e., one thousand paths---the probability of state as a function of time curves were indistinguishable from those in Figure 9.3. A random sample of 1000 paths also produced an excellent fit, as expected. A random sample of 100 paths produced estimates typically in error by 0.01 to 0.02, with occasional values in error by as much as 0.05.

Although the derived process has many more states than the original process (360 versus 6), in many respects the length of the calculations themselves do not increase in proportion. For the six-state process, the products of powers of the 6×6 transition matrix are multiplied to obtain probabilities of state as a function of time. In the derived process, matrix multiplication was unnecessary because the probabilities at a state in this process are themselves the probabilities of being in a state of the original process at the corresponding time; hence, the probabilities in this process can be calculated by simple recursion methods because of the absence of loops in the transition diagram.

D. Summary

A six-state dynamic probabilistic non-Markov process was defined by a PST diagram in order to have a known process for experimentation. Six different methods were used to determine the parameters of a Markov or semi-Markov model to approximate the actual process. Of special interest was the dynamic behavior of the approximating processes as represented by

* Quantization is required which introduces some small errors.

the curves showing $P(\text{state})$ as a function of time for each of the six states and all times from 0 to 60.

The findings are the following:

- $P(\text{state})$ as a function of time fit the actual process rather poorly for Markov chain Methods (1) through (4) and fit quite well for semi-Markov and nonstationary Methods (5) and (6).
- The best fits were obtained by using auxiliary states. (Markov chain methods used auxiliary states to better represent waiting times in a state, and the semi-Markov model and nonstationary model used auxiliary states to reduce dependency.)
- $P(\text{state})$ as a function of time curves from the approximating Markov chains were quite smooth and approached their final values asymptotically; the sharp corners and abrupt cutoffs of the actual process were not well-approximated.
- The least squares method, which used the actual $P(\text{state})$ as a function of time as inputs, gave parameters that implied a value of probability of kill (p_K) and differed appreciably from the true value of p_K . This suggests that estimators which are best for estimating p_K may not be best for estimating $P(\text{state})$ as a function of time.

X MARKOV MODELS THAT INCLUDE FALSE CONTACTS

A. Introduction

Traditional conditional probability models of ASW system effectiveness do not explicitly include false contact considerations such as the detection and prosecution of false contacts. One of the objectives of the current study is to develop models that include false contacts in the dynamic ASW environment.

Three operational false contact models that generalize the traditional conditional probability models are defined in this section. Two different approaches to defining a state space are used; in both cases, a single detection opportunity of an ASW unit on an enemy submarine is assumed.* Conventional ASW screening by destroyers and use of destroyers in an antisubmarine barrier are examples of tactical situations to which the operational models apply.

The first model is a Markov chain model that generalizes the traditional conditional model in the simplest possible way. The second model is a modification of the first, but it uses a continuous time, competing process formulation for transitions into and out of the prosecute false contact state. Tables are given to show how false contacts degrade ASW effectiveness for the second model. The third model uses a state space whose states are ordered pairs of the original states. The first element of the pair describes the ASW units situation with respect to false contacts, and the second element describes the situation with respect to an actual submarine.

*The generalization to multiple-opportunity situations appears to be straightforward.

B. A Markov Chain Model for an ASW Unit in the Presence of False Contacts

The diagram in Figure 10-1 shows the states for a Markov model of a single ASW unit when false contacts may be present. It is the simplest, meaningful generalization of the traditional conditional probability model with the five states designated. It should be noted that the diagram has a single loop. Two states have been added to the traditional model; they are succinctly designated as prosecute false contact and opportunity with false contact.

Fuller definitions of these additional states are:

- Detect and Prosecute False Contact: Detect a contact that will later be found to be false and take appropriate action to prosecute the contact.
- Opportunity with False Contact: An actual submarine arrives and presents a Detection Opportunity while a False Contact is being prosecuted.

Therefore the prosecute false contact state is terminated in two ways:

- When the contact has been classified as a false contact and the ASW unit has returned to the screen.
- When an actual submarine arrives that either presents a detection opportunity or would have presented a detection opportunity to the unit if it were not prosecuting a false contact.

The model applies to situations in which there is a single opportunity to detect a submarine. For example, a destroyer in the traditional screening role with the mission of detecting, classifying, attacking, and killing a penetrating enemy submarine fits this model. A second example is a barrier operation involving a surface ship patrolling one portion of the barrier. An ASW unit in either of these contexts may be more than a single ship; for example, it may be a destroyer and a LAMPS helicopter.

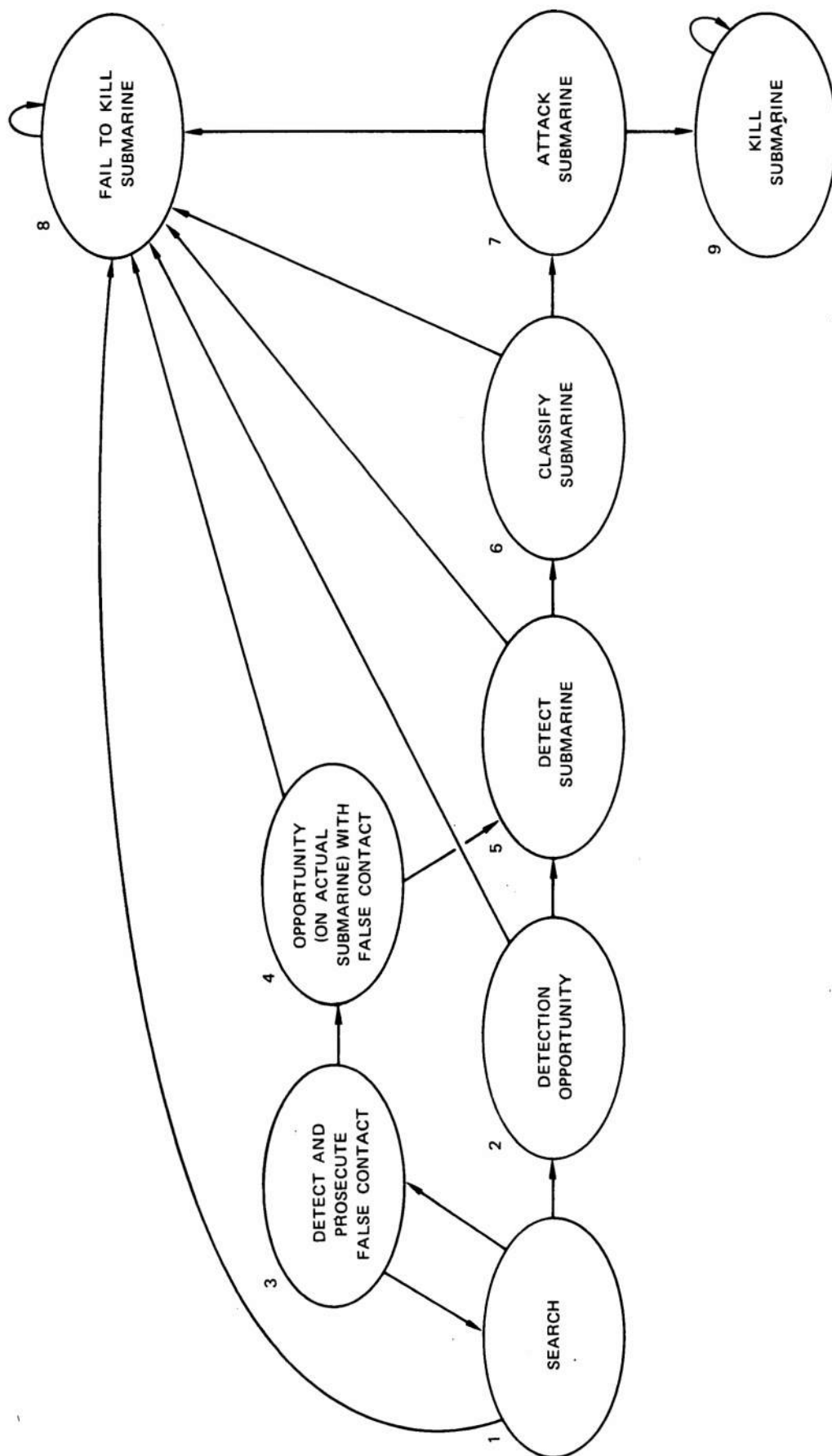


FIGURE 10.1 TRANSITION DIAGRAM FOR FALSE CONTACT MODELS 1 AND 2

State transitions will be explained next. It is assumed that the starting state is search (state 1). While searching, false contacts may from time to time be detected and prosecuted, causing the state to change to state 3 (prosecute false contact). The transition from state 1 to state 3 will depend on the definition of false contact and the ship detecting may or may not make an action appropriate to an actual submarine.

Of the two opportunity states, state 2 (opportunity) is defined in the usual way on the basis of a maximum detection range for the ASW unit. State 4, opportunity with false contact (defined above), is the second opportunity state. A transition from state 3 to state 4 occurs when a penetrating submarine comes within the maximum detection range while the ship is prosecuting a false contact. *

State 3 also changes to state 4 when a submarine would have presented a detection opportunity to the ASW unit had it not changed position to prosecute a false contact. It will be necessary to be somewhat arbitrary in order to make this definition more precise. If a destroyer has an assigned area to search, the submarine may be said to present an opportunity with false contact when it arrives into this assigned area, for example. If the submarine is not detected while in an opportunity state, then the ASW unit fails to detect and hence enters the fail state. It is assumed that once opportunity passes there will not be another chance. If, however, detection of the submarine occurs (the assumption here is that detection means detects and prosecutes), then the state changes to detect just as though the false contact had not been made. From detection onward, the transitions are the same as those in the traditional model. Failure can occur at any time following detection, and classification and attack are necessary conditions for kill. The possibility of counter-attack is not considered in this model.

For the present, it is assumed that data are available to allow the estimation of transition probabilities from a set of tactical ASW exercises or incidents. Let n_{ij} be the observed number of transitions from state i to state j . Then the transition probabilities not involved with the loop are estimated in the traditional way:

$$\hat{p}_{56} = P(C|D) = \frac{\text{number of classifications}}{\text{number of detections}} = \frac{n_{56}}{(n_{56} + n_{58})}$$

$$\hat{p}_{67} = P(A|C) = \frac{\text{number of attacks}}{\text{number of classifications}} = \frac{n_{67}}{(n_{67} + n_{68})}$$

$$\hat{p}_{79} = P(K|A) = \frac{\text{number of kills}}{\text{number of attacks}} = \frac{n_{79}}{(n_{79} + n_{78})} \quad .$$

Each of these estimators is an instance of the use of the general formula

$$\hat{p}_{ij} = n_{ij} / \sum_{k=1}^9 n_{ik} = n_{ij} / n_i \quad .$$

When care is taken to count transitions properly, this formula holds for all transitions including those involved with the loop. The denominator n_i represents the total number of transitions out of a state. For the transient states (i.e., all states except kill and fail), n_i also equals the number of transitions into the state. Thus, the number of transitions into or out of search is

(number of exercises beginning in search) + (number of times false

$$\text{contacts are prosecuted to completion}) = n_{01} + n_{31} \quad ,$$

where n_{01} is the number of exercises starting in Search. Since a given trial may result in 0, 1, 2, ..., false contact prosecutions, the total number of entries into Search may have any nonnegative value. Similarly, the total number of entries into state 3 (detect and prosecute false contact) will be the denominator of the two transition probabilities corresponding to the two ways of leaving this state.

Given the estimates of $\hat{p}_{ij} = P[S(n+1) = j | S(n) = i]$ for all (i,j) branches on the state diagram, Mason's rule of Appendix D for calculating transmissions in linear systems can be applied to give the formula for the desired estimate of the probability of kill. [Matrix inversion to obtain the absorption matrix $A = (I - Q)^{-1}R$ is equivalent to this.] When transition data are derived from a set of complete exercises, the theorem in Appendix E ensures that this estimate is identical to the simpler estimate (number of kills/number of exercises).

Mason's rule requires the calculation of loop transmissions and open path transmissions. Since there is a single loop with transmission $\hat{p}_{13}\hat{p}_{31}$, the graph determinant is simply

$$\Delta = 1 - \hat{p}_{13}\hat{p}_{31}.$$

There are two open paths from search to kill. The first path is Search $\rightarrow$ Prosecute False Contact $\rightarrow$ Opportunity with False Contact $\rightarrow$ Detect $\rightarrow$ Classify $\rightarrow$ Attack $\rightarrow$ Kill with transmission equal to the product of the conditional probabilities along the path $\hat{p}_{13}\hat{p}_{34}\hat{p}_{45}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79}$. The second path, Search $\rightarrow$ Opportunity $\rightarrow$ Detect $\rightarrow$ Classify $\rightarrow$ Attack $\rightarrow$ Kill, has transmission $\hat{p}_{12}\hat{p}_{25}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79}$. These two open paths have the detect, classify, attack, and kill states in common so the last three transition probabilities can be factored out. The formula for $\hat{p}_K$ which results from

Mason's rule is therefore

$$\hat{p}_K = \frac{[(\hat{p}_{13}\hat{p}_{34}\hat{p}_{45}) + (\hat{p}_{12}\hat{p}_{25})] \cdot \hat{p}_{56} \cdot \hat{p}_{67} \cdot \hat{p}_{79}}{1 - \hat{p}_{13}\hat{p}_{31}} \quad (10.1)$$

Because of the simplicity of the loop structure, this formula can either be interpreted more directly, or, alternatively, be derived from basic probability principles. Examination of the transition diagram shows that the numerator is the probability of transmitting directly from the search state to the kill state, without returning to the search state. The loop product $\hat{p}_{13}\hat{p}_{31}$ in the denominator is the probability that a given trial starting in search will result in one or more loops (i.e., returns to search). Therefore, the graph determinant Δ , with value $1 - \hat{p}_{13}\hat{p}_{31}$, is the probability of no return to search for a given trial. Conditioning on the number of loops (k) before a series of transitions finally leads to one of the absorbing states,

$$\begin{aligned} P(\text{Kill}) &= \sum_{k=0}^{\infty} P(\text{Kill} | \text{path begins with } k \text{ loops}) \\ &\quad \times P(\text{path begins with } k \text{ loops}) \quad (10.1a) \end{aligned}$$

The first term in the production on the right, which is actually independent of k , is the numerator in Eq. (10.1). The second term in the product is simply $(\hat{p}_{13}\hat{p}_{31})^k$. Therefore Eq. (10.1a) becomes

$$\begin{aligned} P(\text{Kill}) &= [\text{numerator of Eq. (10.1)}] \times \sum_{k=0}^{\infty} (\hat{p}_{13}\hat{p}_{31})^k \\ &= \text{numerator} / (1 - \hat{p}_{13}\hat{p}_{31}) \quad (10.1b) \end{aligned}$$

which is identical to Eq. (10.1).

An algebraically equivalent form separates the probability estimate Eq. (10.1) into two components.

$$\hat{p}_k = \frac{\hat{p}_{13}\hat{p}_{34}\hat{p}_{45}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79}}{1 - \hat{p}_{13} \cdot \hat{p}_{31}} + \frac{\hat{p}_{12}\hat{p}_{25}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79}}{1 - \hat{p}_{13}\hat{p}_{31}} \quad (10.2)$$

In this form the first term depends only on false contacts and the second is the usual estimate divided by $1 - \hat{p}_{13}\hat{p}_{31}$, which may be considered to be a false contact "correction term."

The basic expression of Eq. (10.1) can be rewritten again to show another possible interpretation. When the search to kill transition is considered in two phases--the first from search to detect and the second from detect to kill--the formula for the probability of kill becomes

$$\begin{aligned} \hat{p}_k &= P(\text{Kill}) = P(\text{Detect}) \times P(\text{Kill}|\text{Detect}) \\ &= \left(\frac{\hat{p}_{13}\hat{p}_{34}\hat{p}_{45}}{1 - \hat{p}_{13}\hat{p}_{31}} + \frac{\hat{p}_{12}\hat{p}_{25}}{1 - \hat{p}_{13}\hat{p}_{31}} \right) \times \hat{p}_{56}\hat{p}_{67}\hat{p}_{79} \quad (10.3) \end{aligned}$$

The effects of false contacts are isolated in the first term representing $P(\text{detect})$, except for the "correction factor" $1 - \hat{p}_{13}\hat{p}_{31}$ in the second term.

Several special cases of Eq. (10.1) are of interest and are discussed as follows:

- When there are no false contacts the estimate of $\hat{p}_{13}$ is zero and the kill probability formula is then reduced to the traditional estimator

$$\hat{p}_k = \hat{p}_{12}\hat{p}_{25}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79} \quad (10.4)$$

- When a false contact prosecution completely precludes detection of an actual submarine, $\hat{p}_{45}$ will be zero; then the

formula becomes

$$\hat{p}_k = \frac{\hat{p}_{12}\hat{p}_{25}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79}}{1 - \hat{p}_{13}\hat{p}_{31}} \quad (10.5)$$

Even though the first term is now zero, the $(1 - \hat{p}_{13}\hat{p}_{31})$ correction term remains in the denominator to modify the traditional estimate.

- If $\hat{p}_{31}$ is zero--that is, if the ASW unit never returns to search following prosecution of a false contact--the denominator disappears and $\hat{p}_k$ is the sum of two terms:

$$\hat{p}_k = \hat{p}_{13}\hat{p}_{34}\hat{p}_{45}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79} + \hat{p}_{12}\hat{p}_{25}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79} \quad (10.6)$$

Alternatively, since $\hat{p}_{34} = 1$ and $\hat{p}_{56}\hat{p}_{67}\hat{p}_{79}$ is common to both terms,

$$\hat{p}_k = \hat{p}_{13}\hat{p}_{45} + \hat{p}_{12}\hat{p}_{25}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79} \quad (10.6a)$$

The first term (in parentheses) shows that there are two exclusive ways to detect; the remaining product is the probability of kill given detection.

- If an actual submarine always arrives while the ASW unit is in Search (and never arrives while prosecuting a false contact) the $\hat{p}_{34}$ and $\hat{p}_{18}$ are both zero. This implies that $\hat{p}_{31} = 1$ and $\hat{p}_{12} = 1 - \hat{p}_{13}$. Putting these into Eq. (10.1) results in

$$\hat{p}_k = \hat{p}_{25}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79} \quad (10.7)$$

and shows that no false contact considerations remain. [The $\hat{p}_{12}$ factor disappeared because it was assumed that the submarine always arrived during search so that the opportunity state (2) was always reached.]

- When false contacts are so frequent that they are effectively always present, then $\hat{p}_{12}$ may be considered to be

zero and the formula becomes

$$\hat{p}_k = \frac{\hat{p}_{13}\hat{p}_{34}\hat{p}_{45}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79}}{1 - \hat{p}_{13}\hat{p}_{31}} \quad (10.8)$$

If $\hat{p}_{18}$ is zero also, then $\hat{p}_{13} = 1$ and Eq. (10.8) becomes

$$\hat{p}_k = \frac{\hat{p}_{34}\hat{p}_{45}\hat{p}_{56}\hat{p}_{67}\hat{p}_{79}}{1 - \hat{p}_{31}} \quad (10.9)$$

C. A Mixed Markov Model for an ASW Unit in the Presence of False Contacts

The Markov chain model in the previous section can be generalized somewhat by passing to the competing process formulation of a continuous time Markov model. This model, which will have transition rates as its only parameters, can be reduced either to the Markov chain model involving only transition probabilities or to a mixed model involving both transition rates and transition probabilities as parameters. The mixed model should be suitable for assessing the impact of false contacts on ASW systems effectiveness from data obtained from exercises in which false contacts were not considered.

In general, the competing process formulation of a continuous time Markov model is as follows. Associated with each branch of the transition diagram (i.e., with each transition) is an exponential distribution with parameter λ_{ij} . The set of λ_{ij} completely characterizes the process, for transitions are determined from the λ_{ij} by the following rules. First selecting a set of candidate transition times $|T_{ij}|$, one value for each branch out of state i . [The candidates are chosen independently of one another, the $(i,j)^{th}$ time being selected at random from an exponential distribution with parameter λ_{ij} .] The next state, j^* , is selected as that state j with the minimum transition time, and the time of the transition is chosen equal to that time. Symbolically, $j^* = j$ such that T_{ij}

is minimized; therefore, T_{ij}^* is the waiting time in i prior to the transition to j^* . It can be shown that the resulting distribution of the time required to transit from i to j is exponential with parameter λ_i , independent of the destination state j , where

$$\lambda_i = \sum_j \lambda_{ij} \quad (10.10)$$

The transition probabilities for the imbedded Markov chain implied by the above rules can be shown to be simply related to the transition rates by

$$p_{ij} = \lambda_{ij} / \lambda_i \quad \text{when } i \neq j \quad (10.11)$$

To apply the competing process concept to the formulation to a false contact model, consider a model with the same set of states as the Markov chain model above. The rules for writing down the Laplace transform of the transmission from state 1 (search) to state 9 (kill) result in Eq. (10.12).

$$t_{19}(s) = \frac{\left[\frac{\lambda_{12}\lambda_{25}}{(\lambda_1+s)(\lambda_2+s)} + \frac{\lambda_{13}\lambda_{34}\lambda_{45}}{(\lambda_1+s)(\lambda_3+s)(\lambda_4+s)} \right] \cdot \left[\frac{\lambda_{56}\lambda_{67}\lambda_{79}}{(\lambda_5+s)(\lambda_6+s)(\lambda_7+s)} \right]}{1 - \frac{\lambda_{13}\lambda_{31}}{(\lambda_1+s)(\lambda_3+s)}} \quad (10.12)$$

The derivation of this formula follows from the use of Mason's rule with an appropriate set of branch transmissions. The state 1-state 2 branch, which formerly had a transmission p_{12} , now has the transmission $\lambda_{12}/(\lambda_1 + s)$. In general, the $i - j$ transition has the transmission $\lambda_{ij}/(\lambda_i + s)$. The Markov chain probability of kill formula is now obtained by setting the Laplace variable s to zero and substituting p_{ij} for λ_{ij}/λ_i throughout the right side of Eq. (10.12).

The objective in this section is to obtain a modified model, however. To this end, the variable s is set to zero and p_{ij} is substituted for λ_{ij}/λ_i for the 5-6, 6-7, and 7-9 transitions only. The result is Eq. (10.13).

$$t_{19}^{(0)} = \frac{\left(\frac{\lambda_{12}\lambda_{25}}{\lambda_1\lambda_2} + \frac{\lambda_{13}\lambda_{34}\lambda_{45}}{\lambda_1\lambda_3\lambda_4} \right) p_{56}p_{67}p_{79}}{1 - \frac{\lambda_{13}\lambda_{31}}{\lambda_1\lambda_3}} \quad (10.13)$$

Since the arrival rate of an actual submarine should not depend on whether the ASW unit is prosecuting a false contact or is in Search, the quantities λ_{12} and λ_{34} may be assumed to be equal. Replacing λ_{34} by λ_{12} and multiplying numerator and denominator by λ_1 results in Eq. (10.14).

$$t_{19}^{(0)} = \frac{\lambda_{12} \left(\frac{\lambda_{25}}{\lambda_2} + \frac{\lambda_{13}}{\lambda_3} \frac{\lambda_{45}}{\lambda_4} \right) p_{56}p_{67}p_{79}}{\lambda_1 - \frac{\lambda_{13}\lambda_{31}}{\lambda_3}} \quad (10.14)$$

To have a more convenient set of parameters to work with, it is now assumed that the probability of detection is degraded by a constant factor f when a false contact is being prosecuted. Specifically, it is assumed that

$$p_{45} = p_{25} \times f \quad (10.15)$$

where f is a fixed number in the interval (0,1). Substitution of Eq. (10.15) together with Eq. (10.11) into Eq. (10.14) gives Eq. (10.16).

$$t_{19}^{(0)} = \frac{\lambda_{12} \left(1 + f\lambda_{13}/\lambda_3 \right) p_{25}}{\lambda_1 - \lambda_{13}\lambda_{31}/\lambda_3} p_{56}p_{67}p_{79} \quad (10.16)$$

From the factorization in Eq. (10.16), False Contacts can be seen to result in a degradation in the transition probability p_{15} in the traditional conditional probability model. Denoting this degraded value by $\tilde{p}_{15}$, we have

$$\tilde{p}_{15} = \frac{\lambda_{12} \left(1 + \frac{f\lambda_{13}}{\lambda_3} \right) p_{25}}{\lambda_1 - \frac{\lambda_{13}\lambda_{31}}{\lambda_3}} \quad (10.17)$$

The value of the degradation factor itself is the ratio

$$d = \frac{\lambda_{12} \left(1 + \frac{f\lambda_{13}}{\lambda_3} \right)}{\lambda_1 - \frac{\lambda_{13}\lambda_{31}}{\lambda_3}} \quad (10.18)$$

Another parameter needs to be introduced for the sake of interpretability. If it assumed that the average time spent prosecuting a false contact in the absence of a submarine arrival ($1/\lambda_{31}$) is a constant k times the average time between arrivals of false contacts in the absence of a submarine arrival ($1/\lambda_{13}$), then

$$k \frac{1}{\lambda_{13}} = \frac{1}{\lambda_{31}} \quad , \quad \text{or } k = \lambda_{13}/\lambda_{31} \quad (10.19)$$

To illustrate the numerical value of the degradation caused by false contacts, let it first be assumed that the value of f is zero. This means that no submarine detection is possible while the ASW unit is prosecuting a false contact. Without loss of generality it may be assumed that the arrival rate of actual submarines in the absence of false contacts (λ_{12}) is unity. Equation (10.18) then becomes

$$d = \frac{1}{\left(1 + \lambda_{18}\right) + \frac{k}{1 + k/\lambda_{13}}} \quad (10.20)$$

when $\lambda_{12} = \lambda_{34} = 1$, $f = p_{45}/p_{25} = 0$, and $k = \lambda_{13}/\lambda_{31}$.

Under these conditions, the degradation factor d is given in Table 10.1 for the parameters indicated when λ_{18} is also zero.

Table 10.1

VALUES OF DEGRADATION FACTOR CAUSED BY FALSE CONTACTS

Parameters		Degradation Factor
k	λ_{13}	d
0.2	∞	0.833
0.2	10	0.836
0.2	5	0.839
0.2	1	0.857
0.4	∞	0.714
0.4	10	0.722
0.4	5	0.729
0.4	1	0.778
0.6	∞	0.625
0.6	10	0.639
0.6	5	0.651
0.6	1	0.727
0.8	∞	0.556
0.8	10	0.574
0.8	5	0.592
0.8	1	0.692
$(\lambda_{12} = \lambda_{34} = 1, f = 0,$ $\lambda_{18} = 0)$		

If we return to the general case where $f > 0$ (arriving submarines may be detected when a false contact is being prosecuted), the formula may be readily derived

$$d = \frac{\lambda_{12} \left[1 + f \frac{1}{(1/k) + (\lambda_{12}/\lambda_{13})} \right]}{(\lambda_{12} + \lambda_{18}) + \frac{\lambda_{12}}{(1/k) + (\lambda_{12}/\lambda_{13})}} \quad (10.21)$$

Once again λ_{12} may be chosen equal to unity to normalize the parameters, resulting in the final expression

$$d = \frac{1 + f \frac{1}{(1/k) + (1/\lambda_{13})}}{1 + \lambda_{18} + \frac{1}{(1/k) + (1/\lambda_{13})}} \quad (10.22)$$

Some special cases of Eq. (10.22) may be noted as a check. In all check cases let λ_{18} --the failure rate when in search--be zero. When $f = 1$, the value of d becomes unity as it should since false contacts do not influence the process. Alternatively, when the value of k is zero, the prosecution of false contacts is instantaneous and results in a degradation factor of unity. A third case, analogous to a queueing system in which the arrival rate equals the departure rate, results when the value of k is unity. For the third case, algebraic manipulation of Eq. (10.22) results in

$$d = 1 - \frac{1 - f}{2 - 1/\lambda_{13}} \quad (10.23)$$

Therefore, as the arrival rate of false contacts λ_{13} increases, the degradation factor approaches the value $d = (1 + f)/2$.

In Eq. (10.22), the parameters are seen to be f , λ_{13} , and the quantity $(1/k) + (1/\lambda_{13})$. Table 10.2 was constructed for two values of f and values of the other parameters as indicated.

Table 10.2

DEGRADATION FACTOR d CAUSED BY THE PRESENCE OF FALSE CONTACTS

Parameters		Degradation Factor d	
$1 + \lambda_{18}$	$\frac{1}{\lambda_{13}} + \frac{1}{k}$	$f = 0.5$	$f = 1.0$
1.001	1	0.7496	0.9995
	3	0.8743	0.99925
	6	0.9278	0.9991
	12	0.9607	0.99908
	24	0.9791	0.99904
1.01	1	0.7463	0.99502
	3	0.8685	0.99256
	6	0.9207	0.9915
	12	0.9527	0.99085
	24	0.9707	0.99049
1.5	1	0.6	0.8000
	3	0.6364	0.7273
	6	0.65	0.7000
	12	0.6579	0.6842
	24	0.6622	0.6494

D. A False Contact Model Formed from a "Product"

The third false contact model uses a different method, which may be called the product method, for defining a set of states. Again it is assumed that at most one detection opportunity is presented on an actual submarine. The first four of the original states--Opportunity, Detect, Classify, and Attack--may be considered to describe the ASW unit's

situation with respect to either an actual submarine or a false contact. If subscripts s and f are used to denote submarine and false contact respectively, then the cartesian product of these sets may be defined by choosing one state from each set and adding the subscript f to the first component and subscript s to the second component. For convenience of notation, the original states may be denoted by their first letters. For example, the product state (O_f, O_s) means there is a detection opportunity on a false contact and also a detection opportunity on an actual submarine.

Another state needs to be added to complete the original actual submarine set. This state, designated N_s , represents no opportunity for detection of an actual submarine. If an ASW unit can handle one actual submarine contact and one false contact simultaneously, then all combinations in the product set of states are possible although some may rarely be observed in actual ASW operations. It is simpler, however, to continue considering all combinations of states rather than to try eliminating those that may be rarely observed. Therefore, the set of states is formed by choosing the first component from the set $\{O_f, D_f, C_f, A_f, K_f\}$ and the second component from the set $\{N_s, O_s, D_s, C_s, A_s\}$, and adding two more states fail (not to kill the actual submarine) and kill (the actual submarine). There are 27 states in all, and the starting state may be taken to be (O_f, N_s) when the submarine is initially beyond the maximum detection range of the ASW unit.

It should be noted that not only the detection of false contacts but also the misclassification of false contacts and attacks on false contacts are possible in this model. A complete sequence of errors of the latter kind are represented by the sequence of states

$$(O_f, N_s), (D_f, N_s), (C_f, N_s), (A_f, N_s), (K_f, N_s) \quad .$$

The last state in this sequence represents the kill of the false contact, which may have been evaluated as kill of an actual submarine during the operation. The other kind of error, calling an actual submarine a false contact, is also possible. These errors are represented in the model by transitions to the fail state from any state with second component D_s (detect actual submarine).

Examples of this application to some of the other states will now be discussed. If the exercise is begun with a destroyer in a barrier search before the actual penetration attempt by a submarine, then the starting state can be taken as (O_f, N_s) since there are always opportunities to detect false contacts. The state (O_f, O_s) would then be entered when the submarine comes within detection range of the destroyer's sensor(s) because Opportunities are then presented for detecting either the submarine or a false contact. Detecting a false contact first would cause a transition to (D_f, O_s) ; and if the submarine moves out of detection range before the destroyer classifies the false contact, the state becomes (D_f, N_s) . If another detection opportunity never materializes on the sub, failure will result. (The exact point at which the state should become F is somewhat arbitrary.)

In general, when an actual submarine is being prosecuted, the Opportunity $\rightarrow$ Detect $\rightarrow$ Classify $\rightarrow$ Attack $\rightarrow$ Kill chain can be followed with the first coordinate always being O_f as the submarine is progressively detected, classified, attacked, and killed. If, for example, a false contact is detected while an attack is under way (weapon in the water) and this causes the destroyer to direct its attention to the false contact (which may be a decoy), then the state becomes (D_f, A_s) . The sub-f sequence for the false contact may follow, but it will be interrupted if kill of the submarine actually occurs. (The state then becomes K to end the exercise.) If the torpedo expires without killing, then the

second component of state becomes N_s if the submarine has passed out of detection range or becomes O_s if it remains within range.

Some examples may be helpful at this point. Consider three hypothetical, reconstructed exercises that evolved as follows.

- The destroyer detects a false contact, classifies it as non-sub, and returns to search. The submarine comes within detection range but is undetected; thus, the mission results in a failure. The state sequence is: $(O_f, N_s), (D_f, N_s), (O_f, N_s), (O_f, O_s), F$.
- The destroyer detects a false contact after searching for some time; while the false contact is being classified, the actual submarine comes within range. The false contact is correctly classified as false and the destroyer returns to search while the submarine is still within range. The submarine passes out of range without being detected. The state sequence is: $(O_f, N_s), (D_f, N_s), (D_f, O_s), (C_f, O_s), (O_f, O_s), F$.
- The destroyer detects a false contact, correctly classifies it, and returns to search. It then detects another false contact, classifies it as a submarine, and performs an attack on the false contact. The destroyer then returns to search, the actual submarine comes within range and is detected. The submarine is classified as a submarine and attack maneuvering begins. The submarine evades successfully before weapon launch, and no further detection opportunity is available against it. The state sequence is: $(O_f, N_s), (D_f, N_s), (C_f, N_s), (O_f, N_s), (D_f, N_s), (C_f, N_s), (A_f, N_s), (O_f, N_s), (O_f, O_s), (O_f, D_s), (O_f, C_s), F$.

Markov models earlier in this report were sufficiently small to permit the calculation of absorption probabilities by inspecting the transition diagram and applying Mason's rule. There are far too many states in this model for this approach to succeed. Instead, the algebraic formula for the absorption matrix A derived in Section V must be used:

$$A = (I - Q)^{-1}R \quad (10.24)$$

where the states are systematically numbered such that the transition probability matrix P can be partitioned in the form

$$P = \begin{pmatrix} Q & R \\ O & I \end{pmatrix} . \quad (10.25)$$

In this model, Q is dimensioned 25 by 25, R is 25 by 2, and the identity matrix is 2 by 2. Calculations for the other measures of effectiveness may also be carried out in matrix terms as shown in Section V.

It is assumed in Eq. (10.24) that transition probability estimates are available from data. If the n_{ij} data are available, the formula $\hat{p}_{ij} = n_{ij}/n_i$ may be used to obtain the elements of Q and R . Other data such as average time-in-state data may also be used to estimate the p_{ij} as discussed in Section VI.

XI NUMERICAL EXPERIMENTATION: EXAMPLE TWO (FALSE CONTACTS)

A. Introduction

This section is concerned with numerical experimentation on the Markov chain false contact model defined in Section X. A PST diagram is used to define a non-Markovian dynamic probabilistic process with the same nine states as used for the Markov chain model. Data of various kinds are generated from the PST diagram and are used to estimate parameters of the Markov model. These parameters are then used to calculate the estimates $[P(\text{state}) \text{ as a function of time}]$ from the Markov model, and comparison of these curves with those from the actual (PST-defined) process reveals the adequacy of the Markov and semi-Markov approximations.

Because the actual process is so strongly non-Markovian, the state space had to be extended in an attempt to "decorrelate" some of the dependencies in the structure. Auxiliary states were systematically added by three different methods. The first method added states to provide "memory" of the previous state. The second method removed the loop by adding a new state each time a state was reentered. The third method added states to provide memory of all previous states.

A summary of the rather lengthy experimentation on this process will now be given. It was shown in Section VII that the Markov chain calculation for the absorption probability ($\hat{p}_k$) is exact when the true transition probabilities are known for the imbedded process even when the true process is non-Markovian. Knowing these true probabilities is equivalent, in terms of sample size, to having an infinite number of samples of the process. In the first set of experiments in Section XI-B, finite samples were first chosen from the true process, and all transition probabilities

were estimated from the finite sample by using the estimator

$$\hat{p}_{ij} = n_{ij} / \sum_k n_{ik} \quad ;$$

finally, the $\hat{p}_k$ formula was used to estimate the true probability of kill. The estimator was verified to be identical in all cases to the simple direct estimator (number of kills)/(number of samples). Next, some simple statistical tests were made on the transition probabilities and the estimator $\hat{p}_k$ and all results agreed with predictions from binomial theory.

The next set of experiments (Section XI-C) concerns methods for choosing the parameters of a Markov or semi-Markov model by using the given set of states in order to fit the dynamic behavior [P(state) as a function of time] to the observed data from the process defined by the PST diagram. Because the theorem on absorption probabilities already has proved that no improvement is possible in estimating the probability of absorption in state 9 (kill) for a set of complete exercises, the deconditioning can help only when the P(state) as a function of time fit is of interest or when not all data are from a set of complete exercises. In the latter case, better estimates of probability of kill as well as dynamic behavior should be possible because of the deconditioning. The following methods were used.

- Markov chain with original states, fitting mean waiting time $\bar{T}_i$ in each state.
- Semi-Markov model, using geometrically distributed T_{ij} .
- Semi-Markov model, using uniformly distributed T_{ij} .
- Semi-Markov model, with transition time density $h_{ij}(\cdot)$ obtained by averaging across the true process.
- Markov chain, with transition probabilities selected so that P(state) as a function of time fits the data in a least squares sense.

For comparison purposes, samples of size 100 were also drawn from the true population and the $P(\text{state})$ at time t estimated by the observed proportion of times the process was in that state at time t .

The next experiments for fitting $P(\text{state})$ as a function of t used altered sets of states. Additional states were introduced systematically in three ways. In the first method (Section XI-D-1) states are redefined to correspond to branches of the original process, which is equivalent to a model with a one-state memory. The second method (Section XI-D-2) removes the loop in the model by considering a return to the search state as a new state on each return. The third method (Section XI-D-3) is again semi-Markov and provides complete memory of the earlier states. The effect in each case was to add more states by "conditioning," thereby increasing the number of transition probabilities to be estimated which in theory should allow one to find a better fit in the $P(\text{state})$ as a function of time sense. These three methods produced a somewhat better fit than those using the original set of states.

In Section XI-E, the Markov property was tested. A comparison of several statistical quantities from the approximating Markov model and the true non-Markovian model were first given. Estimates of the probability of occupying any state not in a loop--including the success state--are found to be exact for any finite set of complete exercise data while estimates of most other quantities differ in the true process and the approximating Markov process. The theorem of Section VII proved in Appendix E shows equality of the measures because the signals at the nodes were closely related to the probability of entering a state in the formulation for the theorem. A chi-squared test for determining whether the entire set of data may be assumed to come from a Markov model concludes the section.

Figure 11.1 shows the PST diagram for the nine-state false contact model of Section X. Not all possible kinds of paths are represented by

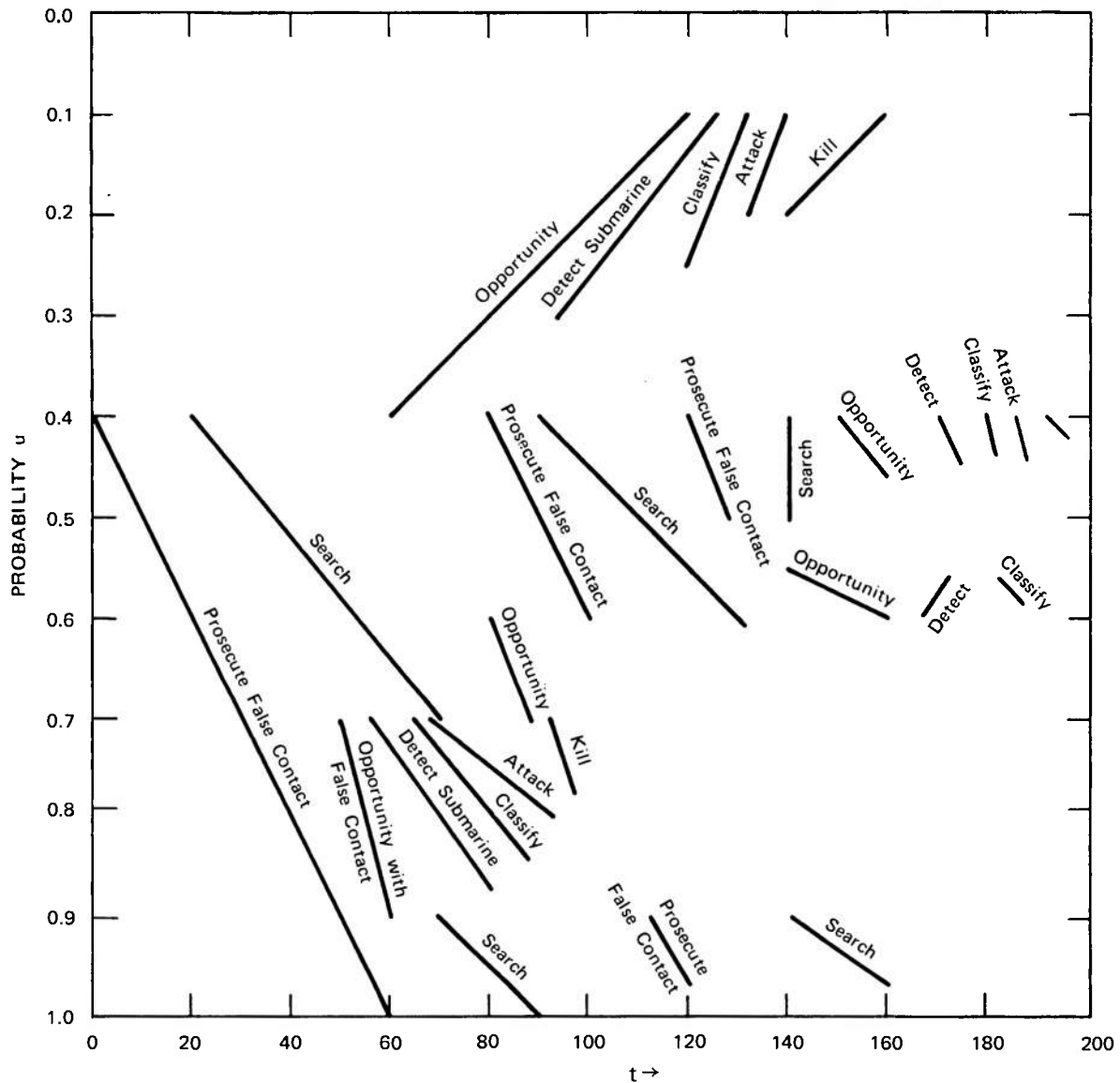


FIGURE 11.1 PROBABILITY STATE TIME DIAGRAM FOR FALSE CONTACT MODEL

the PST diagram (the 1-2-5-6-7-8 path is missing, for example). A maximum of three trips around the single loop is implied by the diagram; they occur for $0.4 \leq u \leq 0.5$. The probability of kill is 0.19 because the kill state is entered in three places ($0.1 \leq u \leq 0.2$, $0.40 \leq u \leq 0.42$,

$0.70 \leq u \leq 0.77$) and the lengths of these intervals adds up to 0.19.

Time varies from $t = 0$ to $t = 200$ in an unspecified set of units.

Several quantities of interest from the actual process are given in the following tables and figures. Figure 11.2 shows the transition diagram for the imbedded process and its transition probabilities that would be obtained from sampling the process infinitely often. These transition probabilities are also shown in tabular form in Table 11.1. Also shown on the branches are the numbers of transitions (based on 1000 equally spaced paths) that determine the true transition probabilities for the imbedded chain. Table 11.2 shows the true probabilities of occupying all states as a function of time; the several Markov approximations to the actual process will be judged by determining how well the approximating $P(\text{state})$ as a function of time curves match these data. Mean times of the i to j transition ($\bar{T}_{ij}$) data are shown in Table 11.3 for all states i and j ; the standard deviations of the T_{ij} are shown in Table 11.4.

B. Monte Carlo Experiments for Estimation of Transition Probabilities

For this set of experiments, the imbedded process is sampled; that is, transition counts n_{ij} are obtained by sampling, but the information on transition times between states is ignored. The natural estimator $\hat{p}_{ij} = n_{ij} / \sum_k n_{ik}$ is used to estimate the transition probabilities p_{ij} . Ten sets of replications of sizes 25 to 100 were independently run and the $\hat{p}_{ij}$ estimates are tabulated in Tables 11.5 and 11.6. A single column of each of these tables represents estimates for a set of replications (or case). The estimates of the measure of effectiveness (probability of terminating in the kill state) resulting from Eq. (10.1) of Section X-B are shown on the bottom row.

The rightmost columns compare the average $\hat{p}_{ij}$'s, where the averaging was over all ten cases, with the true transition probabilities. The

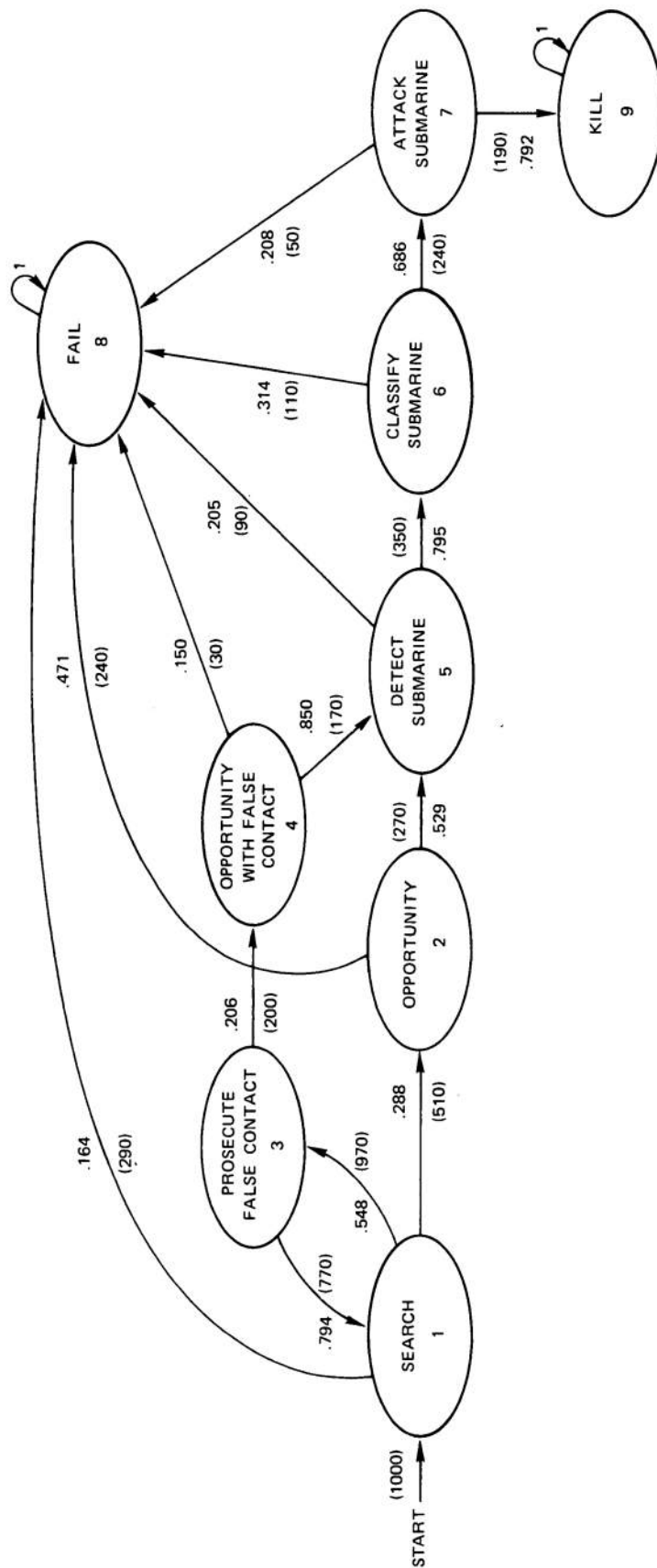


FIGURE 11.2 TRANSITION DIAGRAM AND PROBABILITIES FOR IMBEDDED PROCESS OF FALSE CONTACT MODEL

Table 11.1

TRANSITION PROBABILITY MATRIX (p_{ij}) FOR IMBEDDED CHAIN

	1	2	3	4	5	6	7	8	9
1	0	0.288	0.548	0	0	0	0	0.164	0
2	0	0	0	0	0.529	0	0	0.471	0
3	0.794	0	0	0.206	0	0	0	0	0
4	0	0	0	0	0.850	0	0	0.150	0
5	0	0	0	0	0	0.795	0	0.205	0
6	0	0	0	0	0	0	0.686	0.314	0
7	0	0	0	0	0	0	0	0.208	0.792
8	0	0	0	0	0	0	0	1.000	0
9	0	0	0	0	0	0	0	0	1.000

estimated $\hat{p}_k$ obtained by using the averaged $\hat{p}_{ij}$ is 0.1806 for 25 replications/case set of data and 0.194256 for the 100 replications/case set of data. The true value of p_k is 0.19.

The last row of each table shows the number of successes for each case; the ratio (number of successes)/(number of trials) may be directly compared with the value of $\hat{p}_K$ computed for the formula for absorption probabilities. The correspondence is exact for the 25 replications cases while some round-off error occurs in the fourth decimal place for the 100 replications cases.

A simple test on the reasonableness of the estimates of $\hat{p}_K$ can be made on the basis of the observation that the variance of a binomial distribution with n trials and probability of success p_K is $p_K(1 - p_K)/n$. In the 25-sample case, the true variance is $0.19(1 - 0.19)/25 = 0.006156$ so that the standard deviation is 0.0783. If a normal approximation to the binomial density is used, about 68 percent of the samples within one standard deviation of the true mean value 0.19 can be expected; that is, in the interval (0.1117, 0.2683). Of the ten estimates, two are outside and eight are inside this interval, which is quite reasonable since $(0.68) \times 10 \cong 7$ were expected. Repeating this procedure for the 100-sample

Table 11.2

PROBABILITY OF BEING IN A STATE AS A FUNCTION OF TIME $\times 1000$

t	State								
	1	2	3	4	5	6	7	8	9
5	950	0	50	0	0	0	0	0	0
10	900	0	100	0	0	0	0	0	0
15	850	0	150	0	0	0	0	0	0
20	800	0	200	0	0	0	0	0	0
25	780	0	220	0	0	0	0	0	0
30	760	0	240	0	0	0	0	0	0
35	740	0	260	0	0	0	0	0	0
40	720	0	280	0	0	0	0	0	0
45	700	0	300	0	0	0	0	0	0
50	680	0	320	0	0	0	0	0	0
55	660	0	240	100	0	0	0	0	0
60	640	0	160	172	28	0	0	0	0
65	645	25	130	137	58	5	0	0	0
70	650	50	100	101	64	27	8	0	0
75	650	75	75	66	70	35	29	0	0
80	650	100	50	30	77	43	50	0	0
85	538	187	75	30	48	52	70	0	0
90	450	250	100	30	30	49	91	0	0
95	400	268	125	30	37	40	48	0	52
100	350	262	150	30	68	40	30	0	70
105	350	256	125	30	99	40	30	0	70
110	350	250	100	30	130	40	30	0	70
115	324	243	101	30	162	40	30	0	70
120	280	237	120	30	193	40	30	0	70
125	243	206	157	30	161	103	30	0	70
130	230	200	170	30	105	165	30	0	70
135	230	200	170	30	80	152	68	0	70
140	230	200	170	30	80	90	130	0	70
145	335	212	53	30	80	90	105	0	95
150	340	225	35	30	80	90	80	0	120
155	315	267	18	30	80	90	55	0	145
160	290	310	0	30	80	90	30	0	170
165	290	310	0	30	80	90	30	0	170
170	290	295	0	30	95	90	30	0	170
175	290	240	0	30	150	90	30	0	170
180	290	240	0	30	150	90	30	0	170
185	290	240	0	30	95	145	30	0	170
190	290	240	0	30	90	110	70	0	170
195	290	240	0	30	90	110	55	0	185
200	0	0	0	0	0	0	0	810	190

Table 11.3

MEAN TRANSITION TIMES ($\bar{T}_{ij}$)

	1	2	3	4	5	6	7	8	9
1	-1.000	61.488	34.857	-1.000	-1.000	-1.000	-1.000	115.593	-1.000
2	-1.000	-1.000	-1.000	-1.000	12.375	-1.000	-1.000	110.140	-1.000
3	25.334	-1.000	-1.000	14.975	-1.000	-1.000	-1.000	-1.000	-1.000
4	-1.000	-1.000	-1.000	-1.000	13.796	-1.000	-1.000	140.725	-1.000
5	-1.000	-1.000	-1.000	-1.000	-1.000	11.010	-1.000	100.882	-1.000
6	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	7.443	80.306	-1.000
7	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	71.878	14.585
8	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	0.	-1.000
9	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	0.

Table 11.4

STANDARD DEVIATIONS OF TRANSITION TIMES (σ_{Tij})

	1	2	3	4	5	6	7	8	9
1	-1.000	36.625	16.979	-1.000	-1.000	-1.000	-1.000	64.134	-1.000
2	-1.000	-1.000	-1.000	-1.000	4.735	-1.000	-1.000	30.010	-1.000
3	7.670	-1.000	-1.000	2.894	-1.000	-1.000	-1.000	-1.000	-1.000
4	-1.000	-1.000	-1.000	-1.000	4.488	-1.000	-1.000	.440	-1.000
5	-1.000	-1.000	-1.000	-1.000	-1.000	2.757	-1.000	26.415	-1.000
6	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	1.455	34.875	-1.000
7	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	49.022	4.480
8	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	1	-1.000
9	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	0.

Table 11.5

ESTIMATES FOR p_{ij} --- 25 REPLICATIONS FOR EACH CASE

p_{ij}	Case										Mean	True
	1	2	3	4	5	6	7	8	9	10		
p_{12}	0.220	0.255	0.233	0.341	0.093	0.342	0.190	0.429	0.368	0.288	0.270	0.288
p_{13}	0.580	0.574	0.581	0.463	0.741	0.526	0.571	0.524	0.500	0.614	0.568	0.548
p_{18}	0.200	0.170	0.186	0.195	0.167	0.132	0.238	0.048	0.132	0.158	0.162	0.164
p_{25}	0.273	0.417	0.400	0.571	0.400	0.538	0.750	0.389	0.571	0.308	0.462	0.529
p_{28}	0.727	0.583	0.600	0.429	0.600	0.462	0.250	0.611	0.429	0.692	0.538	0.471
p_{31}	0.862	0.815	0.720	0.842	0.725	0.650	0.708	0.773	0.684	0.914	0.769	0.794
p_{34}	0.138	0.185	0.280	0.158	0.275	0.350	0.292	0.227	0.316	0.086	0.231	0.206
p_{45}	1.000	1.000	1.00	1.00	0.818	0.857	1.000	0.800	1.000	0.667	0.914	0.850
p_{48}	0	0	0	0	0.182	0.143	0	0.200	0	0.333	0.086	0.150
p_{56}	1.000	0.900	0.727	0.909	0.727	0.692	0.769	0.909	0.786	1.000	0.842	0.795
p_{58}	0	0.100	0.273	0.091	0.273	0.308	0.231	0.091	0.214	0	0.158	0.205
p_{67}	0.714	0.556	0.250	0.600	0.500	0.667	0.800	0.800	0.727	1.000	0.661	0.686
p_{68}	0.286	0.444	0.750	0.400	0.500	0.333	0.200	0.200	0.273	0	0.339	0.314
p_{78}	0	0.400	0.500	0.333	0	0.167	0	0.250	0.375	0.500	0.252	0.208
p_{79}	1.000	0.600	0.500	0.667	1.000	0.833	1.000	0.750	0.625	0.500	0.747	0.792
$\hat{p}_k$	0.200	0.120	0.040	0.160	0.160	0.200	0.320	0.240	0.200	0.120	0.1806	0.190
Number of Successes	5.0	3.0	1.0	4.0	4.0	5.0	8.0	6.0	5.0	3.0		

Note: Formula for estimating kill probability: $\hat{p}_k = \frac{(p_{12}p_{25} + p_{13}p_{34}p_{45})p_{56}p_{67}p_{79}}{1 - p_{13}p_{31}}$

Table 11.6
ESTIMATES FOR P_{ij} --100 REPLICATIONS FOR EACH CASE

	1	2	3	4	5	6	7	8	9	10	Mean	True
P_{12}	0.289	0.299	0.353	0.300	0.288	0.344	0.318	0.270	0.254	0.247	0.2962	0.288
P_{13}	0.572	0.490	0.500	0.589	0.554	0.550	0.542	0.509	0.554	0.581	0.5441	0.548
P_{18}	0.139	0.210	0.147	0.111	0.158	0.106	0.140	0.221	0.192	0.172	0.1596	0.164
P_{25}	0.558	0.511	0.583	0.614	0.604	0.548	0.509	0.614	0.467	0.510	0.5518	0.529
P_{28}	0.442	0.489	0.417	0.386	0.396	0.452	0.491°	0.386	0.533	0.490	0.4482	0.471
P_{31}	0.777	0.740	0.824	0.804	0.824	0.808	0.814	0.759	0.786	0.852	0.7988	0.794
P_{34}	0.223	0.260	0.176	0.196	0.176	0.192	0.186	0.241	0.214	0.148	0.2012	0.206
P_{45}	0.913	0.900	1.000	0.773	0.833	0.947	0.889	0.950	0.905	0.882	0.8992	0.850
P_{48}	0.087	0.100	0	0.227	0.167	0.053	0.111	0.050	0.095	0.118	0.1008	0.150
P_{56}	0.880	0.667	0.860	0.885	0.660	0.865	0.733	0.696	0.800	0.850	0.7896	0.795
P_{58}	0.120	0.333	0.140	0.115	0.340	0.135	0.267	0.304	0.200	0.150	0.2104	0.205
P_{67}	0.614	0.607	0.605	0.717	0.871	0.667	0.758	0.625	0.781	0.618	0.6863	0.686
P_{68}	0.386	0.393	0.395	0.283	0.129	0.333	0.242	0.375	0.219	0.382	0.3137	0.314
P_{78}	0.259	0.235	0.192	0.182	0.148	0.167	0.240	0.200	0.400	0.238	0.2261	0.208
P_{79}	0.741	0.765	0.808	0.818	0.852	0.833	0.760	0.800	0.600	0.762	0.7739	0.792
$\hat{P}_k$	0.20015	0.12996	0.2101	0.27	0.2299	0.2496	0.19003	0.1601	0.1500	0.15996	0.194256	0.190
Number of Successes	20	13	21	27	23	25	19	16	15	16		

cases gives an interval (0.1508, 0.2292) with estimates for six cases lying inside the interval and estimates for four cases lying outside the interval, which is again satisfactory.

Similar distribution tests were also made on the distribution of the transition probability estimates $\hat{p}_{ij}$. The variance is more difficult to determine this time, however, since the number of samples is random. If N is the random number of samples available to estimate $\hat{p}_{ij}$,

$$\text{var } \hat{p}_{ij} = E[\text{var}(\hat{p}_{ij}|N)] + \text{var}[E(\hat{p}_{ij}|N)] = E[\text{var}(\hat{p}_{ij}|N)]$$

because $E(\hat{p}_{ij}|N) = p_{ij}$, a constant. With the use of an average value assumption, the variance of $\hat{p}_{ij}$ is approximately

$$\text{var } \hat{p}_{ij} = p_{ij}(1 - p_{ij})/\bar{N} \quad ,$$

where $\bar{N}$ is the average value of N .

Table 11.7 summarizes the results of the distribution calculations for the 100-replication cases. The true values of p_{ij} are shown in the left column, the next column gives the sample standard deviations ($\hat{\sigma}_{ij}$) of $\hat{p}_{ij}$, while the last column shows the numbers of cases (out of ten) whose estimates fell inside the estimated one-sigma band about p_{ij} . The average number lying inside was 7.2, compared with the theoretical expected value of 6.8. It is of interest to note that the variance of $\hat{p}_k$ is 0.0392, which is smaller than the estimated sigmas--nine of the twelve $\hat{p}_{ij}$ from which it is computed.

C. Approximating the Actual Process by Markov and Semi-Markov Processes

Several attempts were made to approximate the dynamic performance of the actual (PST-defined) process by selecting parameters for a Markov or

Table 11.7

NUMBERS OF SAMPLES WITHIN THE ONE-SIGMA INTERVAL

Transition (i,j)	True p_{ij}	Standard Deviation $\hat{\sigma}_{ij}$	Number Samples Inside ($p_{ij} - \hat{\sigma}_{ij}$, $p_{ij} + \hat{\sigma}_{ij}$)
12	0.288	0.0340	7
13	0.548	0.0340	7
18	0.164	0.0340	6
25	0.529	0.0699	7
28	0.471	0.0699	7
31	0.794	0.041	8
34	0.206	0.041	8
45	0.850	0.0798	7
48	0.150	0.0798	8
56	0.795	0.0609	6
58	0.205	0.0609	2
67	0.686	0.0784	6
68	0.314	0.0784	7
78	0.208	0.0828	9
79	0.792	0.0828	9
			average = 7.2

Note: Expected number of samples inside the interval is 6.8 based on normal approximation.

The variance of $\hat{p}_k$ is 0.0392.

semi-Markov process by methods discussed in Section VI. Unless otherwise indicated, an unlimited amount of data were assumed to be available from the actual process.

Since the $P(\text{state})$ curves characterize the dynamics of the process, the approximating Markov process should have similar $P(\text{state})$ as a function of t curves. The true probability of state as a function of time relationships, which were shown in tabular form in Table 11.2, are shown in graphical form in Figure 11.3. The Markov approximations are expected to have difficulty achieving a fit to these rather irregular curves.

1. Markov Chain Model with Matching Average Waiting Time

In this first attempt to match the dynamic behavior of the actual (PST-defined) process, the simple method of estimating the p_{ij} for a Markov chain used the actual values of the transition probabilities for the imbedded chain together with the average waiting time $\bar{T}_i$ in each state i . Mean times were found by a weighted average across rows in Table 11.3. The method, which gave a poor fit in the earlier numerical example in Section IX, gave an even poorer fit in this example. The approximating $P(\text{state})$ as a function of t curves were so poor that they were not plotted. Evidently, better methods are needed.

2. The First Semi-Markov Approximation: Geometrically Distributed T_{ij}

The second attempt at fitting $P(\text{state})$ as a function of time employed a semi-Markov model with geometrically distributed transition times T_{ij} . The parameter was chosen to match the known mean transition times $\bar{T}_{ij}$ from Table 11.3. As in Subsection 1 above, the time transition probabilities for the imbedded process were used for the transition behavior.

This model was expected to be superior to that in Subsection 1 because the mean waiting time in state i was now made dependent on the destination state j . These mean times were given in Table 11.3. The

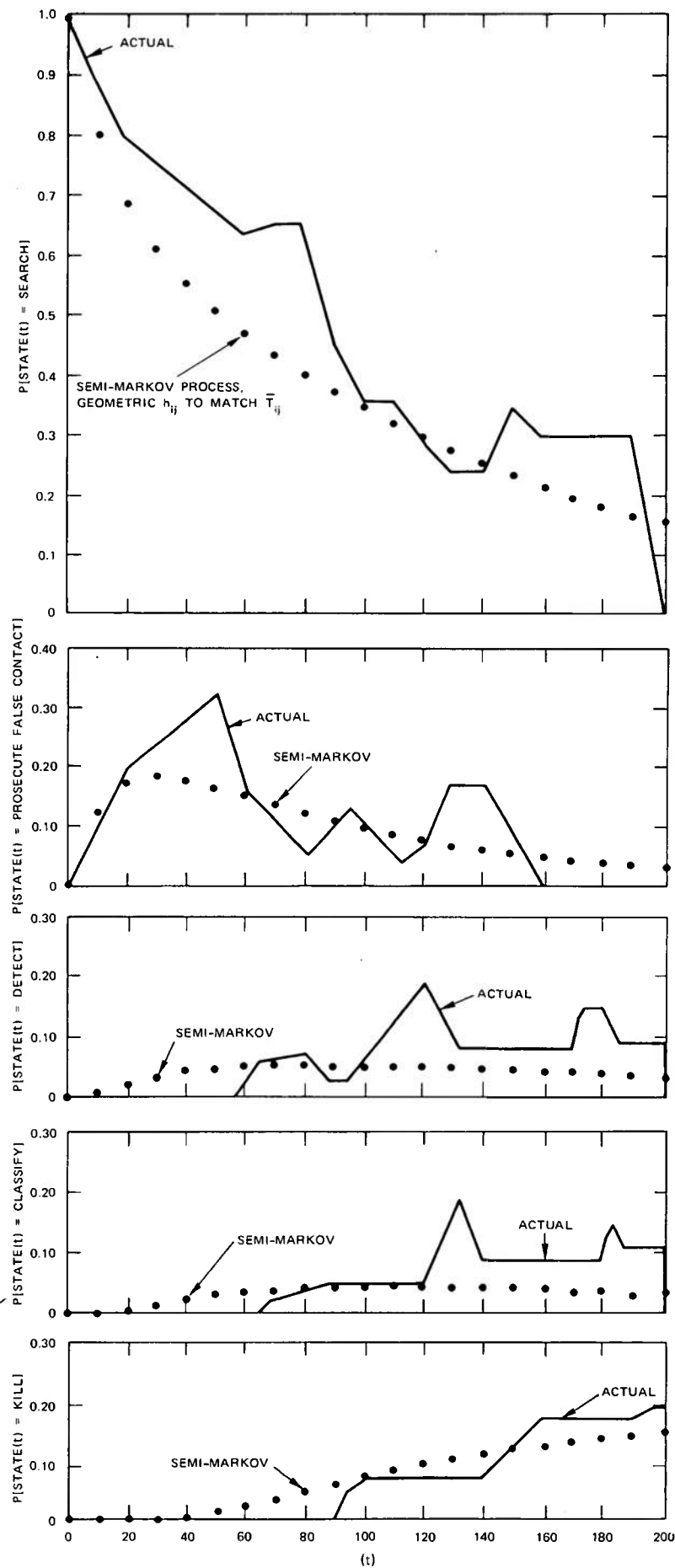


FIGURE 11.3 COMPARISON OF $P(\text{STATE})$ AS A FUNCTION OF TIME CURVES FOR ACTUAL PROCESS AND SEMI-MARKOV, GEOMETRIC h_{ij} APPROXIMATION (FIVE STATES)

comparison between the true $P(\text{state})$ as a function of time curves and the ones found from this approximate semi-Markov model is shown in Figure 11.3.

The formula for determining $P(\text{state})$ at time n is given in Volume II of Ref. 5, p. 585:

$$\phi_{ij}(n) = \delta_{ij} P(T_i > n) + \sum_{k=1}^N p_{ik} \sum_{m=1}^n h_{ik}^{(m)} \phi_{kj}(n-m)$$

for $i, j = 1, 2, \dots, N$; $n = 0, 1, 2, \dots$. Here $\delta_{ij} = 1$ if $i = j$ and 0; otherwise, T_i is the (random) waiting time in state i given by

$$P(T_i > n) = \sum_{m=n+1}^{\infty} \sum_{j=1}^N p_{ij} h_{ij}^{(m)} = 1 - \sum_{m=0}^n \sum_{j=1}^N p_{ij} h_{ij}^{(m)}$$

$$\phi_{ij}(n) = P(\text{state at } n = j \mid \text{state } i \text{ entered at time zero})$$

$$h_{ij}^{(m)} = P(\text{transition from } i \text{ to } j \text{ occurs at time } m \mid \text{transition}$$

is to j from i) .

3. The Second Semi-Markov Approximation: Uniform Distribution

The next semi-Markov approximation assumed a uniform distribution for each transition time T_{ij} . The lower and upper limits of this distribution were selected to match the mean and variance of T_{ij} for the true process. If the lower and upper limits are denoted by (a_{ij}, b_{ij}) --the mean transition time by $\bar{T}_{ij}$ and the variance of this time by v_{ij} --the relations are

$$a_{ij} = m_{ij} - \sqrt{3v_{ij}}$$

$$b_{ij} = m_{ij} + \sqrt{3v_{ij}} \quad ,$$

where a_{ij} is replaced by zero if negative. The fit obtained with this approximation is shown in Figure 11.4 and seems generally superior to that obtained by matching only the single parameter $\bar{T}_{ij}$.

4. The Third Semi-Markov Approximation: Averaged h_{ij}

The third semi-Markov approximation entailed the calculation of density functions $h_{ij}(\cdot)$ by appropriate conditioning methods on the density functions implied by the PST diagram. As suggested by the notation, one density function was chosen for each allowed transition. Because of the complexity of the ways some of the densities were formed, no general pattern or form of density could be imposed in all cases such as was done in the previous semi-Markov model.

A comparison of the true and estimated $P(\text{state})$ as a function of time curves is shown in Figure 11.5. The fit is somewhat improved over either of the earlier two semi-Markov approximations but is still far from perfect. The reasons for the imperfections are that there are correlations in the true process that are not reflected in the semi-Markov model; as flexible as semi-Markov models are, they still have strong assumptions of independence of all densities $h_{ij}(\cdot)$ for each transition.

D. Approximations Using Auxiliary States

Three different ways to add auxiliary states for improvement in the quality of the $P(\text{state})$ as a function of time curves were formulated and tested numerically. Some improvement was noted over the previous methods that used the original state space. By the end of this experimentation,

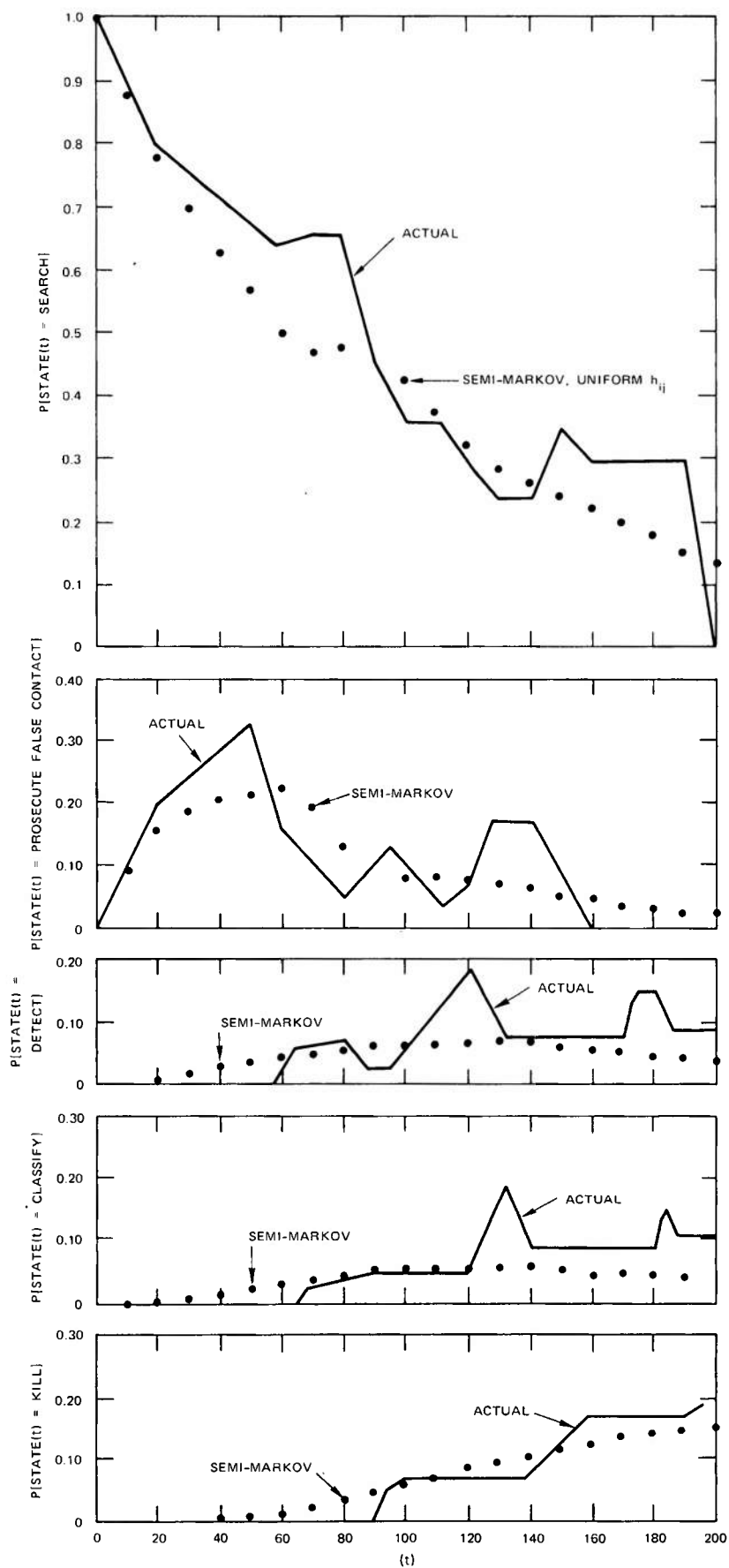


FIGURE 11.4 COMPARISON OF $P(\text{STATE})$ AS A FUNCTION OF TIME CURVES FOR ACTUAL PROCESS AND SEMI-MARKOV, UNIFORM h_{ij} APPROXIMATION (FIVE STATES)

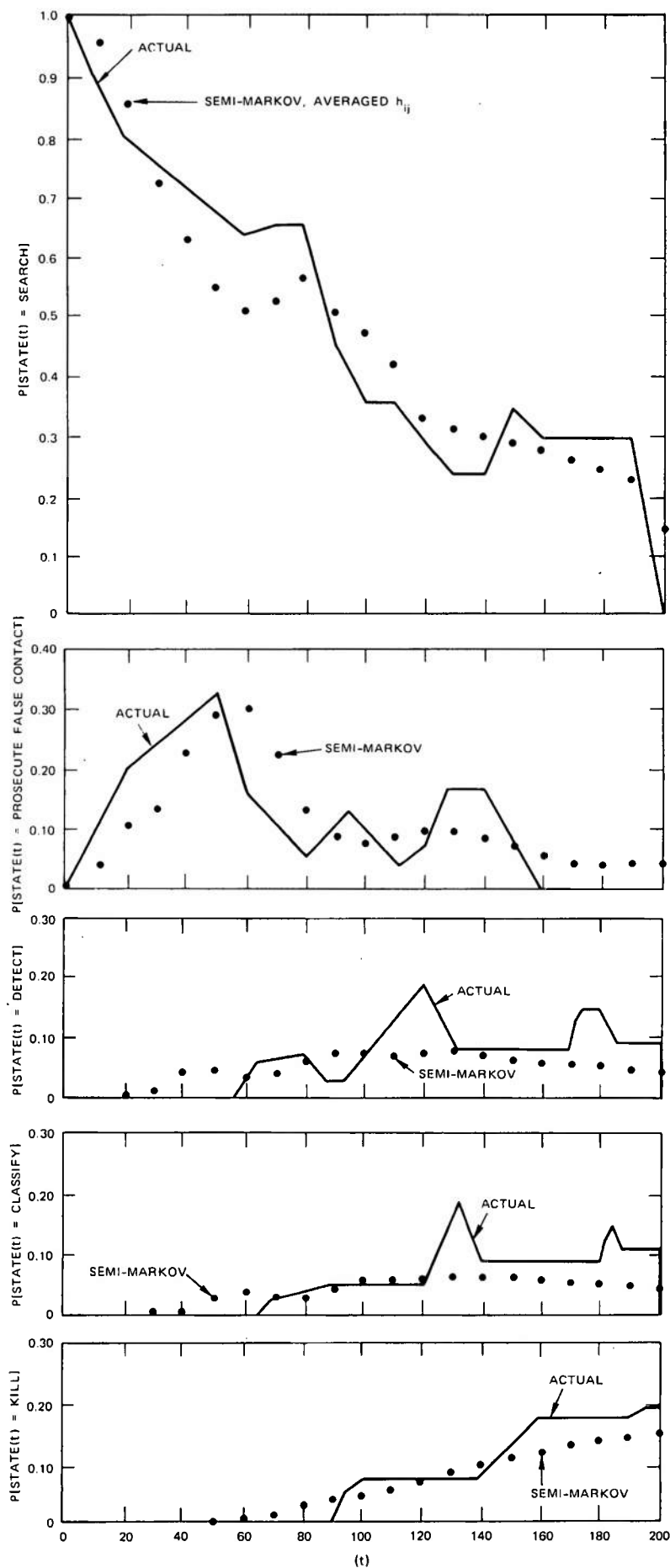


FIGURE 11.5 COMPARISON OF $P(\text{STATE})$ AS A FUNCTION OF TIME CURVES FOR ACTUAL PROCESS AND SEMI-MARKOV, AVERAGED h_{ij} APPROXIMATION (FIVE STATES)

the nature of the difficulty with the actual process was quite clear; time itself would have to enter the state definition.

1. The Fourth Semi-Markov Approximation: One-State Memory

A state transition diagram for the actual process that effectively "remembers" the previous state is shown in Figure 11-6. On this diagram, the nodes are labeled with a pair of integers--the integer to the far right represents the current state, and the integer to the left represents the last state. It may be verified that the states of this process are in one to one correspondence with the branches of the original process whose transition diagram is shown in Figure 11.2. There are now 16 states instead of nine and seven absorbing states instead of two. The probability of absorption in state 8 (fail) is now the sum of being absorbed in states 18, 28, 48, 58, 68, and 78 since there are six states from which the fail state can be entered.

Transition probabilities are again shown on branches in the transition diagram. Several of the transition probabilities are identical to those in the original process, but six differ as a result of the conditioning on the previous state. The transition probabilities that differ are shown with superscripts on the figure together with their values in the actual process.

A new formula for the probability of being absorbed in the Kill state is required; the formula is

$$p_k = p_{12}^{(1)} p_{25}^{(1)} p_{56}^{(1)} p_{67}^{(1)} p_{79}^{(1)} + \frac{p_{13}^{(1)} p_{31}^{(2)} p_{12}^{(1)} p_{25}^{(1)} p_{56}^{(1)} + p_{13}^{(1)} p_{34}^{(1)} p_{45}^{(2)} p_{56}^{(2)} p_{67}^{(2)} p_{79}^{(2)}}{1 - p_{13}^{(2)} p_{31}^{(2)}} .$$

(The first term has no denominator since the open path corresponding to the numerator does not touch the single loop in the diagram, and this makes its path determinant unity.)

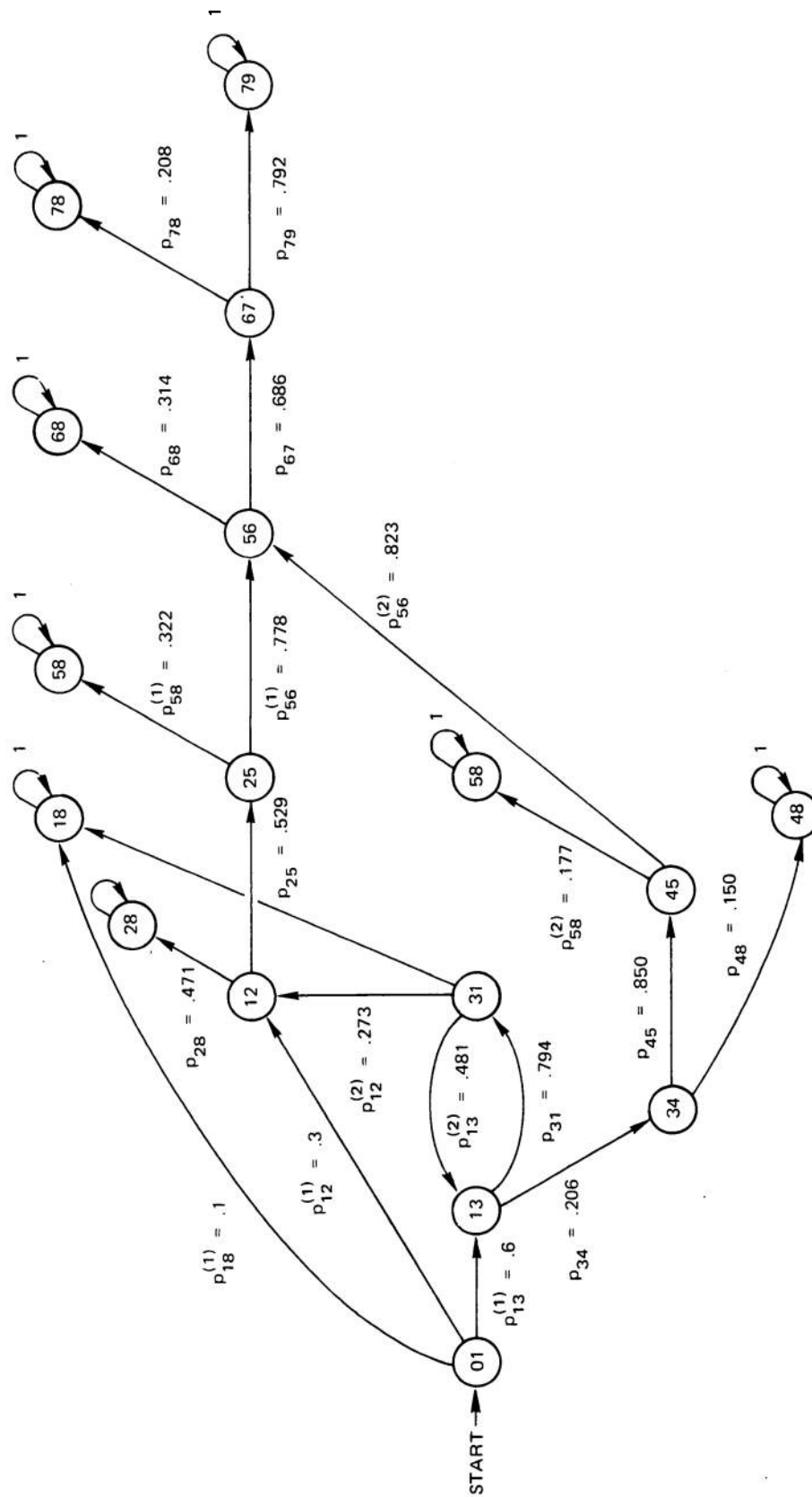


FIGURE 11.6 TRANSITION DIAGRAM AND PROBABILITIES FOR ONE-STATE MEMORY MODEL

A comparison between the true probabilities of state as a function of time and those predicted by the semi-Markov model with the averaged $h_{ij}(\cdot)$ calculated for the states in Figure 11.6 is shown in Figure 11.7. The fit is superior to any other obtained for states search, detect, classify, and kill and it is roughly the same for other states. The reason for the lack of a dramatic improvement is that the deconditioning implied by going to a one-state memory removed some dependencies, but many other dependencies were unaffected.

A rough measure of the goodness of fit of the predicted curves to the actual curves was adopted so that the various fitting methods could be compared systematically. The results tabulated in Table 11.8 show that the methods gave successively better results for all states except the prosecute false contact state. Why the measures do not decrease monotonically for this single state is unknown.

The last row of Table 11.8 shows the measures obtained for the direct estimation method. The probability of occupying a state at a time t was estimated simply by the number of times the state was occupied at time t divided by the number of replications (100). Values shown in the table are the average of two sets of 100 Monte Carlo replications. Although the sample size is small (two sets), this simple estimator may be superior to those obtained by the other methods. In the example under study, this is perhaps to be expected because of the irregularities in the true $P(\text{state})$ as a function of time curves, and the direct estimator is able to "track" these irregular curves better than the more highly structured Markov and semi-Markov methods.

2. A Loopless Version of the False Contact Model

In the subsection above, a systematic method was used to increase the number of states in order to decondition the relationships

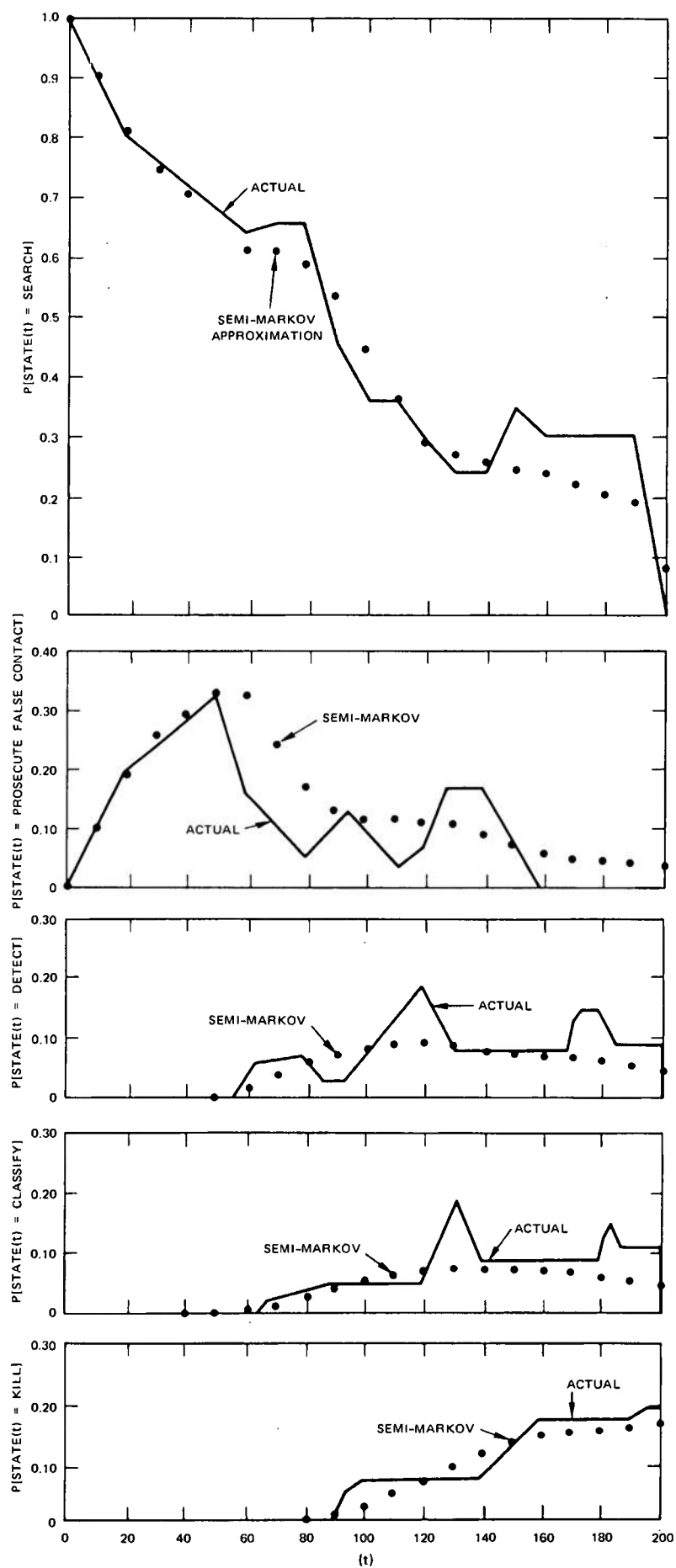


FIGURE 11.7 COMPARISON OF $P(\text{STATE}(t))$ AS A FUNCTION OF TIME CURVES FOR ACTUAL PROCESS AND SEMI-MARKOV, AVERAGED h_{ij} WITH ONE-STATE MEMORY

Table 11.8

MEASURES OF DYNAMIC FIT*

Processes	Search	Prosecute False Contact	Detect	Classify	Kill
Semi-Markov Process					
Matched mean time T_{ij}	109	45	47	33	28
Match mean and variance by uniform distribution	87	55	36	28	24
Averaged h_{ij}	80	64	35	24	24
One-state memory	46	56	28	16	17
Markov Chain					
100 replications (average of two cases)	40	--	22	--	7
Markov					
Least squares	87	66	43	37	61

* The measure is:

$$\text{constant} \times \sum_{n=1}^{10} |P(\text{state} = k \text{ at time } 20n) - \hat{P}(\text{state} = k \text{ at time } 20n)|$$

where $P(\cdot)$ is actual and $\hat{P}(\cdot)$ is predicted.

somewhat so that a better fit to the $P(\text{state})$ as a function of time relations might be attained. In that model, the previous state was "remembered" and the loop between the (original) states search and prosecute false contact was retained. This section shows how the loop was removed

by the addition of auxiliary states because this feature of the model seemed to be causing problems.

The loop is removed by treating each new entry into state 1 or state 3 as entry into a new state. When the notation is used resembling the one in the model that remembered the previous state, the state labeled 13 means "the state is 3, with entry from state 1"; the entry 131 means "the state is 1, with entry from state 31." In general, the integers that are read from left to right give the path traversed to arrive at the state represented by the last integer on the right. Figure 11.8 shows the transition diagram and the true values of the transition probabilities. It can be noted that the three ways of entering state 2 (opportunity) from state 1 (detect) have different transition probabilities. (The branches have probabilities 0.250, 0.185, 0.600.) Similarly, transitions to state 8 (fail) from state 1 (search) show differing probabilities. (The branches have probabilities 0.100, 0.075, 0.445, 0.400.) These are indications of the non-Markovian nature of the model and also show that some "deconditioning" is accomplished by this change of state space. The derived $P(\text{state})$ as a function of time curves are disappointingly similar to the last several sets and are therefore not plotted.

3. A Semi-Markov Model with Complete Memory

Considerable effort was devoted to fitting $P(\text{state})$ as a function of time for the false contact example before it was realized that at least part of the difficulties resulted from the imbedded process not being a Markov chain. To overcome this difficulty, a method for defining auxiliary states was found, the new state space automatically provided a complete memory of earlier states.

The PST diagram itself can be used to define the new state space. Each line segment on the PST diagram that defines a state change

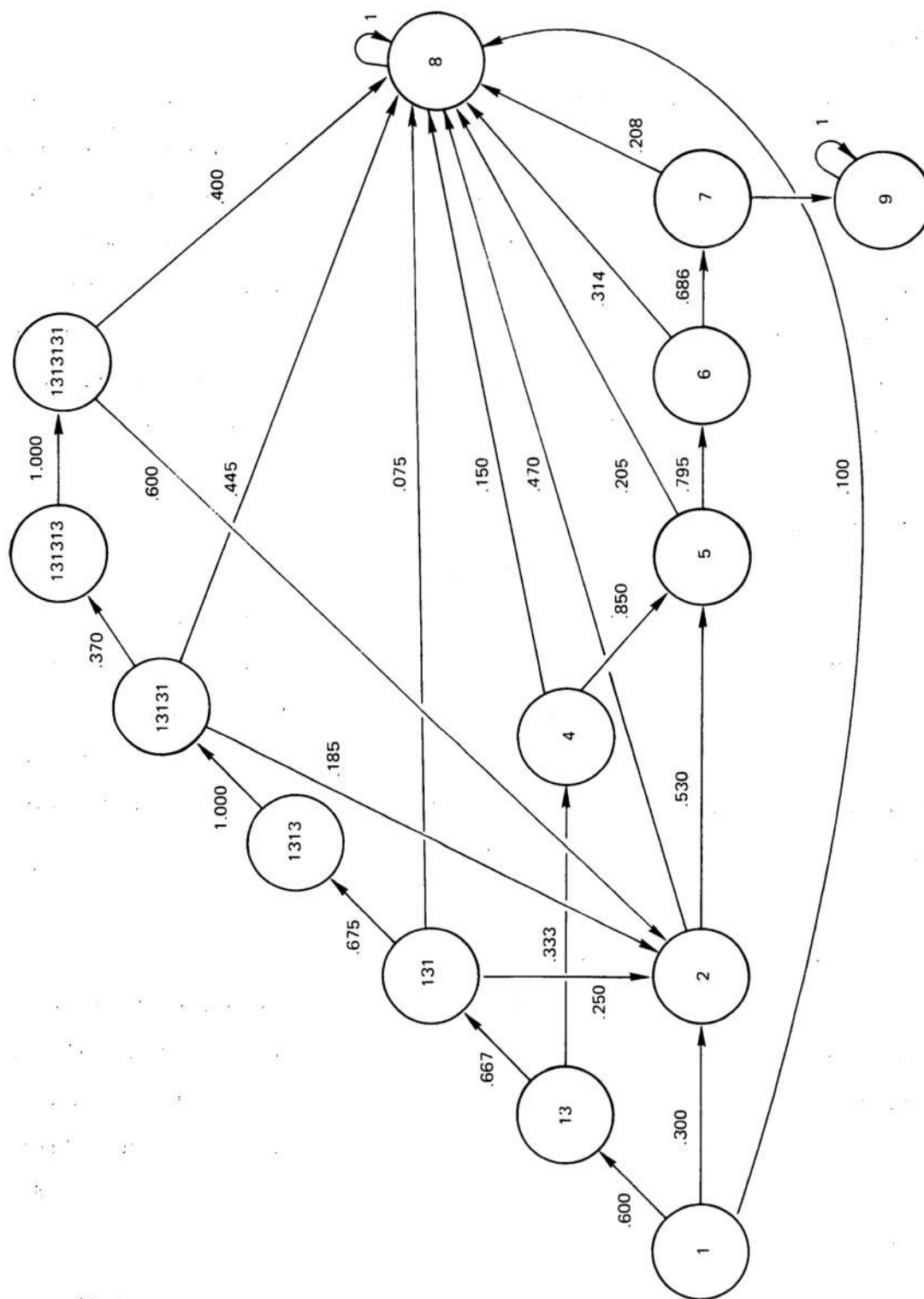


FIGURE 11.8 TRANSITION DIAGRAM FOR LOOPLESS MODEL

can be identified with a state in the new semi-Markov process. The labeling of the new states is shown in Figure 11.9; there are 47 states in all. The transition diagram with these states and with the transition probabilities computed directly from the PST diagram are shown in Figure 11.10.

The transition diagram has a single path for each separate way of reaching an absorbing state from the starting state (1a). Paths in the actual process are therefore in 1 to 1 correspondence with those on the transition diagram. The probability of a path on the transition diagram is the product of the transition probabilities along the path, and these path probabilities agree for all paths in the actual process and in the semi-Markov model approximating it when infinite data are available to estimate the transition probabilities.

Although the PST diagram was used here to generate the new state space, it should be noted that observed path data can be used for this purpose. A computer program can be written to generate the states and to do the necessary transition counting in terms of these states if desired.

A comparison may now be made between the $P(\text{state})$ as a function of transition number n for the actual and approximating imbedded Markov chains. Table 11.9 shows the data. The fit is adequate but not perfect because of the presence of the loop in the imbedded (actual) process.

After the imbedded process is reasonably well-approximated, the dynamics can be added. A uniform distribution that represents the conditional writing time is associated with each transition (branch) on Figure 11.10. These uniform distributions, with parameters taken directly from the PST diagram, were used together with the transition probabilities in a semi-Markov model to calculate $P(\text{state})$ as a function of time curves to compare with those from the actual process. Figure 11.11 compares the

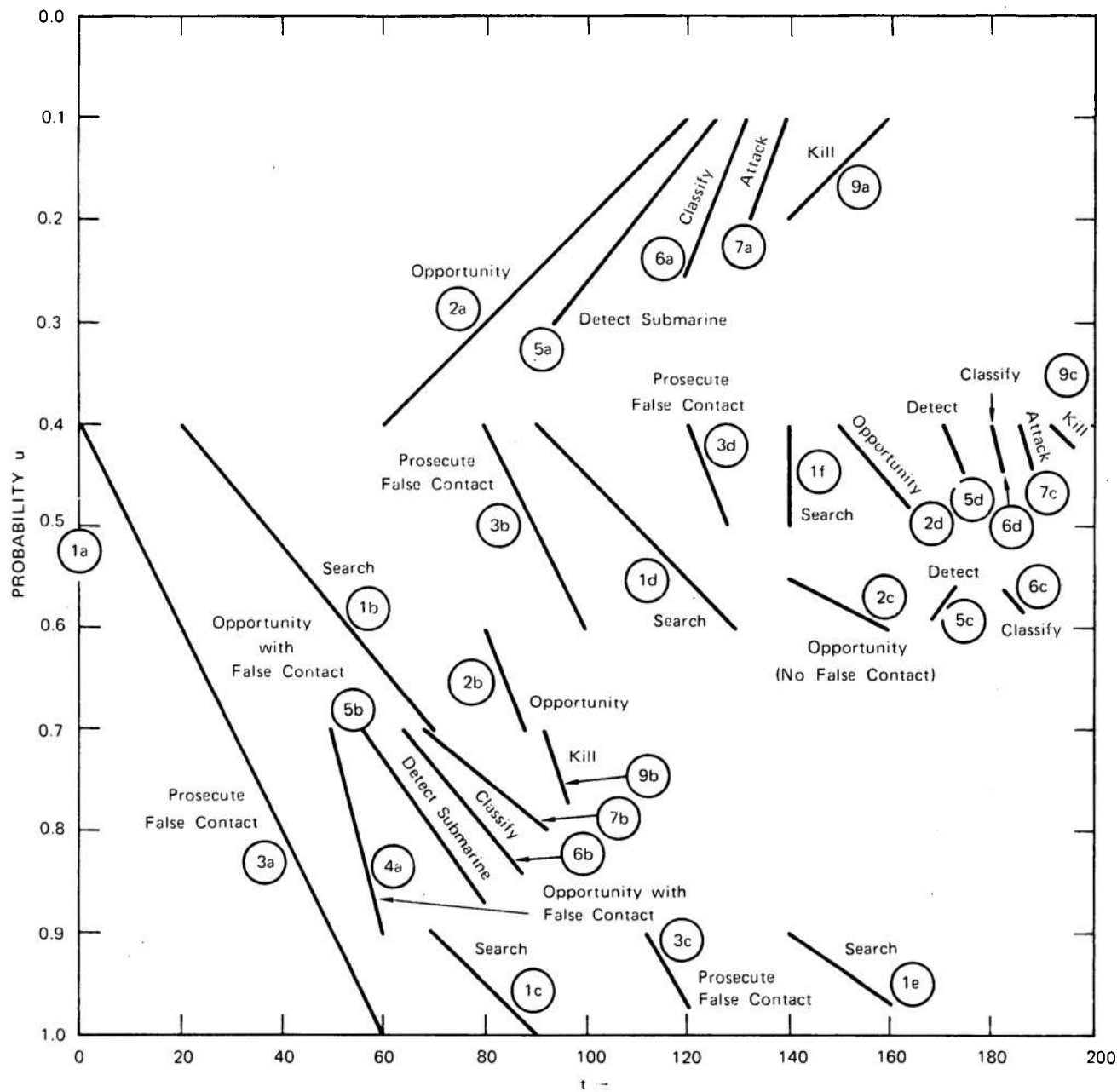


FIGURE 11.9 NEW STATE DEFINITIONS FOR SEMI-MARKOV MODEL WITH COMPLETE MEMORY

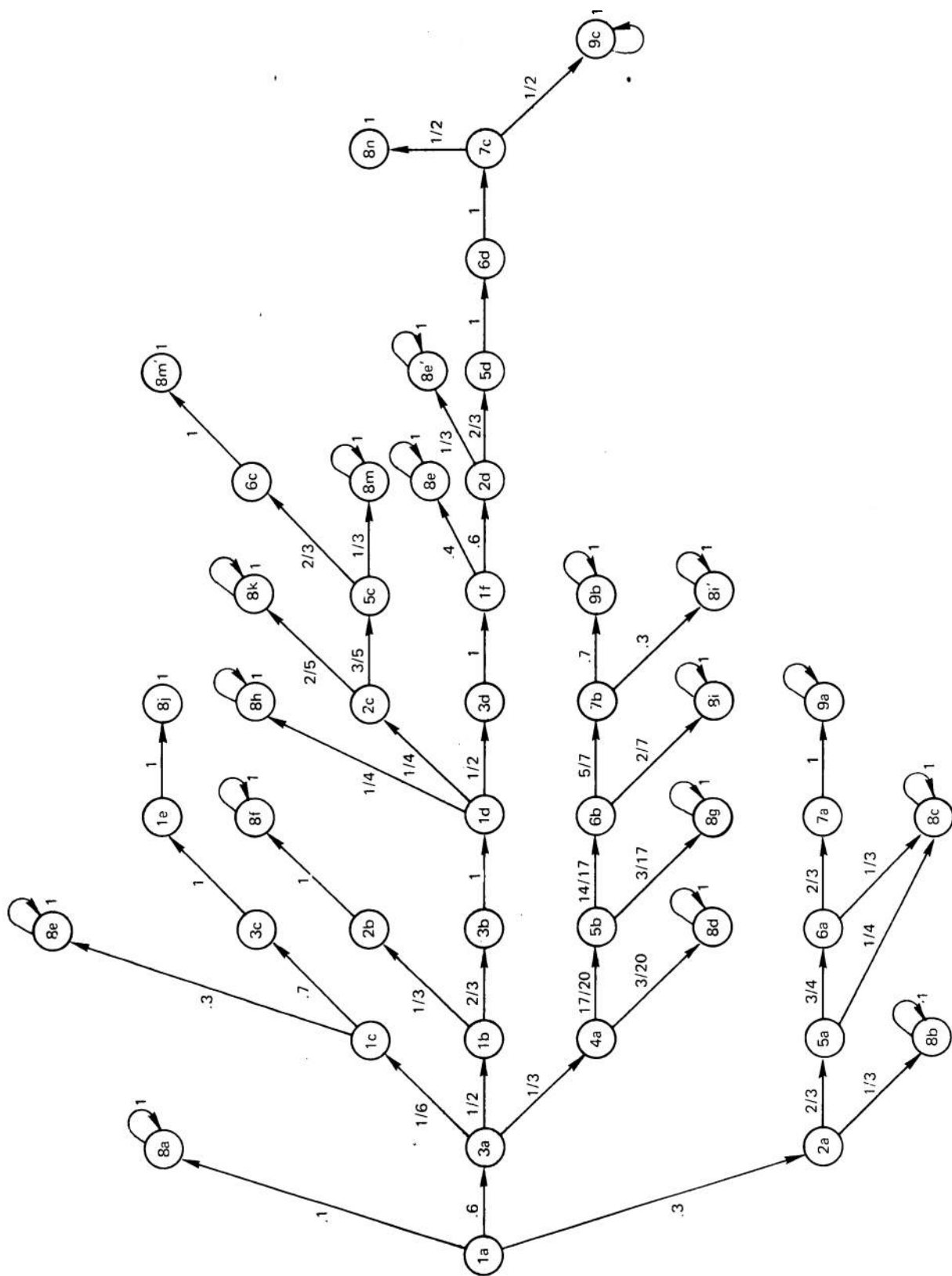


FIGURE 11.10 TRANSITION DIAGRAM WITH NEW STATES AND TRANSITION PROBABILITIES

Table 11.9
COMPARISON OF ACTUAL AND MARKOV P(STATE) AS A FUNCTION FOR IMBEDDED CHAIN

Step	State								
	1	2	3	4	5	6	7	8	9
0	1.00	0	0	0	0	0	0	0	0
1	0	0.30	0.288	0	0	0	0	0.10	0.164
2	0.40	0.435	0	0.20	0.152	0	0	0.20	0.300
3	0	0.10	0.125	0	0.17	0.096	0	0.31	0.419
4	0.27	0.189	0	0.049	0	0.14	0.083	0.49	0.536
5	0	0.05	0.055	0	0	0	0.10	0.65*	0.629*
6	0.10	0.082	0	0.021	0.03	0	0	0.70	0.691
7	0	0.06	0.024	0	0	0.02	0	0.75	0.731
8	0	0.036	0	0.009	0.04	0	0	0.79	0.758
9	0	0	0.010	0	0	0.04	0	0.79	0.776
10	0	0.016	0	0.004	0	0	0.04	0.79	0.788
11	0	0	0.004	0	0	0	0	0.81	0.795

* 0.65 = approximate; 0.629 = actual.

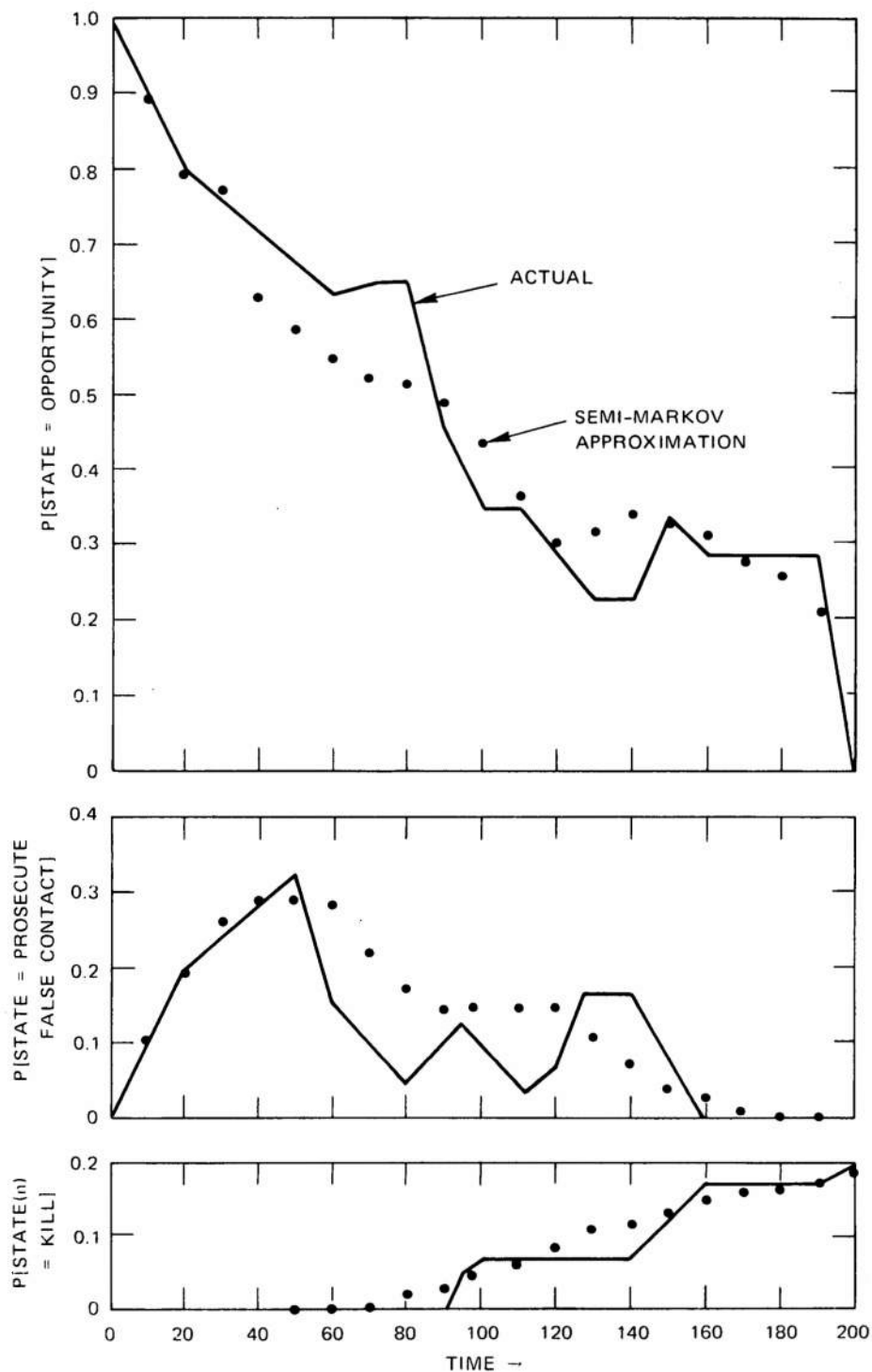


FIGURE 11.11 COMPARISON OF $P(\text{STATE})$ AS A FUNCTION OF TIME CURVES FOR ACTUAL PROCESS AND 47 STATE SEMI-MARKOV PROCESS APPROXIMATION

results for the search, prosecute false contact, and kill states. The fit appears to be better than that obtained in earlier sections because the approximating process is "responding" for the first time to some of the irregularities in the actual process. The curves still were not as good as expected. After more experimentation, the principal difficulty causing all approximations to be poor finally became evident; the problem can be seen most easily on the PST diagram.

Consider the transition from state 1a to state 3a (0.40 to 1.0). Each of these transitions is given the same distribution of waiting time in state 1a, regardless of transitions later on. It is apparent from the PST diagram that this wide distribution (transition time ranged from 0 to 60) was distorting the time behavior for all later transitions. For example, the transitions to the kill state for the range of u from 0.40 to 0.42, which have an actual 1a to 3a transition time varying from zero to approximately two time units, have transition times over the full range from 0 to 60 time units in the 47-state semi-Markov approximation. This phenomenon, repeated in many places, is causing the poor $P(\text{state})$ as a function of time fits for all of the approximations considered. It is believed that time-in-state itself must be incorporated into the state definition in order to overcome this difficulty. Although this insight comes from the form of the PST diagram itself, it is clear that analysis of the path data also could reveal the nature of this obstacle and suggest a way to remove it.

E. Testing the Markov Property

Since the application of Markov chain models is being considered for a variety of ASW tactical analyses, it is of interest to know how the hypothesis of "Markovianness" can be tested. When the time behavior of the process is ignored and only the imbedded Markov chain is observed, a

rather simple test is possible when data are available on entire paths through the Markov transition diagram. The idea is that when a Markov chain with known transition probabilities is hypothesized the path probabilities can be calculated from the formula

$$P(\text{path}) = \text{product of transition probabilities along the path} \quad .$$

To obtain this relationship, the Markov property (lack of memory at a state) is invoked.

Consider a Markov chain with a single starting state and two absorbing states. Let the paths in a Markov chain be labeled according to some arbitrary numbering system--such as numbering according to their probabilities. Since there are infinitely many distinct paths possible whenever the transition diagram has loops, it will be necessary to combine all paths beyond some number, say $k - 1$. Then each of the infinitely many paths falls into one of k classes; the first $(k - 1)$ paths each has its own class, and the k^{th} class has all the remaining paths.

The classical chi-squared goodness of fit test may now be applied. Let there be n independent trials, each resulting in a path through the system. Let p_i be the probability of observing a path in class i ($i = 1, 2, \dots, k$) and f_i be the observed number of sample paths falling in class i . Then according to Ref. 7 the quantity

$$\chi_{k-1}^2 = \sum_{i=1}^k \frac{(f_i - n p_i)^2}{n p_i}$$

is asymptotically distributed as chi-squared with $k - 1$ degrees of freedom.

In the false contact model defined by the PST diagram in Figure 11.1, the paths and their probabilities can be determined by straightforward but tedious calculations. In the PST diagram, the actual probabilities are

determined by finding the total length of the intervals in which the random variable u determines a path of the designated type. For example, the path detect-fail (designated 18) is generated in the single interval $0 \leq u \leq 0.10$ so that $P(\text{path } 18) = 0.10$.

Table 11.10 compares the actual path probabilities with the probabilities that are derived from a Markov model. Inspection of this table shows two things of interest. The first is that the actual path probabilities (which reflect the Markov assumption) are in general very different from the probabilities the paths must have if the process is truly Markov with transition probabilities obtained from infinite sampling. The second aspect of interest is that the sum of the path probabilities over the paths observable from the actual process is only 0.6177; missing paths have probability $1 - 0.6177 \cong 0.38$. Because there are infinitely many paths with positive probability, it was known in advance that the sum over any finite number of paths would be less than unity but it is somewhat surprising to see the sum so much less than unity in this example.

Applying the chi-squared test to this example using data from Table 11.10 will always result in failure of the model to pass the test whenever the sample size is around 100. To see this, consider 22 classes into which the result of an experiment may fall: the first 21 classes correspond to the 21 possible paths enumerated in the table and all other possible paths comprise the remaining class. In terms of the chi-squared formula, the value of k is 22, and the last term of the sum has value

$$\left(0 - n p_{22}\right)^2 / n p_{22} = n p_{22} = \text{expected number of paths in}$$

the unobserved class .

Table 11.10

COMPARISON OF PATH PROBABILITIES

Path	Actual Probability of Path	Markov Chain Probability
1 8	0.10	0.1640
1 2 8	0.10	0.13565
1 2 5 8	0.05	0.031248
1 2 5 6 8	0.05	0.038032
1 2 5 6 7 9	0.10	0.06581
1 3 1 3 1 3 1 2 5 6 7 9	0.02	0.0068272
1 3 1 3 1 3 1 2 5 6 7 8	0.02	0.001793
1 3 1 3 1 3 1 2 8	0.02	0.014073
1 3 1 3 1 3 1 8	0.04	0.0011808
1 3 1 3 1 8	0.12	0.0096796
1 3 1 3 1 2 5 6 8	0.02	0.007200
1 3 1 3 1 2 8	0.02	0.0045591
1 3 1 3 1 2 5 8	0.01	0.00093462
1 3 1 2 8	0.10	0.059022
1 3 4 5 6 7 9	0.07	0.001420
1 3 4 5 6 7 8	0.03	0.000658
1 3 4 5 6 8	0.04	0.001449
1 3 4 5 8	0.03	0.001190
1 3 4 8	0.03	0.016933
1 3 1 8	0.03	0.071358
	$\Sigma = 1.00$	$\Sigma = 0.664045$

If there are 100 samples ($n = 100$), the value of $n p_{22}$ is

$$100 \times (1 - 0.6177) \cong 38 ,$$

which corresponds to a significance level of 0.99 at $k - 1 = 21$ degrees of freedom. The remaining terms in the sum are all nonnegative so that chi-squared exceeds 38, hence the hypothesis is rejected at the 0.99 level. Since the sums of squares will be appreciable, it can be anticipated that a sample size of much less than 100 will practically guarantee failure of the test. This shows the process to be so badly non-Markovian that the hypothesis that path data generated by the process could have arisen from sampling a Markov chain is always rejected for samples of size 100 or greater. The chi-squared test thus serves a useful purpose in this example by alerting the analyst to the strong possibility that the model to which Markov methods are being applied is actually non-Markovian.

Many other chi-squared tests also can be devised for Markovianity of the process. A test entailing the observed proportions of the time the process is in state j at t [$y_j(t)$] and the expected proportions $p_j(t)$ is readily defined by noting that if there are n samples of the complete process, then $n p_j(t)$ is the expected number of samples for which the state is j at time t and $n y_j(t)$ is the observed number at time t . When all times and all r states are summed, a random variable $\chi^2_{(r-1)T}$ that is asymptotically chi-squared with $(r - 1)T$ degrees of freedom results when the process is Markovian.

$$\chi^2_{(r-1)T} = \sum_{t=1}^T \sum_{j=1}^r \frac{[n y_j(t) - n p_j(t)]^2}{n p_j(t)} = \sum_{t=1}^T \sum_{j=1}^r \frac{n[y_j(t) - p_j(t)]^2}{p_j(t)} .$$

This formulation assumes that all $p_j(t)$ are nonzero. Practical experience with the chi-squared test suggests that the classes into which observations

may fall should be such that at least five observations are expected in each class. These considerations must be considered in the use of this test, and some alteration of the summation region and an adjustment in the number of degrees of freedom will result.

XII MARKOV MODELS AND MONTE CARLO SIMULATION

A. Introduction

This section shows how Markov models can be used in conjunction with a Monte Carlo simulation model for one of several purposes. One reason to consider a Markov model in a context in which a simulation model exists may be simply to explain and interpret the simulation model: suitably choosing a state space and recording transition counts and transition times may make it possible to gain insight into the simulated problem that could not be gained by examination of the end results of the simulation alone. A second reason may be to build an auxiliary model that is easier to use for some purposes other than the simulation itself, such as sensitivity analysis. In some situations, it may be possible to replace the simulation model entirely by the Markov model once the simulated system is understood. A third reason that a Markov model may be desirable relates to validation of the simulation. Because the $P(\text{state})$ as a function of time relationships effectively characterizes the model dynamics and a valid Monte Carlo model should adequately model the dynamics of the tactical situation as well as the end outcomes, the simulation may be validated by a comparison of the $P(\text{state})$ as a function of time curves that it generates with the same relationships estimated from operational data.

In this section some experimentation has been done using data from a Monte Carlo simulation model developed by NWRC for studying the effectiveness of acoustic deception devices (ADDs) in a transit area scenario. The simulation data were generated to:

- Determine whether the simulation model could be adequately represented by a Markov model.
- Estimate the parameters of the Markov model.
- Determine properties of the Markov model that could be compared with the corresponding properties of the simulation.
- Show how some sensitivity analysis may be performed by using the Markov model with its estimated parameters.

B. The Simulation Model

The simulation model selected is described in detail in Ref. 10. In outline, a single submarine and a single High Value Unit (HVV) are contained within an objective area represented by a circle with radius 200 nmi. The HVV or its supporting forces may deploy various numbers of acoustic deception devices so that submarine detection and classification of the HVV will be more difficult. The submarine searches until it detects either the HVV or an ADD; detecting an ADD "captures" the submarine for a period of time called "capture time." During the capture time, the submarine closes the ADD to investigate it further and then returns to search at the end of the capture time period. All units have prescribed forms of random motion consisting of straightline tracks with random course changes superimposed. Capture may occur any number of times before detection of the HVV occurs, and the model stops when the HVV has been detected. Principal inputs to the model are speeds of the HVV, submarine, ADD, length of the straightline track between random course changes, sonar detection ranges of the HVV and the ADDs, and a parameter essentially describing the length of time that a submarine "remembers" a previously detected ADD. Parameters selected correspond to those shown on Figure 5-2 of Ref. 10; they are:

Number of ADDs = 1

Capture time = 20 hours

Detection ranges (on both the HVU and ADD) = 60 nmi

Memory time = 12 hours

Speed of all units = 10 knots

Length of straightline track segments = 50 nmi.

C. Selection of States for the Markov Model

The states originally selected were those used in an analytical semi-Markov model of the same scenario as described in Ref. 6. States in this three-state model were designated search, capture, and detection (of the HVU), with detection being an absorbing state and search the starting state. However, examination of the data soon revealed that there should be two search states--initial search and search--because of the considerable difference in the parameters for initial searches and later searches. The transition diagram for the imbedded Markov chain is shown in Figure 12.1. (See also Appendix F.)

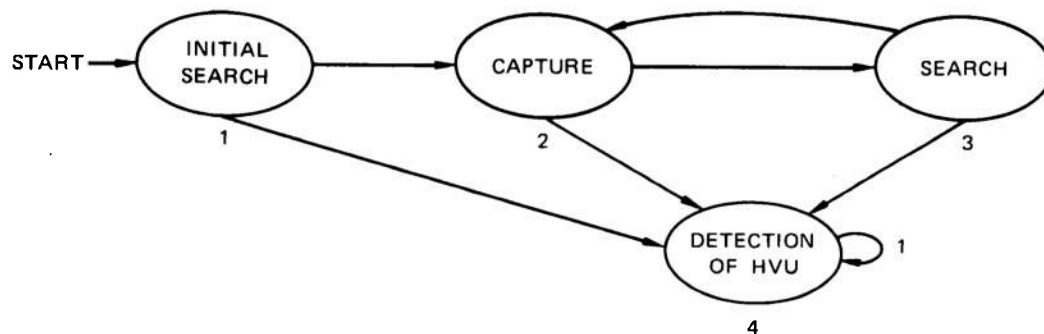


FIGURE 12.1 IMBEDDED MARKOV CHAIN FOR SIMULATION MODEL

Since the simulation model did not employ Markovian assumptions, it was decided that a test of the reasonableness of a Markov approximation should be performed. This was accomplished by the same method used to obtain a loopless version of the false contact model in Section XI. Basically, the loop in Figure 12.1 is unwound by considering successive

entries into capture and search as new states as shown in Figure 12.2. For convenience, the states are numbered as shown.

In a Markov model, the transition times between states are statistically independent of one another. If the simulation model is Markov, it should then exhibit this property also; therefore, a crude test of Markovianness can be formulated on the basis of this assumption.

Let $\bar{T}_i$ be the average time that state i is first entered, and $\bar{T}_{ij}$ = mean time spent in state i , given that the next transition is to state j . Observing that each capture-to-search transition should have the same mean time $\bar{T}_c$ and each search-to-capture transition should have the same mean time $\bar{T}_s$, independence of transitions implies

$$\bar{T}_2 = \bar{T}_{12}$$

$$\bar{T}_3 = \bar{T}_{12} + \bar{T}_c$$

$$\bar{T}_4 = \bar{T}_{12} + \bar{T}_c + \bar{T}_s$$

$$\bar{T}_5 = \bar{T}_{12} + \bar{T}_c + \bar{T}_s + \bar{T}_c = \bar{T}_{12} + 2\bar{T}_c + \bar{T}_s$$

$$\bar{T}_6 = \bar{T}_{12} + 2\bar{T}_c + 2\bar{T}_s$$

$$\vdots$$

Inspecting $\bar{T}_n$ for even n shows that the quantity $(\bar{T}_c + \bar{T}_s)$ is added to $\bar{T}_{n-2}$ for each successive capture-search-capture cycle. Therefore, the even values of $\bar{T}_n$ should plot in a straight line as a function of n .

Figure 12.3 shows a plot of $\bar{T}_n$ as a function of n computed from 1000 replications of the simulation. The plotted values for even-numbered

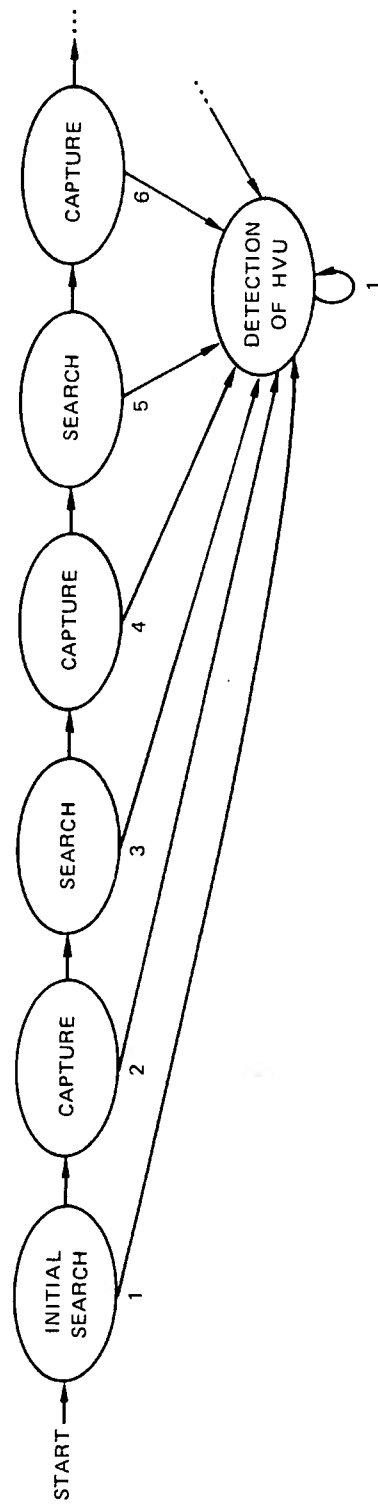


FIGURE 12.2 UNWINDING THE IMBEDDED MARKOV CHAIN

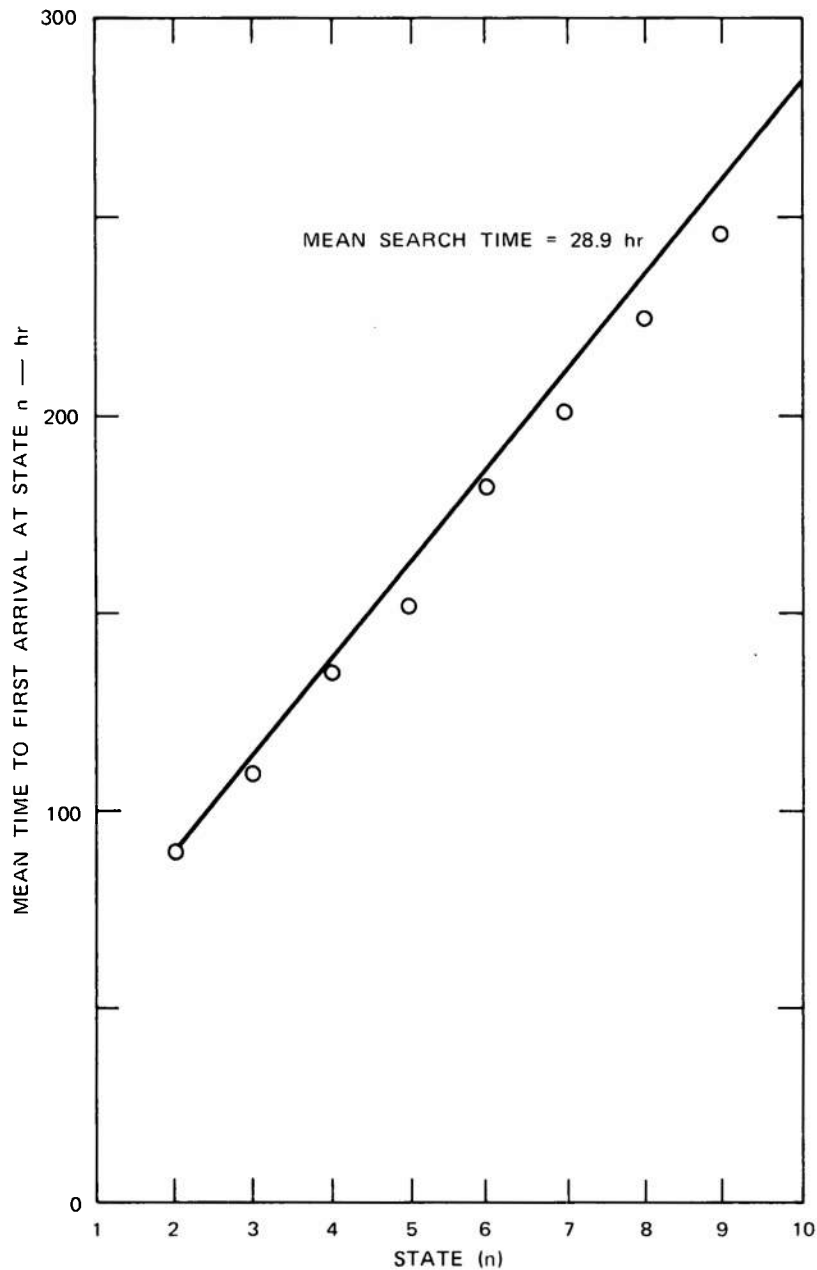


FIGURE 12.3 MEAN TIMES OF FIRST ENTRY INTO STATE i ($\bar{T}_i$)

states fall very nearly on a straight line, which tends to confirm the assumption of independence of transition times. A straight line has been drawn through the initial point to represent a mean search time of 28.9 hours and mean capture time of 20 hours (mean search time was calculated from the same set of replications). Other sets of replications also gave

this same near-linear pattern and agreement between the $\bar{T}_i$ data points and the continuous line found by averaging search times.

The complete output from the set of 1000 replications is shown in tabular form in Table 12.1. $\bar{T}_i$ and $\bar{T}_{i,i+1}$ have already been defined, $\text{var } T_i$ is the variance of T_i , and $\text{var } T_{i,i+1}$ is the variance of $T_{i,i+1}$. The number of replications entering state i is N_i , and the values decrease because of detections of the HVU. The last column is the estimate of the transition probability $p_{i,i+1}$ obtained from the ratio N_{i+1}/N_i .

Table 12.1

COMPUTER SIMULATION OUTPUTS

State	State Number	$\bar{T}_i$	$\text{var } T_i$	$\bar{T}_{i,i+1}$	$\text{var } T_{i,i+1}$	N_i	$\hat{p}_{i,i+1}$
S	1	0	0	0	0	1000	0.473
C	2	89.5	9,568	89.5	9568	473	0.922
S	3	109.5	9,843	20.0	0	436	0.706
C	4	134.8	11,952	27.0	2030	308	0.948
S	5	152.0	11,891	20.0	0	292	0.733
C	6	182.0	15,118	30.1	3539	214	0.953
S	7	200.5	14,752	20.0	0	204	0.819
C	8	224.5	16,674	30.9	3009	167	0.940
S	9	245.4	17,296	20.0	0	157	--

Mean times to first reach state i should also be obtainable by accumulating the values in the $\bar{T}_{i,i+1}$ column as shown in Table 12.2, where the $\bar{T}_i$ are repeated for comparison.

The two estimates for $\bar{T}_i$ do not agree because they are over different samples; the $\bar{T}_i$ are averaged over only those cases reaching state i , while

Table 12.2

COMPARING MEANS AND VARIANCES

State Number	$\bar{T}_i$	$\Sigma \bar{T}_{i,i+1}$	$\text{var } \bar{T}_i$	$\sum_{i=1}^{n-1} \text{var } T_{i,i+1}$
1	0	0	0	0
2	89.5	89.5	9,568	9,568
3	109.5	109.5	9,843	9,568
4	134.8	136.5	11,952	11,598
5	152.0	156.5	11,891	11,598
6	182.0	186.6	15,118	15,137
7	200.5	206.6	14,752	15,137
8	224.5	237.5	16,674	18,146
9	245.4	257.0	17,296	18,146

the $\bar{T}_{i,i+1}$ are obtained by averaging over all transitions between i and $i + 1$. (Since more data are used to estimate the values in the second column, their variances should be somewhat smaller.)

Several other tests of the reasonableness of the Markov assumption can be made from the data in Table 12.1. Since the random variables $T_{i,i+1}$ are assumed to be independent, their variances should also be additive; that is,

$$\text{var}(T_n) = \sum_{i=1}^{n-1} \text{var } T_{i,i+1} .$$

The columns on the right side of Table 12.2 compare the two estimates of $\text{var } T_i$, but they do not agree for the same reasons that the means do not agree. A rough estimate of the average variance per cycle is found by averaging the last three values in the $\text{var } T_{i,i+1}$ column; the average

variance is $(2030 + 3539 + 3009)/3 \cong 2859$ and the average standard deviation is therefore $\sqrt{2859} \cong 53.5$.

If the waiting time in the search state were exponentially distributed, this standard deviation would be expected to be about equal to the mean value of 28.9 hours. The disparity between these values indicates that the waiting time densities are not geometric as they are in a Markov chain. (Despite this, a Markov chain approximation works quite well.)

D. Estimating the Transition Probabilities from Transition Data

On the basis of the $\hat{p}_{i,i+1}$ column in Table 12.1, the alternating values 0.922, 0.948, 0.953, and 0.940 should all be estimates of

$$P[S(n+1) = \text{search} | S(n) = \text{capture}] \equiv p_{23} \equiv p_{s/c} .$$

The values cannot be averaged to obtain $\hat{p}_{23}$ because different sample sizes are entailed. Similarly, the alternating values 0.706, 0.733, and 0.819 all estimate

$$P[S(n+1) = \text{capture} | S(n) = \text{search}] \equiv p_{32} \equiv p_{c/s} .$$

The value at the top of the column (0.473) is an estimate of

$$P(\text{next state is capture} | \text{this state is initial search}) \equiv p_{12} .$$

From symmetry of the HVU and ADD with respect to the submarine's Initial Search, this value is expected to be close to $1/2$.

The weighted estimates of $p_{s/c}$ and $p_{c/s}$ may be found most easily by using the N_i column of Table 12.1.

$$\hat{p}_{s/c} = \frac{436 + 292 + 204 + 157}{473 + 308 + 214 + 167} \cong 0.9372$$

$$\hat{p}_{c/s} = \frac{308 + 214 + 167}{436 + 292 + 204} \cong 0.7393$$

Therefore, the transition diagram for the imbedded chain is that shown in Figure 12.4.

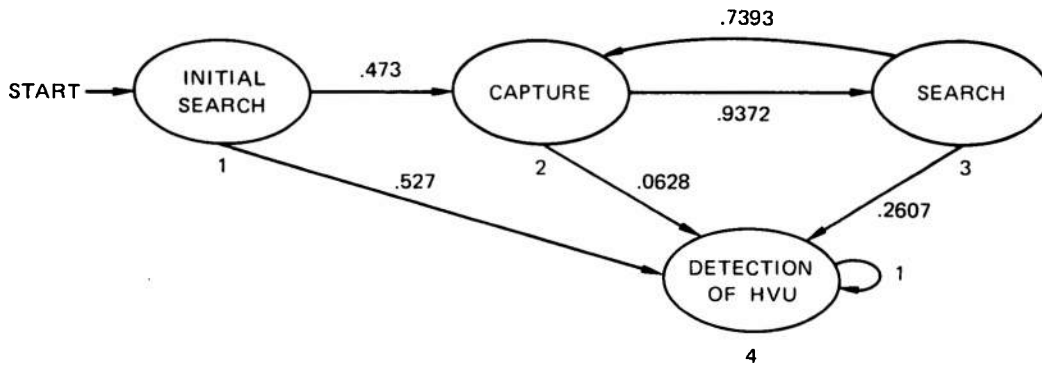


FIGURE 12.4 TRANSITION DIAGRAM FOR IMBEDDED CHAIN OF SIMULATION MODEL

From here it is a short step to the Markov chain approximation. Weighting the average times T_i by the N_i and normalizing give an average duration of search of 28.9 hours. Therefore, the self-loop p_{33} should have a value $1 - 1/28.9 = 0.9654$ to give the correct average waiting time in state 3 (search). The p_{11} that gives an average of 89.5 hours in state 1 is $1 - 1/89.5 = 0.9888$. Similarly, since the average time in the capture state (2) is the constant capture time of 20 hours, the value of p_{22} is $1 - 1/20 = 0.95$. Normalizing the probabilities shown on Figure 12.4 for the imbedded chain results in the transition diagram for the Markov chain approximation as shown in Figure 12.5.

With the transition probability estimates shown in Figure 12.5, the $P(\text{state})$ as a function of time estimates can be found by raising the

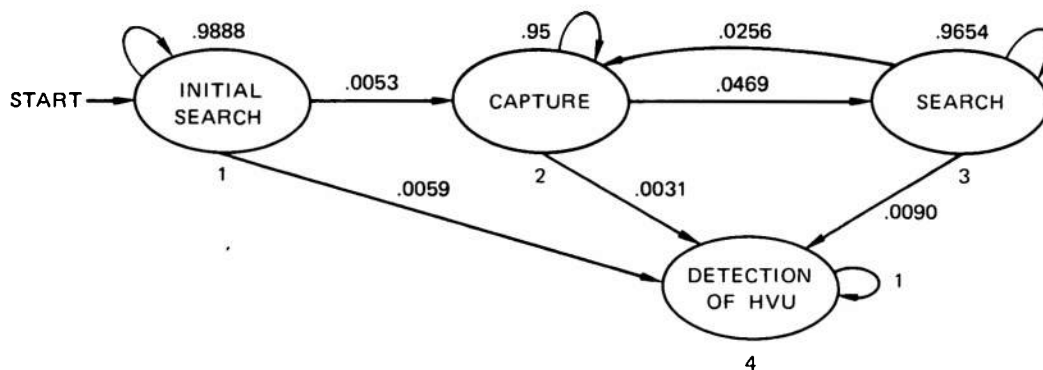


FIGURE 12.5 TRANSITION DIAGRAM FOR MARKOV CHAIN APPROXIMATION

transition matrix P to successive powers. Figure 12.6 shows the following principal curve of interest:

$$\hat{P}_M(t) \equiv P(\text{HVU is not detected by time } t) = 1 - P(\text{absorption in state 4 by time } t)$$

Also shown on Figure 12.6 is another estimate of this probability obtained from the ratios

$$\hat{P}_S(t) = P(\text{HVU is not detected by } t) = \frac{\text{number of replications for which HVU is not detected at } t}{\text{number of replications}}$$

[$\hat{P}_S(t)$ was calculated from an independent set of 300 replications, the 1000 replication set used to obtain $\hat{P}_M(t)$ could not be used due to computer limitations.]

The correspondence between $\hat{P}_M(t)$ and $\hat{P}_S(t)$ is quite good. It shows that the Markov chain approximation may be adequate even though the assumption of exponential holding times is not satisfied at the capture and search states.

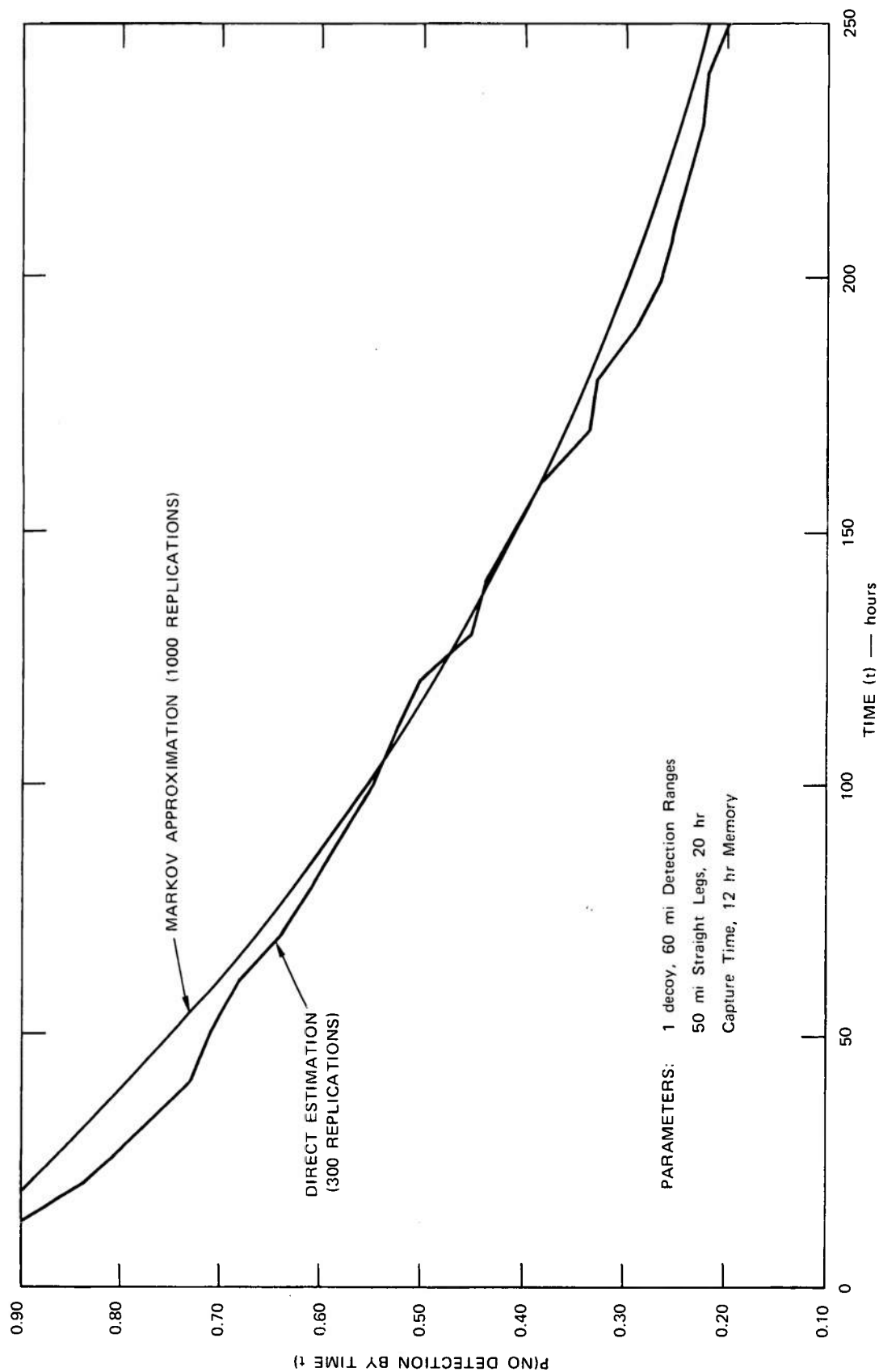


FIGURE 12.6 COMPARISON OF P(NO DETECTION BY TIME t) FOR SIMULATION MODEL AND MARKOV CHAIN WITH PARAMETERS ESTIMATED FROM 1000 INDEPENDENT TRIALS

The average time-in-state (transmission) matrix is easily found to be

$$T = (I - Q)^{-1} = \begin{pmatrix} 89.2857 & 30.9302 & 41.9256 \\ 0 & 65.3619 & 88.5976 \\ 0 & 48.3603 & 94.4537 \end{pmatrix} .$$

The (1,1) element is the average duration of the initial search, and it agrees quite well with the value 89.5 hours shown in Table 12.2. The average time in the capture state is the (1,2) element (30.9302 hours). Therefore, an estimate of the average number of captures per trial is

$$\frac{\text{average time in capture}}{\text{average duration of a single capture}} = \frac{30.9302}{20} \cong 1.5 .$$

Since there is only a single absorbing state, the absorption matrix $A = (I - Q)^{-1}R$ must have all components equal to one; this is easily verified numerically. The average time till detection matrix is

$$(I - Q)^{-1}(I - Q)^{-1}R = TA ,$$

and no normalization is necessary because of the single absorbing state.

Using values from Figure 12.5, we obtain

$$TA = \begin{pmatrix} 162.142 \\ 153.959 \\ 142.814 \end{pmatrix} = \begin{pmatrix} \text{mean time to Detect starting in state 1} \\ \text{mean time to Detect starting in state 2} \\ \text{mean time to Detect starting in state 3} \end{pmatrix}$$

so that the mean time to detection of the HVU is about 162 hours. (The sum of the elements in the first row is also 162.142 hours because there is only one absorbing state.)

It is informative to consider the individual terms of the inner product of the first row of $(I - Q)^{-1}$ and R , for these terms are the average numbers of detections from states 1, 2, and 3, respectively. Thus,

Average number of HVU

detections from initial search = $89.2857 \times 0.0059 \cong 0.527$

Average number of

detections from capture = $30.9302 \times 0.0031 \cong 0.096$

Average number of

detections from search = $41.9256 \times 0.0090 \cong 0.377$.

(The sum of these elements is unity, the probability of absorbing in state 4.) Although there are 1.5 captures on the average, slightly more than half of the HVU detections occurred without a capture. The average number of HVU detections from initial search (0.527) is seen to agree with the 1-to-4 transition probability for the imbedded chain of Figure 12.4.

E. Estimating the Parameters from Average Time-in-State Data

Section VI-C-2 showed that estimates $\hat{t}_{ij}$ of the average time spent in each state j (given starting state i) can be used to estimate transition probabilities. A set of 50 replications was run to obtain the data shown in Table 12.3. All data were generated starting in state 1; however, and the $\hat{t}_{ij}$ must be obtained for starting states 2 and 3 also. Removing the transitions from state 1 to state 2 made it possible to estimate the $\hat{t}_{2j}$ with data from the same set of replications; similarly, the $\hat{t}_{3j}$

Table 12.3

SIMULATION OUTPUT WHEN USING AVERAGE TIME-IN-STATE DATA

State	$\bar{T}_{i,i+1}$	N_i
1	0	50
2	76.9	27
3	20.0	26
4	16.1	24
5	20.0	22
6	36.8	15
7	20.0	13
8	15.6	11
9	20.0	8
10	21.1	5
11	20.0	5
12	12.5	2
13	20.0	1
14	18.5	1

were estimated by removing all transitions before the first entry into state 3.

The estimated transition matrix $\tilde{T}$ of average times-in-state is

$$\tilde{T} = \begin{pmatrix} 76.9 & 34 & 33.6 \\ 0 & 63 & 62.2 \\ 0 & 44.6 & 64.6 \end{pmatrix} = (I - \tilde{Q})^{-1}$$

from which $\tilde{Q}$ is determined to be

$$\tilde{Q} = (I - \tilde{T})^{-1} = \begin{pmatrix} 0.986996 & 0.006939 & 0.000111 \\ 0 & 0.950288 & 0.047801 \\ 0 & 0.034422 & 0.951376 \end{pmatrix}.$$

The $\tilde{R}$ is determined so that the sum of the elements in each row of $\tilde{P}$ is unity. Thus, the estimated transition matrix $\tilde{P}$ is

$$\tilde{P} = \begin{pmatrix} \tilde{Q} & \tilde{R} \\ 0 & I \end{pmatrix} = \left(\begin{array}{ccc|c} 0.986996 & 0.006939 & 0.000111 & 0.005954 \\ 0 & 0.950288 & 0.047801 & 0.001911 \\ 0 & 0.034422 & 0.951376 & 0.014202 \\ \hline 0 & 0 & 0 & 1 \end{array} \right).$$

[A calculation can be made to see whether $(I - \tilde{Q})^{-1}\tilde{R}$ equals $(1,1,1)^T$; the result is $(1.0000238, 1.0037574, 1.0026789)^T$ which is satisfactorily close.]

Raising $\tilde{P}$ to the n^{th} power and plotting $1 - \tilde{p}_{14}^{(n)}$ as a function of time n produces the results in Figure 12.7.

Another estimate of this same curve for $P(\text{no detection by time } n)$ was made by using the transition data and average waiting time data as in the earlier 1000 replication case. The estimated transition matrix $\hat{P}$ is

$$\hat{P} = \begin{pmatrix} 0.9870 & 0.0070 & 0 & 0.0060 \\ 0 & 0.95 & 0.0441 & 0.0059 \\ 0 & 0.0357 & 0.9539 & 0.0104 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When $\hat{P}$ is used to determine $1 - \hat{p}_{14}^{(n)} = P(\text{no detection of HVU by } n)$, the curve plots essentially on top of the $\tilde{P}$ -derived curve just found as shown in Figure 12.7. In this case, the two methods of estimating $P(\text{no detection of the HVU by time } n)$ appear to be virtually equivalent. (Whether this will also be true in other contexts is unknown. It is easy

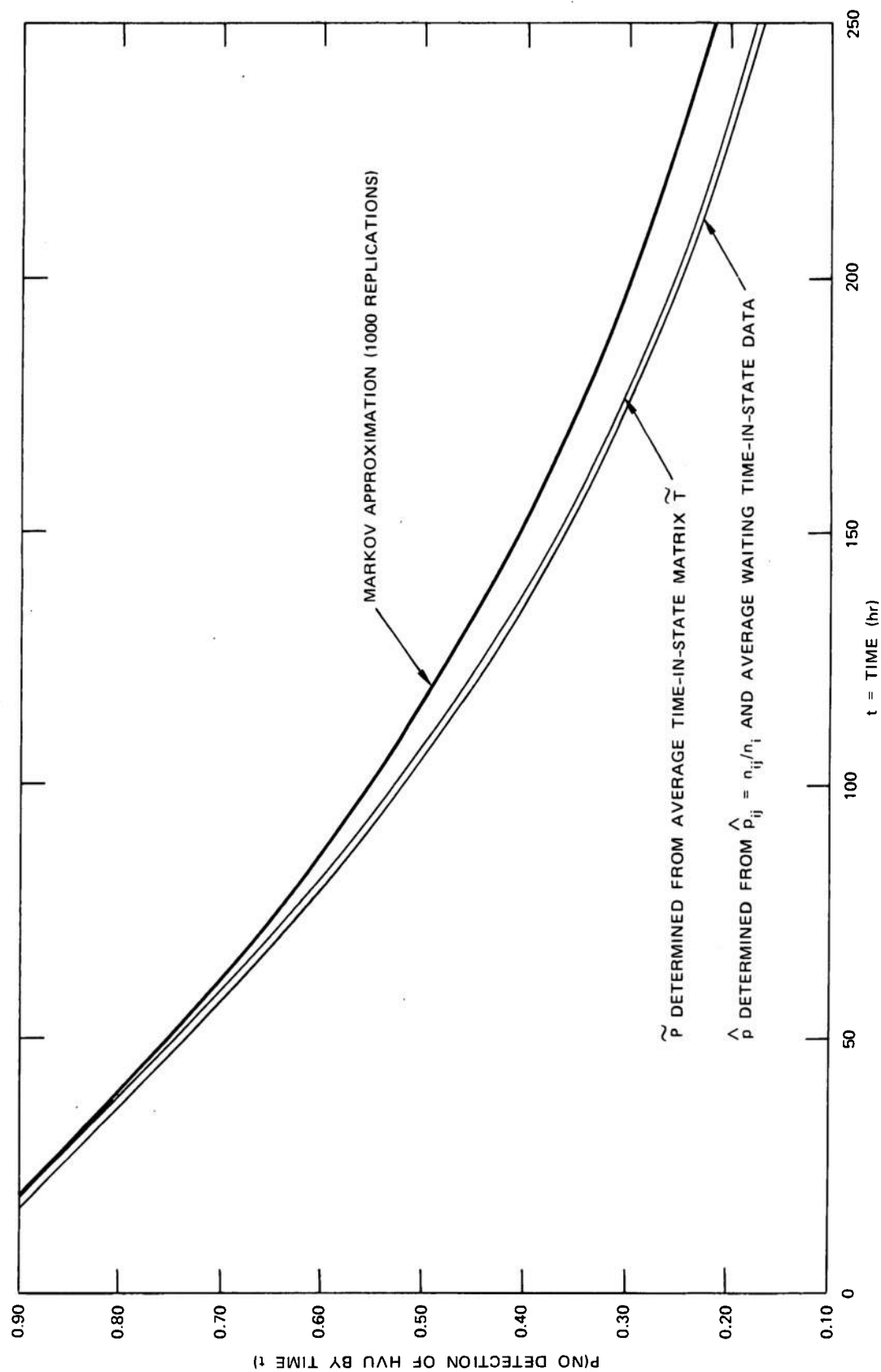


FIGURE 12.7 COMPARISON OF $P(\text{NO DETECTION BY TIME } t)$ FOR TRANSITION PROBABILITIES ESTIMATED FROM TRANSITION DATA AND AVERAGE TIME-IN-STATE DATA

to show by examples that the $\hat{p}_{ij}$ and $\tilde{p}_{ij}$ are not identical when estimated from the same data.)

F. Average Time Until Detection of the High Value Unit

In both the simulation model and Markov model, the HVU is ultimately detected by the submarine. Therefore, the mean time until detection is implicitly conditioned on detection ultimately occurring. Since the starting state is 1, the Markov chain model gives

$$\begin{aligned}\bar{T}_D \equiv \text{mean time until detection} &= \left[(I - Q)^{-1} \right]_1 = \left[(I - Q)^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]_1 \\ &= \text{sum of elements of first row of } (I - Q)^{-1} \quad . \quad (12.1)\end{aligned}$$

A more general formula for mean time until detection can be readily derived for a semi-Markov model. The mean time to detect depends only on the mean times of the transitions ($\bar{T}_{ij}$) and the transition probabilities p_{ij} for the imbedded chain. (This can be seen by conditioning on the path taken and noting that the mean time for each path is the sum of the mean transition times along that path.) Therefore, the transition time from i to j can be considered to be concentrated at the mean, resulting in a z transform of the form $p_{ij} z^{\bar{T}_{ij}}$ on the (i,j) branch.

Dropping all the "bars" over the mean times $\bar{T}_{ij}$ is typographically convenient. Figure 12.8 shows the transition diagram and z transforms of the transition time densities.

The transmission $g(z)$ from state 1 to state 4 is required to calculate $\bar{T}_D$.

$$g(z) = p_{14}z^{T_{14}} + \frac{p_{12}z^{T_{12}} \left(p_{24}z^{T_{24}} + p_{23}p_{24}z^{T_{23}+T_{34}} \right)}{1 - p_{23}p_{32}z^{T_{23}+T_{32}}} \quad (12.2)$$

Differentiating with respect to z , setting $z = 1$, and rearranging gives the average time until detection $\bar{T}_D$.

$$\begin{aligned} \bar{T}_D = g'(z = 1) = & p_{14}T_{14} + p_{12}T_{12} + \frac{p_{12}p_{24}}{1 - p_{23}p_{32}}T_{24} + \frac{p_{12}p_{23}}{1 - p_{23}p_{32}}T_{23} \\ & + \frac{p_{12}p_{23}p_{32}}{1 - p_{23}p_{32}}T_{32} + \frac{p_{12}p_{23}p_{34}}{1 - p_{23}p_{32}}T_{34} \quad (12.3) \end{aligned}$$

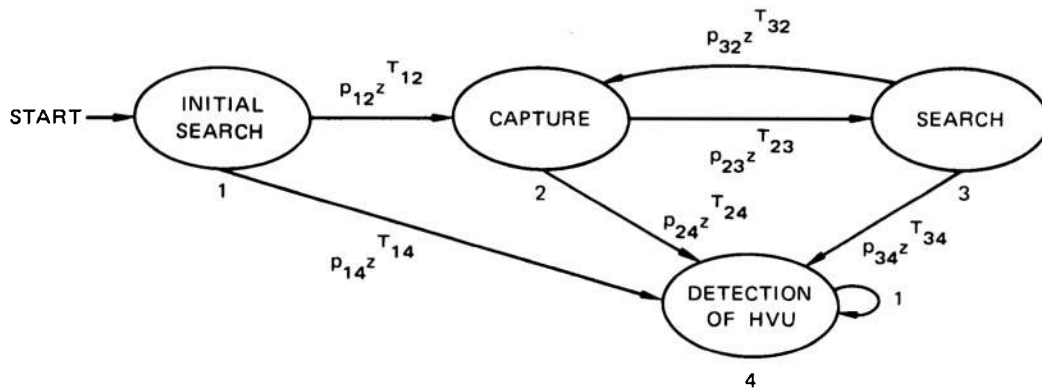


FIGURE 12.8 SEMI-MARKOV MODEL WITH ALL TRANSITIONS AT THEIR AVERAGE TIMES

The multipliers of the average times T_{ij} are easily interpreted here. The coefficient of T_{14} is p_{14} , the average number of times that the $1 \rightarrow 4$ transition is made. Similarly, p_{12} (the coefficient of T_{12}) is the average number of times that the $1 \rightarrow 2$ transition is made. In general, the coefficient of T_{ij} is the average number of i, j transitions. Therefore, the average time until detection is a sum over all transitions

$i \rightarrow j$, of products of the form

(average number of $i \rightarrow j$ transitions) $\times$ (average time required for an
 $i \rightarrow j$ transition) .

Differentiating the $\bar{T}_D$ formula of Eq. (12.3) with respect to the T_{ij} gains some insight into the sensitivity of $\bar{T}_D$ to the T_{ij} . Of special interest are the average capture time (given a return to search) T_{23} and average search time (given a return to capture) T_{32} . We have

$$\frac{\partial \bar{T}_D}{\partial T_{23}} = \frac{p_{12}p_{23}}{1 - p_{23}p_{32}} \quad (12.4)$$

$$\frac{\partial \bar{T}_D}{\partial T_{34}} = \frac{p_{12}p_{23}p_{34}}{1 - p_{23}p_{32}} \quad (12.5)$$

Thus, $p_{12}p_{23}/(1 - p_{23}p_{32})$ is the increase in the average time until detection ($\bar{T}_D$) per hour increase in average capture time (T_{23}) for small changes in T_{23} ; all other parameters remain constant. Similarly, a one hour increase in T_{34} induces an increase of $p_{12}p_{23}p_{34}/(1 - p_{23}p_{32})$ hours in $\bar{T}_D$; all other parameters remaining constant. Using the p_{ij} found for the 1000 replication case (Figure 12.4) gives estimates

$$\frac{\partial \bar{T}_D}{\partial T_{23}} \cong 1.45 \quad \text{hours/hour}$$

$$\frac{\partial \bar{T}_D}{\partial T_{34}} \cong 0.3 \quad \text{hours/hour} \quad .$$

In the simulation model, the Capture $\rightarrow$ Detect HVU transition occurs at the end of capture if the HVU is within detection range. Therefore,

$T_{24} = T_{23}$ so that the T_{23} term in Eq. (12.3) becomes

$$\frac{p_{12}(p_{24} + p_{23})}{1 - p_{23}p_{32}} T_{23} = \frac{p_{12}T_{23}}{1 - p_{23}p_{32}} \quad (12.6)$$

which equals the average number of times the submarine is captured. The sensitivity analysis then simply shows that each hour increase in T_{23} implies an increase of $p_{12}/(1 - p_{23}p_{32}) =$ average number of captures $\bar{T}_D$. For this example, the average number of captures is 1.54 so that each hour increase in average capture time increases $\bar{T}_D$ by this amount.

XIII COMBINING PARAMETER ESTIMATES

A. Introduction

Section IV asserted that Markov models have a convenient structure for bringing together parameter estimates obtained from various sources into a single effectiveness model. When more than a single estimate of a parameter are available, some method is required to combine the separate estimates into an overall parameter estimate. This section shows how estimates may be combined in an optimum linear way when the estimates are statistically independent and their variances are known.

Other topics in this section deal with variance considerations in Markov chain models when the probability of success is the measure of interest. Variance calculations are first reviewed for a simple Bernoulli model and then extended for a more complex model that has some relevance for exercise design. The section concludes with a model for determining the amount of effort that should be devoted to sampling at a state in a Markov chain model of a simple situation when the costs of sampling are considered. Although the model is too simple to be realistic, it illustrates how the available effort may be theoretically apportioned by designing exercises to minimize the variance of the estimator for the probability of success.

B. Combining Independent, Unbiased Estimates

Suppose X_1 and X_2 are independent estimates of the same quantity M , and that each is unbiased with common mean M . Variances of these random variables are assumed to be different, however, and they are denoted by σ_1^2 and σ_2^2 , respectively.

Elementary calculus can be used to combine these two estimates in the best linear way. Specifically, if the improved estimate Y is defined as a linear combination of X_1 and X_2 by

$$Y = aX_1 + (1 - a)X_2 \quad (13.1)$$

for some fixed nonnegative a , then a can be found to make the variance of Y a minimum. For any a , Y is an unbiased estimate of M , because

$$\begin{aligned} E(Y) &= E[aX_1 + (1 - a)X_2] = aE(X_1) + (1 - a)E(X_2) \\ &= aM + (1 - a)M = M \end{aligned} \quad (13.2)$$

Since a is a constant and X_1 , X_2 are independent, the variance of Y is

$$\begin{aligned} \text{var } Y &= a^2 \text{var } X_1 + (1 - a)^2 \text{var } X_2 \\ &= a^2 \sigma_1^2 + (1 - a)^2 \sigma_2^2 \end{aligned} \quad (13.3)$$

Differentiating Eq. (13.3) with respect to a , setting the derivative equal to zero, and solving for a gives the optimal weights a and $(1 - a)$

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \quad 1 - a = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \quad (13.4)$$

That is, an estimate is weighted in proportion to the variance of the other estimate when there are two variables. From Eqs. (13.1) and (13.4) the weighting formula is

$$Y = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} X_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} X_2 \quad (13.5)$$

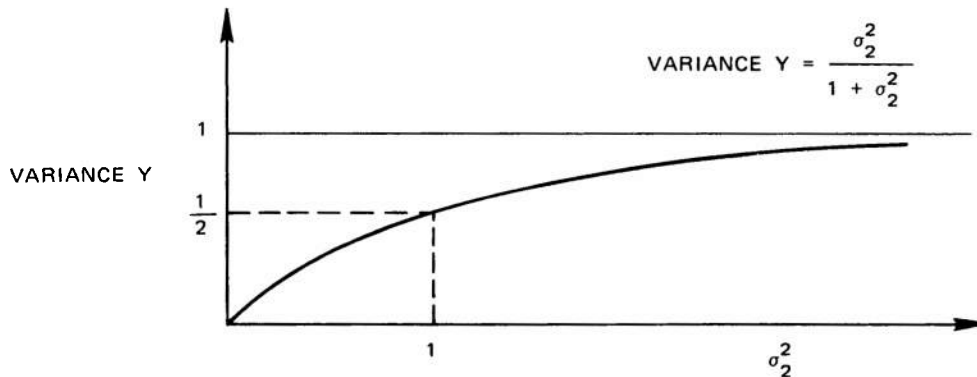
The variance of Y for this value of a is found from Eqs. (13.3) and (13.4)

$$\text{var } Y = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad (13.6)$$

Without loss of generality, we may assume that $\sigma_1^2 < \sigma_2^2$, so that

$$\frac{\sigma_1^2}{2} < \text{var } Y = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} < \frac{\sigma_2^2}{2} \quad (13.7)$$

If σ_1^2 is arbitrarily fixed at unity, then the resultant variance of Y may be plotted versus σ_2^2 :



The diagram shows that:

- The variance of the combined estimate is always less than the variance of the best estimator.
- The overall variance is halved when variances are equal.

The above derivation can easily be generalized to three statistically independent estimators of the same quantity M. If X_1, X_2, X_3 are estimates with mean M and variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$, then the appropriate linear combination is

$$Y = aX_1 + bX_2 + (1 - a - b)X_3 \quad \text{for } 1 \geq a, b \geq 0 \quad (13.8)$$

The optimal weights turn out to be

$$a = \sigma_{23}^2 \sigma_{12}^2 / D$$

$$b = \sigma_{13}^2 \sigma_{12}^2 / D$$

$$1 - a - b = \sigma_{12}^2 \sigma_{23}^2 / D, \quad (13.9)$$

where D is the normalizing constant $(\sigma_{12}^2 \sigma_{23}^2 + \sigma_{13}^2 \sigma_{23}^2 + \sigma_{12}^2 \sigma_{13}^2)$. Thus, a weight is the product of the other pair of weights, normalized. (The generalization to four or more estimates will not be done here. The formula may entail more complex functions than might be guessed from extrapolating the cases above.)

C. Application to Estimation of Transition Probabilities

Let n_{ij} be the number of transitions from i to j observed from N trials of a Markov chain, and let n_i be the number of transitions out of state i : $n_i = \sum_j n_{ij}$. Then the natural estimator for the transition probability p_{ij} is $\hat{p}_{ij} = n_{ij} / n_i$. Suppose two sets of numbers of transition data are denoted by superscripts 1 and 2. Then the intuitive way to combine these data is to simply add the numbers of transitions for corresponding i and j

$$\tilde{n}_{ij} = n_{ij}^{(1)} + n_{ij}^{(2)} \quad \text{for all } i, j. \quad (13.10)$$

Then the corresponding estimators of transition probabilities $\tilde{p}_{ij}$ are

$$\tilde{p}_{ij} = \frac{n_{ij}^{(1)} + n_{ij}^{(2)}}{\sum_j [n_{ij}^{(1)} + n_{ij}^{(2)}]} = \frac{n_{ij}^{(1)}}{n_i^{(1)} + n_i^{(2)}} + \frac{n_{ij}^{(2)}}{n_i^{(1)} + n_i^{(2)}}. \quad (13.11)$$

Some algebraic manipulation puts this in the form where weights w_1 and w_2 are identifiable as are the estimates for the first and second data sets.

$$\begin{aligned}\tilde{p}_{ij} &= \left[\frac{n_{ij}^{(1)}}{n_i^{(1)}} \times \frac{n_i^{(1)}}{n_i^{(1)} + n_i^{(2)}} \right] + \left[\frac{n_{ij}^{(2)}}{n_i^{(2)}} \times \frac{n_i^{(2)}}{n_i^{(1)} + n_i^{(2)}} \right] \\ &\equiv \frac{n_{ij}^{(1)}}{n_i^{(1)}} w_1 + \frac{n_{ij}^{(2)}}{n_i^{(2)}} w_2\end{aligned}\quad (13.12)$$

The question is "do the weights w_1 , w_2 agree with the a and $(1 - a)$ determined earlier?" Fortunately, the answer is yes.

To see this, recall that the formula for the variance of the sample success probability estimator in N Bernoulli trials is $p(1 - p)/N$, or pq/N when $q = 1 - p$. Applying this here gives

$$\text{var } \hat{p}_{ij}^{(1)} = p_{ij} q_{ij} / n_i^{(1)} \quad , \quad \text{var } p_{ij}^{(2)} = p_{ij} q_{ij} / n_i^{(2)} \quad . \quad (13.13)$$

The multiplier for the second variable $(1 - a)$ is therefore

$$\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = \frac{p_{ij} q_{ij} / n_i^{(1)}}{\left[p_{ij} q_{ij} / n_i^{(1)} \right] + \left[p_{ij} q_{ij} / n_i^{(2)} \right]} = \frac{n_i^{(2)}}{n_i^{(1)} + n_i^{(2)}} \quad (13.14)$$

which agrees with w_2 . Since $w_1 + w_2 = 1$, the value of w_1 will agree also; therefore, for the special case of combining transition information from independent samples, the simple "add transitions" rule is optimal for obtaining estimates of p_{ij} with minimum variance.

When estimates of transition probabilities are combined that are not determined from ratios n_{ij}/n_i , the new estimates may not sum to unity. For example, suppose transitions can be made from state 1 to states 2 or 3 in a larger process that need not be specified. Then, if two estimates

of p_{12} are (0.2, 0.3) and of p_{13} are (0.8, 0.7) with corresponding variances (1,2) for the p_{12} estimates and (3,4) for the p_{13} estimates, then the weighted estimates from Eq. (13.5) are

$$\tilde{p}_{12} = \frac{2}{1+2} (0.2) + \frac{1}{1+2} (0.3) = \frac{0.7}{3}$$

$$\tilde{p}_{13} = \frac{4}{3+4} (0.8) + \frac{3}{3+4} (0.7) = \frac{3.2 + 2.1}{7} = \frac{5.3}{7} .$$

The weighted estimates do not sum to unity:

$$\tilde{p}_{12} + \tilde{p}_{13} = \frac{20.8}{21} \neq 1 .$$

A slightly revised optimization procedure can force the weighted probabilities to be unity. Let X_1 and X_2 each have mean p_{12} and suppose $\text{var } X_1 = \sigma_1^2$, $\text{var } X_2 = \sigma_2^2$; similarly, let Y_1 and Y_2 each have mean p_{13} , with respective variances σ_3^2 and σ_4^2 . Weighting factors a and b are then sought such that

$$\tilde{p}_{12} = aX_1 + (1-a)X_2 \quad , \quad \tilde{p}_{13} = bY_1 + (1-b)Y_2 \quad , \quad (13.15)$$

$$\tilde{p}_{12} + \tilde{p}_{13} = 1 \quad , \quad \text{and } \left(\text{var } \tilde{p}_{12} + \text{var } \tilde{p}_{13} \right) \text{ is a minimum} .$$

Forming the Lagrangian, taking partial derivatives with respect to a and b , setting these derivatives to zero, and solving together with

$$\tilde{p}_{12} + \tilde{p}_{13} = 1 \text{ results in}$$

$$a = b = \frac{\sigma_2^2 + \sigma_4^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2} \quad (13.16)$$

when $X_1 \neq X_2$ and $Y_1 \neq Y_2$. The resulting variance of $\tilde{p}_{12}$ is

$$\text{var } \tilde{p}_{12} = \left[\frac{1}{1 + (\sigma_1^2 + \sigma_3^2)/(\sigma_2^2 + \sigma_4^2)} \right]^2 \sigma_1^2 + \left[\frac{1}{1 + (\sigma_2^2 + \sigma_4^2)/(\sigma_1^2 + \sigma_3^2)} \right]^2 \sigma_2^2 . \quad (13.17)$$

The extension to three probabilities (not given here) results in a system of four linear equations in four unknowns for the optimal linear weights.

D. The Variance of the Probability of Success

Many questions arise when looking into the variance of the probability of success p_s . On the one hand, we can assume that there is some probability of success p_s --fixed but not known-- such that success is determined by tossing a coin with sides labeled success and fail and which comes up success with probability p_s and fail with probability $1 - p_s$. This Bernoulli model yields a variance of $p_s(1 - p_s)/N$ for N independent trials, where a trial corresponds to a complete exercise.

Other models, however, give other variances. Assume, for example, some value of success probability p_s exists for each fixed set of environmental parameters, but the environment varies randomly. Then p_s is a random variable with a mean and variance, and its mean value $\bar{p}_s$ is the average probability of success. Dropping the subscript s for simplicity, assume p is a random variable with mean $\bar{p}$ and variance v . The mean and variance of the estimator $\hat{p} = (\text{number of successes})/N$ can be readily worked out. Choose N_1 values of p at random, independently from each other. For each p_i , perform N_2 trials (exercises) each resulting in success or failure. Let S_i = number of successes using p_i . Then S , the number of successes in $N_1 \times N_2$ trials, is

$$S = S_1 + S_2 + \dots + S_{N_1} . \quad (13.18)$$

Thus, the estimate of (average) probability of success is $\hat{p} = S/(N_1 N_2)$ since there are $N_1 N_2$ trials in all. The mean value of $\hat{p}$ is

$$E(\hat{p}) = E\left[\sum_{i=1}^{N_1} S_i / (N_1 N_2)\right] = \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} E(S_i) \quad (13.19)$$

But,

$$E(S_i) = E[E(S_i | p)] = E(N_2 p) = N_2 E(p) \quad (13.20)$$

Hence,

$$E(\hat{p}) = \frac{N_1 \times N_2 E(p)}{N_1 N_2} = E(p) = \bar{p} \quad (13.21)$$

so that $\hat{p}$ is unbiased as anticipated.

Similarly, by conditioning on p and using the general formula

$$\text{var } \hat{p} = [E(\text{var } \hat{p} | p)] + \text{var}[E(\hat{p} | p)] \quad (13.22)$$

one obtains after some algebra

$$\text{var } \hat{p} = \frac{\bar{p}(1 - \bar{p})}{N_1 N_2} + \frac{\text{var } p}{N_1} \frac{N_2 - 1}{N_2} \quad (13.23)$$

Note, this is not equal to the variance obtained when $\bar{p}$ replaces p in the Bernoulli model; in fact, the Bernoulli model estimate gives only the first term $\bar{p}(1 - \bar{p})/N_1 N_2$. The actual variance is larger by the non-negative quantity $\text{var } p \times (N_2 - 1)/N_1 N_2$.

If this model were to be used for exercise design purposes, some very unrealistic designs would result. Suppose, for example, that the k exercises can be performed in all, and that the design problem consists

of choosing N_1 and N_2 to minimize the variance in Eq. (13.23). The second term in Eq. (13.23) is zero when N_2 is chosen unity; that is, when only a single exercise is performed for each environment. The trouble with the model is that the exercise costs are not considered, and other benefits from the exercises are ignored.

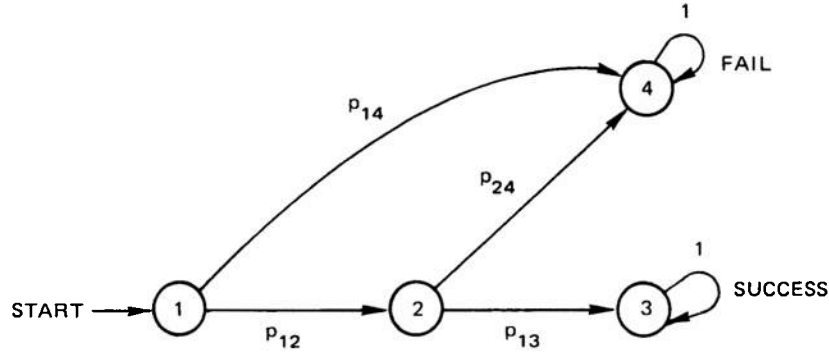
A slight generalization of the above model is easy to derive. Suppose that N separate environments can be independently chosen, where the probability of success is p_i for the i^{th} environment. Perform N_i independent trials in environment i , and let S_i be the number of successes. The total number of trials is $\sum_{i=1}^N N_i$, which is denoted by N_o . As before, the estimate $\hat{p} = \sum_{i=1}^N S_i / N_o$ is an unbiased estimate of the average probability of success $\bar{p}$. The variance of $\hat{p}$ works out to be

$$\text{var } \hat{p} = \text{var} \left(\sum_{i=1}^N S_i / N_o \right) = \frac{\bar{p}(1 - \bar{p})}{N_o} + \text{var } p \left(\frac{\sum_{i=1}^N N_i^2 - \sum_{i=1}^N N_i}{N_o^2} \right) \quad (13.24)$$

Once again the N_i should all be chosen equal to unity in order to minimize the variance of $\hat{p}$ because this choice makes the second term zero when the sum of the N_i is a constant. When these N_i are chosen, the variance is the $\bar{p}(1 - \bar{p})/N_o$, which is the Bernoulli formula for variance with $\bar{p}$ replacing p .

E. Optimal Allocation of Sampling in a Simple Markov Chain

A simple Markov chain model with four states and constant (but unknown) transition probabilities is defined here to illustrate how exercise effort may be allocated in principle when exercise costs are considered and when minimizing the variance of the estimator for the probability of success $\hat{p}_s$ is desirable. The transition diagram with transition probabilities on the appropriate branches is shown below:



Because there are no loops in the chain, the probability of success is $p_s = p_{12}p_{23}$ when the process always starts in state 1. The estimate of p_s will be the product $\hat{p}_{12}\hat{p}_{23}$, where the carets denote estimates of p_{12} and p_{13} .

Although the ASW operations being modeled always start in state 1, we assume that it is possible to perform experiments (exercises) that start at either state 1 or state 2 and proceed for only a single transition. Let the cost of an experiment starting at state 1 be c_1 , the cost of an experiment starting at state 2 be c_2 , and the total resources available be c in commensurable units. Suppose that estimates $p_{12}^{(1)}$ and $p_{23}^{(1)}$ are available for the quantities p_{12} and p_{13} . Variances of these estimates are assumed to be available also, with respective values of $v_{12}^{(1)}$ and $v_{23}^{(1)}$. The experimental design problem is: Choose the number of experiments starting in state 1 (n_1) and number of experiments starting in state 2 (n_2) such that the predicted variance of the estimate of p_s is minimized subject to the cost constraint $c_1n_1 + c_2n_2 = c$.

The idea is to obtain estimates $p_{12}^{(2)}$ and $p_{23}^{(2)}$ from the $n_1 + n_2$ new experiments, weight these estimates in the optimal linear manner as derived in Subsection B, and minimize the variance of this weighted estimator. Some assumptions must be made about the variances this new set of transition probability estimators because they depend on the unknown values of the true transition probabilities.

If an average value approach is used, the predicted mean and variances of $p_{12}^{(2)}$ and $p_{23}^{(2)}$ (the estimates obtained from n_1 trials starting at state 1 and n_2 trials starting at state 2) are:

$$\text{var } p_{12}^{(2)} \cong p_{12}^{(1)} [1 - p_{12}^{(1)}] / n_1 \quad (13.25)$$

$$\text{var } p_{23}^{(2)} \cong p_{23}^{(1)} [1 - p_{23}^{(1)}] / n_2 \quad (13.26)$$

$$E[p_{12}^{(2)}] \cong p_{12}^{(1)} \quad (13.27)$$

$$E[p_{23}^{(2)}] \cong p_{23}^{(1)} \quad (13.28)$$

Letting $\bar{p}_{12}$ and $\bar{p}_{23}$ be the predicted weighted estimators for p_{12} and p_{23} , Eq. (13.6) yields the predicted variance

$$\text{var } \bar{p}_{12} = \frac{v_{12}^{(1)} v_{12}^{(2)}}{v_{12}^{(1)} + v_{12}^{(2)}} = \frac{1}{\frac{1}{v_{12}^{(1)}} + \frac{n_1}{p_{12}^{(1)} [1 - p_{12}^{(1)}]}} \quad (13.29)$$

and a predicted mean

$$\bar{p}_{12} = a p_{12}^{(1)} + (1 - a) p_{12}^{(2)} = \frac{v_{12}^{(2)} p_{12}^{(1)} + v_{12}^{(1)} p_{12}^{(2)}}{v_{12}^{(1)} + v_{12}^{(2)}} \quad (13.30)$$

Similar expressions hold for the $2 \rightarrow 3$ transition, with subscripts 12 replaced by 23 and n_1 replaced by n_2 . All superscripts may now be dropped from here on since they are all (1).

The variance of the product of two independent random variables (X and Y) is easily shown to be

$$\text{var } (XY) = E(X^2)E(Y^2) - E^2(X)E^2(Y) \quad . \quad (13.31)$$

Applying this formula to $\hat{p}_s = \hat{p}_{12}\hat{p}_{13}$ gives

$$\text{var } (\hat{p}_s) = E(\bar{p}_{12}^2)E(\bar{p}_{23}^2) - E^2(\bar{p}_{12})E^2(\bar{p}_{23}) \quad . \quad (13.32)$$

Equations (13.29), (13.30), and (13.32) imply

$$\text{var } \hat{p}_s = \left[\frac{1}{\frac{1}{v_{12}} + \frac{n_1}{p_{12}(1-p_{12})}} + p_{12}^2 \right] \left[\frac{1}{\frac{1}{v_{23}} + \frac{n_2}{p_{23}(1-p_{23})}} + p_{23}^2 \right] - [p_{12}p_{23}]^2 \quad . \quad (13.33)$$

Expression (13.33) is to be minimized with respect to n_1, n_2 subject to the cost constraint

$$c_1 n_1 + c_2 n_2 = c \quad (13.34)$$

and nonnegativity constraints

$$n_1, n_2 \geq 0 \quad . \quad (13.35)$$

For convenience, define new variables $a_{12}, b_{12}, a_{23}, b_{23}$ by

$$a_{12} = p_{12}(1 - p_{12})$$

$$b_{12} = p_{12}(1 - p_{12}/v_{12})$$

$$a_{23} = p_{23}(1 - p_{23})$$

$$b_{23} = p_{23}(1 - p_{23})/v_{23} \quad .$$

Equation (13.33) then takes on the simpler form of Eq. (13.36)

$$\text{var } \hat{p}_s = \left(\frac{a_{12}}{b_{12} + n_1} \right) \left(\frac{a_{23}}{b_{23} + n_2} \right) + p_{23}^2 \frac{a_{12}}{b_{12} + n_1} + p_{12}^2 \frac{a_{23}}{b_{23} + n_2} \quad . \quad (13.36)$$

The usual procedure for constrained optimization may then be followed. Let λ be the Lagrange multiplier for the cost constraint in Eq. (13.34). Differentiating the Lagrangian $\text{var } \hat{p}_s + \lambda(c_1 n_1 + c_2 n_2)$ partially with respect to n_1 and n_2 , equating the partials to zero, and doing some algebra results in

$$\frac{a_{23}}{c_2} \frac{c_1}{a_{12}} \left[a_{12}(b_{12} + n_1) + p_{12}^2 (b_{12} + n_1)^2 \right] = \left[a_{23}(b_{23} + n_2) + p_{23}^2 (b_{23} + n_2)^2 \right] \quad . \quad (13.37)$$

Replacing n_2 by $(c - c_1 n_1)/c_2$ on the right hand side (RHS) of Eq. (13.37) and doing some algebraic manipulation yields

$$\text{RHS of Eq. (13.37)} = R_o + R_1 n_1 + R_2 n_1^2 \quad , \quad (13.38)$$

where

$$R_o = a_{23} b_{23} + a_{23} \frac{c}{c_2} + p_{23}^2 \left(b_{23} + \frac{c}{c_2} \right)^2 \quad (13.39)$$

$$R_1 = -a_{23} \frac{c_1}{c_2} - 2p_{23}^2 \left(b_{23} + \frac{c}{c_2} \right) \frac{c_1}{c_2} \quad (13.40)$$

$$R_2 = p_{23}^2 \left(c_1 / c_2 \right)^2 \quad . \quad (13.41)$$

Similarly, the left hand side (LHS) of Eq. (13.37) can be expanded to give

$$\text{LHS of Eq. (13.37)} = L_o + L_1 n_1 + L_2 n_1^2, \quad (13.42)$$

where

$$L_o = \frac{a_{23}}{a_{12}} \frac{c_1}{c_2} \left(a_{12} b_{12} + p_{12}^2 b_{12}^2 \right) \quad (13.43)$$

$$L_1 = \frac{a_{23} c_1}{a_{12} c_2} \left(a_{12} + 2 p_{12}^2 b_{12} \right) \quad (13.44)$$

$$L_2 = \frac{a_{23} c_1}{a_{12} c_2} p_{12}^2. \quad (13.45)$$

Equation (13.37) now becomes

$$L_o + L_1 n_1 + L_2 n_1^2 = R_o + R_1 n_1 + R_2 n_1^2 \quad (13.46)$$

from which n_1 may be found by the quadratic formula. In general, n_1^* will not be an integer; but it will be necessary to round to an integer. If n_1^* is negative, it should be replaced by zero; if $n_2^* = (c - c_1 n_1^*)/c_2$ is negative, the n_2^* should be replaced by zero and n_1^* should be recalculated from $n_1^* = (c/c_2)$.

Table 13.1 shows some results from the optimization procedure. Initial data were calculated as though they were from complete trials with equivalent sample sizes (ESS). The final column shows the ratio of the final standard deviation to the initial standard deviation. Since cost ratios c_1/c and c_2/c were used as parameters, the cost constraint was $(c_1/c) + (c_2/c) = 1$.

Table 13.1

TYPICAL VALUES OF OPTIMAL N_1^*, N_2^*
(Not Rounded)

p_{12}	p_{23}	ESS	c_1/c	c_2/c	N_1^*	N_2^*	$\sigma_{\text{Final}}/\sigma_{\text{Initial}}$
.500	.500	25	.001	.001	813.733	186.367	.235
.500	.500	25	.001	.010	624.008	37.599	.418
.500	.500	25	.001	.025	1000.000	0.	.811
.500	.500	25	.001	.100	1000.000	0.	.811
.500	.500	25	.010	.001	72.174	278.261	.334
.500	.500	25	.010	.010	55.463	44.537	.495
.500	.500	25	.010	.025	46.439	21.424	.597
.500	.500	25	.010	.100	57.945	4.206	.768
.500	.500	25	.025	.001	29.576	260.609	.423
.500	.500	25	.025	.010	19.693	50.767	.559
.500	.500	25	.025	.025	15.614	24.386	.652
.500	.500	25	.025	.100	16.407	5.898	.805
.500	.500	25	.100	.001	6.894	310.569	.530
.500	.500	25	.100	.010	2.469	75.313	.626
.500	.500	25	.100	.025	.335	38.659	.697
.500	.500	25	.100	.100	0.	10.000	.837
.500	.500	100	.001	.001	670.708	329.292	.361
.500	.500	100	.001	.010	477.483	52.252	.618
.500	.500	100	.001	.025	541.996	18.320	.733
.500	.500	100	.001	.100	1000.000	0.	.832
.500	.500	100	.010	.001	56.162	438.383	.529
.500	.500	100	.010	.010	26.446	73.554	.729
.500	.500	100	.010	.025	28.085	28.766	.826
.500	.500	100	.010	.100	84.860	1.514	.909
.500	.500	100	.025	.001	18.277	543.082	.580
.500	.500	100	.025	.010	0.	100.000	.744
.500	.500	100	.025	.025	0.	40.000	.838
.500	.500	100	.025	.100	14.690	6.328	.939
.500	.500	100	.100	.001	.414	958.624	.602
.500	.500	100	.100	.010	0.	100.000	.744
.500	.500	100	.100	.025	0.	40.000	.838
.500	.500	100	.100	.100	0.	10.000	.943
.500	.800	25	.001	.001	905.476	94.524	.237
.500	.800	25	.001	.010	1000.000	0.	.588
.500	.800	25	.001	.025	1000.000	0.	.588
.500	.800	25	.001	.100	1000.000	0.	.588
.500	.800	25	.010	.001	85.778	142.223	.419
.500	.800	25	.010	.010	78.758	21.242	.531
.500	.800	25	.010	.025	79.357	8.257	.598
.500	.800	25	.010	.100	100.000	0.	.681
.500	.800	25	.025	.001	34.505	137.382	.552
.500	.800	25	.025	.010	30.331	24.173	.641
.500	.800	25	.025	.025	29.560	10.440	.696
.500	.800	25	.025	.100	39.341	.165	.765
.500	.800	25	.100	.001	8.430	157.006	.719
.500	.800	25	.100	.010	6.281	37.191	.782
.500	.800	25	.100	.025	5.544	17.925	.824
.500	.800	25	.100	.100	6.866	3.134	.887

XIV POSSIBLE OPERATIONAL APPLICATIONS

A. Effectiveness Applications

1. False Contacts

The nine-state false contact model developed in Section X can be used to replace the traditional, conditional probability model based on the sequence Opportunity → Detect → Classify → Attack → Kill in a false contact environment. When data from complete exercises are used to estimate parameters, the value of the effectiveness measure (probability of kill) will be unchanged but the influence of false contacts will have been made explicit. Among other advantages, this explicitness enables one to do sensitivity analysis by varying the average time to prosecute a false contact to determine the impact on effectiveness.

The 27-state false contact model, which includes attacks on false contacts, may also be used to replace the traditional, conditional probability model. With this model, weapon expenditures on false contacts can be estimated as well as the probability of killing the actual submarine.

Many other forms of false contact models may be formulated by using Markov methods. Further, such things as the kind of the false contact and alertment status in the state definitions can also be distinguished.

2. Countermeasure Environments

Markov models may be defined for complex mission environments that entail the use of certain kinds of countermeasures. This form of

model should be well-suited to model countermeasure environments because of the crucial importance of information in these environments and the fact that states of a Markov model are readily definable in terms of information. When states have been satisfactorily defined, countermeasures can influence events directly associated with the basic model parameters in two ways:

- The countermeasures influence the next information state in a manner directly associated with the transition probabilities p_{ij} (for $i \neq j$).
- The mean time in an information state is directly associated with the holding time parameter p_{ij} .

An example of this concept used in analytical countermeasure modeling is the semi-Markov model developed in Ref. 6. In this model, the important information state was designated capture and the mean time in this state (capture time) was the principal parameter used to characterize an acoustic decoy.

The UPTIDE series of exercises provides an example of an operational environment in which states have been tentatively defined and data have been gathered on the time spent in these states. Two sets of states are defined: one for the Blue forces and the other for an Orange submarine. The High Value Target (HVT) information states are defined in terms of the existence of false/valid contacts.

- No false contacts and no valid contacts
- False contacts and no valid contacts
- Valid contacts and no false contacts
- Valid contacts and false contacts.

States for an Orange submarine were defined in terms of both submarine information and the submarine's activity relative to the Blue forces. Labels on some of the states were Lured, Deceived, Evading, Search,

Confused, and Had Knowledge of the HVT. Several combinations of these states, such as Lured and Deceived, were also used.

Apparently little has been done with the collected data beyond the tabulation of the percentages of time spent in the various states and the number of entries into each state. Because transition data were not tabulated for the exercises, it is difficult to determine how well a Markov model fits the data or to perform a sensitivity analysis.

As they are currently defined, these states are probably unsuitable for effectiveness modeling. The state space should be more detailed to include the traditional functions detect, classify, and attack and further refinements such as a breakdown of detection into first CZ detections and record CZ detections. Separate states should be provided to distinguish these differing types of detection opportunities for the HVT, other surface ships, and acoustic decoys. Classification of surface targets by submarines should also be represented in the state definitions. Because submarines may erroneously fire on acoustic decoys, the state space should reflect this potentially important contribution of decoys to overall ASW systems effectiveness.

If a Markov model were successfully fitted to UPTIDE data, some important sensitivity analyses could be performed and quantitative trade-offs could be established. A sensitivity analysis similar to that performed in Section XII could give an estimate of the number of hours increase in average survival time for each hour increase in mean capture time. Examples of interesting trade-offs are:

- The relationship between the range at which an acoustic decoy is detectable (an acoustic power consideration) and the length of time that a submarine is captured (a fidelity consideration).

- The relationship between the range at which the HVT can be detected and the probability of its being misclassified.
- The relationship between the probability of survival of the HVT and other quantities such as the mean time acoustic decoys were operative.

B. Other Applications

Throughout this report, the emphasis has been on ASW effectiveness modeling. However, applications of the methodology to real naval problems should not be limited to effectiveness models because they may also be applicable to important problem aspects related or ancillary to effectiveness. Markov models are potentially valuable in the four areas discussed below. These areas include a dynamic model of the:

- Range between a submarine and a HVU
- Electromagnetic radiations from an aircraft carrier
- Submarine classification process
- Submarine exposure.

1. Distance Between a Submarine and a High Value Target

Because so many tactical interactions are ultimately a function of range, one quantity of special interest to model dynamically is the distance between two major participants such as the enemy submarine and an aircraft carrier in a given scenario. A dynamic model of this range may be useful in several applications; in particular, the model may be a useful component in the design of a real-time tactical decision-making aid for the Officer in Tactical Command (OTC). The model may provide useful quantitative answers to the following questions. All questions will assume that some information on the current range to the submarine is available and expressible as a probability distribution.

- What is the mean length of time until the submarine first comes within two convergence zones of the carrier? One convergence zone? Expected range of detection via direct path?
- What is the probability that the submarine will not come within a specified distance (say 30 mi) of the carrier in the next 24 hour period?
- What is the probability distribution of the current range? (From this distribution, such quantities as the median current range, the probability that the range exceeds that at which a UHF transmission could be detected by the submarine, and the probability that the submarine is beyond the range of periscope detection can be determined.)

2. Radiation from an Aircraft Carrier

One may consider the detectable electromagnetic radiation from an aircraft carrier as a random process. A model of this process may be a useful component in a larger model in which the carrier is detectable by any one of several sensors. (In principle, all forms of detection can be considered in a single model by relating sonar detectability to carrier speed and screw usage and periscope detectability to time of day.) A state of detectable radiation would be defined in a Markov model in terms of the radiating equipment: State 1 may be occupied when there is complete EMCON, state 2 may be occupied when HF radio is being used, state 3 may be occupied when the surface search radar is energized, and so forth. Obviously, many combinations of radiators must also be considered.

Inherent in this model are such quantities as the mean total time each sensor is used during the exercise, the mean time between successive uses of a given sensor, and the mean length of each kind of transmission.

3. Submarine Classification Process

This model would be a dynamic model with absorbing states representing the final classification used to determine the parameters in a confusion matrix. (Indeed, the absorption probabilities found from the dynamic model will be closely related or identical to entries in the confusion matrix.) Dynamic quantities such as the mean time to classify a HVT correctly, the probability of ever holding an erroneous classification, and the like should be readily computable from the model. States will bear such labels as "possible carrier," "probable carrier," "certain carrier" in this model.

4. Submarine Exposure

This model relates to detection in a manner similar to the aircraft carrier radiation model. States would be defined in terms of detectability of the submarine; exposure of the attack periscope would correspond to occupying a certain state, exposure of ECM masts another state, and so forth.

C. Recommendations

(1) A research project to apply the dynamic Markov modeling methods developed during this study should be undertaken in an area of current interest to the Navy. The model should be developed for a class of tactical problems where wholly satisfactory measures of effectiveness have not yet been developed because of the dynamic nature of the problem, the complexity of the problem, or the difficulty of relating lower level performance measures to the overall effectiveness measures. The Naval Analysis Programs Office of the ONR could sponsor such research as a continuation of the Queueing Methodology study.

(2) The technique of using a Markov model to compare dissimilar models or to assist in validating a simulation model should be suggested to Navy analysts concerned with simulation validation and effectiveness modeling.

Appendix A

RELATED ASW MODELING

Appendix A

RELATED ASW MODELING

1. General Discussion

A survey of recent analytical ASW modeling that is related to the current SRI research is presented in this appendix. The relationship between these works and the SRI research is also discussed.

Aside from the direct applicability to ASW situations as depicted in this report, these techniques are in some ways similar to but in other respects generalizations of applications of queueing and Markov or semi-Markov analysis appearing in other recent work in the ASW field.

Much of the earlier work in ASW analysis has centered around computation of total system effectiveness as compounded by a host of influencing factors that in a sense may be represented by a series of necessary stages for success of an ASW mission in a logical sequential order. That is, an approximate description of ASW functions is the traditional series of stages in so-called WSE models:

Opportunity → Detect → Classify → Localize → Attack → Kill .

As shown in Section III, this series leads naturally to a Markov presentation from the assumption that all transitions in the series are independent of previous ones and additional exit transitions to a fail state are available. This traditional series of stages produces a simple formula for the probability of success:

$$P_{s|o} = P_D \cdot P_C \cdot P_L \cdot P_A \cdot P_K .$$

At this point, an explanation of the gross effect of dynamic interactions on the nature of mathematical models as frequently used in ASW analysis seems worthwhile. A specific example is the simple concept of a multiple server queue used in Conolly's ASW model.^{1,1} The only reason that a queue has to have a waiting line (in the ASW case, this would consist of contacts not yet detected or pursued) is that dynamic interaction occurs. That is, if arrivals are distributed (in probability) so that they can accumulate to more than the "servers" can handle in some given period of time, then some arrivals would have to wait before beginning service. Conolly assumes an infinite amount of service facilities (prosecuting destroyers) and therefore experiences no waiting for service, but he allows service time to depend on number of arrivals still in the system. More generally, crisis situations in ASW contact prosecution may entail excessive congestion at certain ASW function levels so that some contacts might have to wait to be served. An accurate measure of threat to forces protected by the ASW team is essentially impossible to determine without dynamic analysis of the degree to which unprosecuted contacts can accumulate, for how long, at what range, and so on. This fact was recognized in the MAFTEP outline that led to the current project and has been slowly permeating recent work on ASW effectiveness modeling.^{1,11-17}

In its early development, dynamic analysis of ASW effectiveness has been mostly of a qualitative nature.¹²⁻¹⁵ A number of new, advanced, analytical approaches for modeling ASW effectiveness have recently been introduced, including decision-theoretic techniques (principally statistical hypothesis testing)^{12,13} and queueing theory.^{11,14,15} To our best knowledge, actual analytical evaluation of ASW models before the current work has found little application of advanced concepts beyond a Markov or semi-Markov process with a few states.^{16,17} The analysis during the first year's work on the current project entailed queueing models represented by highly dimensional vector Markov processes along with flow

graph techniques and recent developments in successive overrelaxation methods.¹

The current work centers mainly on discrete-time Markov and semi-Markov modeling with moderate numbers of states, and the analysis indicates that considerably higher dimensions could be used without stepping beyond the bounds of computational feasibility. Advanced techniques used in the current work include flow graph methods for discrete-time semi-Markov processes (using Z-transforms), special matrix methods or quadratic programming for least squares fitting of Markov process parameters to observed data, and concrete examples of ASW models to demonstrate applicability of these techniques.

The new developments of the current study relate to earlier ASW analyses in two principal ways:

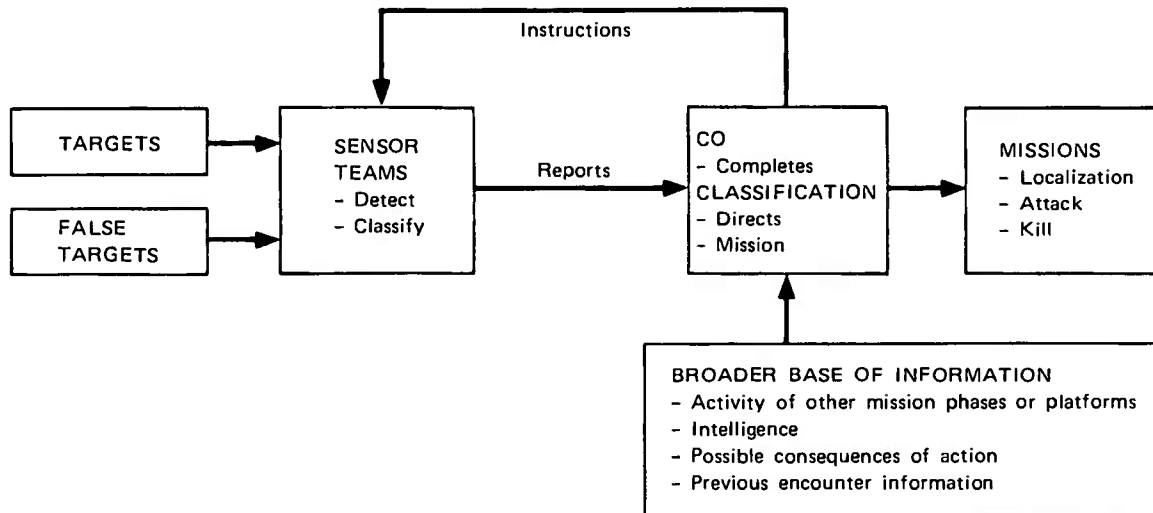
- Interpretation and means of generalizing their approaches in terms of more advanced techniques.
- Motivation for and justification of Markov techniques applied to ASW analysis, as performed under the present contract.

As expounded above, the general theory of Markov and semi-Markov processes leads to very powerful tools for stochastic process analysis, somewhat analogous to the technique of linearization in the areas of mathematical programming, differential equations, and operations research. In spite of much advanced formulation of new concepts in all these areas, about the only way to obtain numerical results effectively is to Markovize a complex analytical formulation. Again, a good example of this is the successful computations in the first year's work on this contract with Markov processes having a thousand states used to analyze an ASW congestion situation originally formulated in terms of queueing theory.

In the sections below, the ASW analyses referred to above will be treated individually to explain in greater detail their relationship to the current work.

2. Wagner ASW Model with False Contact Considerations

Daniel H. Wagner Associates recently concluded an investigation into the possibilities of modeling ASW performance with primary emphasis on measuring effects of false contacts.^{12,13} The Wagner ASW model subdivides ASW operations into the following two categories: detection and classification, and command level decisions and actions. The interplay between these two areas may be viewed as a dynamical feedback process,^{12,13} with updated classification information proceeding to the command and action model and sensor team instructions feeding back to the sensor model, as indicated in Figure A.1.



SOURCE: References 12 and 13.

FIGURE A.1 BASIC STRUCTURE OF WAGNER ASW MODEL

Within the Sensor Team block of the flow diagram in Figure A.1, the Wagner report describes in greater detail how information processing

interacts as the various types of sensor teams and units among one type function together. This is depicted in Figure A.2. Basically, these

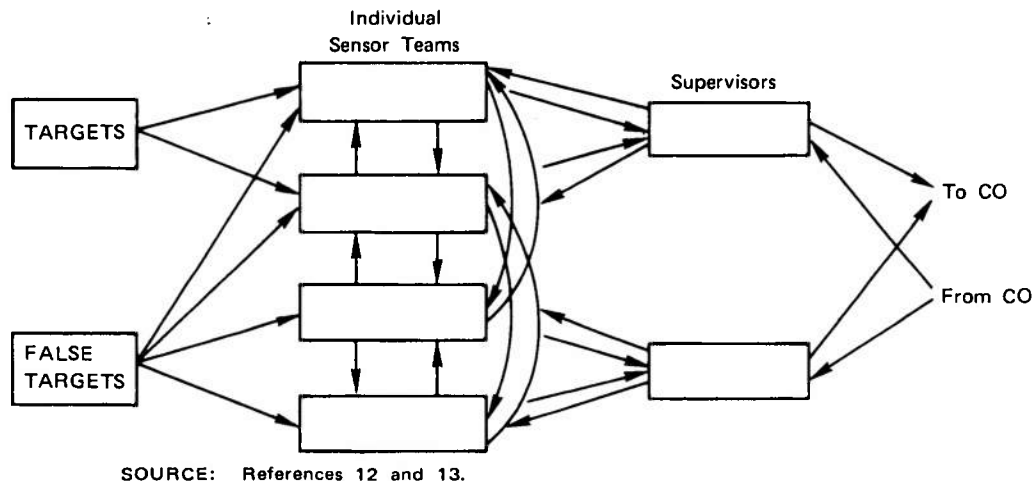


FIGURE A.2 INTERACTION WITHIN THE "INNER SANCTUM" OF THE SENSOR TEAMS

interactions are in the form of reports of contacts made or updatings of characteristics of earlier reported contacts for the benefit of other sensor teams and the supervisors in making or updating their classification decisions. Thus, the arrows in Figure A.2 may be interpreted as indicating flow of information on a statistical basis. This viewpoint gives the first indication of a relationship with the analysis of the current project; that is, the Wagner model may be regarded as (or at least approximated by) a Markov process, or perhaps more generally a semi-Markov process, whose states are described by levels of information content and classification pertaining to a specified number of contacts. In the first year's work on this contract, four levels of action on up to four contacts on actual submarines in the system were considered in a simplified dynamical ASW model.

The Wagner model does not explicitly discuss the dynamics of how various contacts' penetration levels interact in the information processing

system of Figures A.1 and A.2. Rather, the primary emphasis is on modeling the decision-making mechanism by (Neyman-Pearson type) statistical hypothesis testing.^{12,13} The extensive formulation of state vectors and their functional interrelationship results in a precise mathematical statement of the relations among the following quantities or logical (Boolean) variables (for hypotheses):

- Detection/Classification Phase
 - At-sea environment factors
 - Classification clues present
 - Perception probabilities for specific objects detected
 - Hypothesis that object observed is of a certain classification
 - Rejection probabilities
- Command/Action Phase
 - Possible tactics of contacts
 - Hypothesis that contact is employing a certain tactic
 - Current classification information
 - Tactical mission decisions.

Many others are omitted here for the sake of brevity.

For the type of approach put forth there, the Wagner report indicates an important application to ASW simulations. Presumably, any dynamical interplay not explicitly considered in the Wagner model itself would be relegated to treatment by the time varying mechanics of the model being simulated. Some dynamical aspects of ASW modeling are introduced in the Wagner analysis in a typical curve for temporal variation in perception level (see Ref. 12, Chapter IV), in discussions pertaining to implementation in simulations (see Ref. 12, Chapter V), and in statements regarding assumed statistics of contact arrivals and classification time (Ref. 13, Appendix B).

A possible useful extension of the Wagner work might be an attempt to carry over the analysis to a dynamic level by suitable embedding of Markov chains as was done on this contract in another context. An outline of how this might be achieved follows below.

The first step necessary for Markov representation of ASW operations as described in the Wagner reports is to decide on what ASW events may be set apart--or perhaps combined with others--to form distinct states of the system.^{12,13} In doing this, it is also necessary to consider whether the system can be logically regarded as transiting from one of these states to another without too much interference from still other omitted factors. These could be, for example, long-term value information about a contact determined early in the detection to prosecution sequence.

The Wagner model was already formulated in terms of a Markov process in the simulation of Ref. 13--Appendix B. In fact, 16 states were clearly delineated, but only simulation was attempted. Our model was very closely related in that progress of contacts "through the system" described the state of the system, and transitions between states were similarly described although we did not consider as many states and transitions for a given contact. The much higher dimensionality in our model arose from combining possible states of various simultaneous contacts. Hence this feature appears to be the next step in generalizing the Wagner model toward a more realistic one, and the efficient computational techniques used in our work--namely, flow graph methods, Successive Over Relaxation methods, or direct Laplace transform evaluation and inversion--would succeed in arriving at solutions for an extended model along the lines of the basic Wagner approach.

3. Operations Research Incorporated ASW Queueing Model

A parallel study to the current contract was aimed at modeling ASW operations of a patrol submarine in terms of queueing concepts and techniques. At its outset, a queueing structure was formulated similar to our unit model of the first year's work on this contract.¹⁴ Later, emphasis changed to modeling of the classification process.^{15,16} An analytical formulation for the probability of classifying by a given time was also worked out, taking multiple contacts into consideration. Some generally useful concepts for ASW modeling introduced by ORI in their model were the "classification window" and the "growth of information" idea for controlling the incidence of actual ability to classify. These concepts, although not formally stated, appear to be the basis of other ASW analyses, as will be indicated further below.

4. Semi-Markov Mission Analysis

Semi-Markov models of aircraft operations in ASW and general target search,^{16,17} were recently formulated with several states to represent various reasonably distinguishable stages of such operations. Laplace transforms were used in both cases in ways similar to the applications on a single unit ASW model.¹ Flow graphs were used to aid formulation and organize computations, as in our work, and simple examples were worked out. Solutions were obtained analytically in closed form, by inversion of some elementary Laplace transforms.

Both these models involve stages of a process (search for targets) that are passed through more or less independently of activity at other stages, thereby leading to a fairly good Markovian model. The situation in Ref. 16 is simplified further than in Ref. 17 by the assumption that only one contact held at any one time. The possibility of false contacts is handled by considering states of "disturbance," which can be interpreted

as either diversion to pursuing an attacker on the target searching unit or changing over to a possible target of perhaps greater importance to the search mission.¹⁷

5. Semi-Markov Models of Search in the Presence of Decoys

J. M. Moore of SRI has recently introduced a three-state and four-state semi-Markov model for a patrolling submarine versus HVTs and decoys operating within a specified area.⁶ As in semi-Markov mission analysis, states were selected to represent well-distinguishable operational configurations with transitions between them being clearly defined and justifiably approximated by Markov-type behavior. Although Moore's model has a small number of states, it is firmly founded on an operational basis, with consideration of a reasonably detailed structure of searcher and evader maneuvers to arrive at formulas for parameters in the semi-Markov model in terms of measurable or at least known or postulated operational data. In this sense, his model is more practical than our unit ASW model of the previous year's work under the current contract; our model's semi-Markov structure was far richer (in fact, up to a thousand states were allowed), but this structure was arrived at by analysis of a postulated operational description that was quite elementary. Our actual operational states were few in number (describing the number of contacts held), and the analytical model's states arose from strongly distinguishable combinations of where individual contacts are located in the model's partitioning of events into stages (detection, localization, and such).

Another important feature of Moore's work is that a parallel simulation study was conducted at SRI¹⁰ for comparison of model accuracy and validation. This simulation model is discussed and used in Section XII of this report.

As with the other related analyses, a significance of our developments using flow graphs and other multidimensional-oriented techniques is a possible approach to computations when Moore's type of model is expanded to greatly larger number of states. From the other direction, Moore's reduction of an operational situation to a low-dimensional model is a valuable technique to maintain closer ties with "the real world" in modeling. A similar approach was begun but not completed in our single unit ASW model when it was found that too many states arose in a Markov model as a result of combinations of possible contact locations.

6. Passive Sonar Classification Model

A mathematical model of contact classification by passive means was developed at SRI and used on several projects in the ASW area.¹⁸ The basis of this model is consideration of a sequence of information-gathering stages related to acoustic and kinematic aspects of passive sonar search. As quality and quantity of information fluctuates stochastically, a tendency toward some postulated and recognizable pattern class may emerge and be interpreted as a cue to update current classification of a possible contact.

This basis is similar to the ORI approach previously referred to and also to our expanded ASW unit model (Section II-C). The key concept in all three cases is that of growth of information as a measure to be related through a dynamical model to a decision process. In our work and the parallel ORI efforts, the dynamical aspects were put in the form of a Markov process. There is ground to believe that the SRI passive sonar classification model could be represented in Markov or semi-Markov form for application in mission analyses along the lines of Refs. 6, 12-17, thereby bringing into such models a more realistic representation of acoustic signal processing. A good approximation to the complex process

described in Ref. 18 might be to consider "lumped states" created by conditions of acoustic activity leading to the "pattern classes" considered there, and to regard changes in recognized patterns as Markov-type transitions, provided that these could be shown to exhibit suitable properties, especially independence of past history of pattern transitions.

7. Conolly Model of ASW in Terms of Congestion Theory

An elementary queueing-type model of ASW operations was proposed by Conolly for the situation of an ASW force attempting to clear a protected area of enemy submarines.¹¹ The model assumed Poisson arrivals, i.e., detections of submarines in general and independent service time distributions (for the contact prosecutions). A method for calculating the probability distribution of time to clear the protected area was derived in closed analytical form. In none of our more complex models were analytical, closed-form solutions ever found to be practical. Numerical solutions, however, were always feasible.

8. SHAREM Analysis--Markov Models of an ASW Screening Mission

A nine-state Markov model of a typical ASW screening scenario was used in post-analysis of a SHAREM exercise sequence.^{19,20} Standard Markov absorbing state probability calculations were used to analytically formulate MOEs and to evaluate sensitivities with respect to transition probabilities. Exponential distributions assumed for contact holding times were found to be quite consistent with exercise observations.²⁰

Appendix B

SELECTION OF PARAMETER ω

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SELECTION OF PARAMETER ω

The range derived for ω was thought to lie in the range $1 < \omega < 2$, when the eigenvalues of the iteration matrix are all real.²¹ However, it was later realized that the optimal parameter ω may sometimes have to be less than 1 when complex eigenvalues exist for the iteration matrix labeled Q in Appendix B-2 of Ref. 1. Since investigation subsequent to initial work revealed the presence of complex eigenvalues, a value of $\omega = 0.9$ was tried, and the number of over-relaxation iterations per integration step was reduced from about 13 to about 6. Some effort was spent on analyzing the relationship between ω and the convergence rate, and Figure B.1 demonstrates the loci of eigenvalues of the iteration matrix $\mathcal{L}_\omega$ (defined further below) as ω changes, assuming an approximate model of the convergence process. That is, we considered a reduced form of the actual matrix of the almost 5000 equations to be solved, namely:

$$Q = \begin{bmatrix} I & A\Delta t & & \\ -I & I & \frac{-A\Delta t}{2} & \\ & -I & I & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

by interpreting the scalar number α as representing the largest (negative) eigenvalue of the Markov process matrix A , multiplied by the time step Δt , and writing

$$Q' = \begin{bmatrix} 1 & \alpha & & & \\ -1 & 1 & \frac{-\alpha}{2} & & \\ & -1 & 1 & \frac{\alpha}{6} & \\ & & -1 & & \frac{-\alpha}{6} \\ & & & -1 & 1 \end{bmatrix}$$

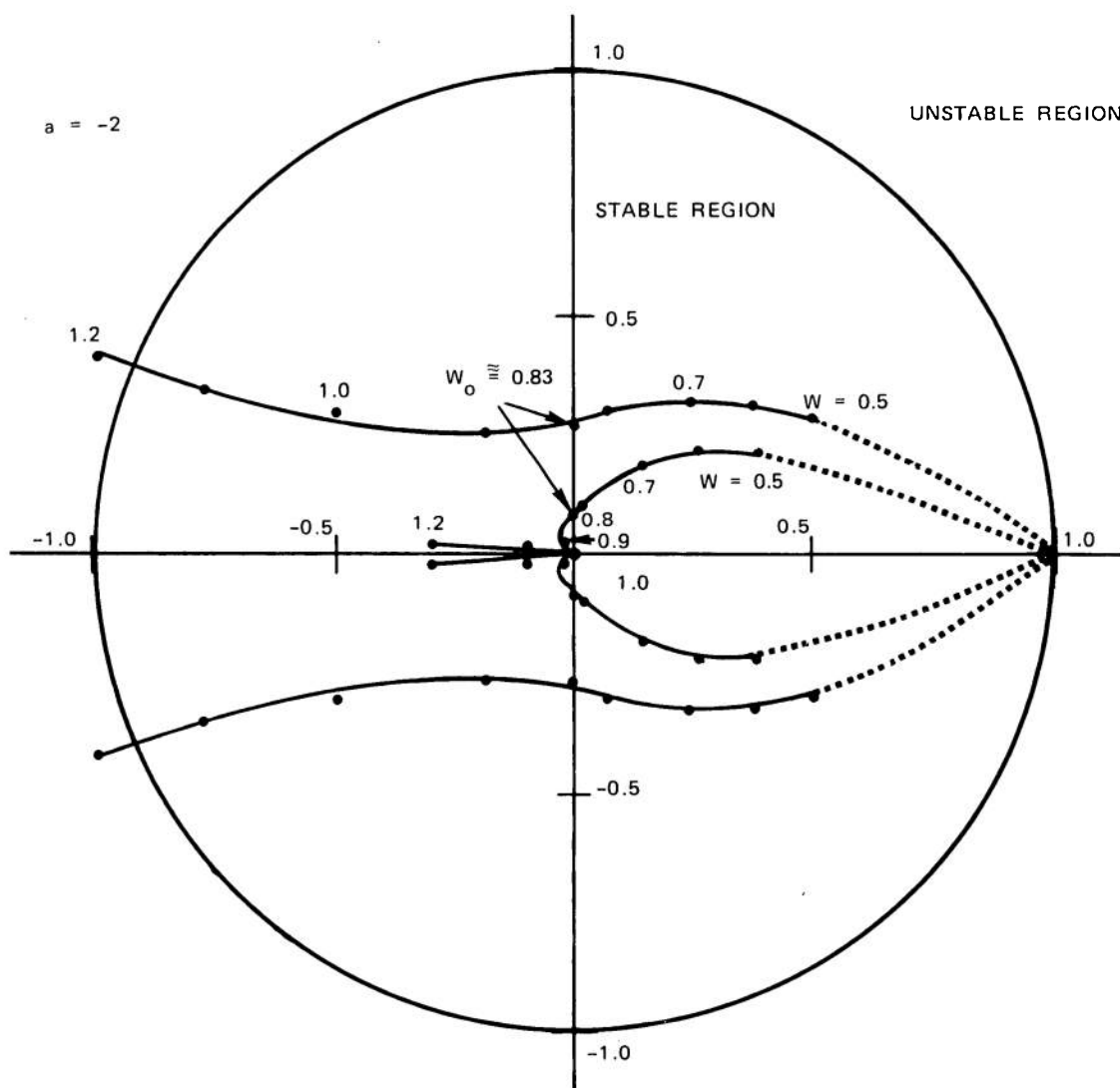
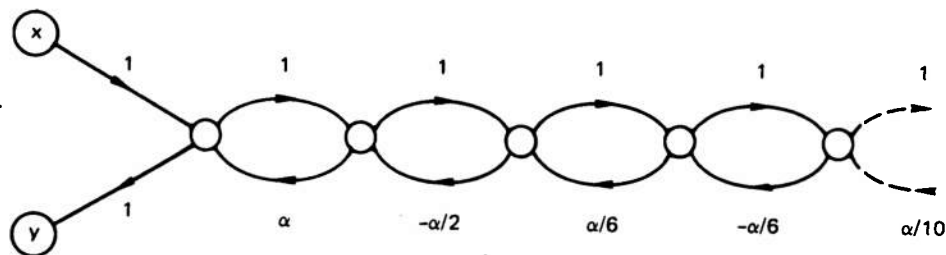


FIGURE B.1 ROOT LOCI FOR VARIOUS W VALUES

This corresponds to taking four terms in the continued fraction expansion of e^{α} and is associated with five nodes of the flow graph



representing $y = e^{\alpha}x$. (We tried using nine terms of this flow graph for better accuracy, but no improvement resulted in overall computational effort.)

Analyzing the five eigenvalues of the over-relaxation iteration matrix $\mathfrak{L}_{\omega}$ and using the above matrix Q' , where

$$\mathfrak{L}_{\omega} = [\omega F - (1 - \omega)D](D + \omega E)^{-1}$$

[see Ref. 1, Eq. (B-8a)] gives an equation for four of these eigenvalues (one of them is $\omega - 1$) as follows:

$$(\lambda + \omega - 1)^2 = \lambda \omega \mu^2, \quad ,$$

where μ is an eigenvalue of the upper and lower triangular parts $E + F$ of the matrix Q in Ref. 1. Hence, after some manipulation, it can be shown that the other four eigenvalues of $\mathfrak{L}_{\omega}$ (being a 5×5 matrix) are given by the solution to the polynomial

$$\begin{aligned} \lambda^4 - \left\{ \frac{\alpha \omega^2}{2} - 4\omega + 4 \right\} \lambda^3 + \left\{ 6(\omega - 1)^2 - \frac{\alpha^2}{12} \omega^4 - \alpha(\omega - 1) \right\} \lambda^2 \\ - \left\{ \frac{\alpha}{2} \omega^2 - 4\omega + 4 \right\} (\omega - 1)^2 \lambda + (\omega - 1)^4 = 0 \end{aligned}$$

When $\omega = 1$, we have the reduced situation

$$\lambda^4 - \frac{\alpha}{2} \lambda^3 + \frac{\alpha^2}{12} \lambda^2 = 0$$

which forces two of the eigenvalues to be 0. This explains why the straightforward "block Jacobi" method (without any relaxation factor, either <1 or >1) is not too good for our problem because it implies degeneracy of eigenvalues. The optimal ω_0 occurs when the coefficients of λ^3 and λ vanish, for then the equation becomes

$$\lambda^4 + \left\{ \frac{34}{3} (\omega - 1)^2 - \frac{8(\omega - 1)^2}{\omega^2} \right\} \lambda^2 + (\omega - 1)^4 = 0 \quad ,$$

where ω is found in terms of α by $\alpha\omega^2/8 = \omega - 1$, i.e.,

$$\omega_0 = \frac{4}{\alpha} (1 \mp \sqrt{1 - \alpha/2})$$

(we take the minus sign to get our $\omega > 0$). For $\omega = \omega_0$, all eigenvalues are of equal, absolute value because they evidently come in squares of conjugate pairs; thus clearly, without having to solve the above equation, the product of the magnitudes is known to be the term $(\omega - 1)^4$. Hence, the optimal convergence rate is $|\omega - 1|$.

Now, the time steps were taken to be $\Delta t = 0.01$, and the highest terms appearing in the large matrix A are -243 on the main diagonal. Hence, the largest eigenvalue (in magnitude) of the matrix $A\Delta t$ can be estimated at -2 . When this value is used, the optimal computed ω_0 is 0.83 . Amazingly, the best convergence obtained experimentally by the integration program was for $\omega \cong 0.9$, with about 5 iterations per step, while $\omega = 1.1$ took 13 steps. Thus, the analytical evaluation of the best ω corroborated the experimental results.

Appendix C

MASON'S RULE

Appendix C

MASON'S RULE

A convenient method is available for determining transmissions (or gains) between any pair of nodes in small flow graphs by inspection. In its algebraic form, the formula resembles Cramer's rule for solving simultaneous linear equations.

Each flow graph has a graph determinant Δ , the value of which remains invariant under various rules for manipulating flow graphs to obtain equivalent flow graphs. To evaluate the graph determinant, it is necessary to find the path transmission of every loop in the flow graph, where a loop is a sequence of (directed) branches from a node back to itself without repeating any other node.* Figure C.1 has three loops:

- Node 1 to node 1, with loop transmission p_{11} .
- Node 1 to node 2, then back from node 2 to node 1, with loop transmission $p_{12}p_{21}$.
- Node 1 to node 2 to node 4 to node 3 back to node 1, with loop transmission $p_{12}p_{24}p_{43}p_{31}$.

The loop transmission is denoted by $L_1, L_2, L_3, \dots, L_n$ in the general case; the graph determinant is then obtained by expanding the product

$$(1 - L_1)(1 - L_2) \dots (1 - L_n)$$

and deleting all terms that contain products of touching loops. In the example of Figure E.1, of the three loops L_1, L_2, L_3 , any selected pair

* A path transmission is the product of the transmissions of the branches comprising the path.

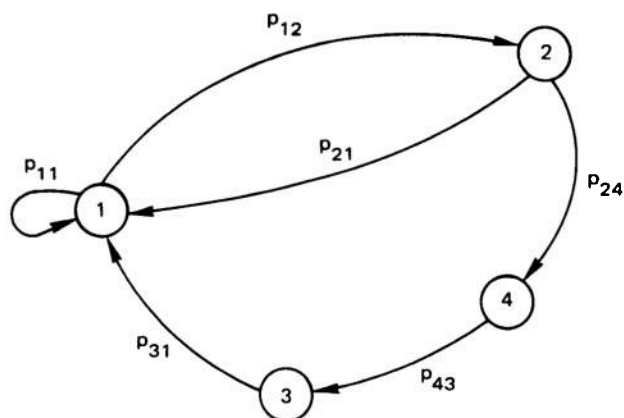


FIGURE C.1 TRANSITION DIAGRAM FOR A TYPICAL MARKOV CHAIN

touch; therefore,

$$\Delta = 1 - L_1 - L_2 - L_3 = 1 - p_{11} - p_{12}p_{21} - p_{12}p_{24}p_{43}p_{31}.$$

An equivalent formula for Δ is

$$\Delta = 1 - \sum_{\text{all loops } k} L_k + \sum_{\text{all pairs of nontouching loops } j \text{ and } k} L_j L_k - \sum_{\text{all triplets of nontouching loops } i, j, k} L_i L_j L_k + \dots$$

Mason's rule for calculating transmissions between selected nodes i and j in an arbitrary flow graph is given by the formula

$$g_{ij} = \frac{\sum_{k=1}^K t_k \Delta_k}{\Delta},$$

where there are K open paths from node i to node j ; t_k is the path transmission of the k^{th} path, Δ is the graph determinant, and Δ_k is the graph determinant of the subgraph corresponding to path k . The k^{th} subgraph is obtained by eliminating the open path k and all loops touching this path; if all loops are killed by an open path, the determinant of its subgraph reduces to unity.

As an example in Figure A.1, the transmission from node 1 to node 2 can be determined to be p_{12}/Δ because there is a single path of transmission p_{12} from node 1 to node 2 and that path kills all the loops so that the determinant of its subgraph is unity. The transmission from node 2 to node 1 is more complex. There are two open paths with transmissions p_{21} and $p_{24}p_{43}p_{31}$, and since each of these open paths kills all the loops in the graph,

$$g_{21} = \frac{(a_{21} \times 1) + (a_{24}a_{43}a_{31}) \times 1}{\Delta}$$

$$= \frac{a_{21} + a_{24}a_{43}a_{31}}{1 - a_{11} - a_{12}a_{21} - a_{12}a_{24}a_{43}a_{31}}$$

Appendix D

LEAST SQUARES AND QUADRATIC PROGRAMMING

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LEAST SQUARES AND QUADRATIC PROGRAMMING

1. Least Squares Programming

The classical least-squares problem can be formulated as follows:

Assume the linear model

$$y = Hx + z \quad , \quad (D.1)$$

where

x is an unknown constant vector ($n \times 1$)

y is an ($m \times 1$) observable random vector

H is an $m \times n$ matrix of full rank

z is an $m \times 1$ error vector such that $E(z) = 0$.

Then the sum of the squared deviations (errors) is

$$z^T z = (y - Hx)^T (y - Hx) \quad . \quad (D.2)$$

Differentiating with respect to x , equating the derivative to zero, and solving for x as in Section VI gives the solution

$$\tilde{x} = \left(H^T H \right)^{-1} H^T y \quad . \quad (D.3)$$

A generalization from Aitken assumes that the covariance matrix of the errors is known

$$E\left(zz^T\right) = V \quad . \quad (D.4)$$

Then the scalar to minimize includes V^{-1} ; specifically minimize

$$(y - Hx)^T V^{-1} (y - Hx) \quad . \quad (D.5)$$

The solution to this generalized least squares problem is

$$\tilde{x} = \left(H^T V^{-1} H\right)^{-1} H^T V^{-1} y \quad . \quad (D.6)$$

This formula may also be used to weight the error components by choosing V to be a suitable diagonal matrix with positive components.

It is sometimes convenient to place additional linear constraints on the unknown variables x in the model consisting of Eqs. (D.1) and (D.4). Assume these constraints are expressible as

$$Ax = k \quad (D.7)$$

where A is a known constant matrix of dimensions $p \times n$, k is a known constant vector of dimensions $p \times 1$, and $\text{rank } A = p < n$. Then the solution to the constrained least squares model in Eqs. (D.1), (D.4), and (D.7) is

$$\hat{x} = \tilde{x} + \left[\left(H^T V^{-1} H\right)^{-1} A^T\right] \left[A \left(H^T V^{-1} H\right)^{-1} A^T\right]^{-1} (k - A\tilde{x}) \quad (D.8)$$

where $\tilde{x}$ is the solution to the unconstrained problem given in Eq. (D.6). If the constraints are not binding, the quantity multiplying $(k - A\tilde{x})$ in Eq. (D.8) will be zero.

The covariance matrix of $\hat{x}$ is

$$\text{cov } \hat{x} = B \left(H^T V^{-1} H \right) B^T \quad (\text{D.9})$$

where

$$B = I - \left(H^T V^{-1} H \right)^{-1} A^T \left[A \left(H^T V^{-1} H \right)^{-1} A^T \right] A \quad (\text{D.10})$$

The constrained least squares formula given here may be found in Ref. 22.

2. Quadratic Programming

Least squares methods given above do not incorporate linear inequality constraints; in particular, the components of the vector x cannot be constrained to be nonnegative. Quadratic programming may be regarded as an extension of constrained least squares in which nonnegativity constraints can be introduced. For purposes here, a quadratic program can be formulated as a minimization problem:

$$\text{minimize } z = x^T D x + c^T x \quad (\text{D.11})$$

subject to constraints

$$Ax = k \quad \text{and} \quad x \geq 0 \quad (\text{D.12})$$

where c and k are known constant vectors and D is a symmetric and positive semidefinite known matrix of constants. (In least squares applications, these conditions on D are automatically satisfied.)

When it is desired to minimize

$$d^2 = (y - Hx)^T (y - Hx) \quad , \quad (D.13)$$

the right hand side is first expanded to obtain

$$d^2 = x^T H^T H x - 2y^T H x + y^T y \quad . \quad (D.14)$$

The matrix D is identifiable as $H^T H$, and the vector c is identifiable as $-2y^T H$. The constant term $y^T y$ may be neglected.

Quadratic programming problems may be solved by modifying a simplex code used for solving ordinary linear programs. The modifications, although few in number, are not easy to implement. A quadratic programming algorithm suitable for small problems was developed during the research; regrettably, it could not be applied to problems of realistic size.

The computational effort for linear programs is known to be roughly proportional to m^3 , where m is the number of rows. A quadratic program with m constraints and n variables requires about the same amount of computational effort as a linear program with $(m + n)$ rows. Typical linear programming applications have about three times as many columns as rows, therefore $(m + n)$ is about 4 m. Cubing this gives the approximation

$$(\text{running time of a quadratic program}) \cong k64 m^3 \quad ,$$

where m is the number of constraints and k is a constant. This relationship should be considered carefully in determining the number of variables and constraints to use in quadratic programming applications.

Appendix E

THEOREMS ON PARAMETER ESTIMATION

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THEOREMS ON PARAMETER ESTIMATION

In Section VII, a set of theorems concerning some basic identities arising in the estimation of parameters of a Markov chain were cited and applied. In this appendix the theorems are precisely stated, proved, and illustrated with a numerical example.

Suppose an arbitrary random process has N states, where states $1, 2, \dots, M$ are transient and states $M + 1, M + 2, \dots, N$ are absorbing. Let data from this process be given for a set of n_{01} complete trials, where complete means each of the n_{01} trials begins in state 1 and continues until absorption occurs in some absorbing state. A path corresponding to a complete trial is therefore an ordered sequence of integers of the form

$$1 \ t_1 \ t_2 \ t_3 \ \dots \ t_n \ a \ ,$$

where each t_j is an integer from 1 to M and a is an integer in the range $M + 1 \leq a \leq N$. The path represents the sequence of states entered when read from left to right--since "a" represents absorption in state a the path is completely described.

The random process is arbitrary. Nevertheless, we will apply a Markov chain model with the same states.

Following Markov chain estimation theory, we suppose transition probabilities p_{ij} are estimated from

$$\hat{p}_{ij} = \begin{cases} n_{ij}/n_i & \text{if } n_i > 0 \\ 0 & \text{if } n_i = 0 \end{cases} \quad (\text{E.1})$$

where

$$n_{ij} = \text{observed number of transitions from } i \text{ to } j$$

$$n_i = \sum_{j=1}^N n_{ij} = \text{number of transitions out of state } i.$$

The n_{ij} are readily computed from the path data by counting transitions. Now we partition $\hat{P}$, the matrix of estimated transition probabilities, as in Section V.

$$\hat{P} = \begin{pmatrix} \hat{Q} & \hat{R} \\ 0 & I \end{pmatrix} \quad (\text{E.2})$$

where $\hat{Q}$ is $M \times M$, $\hat{R}$ is $M \times (N - M)$, and I is the identity matrix of dimension $(N - M) \times (N - M)$.

According to Markov theory (Section V-D), the true absorption probabilities a_{ik} are found by calculating $A = (a_{ik})$

$$A = (I - Q)^{-1}R \quad (\text{E.3})$$

when Q and R are the true transition probabilities.

1. Theorem 1

When Q and R in Eq. (E.3) are replaced by their estimates from Eq. (E.2), then the first row of the estimator $\hat{A}$,

$$\hat{A} = (I - \hat{Q})^{-1}\hat{R} \quad , \quad (\text{E.4})$$

is identical to the row vector of ratios

$$\left(\frac{S_{M+1}}{n_{01}}, \frac{S_{M+2}}{n_{01}}, \dots, \frac{S_N}{n_{01}} \right),$$

where S_j = number of times (out of n_{01} trials) that the process is observed to be absorbed in state j ($j = M + 1, \dots, N$). In other words, the Markov estimates of absorption probabilities are identical to the ratios (number of times process is absorbed in j /number of trials), which are computable directly from the given data by ignoring all internal transitions and simply counting numbers of entries into the respective absorbing states.

a. Proof

Let state 0 be an artificial state from which transitions to state 1 (the starting state) are always made. Using the n_{ij} as defined above, a "balance equation" Eq. (E.5) can be written at each node

$$\begin{aligned} n_{01} + \sum_{k=1}^M n_{k1} &= \sum_{k=1}^N n_{1k} \quad (\text{at state 1}) \\ \sum_{k=1}^M n_{ki} &= \sum_{j=1}^N n_{ij} \quad (\text{at state } i = 2, 3, \dots, M) \end{aligned} \quad (E.5)$$

These relations merely state that every sample path entering a transient state also leaves it. By summing the number of entries into an absorbing state over absorbing states, we obtain Eq. (E.6)

$$\sum_{j=M+1}^N \left(\sum_{i=1}^M n_{ij} \right) = n_{01} \quad (E.6)$$

This relation says that all paths terminate in some absorbing state.

Define

$$x_0 = n_{01} \quad (E.7)$$

$$x_i = \sum_{j=1}^N n_{ij} = \text{number of transitions } \underline{\text{out}} \text{ of (or into) state } i$$

$$i = 1, 2, \dots, M \quad (E.8)$$

$$x_i = \sum_{k=1}^M n_{ki} = \text{number of transitions } \underline{\text{into}} \text{ state } i$$

$$i = M + 1, M + 2, \dots, N \quad (E.9)$$

Then rewrite Eq. (E.5) to obtain Eq. (E.10).

$$x_0 + \sum_{k=1}^M \frac{n_{ki}}{x_k} x_k = x_1$$

$$x_i = \sum_{j=1}^M \left(\frac{n_{ji}}{x_j} \right) x_j \quad \text{at states } i = 2, 3, \dots, M \quad (E.10)$$

Equation (E.6) becomes

$$\sum_{j=M+1}^N x_j = n_{01} = x_0 \quad (E.11)$$

Let

$$f_{ji} = \frac{n_{ji}}{x_j} = \frac{n_{ji}}{\sum_{i=1}^N n_{ji}} n_{ji} \quad \text{for } i = 1, 2, \dots, N; j = 1, 2, \dots, M \quad (E.12)$$

Then Eqs. (E.10) and (E.11) become Eq. (E.13)

$$x_1 = x_0 + \sum_{j=1}^M f_{ji} x_j$$

$$x_i = \sum_{j=1}^M f_{ji} x_j \quad (i = 2, 3, \dots, M)$$

$$x_0 = \sum_{j=M+1}^N x_j \quad \text{from Eq. (E.11)} \quad (\text{E.13})$$

and from the definitions of x_i in Eq. (E.9) for absorbing states

$$x_i = \sum_{k=1}^M \frac{n_{ki}}{x_k} x_k = \sum_{k=1}^M f_{ki} x_k \quad (i = M+1, M+2, \dots, N) \quad (\text{E.14})$$

Combining Eqs. (E.13) and (E.14) gives Eqs. (E.15) and (E.16).

$$x_1 = x_0 + \sum_{j=1}^M f_{j1} x_j \quad (\text{E.15})$$

$$x_i = \sum_{j=1}^M f_{ji} x_j \quad (i = 2, 3, \dots, M, \dots, N) \quad (\text{E.16})$$

But Eqs. (E.15) and (F.16) are precisely the equations that are solved using the flow-graph approach to absorption probabilities when the true transition probabilities p_{ij} are replaced by the f_{ij} (Ref. 5, pp. 161-162, 200). Therefore, the solution of the flow-graph equations, which are transmissions in the flow graph, are the ratios x_j/x_0 when all other variables are eliminated. Choosing one absorption state (say N) as the Success state, the ratio x_N/x_0 is also the direct estimate given in Eq. (E.17).

$$\frac{x_N}{x_0} = \frac{\sum_{i=1}^M n_{iN}}{n_{01}} = \frac{\text{number of paths terminating in state N}}{\text{total number of paths}} \quad (\text{E.17})$$

Therefore, the estimate of success probability obtained when using the flow-graph approach is identical to the simpler estimate on the right side of Eq. (E.17). Since the flow-graph approach can be easily shown to give the same result as the matrix inversion method of Eq. (E.4), the proof is complete.

b. Example of Theorem 1

In a four state Markov chain model, let states 3 and 4 be absorbing and states 1 and 2 be transient. Suppose the given set of complete trials (paths) are given by

11123

12224

12123

1213

14 .

These path data are generated by an arbitrary process, not necessarily Markov.

The numbers of transitions implied by these paths are shown on the transition diagram of Figure E.1. The estimated transition matrix $\hat{P}$ therefore is:

$$\hat{P} = \left(\begin{array}{cc|cc} \frac{2}{9} & \frac{5}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{2}{7} & \frac{2}{7} & \frac{2}{7} & \frac{1}{7} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) = \left(\begin{array}{c|c} \hat{Q} & \hat{R} \\ \hline 0 & I \end{array} \right) .$$

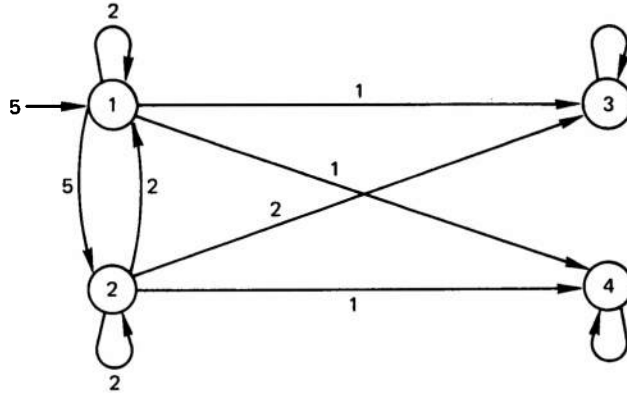


FIGURE E.1 NUMBER OF TRANSITIONS FOR A FOUR-STATE MARKOV CHAIN

$I - \hat{Q}$ and $(I - \hat{Q})^{-1}$ are readily calculated

$$I - \hat{Q} = \begin{pmatrix} \frac{7}{9} & -\frac{5}{9} \\ -\frac{2}{7} & \frac{5}{7} \end{pmatrix}$$

$$(I - \hat{Q})^{-1} = \frac{63}{25} \begin{pmatrix} \frac{5}{7} & \frac{5}{9} \\ \frac{2}{7} & \frac{7}{9} \end{pmatrix} = \begin{pmatrix} \frac{9}{5} & \frac{7}{5} \\ \frac{18}{25} & \frac{49}{25} \end{pmatrix}.$$

The estimated absorption matrix $\hat{A}$ is

$$\hat{A} = (I - \hat{Q})^{-1} \hat{R} = \begin{pmatrix} \frac{9}{5} & \frac{7}{5} \\ \frac{18}{25} & \frac{49}{25} \end{pmatrix} \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{16}{25} & \frac{9}{25} \end{pmatrix}.$$

Having made these Markov calculations, we observe that the (1,1) element of $\hat{A}$ is identical to the ratio

$$\frac{\text{number of paths terminating in state 3}}{\text{number of paths}} = \frac{3}{5}$$

and the (1,2) element is identical to the ratio 2/5 as guaranteed by Theorem 1.

The elements of $(I - \hat{Q})^{-1}$ calculated in Example 1 can be used to illustrate Theorem 2 which is stated below. Recall from Markov theory

that the (i,j) element of $(I - Q)^{-1}$ is the mean number of entries into state j before absorption, given that the process starts in state i .

The first row of $(I - \hat{Q})^{-1}$, which corresponds to starting in state 1, is $(9/5, 7/5)$ from the calculations in Example 1. By counting the number of ones in the five paths of Example 1, we see that $(3 + 1 + 2 + 2 + 1) = 9$ entries into state 1. The average number of entries into state 1 is therefore $9/5$, which is identical to the $(1,1)$ entry of $(I - \hat{Q})^{-1}$. Examining the other transient state, we find that there are $(1 + 3 + 2 + 1) = 7$ transitions into state 2, for an average of $7/5$ entries into state 2 before absorption. Since $7/5$ is also the $(1,2)$ element of $(I - \hat{Q})^{-1}$, it appears that another algebraic identity, similar to that in the theorem, has been found. Further numerical experimentation leads to Theorem 2.

2. Theorem 2

Under the conditions of Theorem 1, the elements of the first row of $(I - \hat{Q})^{-1}$ are identical to the basic estimates obtained directly from the path data by counting transitions into the respective transient states; that is,

$$(I - \hat{Q})_{1,j}^{-1} = \frac{\text{number of entries into state } j}{\text{number of paths } \binom{n}{01}} \quad (j = 1, 2, \dots, M) \quad .$$

a. Proof

The proof of Theorem 2 is contained in Theorem 1. It is only necessary to note that the ratios x_j/x_0 ($j = 1, 2, \dots, M$) are the mean numbers of entries into transient states. These ratios, from Markov theory, equal the flow-graph transmissions that in turn are the elements in the first row of $(I - \hat{Q})^{-1}$. It can be noted that Theorems 1 and 2 can be unified into a single theorem which states that the Markov-derived mean

number of entries into all states is identical to the direct estimates from the data; this unification is possible because the mean number of entries into an absorption state equals the probability of absorption into that state.

b. Example of Theorem 2

Data from the previous example can be used to illustrate Theorem 2. By direct count from the five paths, there are

$$(3 + 1 + 2 + 2 + 1) = 9 \text{ entries into state 1}$$

$$(1 + 3 + 2 + 1) = 7 \text{ entries into state 2.}$$

The first row of $(I - \hat{Q})^{-1}$ is $(9/5, 7/5)$ and thus verifies the identity.

Theorem 2 concerns the mean number of entries into each state before absorption while Theorem 1 concerns the absorption probabilities. Theorem 3 is related to another measure of effectiveness: the mean time to absorption conditioned on the starting state and the absorbing state.

3. Theorem 3

Assume path data and Markov chain model as in the theorem. Choose any absorbing state (say the failure state f) and consider estimating the mean time to absorption in that state, conditioned on starting in state 1 and absorption in that state. Use only those paths terminating in the designated absorbing state f to estimate the p_{ij} . Then the Markov formula for conditional mean time to absorption in state f , given start in state 1, gives identically the same mean time to absorb as does the direct estimate from the path data. Symbolically, the $(1, f)$ element of

$$(I - \hat{Q})^{-1}(I - \hat{Q})^{-1}\hat{R} \quad ,$$

when divided by $\hat{a}_{1f} = (1,f)$ element of $\hat{A}$, is identical to

$$\sum_{\substack{\text{all paths} \\ \text{absorbing in } f}} \frac{\text{number of transitions before absorption in } f}{\text{number of paths absorbing in } f} .$$

a. Proof

Since only data leading to the absorption state f are used to estimate $\hat{Q}$ and $\hat{R}$, the absorption probability estimator $\hat{A}$ is of the form

$$\begin{pmatrix} \bigcirc & \vdots & \bigcirc \\ & \vdots & \\ & \vdots & \end{pmatrix} ;$$

that is, $\hat{A}$ has a single column of ones in the column corresponding to absorption state f and has zeroes elsewhere.

An examination of the formula $(I - \hat{Q})^{-1}(I - \hat{Q})^{-1}\hat{R} = (I - \hat{Q})\hat{A}$ clearly shows that the column of ones in $\hat{A}$ merely serves to sum the elements of rows of $(I - \hat{Q})^{-1}$. This summation represents the total number of transitions before absorption, the conditional mean time to failure in this context. Symbolically,

(conditional mean time until failure)

$$\begin{aligned} &= \sum_{j=1}^M (\text{mean numbers of entries into state } j \text{ before absorption}) \\ &= \sum_{j=1}^M \hat{t}_{1j} = \sum_{j=1}^M (I - \hat{Q})_{1,j} . \end{aligned}$$

b. Example of Theorem 3

Data from the absorption probability example above will also serve to illustrate the identity in Theorem 3 provided that either state 3 or state 4 is selected as an absorbing state. Selecting state 3 arbitrarily, the path data are

11123

12123

1213

from which $\hat{P}$, $\hat{Q}$, and $\hat{R}$ are readily found

$$\hat{P} = \begin{pmatrix} \hat{Q} & \hat{R} \\ 0 & I \end{pmatrix} = \left(\begin{array}{cc|c} \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array} \right)$$

$$(I - \hat{Q}) = \begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

and hence,

$$(I - \hat{Q})^{-1} = \frac{7}{3} \begin{pmatrix} 1 & \frac{4}{7} \\ \frac{1}{2} & \frac{5}{7} \end{pmatrix} = \begin{pmatrix} \frac{7}{3} & \frac{4}{3} \\ \frac{7}{6} & \frac{5}{3} \end{pmatrix} .$$

The absorption matrix $(I - \hat{Q})^{-1}\hat{R}$ consists of all ones because only one absorbing state exists. Therefore,

$$(I - \hat{Q})^{-1}(I - \hat{Q})^{-1}\hat{R} = (I - \hat{Q})^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{3} \\ \frac{17}{6} \end{pmatrix} .$$

The first component (11/3) agrees exactly with the mean number determined from the path data. (Path 11123 requires 4 transitions, path 12123 also requires 4 transitions, and path 1213 requires 3 transitions for a total of eleven transitions in three trials.) The identity is therefore verified in this case.

Appendix F

DETERMINING A SET OF STATES

Appendix F

DETERMINING A SET OF STATES

It was indicated in Section XII-C that the choice of a set of states is likely to be the most difficult part of the application of Markov models. Some discussion of this problem aspect is therefore needed.

Requirements on the state space (set of states) imposed from the mathematical definition of state are:

- States should be defined to be mutually collective and exhaustive; that is, at each instant of time, the process should be in only one state.
- The next transition of the process should depend only on the current state and not on how this state was reached.

The first requirement is not difficult to achieve from the standpoint of logic alone. In a complex realistic environment, however, it may be difficult to ensure consistency. Not much can be said in general about this requirement except that concepts from elementary set theory may be useful. For example, if it is desirable to have states A and B defined (for operational reasons) such that it is possible to be in both states A and B simultaneously, then it may be desirable to define a new state meaning: the process is in both states A and B. The previous meaning of state A must then be modified to mean: in state A and not state B; a similar modification would be required for state B.

The second requirement--paraphrased as: the future should depend only on the present state and not on the past states--is probably impossible to achieve exactly in any real application. When the state is search, for example, no other information about the previous states

occupied or other variables is assumed to be needed to determine the future course of the process.

In most analytical applications, the state space is selected on the basis of information variables as opposed to other possibilities such as geometrical variables (range, bearing, speed). This will probably remain true for operational applications as well. However, the possibility of using both information and geometrical variables should be kept open until it is reasonably certain that geometry need not enter. To keep the number of states from getting out hand, it may be possible to introduce geometry by distinguishing a small number of levels of important variables such as range. For example, a state may be labeled by the pair (Search, long range) to indicate that the (original) state is Search and the actual range between the searcher and the target is greater than some specified threshold.

In applications, an analyst's experience and judgment must ultimately be used to obtain definitions for a set of states. A reasonably systematic approach to assist in state definition is outlined as follows.

- Determine a set of dynamic variables from which states may be defined. At least two major categories of variables will be required: geometry/kinematic variables and information variables.
- Choose a set of states that are mutually exclusive and collectively exhaustive and can be determined from the values of the dynamic variables.
- For a selected set of states:
 - Use fleet exercise data from both exercise reports and other historical data to determine sequence of states entered and the times of entry into states. (The sequence and associated times of entry can be considered a "path" through the states.)
 - Perform tests to determine whether the observed paths may be assumed generated by a Markov or semi-Markov process.

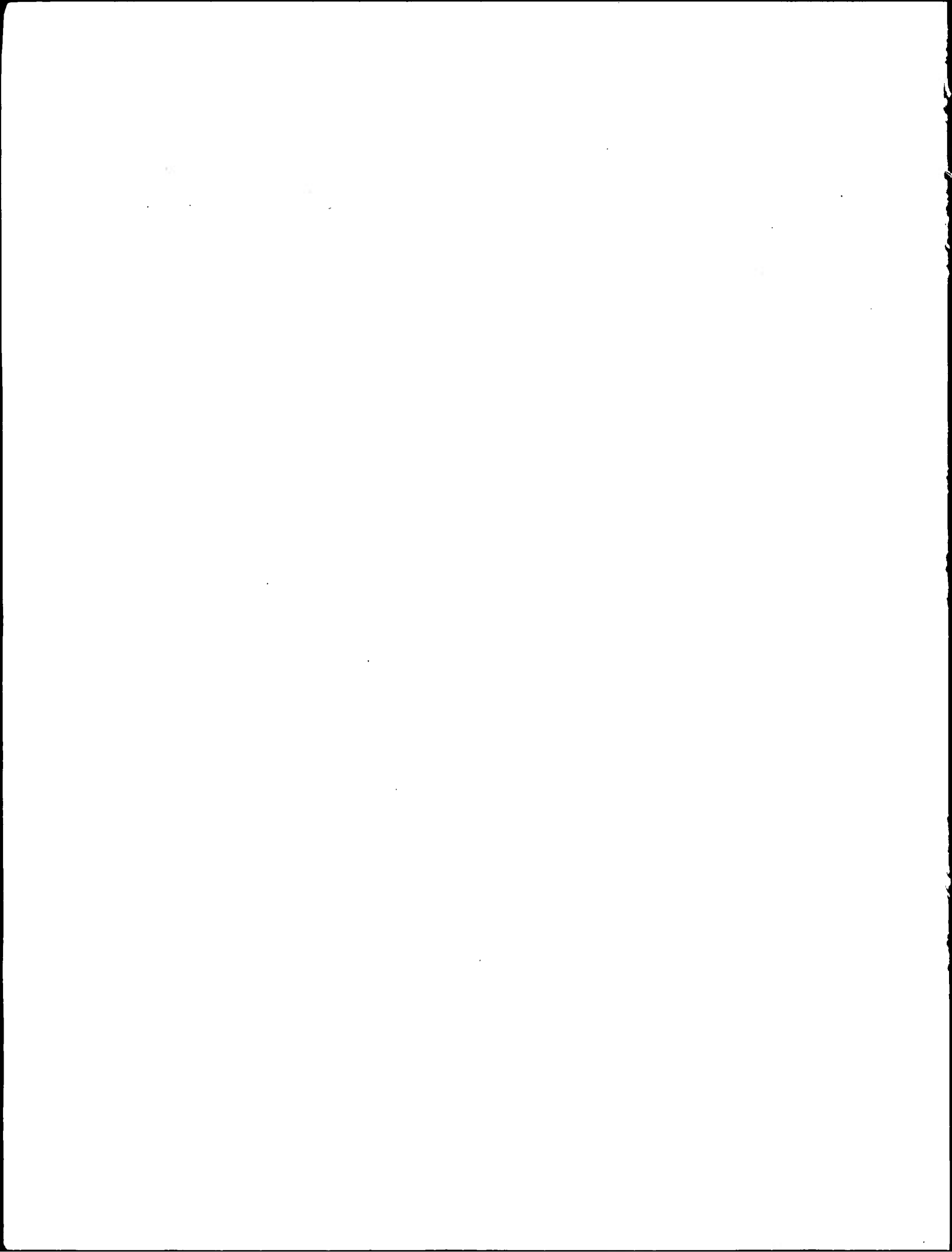
- When the Markov assumption is justified:
 - Estimate parameters of the Markov model from the path data, using both standard methods and methods in Section VI.
 - Run the Markov model to determine properties of the output. Specifically, determine the probability of terminating in a Success state, the mean time to Success (or to Fail), and probability of occupying each state as a function of time for all states.

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