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# STOCHASTIC ATTRITION MODELS OF LANCHESTER TYPE

Alan F. Karr

June 1974



INSTITUTE FOR DEFENSE ANALYSES  
PROGRAM ANALYSIS DIVISION

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combatants, but are as free from restrictive physical interpretation as possible. Each attrition process is a regular step Markov process and is characterized in terms of its jump function, transition kernel, and infinitesimal generator. Also included are several taxonomies of the processes presented, an Appendix of probabilistic technicalities and proofs, and many interpretive remarks. In particular a clarification of the square law-linear law and area fire-point fire dichotomies is given.

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## I. INTRODUCTION AND PURPOSE

The goal of the research effort reported here has been to derive a class of stochastic attrition models from probabilistic assumptions on the behavior of individual combatants and on the interactions among them. Our interest is in stochastic analogs of a family of deterministic attrition models commonly called "Lanchester attrition models", after their originator F. W. Lanchester. These deterministic models are discussed in Section II.

There are several reasons for undertaking such an effort. Previous research on stochastic Lanchester models, as surveyed in Section III, has been concerned mostly with certain computational problems and in general imposes by hypothesis the form of the attrition process, rather than deriving that form from more elementary assumptions. In some cases, therefore, our derivations lead to known and studied processes; the point is that instead of arbitrarily imposed processes we deal with the consequences of elementary and physically meaningful hypotheses. In the terminology of economics we employ a "micro" rather than a "macro" approach, stating assumptions about individual combatants rather than about the overall form of the attrition process.

There should be a general preference for stochastic rather than deterministic attrition models. A stochastic model is more general, more flexible, more realistic, and better founded, and always provides, through expectations of its outputs, scalar characterizations of the system being modeled. Deterministic attrition models of Lanchester type are represented by differential equations and require allowing noninteger numbers of combatants, while the stochastic models we present here have only integer (but vector-valued) states.

The reasons for wishing to derive models from elementary assumptions are also several. First, one wants to know if there exists a set of assumptions from which a known process can be derived and, if so, what physical situations are consistent with the assumptions. Understanding of the model and its possible applicability are enhanced when assumptions are explicitly stated. Different models of combat can be compared in a reasonable manner on the basis of underlying assumptions (as well as by their relationship to historical data) rather than on the untenable basis of outcomes, implications, and heuristic judgments.

Once sets of assumptions are in hand, new models may be created by generalizing or weakening certain assumptions. For example, we have found that a certain stochastic Lanchester model frequently used for describing combat between heterogeneous forces is not, based on underlying assumptions, the appropriate generalization of the corresponding homogeneous model. The effect of weakening unrealistic and untenable assumptions can be explored in a sensible way only if it is realized what those assumptions are.

Finally, once underlying assumptions are found for a family of related processes, one may seek general structural characteristics, unifying taxonomies, and general computational approaches such as we present in Section IV.

The final section of this paper is a compendium of the processes derived so far, giving for each process one family of assumptions from which it can be derived and a probabilistic characterization of the process. It may be that certain of these processes can be derived from alternative sets of assumptions, but we believe that our families of assumptions are essentially unique.

Probabilistic technicalities and proofs of our results appear in the Appendix.

The stochastic attrition processes discussed here are all time-dependent dynamic models with a continuous time parameter. In Karr (1972a, 1972b, 1973, 1974) similar derivations are given for a class of static and discrete time attrition models.

## II. DETERMINISTIC LANCHESTER ATTRITION THEORY

Consider a combat between two homogeneous forces, Blue and Red, and denote by  $b(t)$  and  $r(t)$  the numbers of Blue and Red survivors at time  $t$  after the combat is initiated. The British engineer F. W. Lanchester (1916) suggested that it is the nature of modern warfare that the instantaneous casualty rate on each side be proportional to the current strength of the opposing side. Lanchester thus proposed the now famous model

$$\begin{aligned} b'(t) &= -c_1 r(t) \\ r'(t) &= -c_2 b(t) \end{aligned} \tag{1}$$

where  $c_1, c_2$  are positive and not necessarily equal. In order that (1) make sense, the functions  $b$  and  $r$  must be allowed to assume arbitrary nonnegative values.

The solution of (1) subject to the initial conditions

$$\begin{aligned} b(0) &= b_0 \\ r(0) &= r_0 \end{aligned}$$

is given by

$$\begin{aligned} b(t) &= b_0 \cosh \lambda t - \alpha r_0 \sinh \lambda t \\ r(t) &= r_0 \cosh \lambda t - \alpha^{-1} b_0 \sinh \lambda t \end{aligned} \tag{2}$$

where

$$\lambda = (c_1 c_2)^{\frac{1}{2}}$$

and

$$\alpha = (c_1/c_2)^{\frac{1}{2}} .$$

$\lambda$  is a measure of the intensity of the engagement and  $\alpha$  of the killing effectiveness (per unit time) of one Red combatant relative to that of one Blue.

The functions  $b$  and  $r$  defined by (2) are of interest only until the time  $\tau = \inf\{t: b(t) = 0 \text{ or } r(t) = 0\}$  at which one side or the other is annihilated.  $\tau$  is infinite if and only if

$$c_1 r_0^2 = c_2 b_0^2$$

in which case

$$\lim_{t \rightarrow \infty} r(t) = \lim_{t \rightarrow \infty} b(t) = 0 ;$$

otherwise one side is annihilated at a finite time and the other has a positive surviving strength.

It follows from (2) that

$$(3) \quad \alpha^2 [r_0^2 - r(t)^2] = b_0^2 - b(t)^2$$

for all  $t$ . In view of (3) the system (1) of differential equations is called Lanchester's "square law" of attrition.

Lanchester also proposed the so-called "linear law" in which each side's casualty rate at any time is proportional to the product of its strength and the strength of the opposition. In differential form this model is given by

$$(4) \quad b'(t) = -k_1 b(t)r(t)$$

$$r'(t) = -k_2 r(t)b(t)$$

where  $k_1$ ,  $k_2$  are positive constants. These are not the constants of (1) and, indeed, must have different units. The exact solution to (4) does not interest us (but note that it is nonnegative, removing one objection to (2)). The condition analogous to (3) is

$$r_0 - r(t) = \text{constant} \times (b_0 - b(t))$$

from which the term "linear law" originates.

Of considerable interest is the distinction between "square law" and "linear law" combat. Historically [see, for example, Bonder (1970, p. 160), Deitchman (1962, p. 818), Dolansky (1964, p. 345), and Hall (1971, p. 8)] the belief has been that the square law describes combat situations in which individual opponents are identified and engaged one-by-one, a situation commonly referred to as "point fire" combat. On the other hand, the linear law has been thought to describe combat processes in which weapons (such as artillery) fire only at an area in which opponents are located ("area fire"). The research presented here leads to a number of relevant conclusions, whose overall effect in the preceding distinction is somewhat ambiguous. Indeed, perhaps the best distinction is the following. "Square law" and "linear law" are terms which distinguish two types of engagement initiation. In the former one side initiates engagements (i.e., fires shots) at a mean rate which is proportional to its own numerical strength and independent of the numerical strength of the opposition, while in the latter each side initiates engagements at a mean rate proportional to the product of its numerical strength and that of the opposition. On the other hand the "area fire" - "point fire" distinction seems to involve mostly what can happen once a shot is fired. Area fire processes allow multiple kills with one shot while point fire processes allow at most one kill by a single shot. This dichotomy is not unambiguous; for example certain processes which seem physically to represent the "firing at an occupied area" aspect usually associated with area fire may in fact have at most one kill per shot, due to dispersion of one side or the other.

Seen in this way the square law-linear law and point fire-area fire are not competing distinctions but, rather, are complementary, so that in fact four classes of processes are defined by this classification scheme.

Let us now consider the case of heterogeneous forces, where each side consists of two or more distinct types of combatants. Suppose there are  $M$  types of Blue weapons (combatants) and  $N$  types of Red weapons and denote by  $b_i(t)$  and  $r_j(t)$  the number of Blue type  $i$  and Red type  $j$  weapons, respectively surviving at time  $t$ . Let  $b(t) = (b_1(t), \dots, b_M(t))$  and  $r(t) = (r_1(t), \dots, r_N(t))$  be the vectors of Blue and Red surviving forces at time  $t$ , respectively.

Since there is no known set of precise and unambiguous assumptions leading by means of a rigorous mathematical derivation to (1) (Weiss (1957) is typical of the ambiguity and vagueness of previous sets of assumptions and the lack of rigor of previous derivations) it is not clear what should be the appropriate generalization to the heterogeneous case. The usual heterogeneous version of (1) is obtained purely formally, by replacing  $b$ ,  $r$ ,  $c_1$  and  $c_2$  there by the vector  $b$ , the vector  $r$  and matrices  $c_1$  and  $c_2$  to obtain the model

$$(5) \quad \begin{aligned} b'_i(t) &= - \sum_{j=1}^N c_1(i, j)r_j(t) \quad , \quad i = 1, \dots, M \\ r'_j(t) &= - \sum_{i=1}^M c_2(j, i)b_i(t) \quad , \quad j = 1, \dots, N \quad , \end{aligned}$$

which is often abbreviated in matrix form as

$$\begin{aligned} b' &= - c_1 r \\ r' &= - c_2 b \quad , \end{aligned}$$

where  $c_1 = [c_1(i, j)]$  and  $c_2 = [c_2(j, i)]$  are  $M \times N$  and  $N \times M$  non-negative matrices, respectively.

Evidently there exist neither a closed-form solution to (5) nor any means of handling negative components. The termination rule so easily described for the homogeneous square law model (1) does not carry over. Another of our discoveries is that the stochastic attrition process whose analog is (5) is not, based on sets of assumptions, the appropriate generalization to the heterogeneous case of the stochastic process analogous to (1). Hence the common practice of calling (5) "heterogeneous Lanchester-square combat" is unjustified. As part of our research we have derived the proper stochastic and deterministic heterogeneous square law models, which are presented in Section V.

Relatively little has been said about heterogeneous versions of the linear law model (4) although it has been suggested by D. Howes (personal communication to L. B. Anderson) that the appropriate form is

$$b'_i(t) = -b_i(t) \sum_{j=1}^N k_1(i, j)r_j(t)$$

(6)

$$r'_j(t) = -r_j(t) \sum_{i=1}^M k_2(j, i)b_i(t)$$

where  $k_1$ ,  $k_2$  are nonnegative  $M \times N$  and  $N \times M$  matrices, respectively. What the motivation for (6) is, other than (4) and intuition, is not clear, but it does turn out that the stochastic version of (6) is the proper generalization, based on assumptions, of the stochastic version of (4).

The symmetry of (1) and (4) is not necessary. One might assume instead that

$$b'(t) = -cr(t)$$

(7)

$$r'(t) = -kb(t)r(t) ,$$

for positive constants  $k$  and  $c$ , which is the so-called "mixed-law" of Lanchester combat. The mixed law was proposed by Deitchman (1962) as a model of guerrilla ambushes, with Blue representing the ambushers and Red the ambushed.

### III. SOME ASPECTS OF PREVIOUS WORK ON STOCHASTIC LANCHESTER PROCESSES

We do not intend here to survey in detail the large collection of previous studies of stochastic models purported to be the proper analogs of the deterministic systems (1), (4), (5), (6) and (7), except by indicating the philosophical and mathematical differences between those approaches and our approach here. The two most extensive published surveys are Dolanský (1964) and Hall (1971); Springhall (1968) also contains a survey and a rather extensive bibliography. Included in the list of references are a number of papers of interest in the historical development of Lanchester theories of combat; in particular, the papers of Brown (1955), Isbell and Marlow (1956) and Weiss (1957) contain the germs of the theory. Sets of assumptions in these early works are imprecise and ambiguous and derivations are, for the most part, incomplete or nonexistent. One purpose of this research is to provide consistent and unambiguous sets of assumptions from which certain stochastic attrition models can be rigorously derived. In doing so we have unified and extended the theory of Lanchester combat models.

In order to understand the attrition processes presented in Section V, some background concerning a certain class of Markov processes, namely regular step processes, is required; we shall also make use of the following discussion in the remainder of this section. Blumenthal and Gettoor (1968), Freedman (1971), and Karlin (1968) are principal references.

Let  $E$  be a countable set. A Markov matrix on  $E$  is a mapping  $P : E \times E \rightarrow [0, 1]$  with the property that

$$\sum_{j \in E} P(i, j) = 1$$

for all  $i \in E$ .

Given a Markov matrix  $P$  on  $E$  such that  $P(i, i) = 0$  for all  $i$  and a function  $\lambda : E \rightarrow [0, \infty)$  there exists a Markov process  $(X_t)_{t \geq 0}$  with state space  $E$  satisfying the following intuitive description. If  $\lambda(i) = 0$ , then once the process enters state  $i$  it remains there forever after, while if  $\lambda(i) > 0$  the process, upon entering state  $i$ , remains there an exponentially distributed time with mean  $1/\lambda(i)$  independent of the past history of the process, whereupon it jumps to another state in  $E$  according to the probability distribution  $P(i, \cdot)$ , independent of the length of its sojourn in state  $i$ .  $(X_t)$  is called the regular step process with jump function  $\lambda$  and transition kernel  $P$ .

The family  $(P_t)_{t \geq 0}$  of Markov matrices on  $E$  defined by

$$P_t(i, j) = P\{X_t = j | X_0 = i\}$$

is called the transition function of  $(X_t)$ . The matrix-valued mapping  $t \rightarrow P_t$  can be shown to be differentiable and the matrix

$$Q = P'_0$$

has the property that  $P'_t = QP_t$  for all  $t$ .  $Q$  is called the infinitesimal generator of the regular step process  $(X_t)$  and is given by

$$(8) \quad Q(i, j) = \begin{cases} -\lambda(i) & \text{if } j = i \\ \lambda(i)P(i, j) & \text{if } j \neq i. \end{cases}$$

$Q$  has the interpretation of specifying the "infinitesimal" or "differential" behavior of the process  $(X_t)$  because for  $j \neq i$ ,

$Q(i, j)$  is the "infinitesimal rate" at which the process  $(X_t)$  moves from state  $i$  to state  $j$  in the sense that

$$P\{X_{t+h} = j | X_t = i\} = Q(i, j) h + o(h)$$

as  $h \rightarrow 0$ . Here  $\lim_{h \downarrow 0} o(h)/h = 0$ . The resemblance to a system of

differential equations is evident, so a given stochastic attrition process is called an analog or version of one of the deterministic models (1), (4), (5), (6), or (7) provided its infinitesimal generator sufficiently resembles the appropriate system of differential equations. Further details are in the Appendix.

For example, the second stochastic attrition process presented in Section IV has infinitesimal generator  $Q$  given by

$$\begin{aligned} Q((i, j); (i, j - 1)) &= ic_1 \\ (9) \quad Q((i, j); (i, j)) &= -(ic_1 + jc_2) \\ Q((i, j); (i - 1, j)) &= jc_2 \end{aligned}$$

where  $c_1, c_2$  are positive constants derived from quantities given in the appropriate family of assumptions. The first equation in (9) says, in the differential interpretation of  $Q$ , that when Blue and Red strengths are  $i$  and  $j$ , respectively, Red casualties (that is the transition from  $j$  Red survivors to  $j - 1$  Red survivors) are occurring at infinitesimal rate  $ic_1$ . Hence there is justification for calling the process whose infinitesimal generator is the  $Q$  of (9), a stochastic homogeneous square law attrition process, because of the clear resemblance between (1) and (9). This justification, incidentally, is far from new, dating at least to Snow (1948).

All previous work on stochastic Lanchester-type processes begins essentially at (9) by imposing as a hypothesis the form of the generator of the process to be studied. Such studies have generally been concerned with computing quantities of interest such as

- (1) expected numbers of survivors at each fixed time;
- (2) distribution and expectation of the time required to reach certain subsets of the state space (such as the set  $\{(i, j): i = 0 \text{ or } j = 0\}$  of absorbing states which is entered when one side or the other is annihilated);
- (3) the probability that Blue wins the engagement by exterminating Red.

Our concern has been directed at a more fundamental problem. It is elementary to show, for example, that there exists a regular step process whose infinitesimal generator is the  $Q$  given in (9), but no one has previously presented a complete and unambiguous set of assumptions on the firing behavior and interaction (or lack thereof) among combatants which entail an attrition process whose infinitesimal generator is the  $Q$  of (9). It is this lack of basic and physically meaningful sets of underlying assumptions that this research attempts to alleviate.

Hence in some cases the processes we derive are known, but not always. In all cases, it is the derivation and the more primitive level of the underlying assumptions which are new. Sometimes one set of assumptions leads the way to a new process (in particular this is the case for the stochastic versions of the heterogeneous square law (5)) and these new processes are also discussed in Section V.

#### IV. STRUCTURE OF THE FAMILY OF PROCESSES DERIVED

We discuss in this section several unifying structures and taxonomies which can be applied to the family of stochastic attrition processes presented in the next section, from which the reader hopefully can obtain both overview and insight.

The first taxonomy classifies the stochastic attrition processes presented here in terms of three criteria:

- (1) Multiple kill (area fire) or single kill (mainly point fire)
- (2) Square law engagement initiation or linear law engagement initiation
- (3) Homogeneous or heterogeneous force compositions

This appears in Table 1 below. The classification as to type of engagement initiation is based on analogy between infinitesimal generators of the process and the various systems of differential equations appearing in Section I.

In each instance the homogeneous model is a special case of the heterogeneous model and each heterogeneous process correctly reduces to the corresponding homogeneous process when each side consists of only one type of combatant.

The second taxonomy is based on the families of assumptions underlying the processes, which are presented in Section V. The taxonomy appears in Table 2, which gives for each of the processes the form of the basic assumptions from which it is derived (except assumption (3) which holds for all of the processes). A typical family consists of

- (1) An assumption that either
  - (a) Times between shots fired by a surviving weapon are independent and identically exponentially distributed with some mean, that when a shot is to occur exactly one opponent is detected, attacked, killed or not, and lost from contact, all instantaneously; or

Table 1. TAXONOMY OF THE PROCESSES

Force Composition

Engagement Initiation	Homogeneous	Heterogeneous
I. Multiple kill (area fire)		
Square law	1) Process A1, a new process with multiple kills and square law engagement initiation	
II. Single kill (mainly point fire)		
Square law	1) Process S1, whose generator is analogous to (1)	1) Process S2, whose generator is analogous to (5) 2) Process S3a, a new process obtained by simple extension of the assumptions of Process S1 3) Process S3, a new family of processes incorporating fire allocation, of which S3a is a special case
Linear law	1) Process L1, whose generator is analogous to (4) 2) Process L2, a new process generalizing Process L1 by the inclusion of engagements of positive duration	1) Process L3, a new process obtained by extension of the assumptions of Process L1 to the heterogeneous case, with generator analogous to (6)
Mixed law	1) Process M1a, a special case of M1, with generator analogous to (7)	1) Process M1, a new process of which a process with generator analogous to (7) is a special case.

Table 2. ASSUMPTIONS OF THE PROCESSES

Process	Assumptions
A1 Homogeneous Square	1a) Mean firing rate is specified. 2) Shot kills binomially distributed number of opponents.
S1 Homogeneous Square	1a) Mean firing rate is specified. 2) Shot kills exactly one opponent with probability $p$ , none with probability $1 - p$ .
S2 Heterogeneous Square	1a) One firing process for each opposing weapon type with rate dependent on target and shooter; all such processes occur simultaneously and independently. 2) Kill probabilities depend on target and attacker.
S3 Heterogeneous Square	1a) Each shooting weapon fires shots at mean rate dependent only on its type. Allocation of fire over opposing weapon types is by prescribed probability distributions. 2) Kill probabilities depend on target and attacker.
S3a Heterogeneous Square	1a) Same as S3 except that fire allocation is by uniform distributions (special case).
L1 Homogeneous Linear	1b) Mean time to make one-on-one detection specified. 2) Kill probability specified.
L2 Homogeneous Linear	1b) As in L1. 2) Engagement is one-on-one, lasts for exponentially distributed duration, ends with death of one, the other, or neither combatant with specified probabilities.
L3 Heterogeneous Linear	1b) Mean detection time depends on target and searcher. 2) Kill probability depends on attacker and target.
M1 Heterogeneous Mixed	1) Each side possesses two weapon types, one of which behaves according to 1a) and the other according to 1b). 2) Kill probability depends on target and attacker.
M1a Homogeneous Mixed	1) One side has weapons described by 12); the single weapon type on the other side is described by 1b). 2) Kill probability specified.

- (b) The time required to detect a particular opponent is exponentially distributed with some mean, different opponents are detected independently, and every opponent detected is instantaneously attacked, killed or not, and lost from contact;
- (2) Specification of necessary conditional probabilities of kill given detection and attack;
- (3) An assumption that firing processes of all combatants are mutually independent. Thus each weapon operates independently of all weapons on the other side and all other weapons on its own side.

In heterogeneous, mixed, area fire (A1) and time-to-kill (L2) models, these assumptions are weakened or modified. The heterogeneous Process S3 requires additional assumptions concerning allocation of fire. In heterogeneous processes, mean rates of fire or detection and kill probabilities depend in general on both target and shooting weapon types.

The universal independence assumption (3), even though it is omitted from Table 2, should not be overlooked. It states that in a probabilistic sense there is no interaction among weapons on a given side and interaction among weapons on opposing sides only when a kill occurs. In particular, none of these models is capable of handling synergistic effects sometimes thought to be important, except perhaps by artificial (and possibly unjustifiable) devices such as modifying the initial numbers of weapons of some types, based on the absence or presence of some other weapon type before applying one of the attrition models presented here.

We have attempted to keep our assumptions as free from restrictive physical interpretation as possible, in order to demonstrate the full range of applicability of each model. For

example consider the assumption (1) of Process S1 (see Section V) which states that times between shots fired by a surviving weapon are independent and identically exponentially distributed. This assumption is compatible with a number of different physical realizations of combat. One can envision combatants as stationary and firing at rates dependent only on their own nature (this seems to be an "area fire" kind of assumption) or as pressing forward in such a way as to maintain a constant mean rate of engagements with the opposition. The point is that our assumptions are not unique in terms of physical situations in which they might be felt to be satisfied and we have endeavored to state them in terms which make it easy to verify their plausibility.

The widely held and already mentioned belief in the correspondence

Square Law  $\longleftrightarrow$  Point Fire

Linear Law  $\longleftrightarrow$  Area Fire

is misconceived. Rather, as we have mentioned earlier, there exists the following two way classification of processes

		Engagement Initiation	
		Square Law	Linear Law
Kill Structure	Multiple		
	Single		

which appears to us to make good sense.

We may look also at the common properties of the family of stochastic attrition processes we have derived. Among these properties are the following:

- 1) All are regular step processes (in particular, all are Markovian);
- 2) Infinitesimal generators resemble the Lanchester differential models of combat;
- 3) All components of sample paths are nonincreasing (we have not included provision for reinforcements);
- 4) All states are either transient or absorbing (the latter represent extermination of one side or the other).

Quantities of interest one seeks to compute for use in modeling attrition would include

- 1) Expected numbers of survivors (and thus expected attrition) at various fixed times after the combat begins;
- 2) The distribution and expectation of first entry times of various subsets of the state space (which might represent, for example, breakoff points or extermination of one side);
- 3) The expected numbers of survivors at random times such as those in 2) above;
- 4) Variances of certain quantities, for use in estimating errors made in computational implementations;
- 5) In heterogeneous models, expected attrition caused by each type of opposition weapon.

An advantage of having a family of attrition processes with some common characteristics is the possibility of developing general computational methods. We will now briefly discuss one which might be used to compute expected attrition. For concreteness, consider a

homogeneous process  $((B_t, R_t))_{t \geq 0}$  where  $B_t$  and  $R_t$  are the numbers of Blue and Red survivors at time  $t$ , respectively, With initial conditions of  $i$  Blues and  $j$  Reds the expected number of Blue survivors at time  $t$  is given by

$$(10) \quad E[B_t | (B_0, R_0) = (i, j)] = \sum_{k=0}^i \sum_{\ell=0}^j k P_t((i, j); (k, \ell))$$

where, as will be recalled,

$$P_t((i, j); (k, \ell)) = P\{(B_t, R_t) = (k, \ell) | (B_0, R_0) = (i, j)\} .$$

But by known properties of infinitesimal generators we have

$$P_t((i, j); (k, \ell)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n((i, j); (k, \ell))$$

where  $Q^n$  is the  $n^{\text{th}}$  power of the infinitesimal generator matrix  $Q$  of the process. Therefore (interchange of the order of summation is justified by boundedness of all quantities) we can write

$$E[B_t | (B_0, R_0) = (i, j)] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^i \sum_{\ell=0}^j k Q^n((i, j); (k, \ell)) .$$

In particular, the infinite sum  $\sum_{n=0}^{\infty}$  is the limit as  $M \rightarrow \infty$  of the

finite sums  $\sum_{n=0}^M$  .

Thus for each  $M$  we have the approximation

$$(11) \quad E[B_t | (B_0, R_0) = (i, j)] \sim \sum_{n=0}^M \frac{t^n}{n!} \sum_{k=0}^i \sum_{\ell=0}^j k Q^n((i, j); (k, \ell)) .$$

When  $M = 1$  this becomes, provided instantaneous multiple kills are precluded,

$$(12) \quad E[B_t | (B_0, R_0) = (i, j)] \sim i - tQ((i, j); (i - 1, j))$$

which is essentially the analogous (deterministic) differential equation. This, incidentally, provides a justification (albeit tenuous) of using the deterministic differential models as approximations to the true expectations obtained from stochastic models.

But why choose  $M = 1$  in (11), once the whole family of approximations (11) is available and justified and when it can even be shown that each is a better approximation than the preceding one? The matrix  $Q$  consists mostly of zeroes [indeed,  $Q((i, j); (k, \ell))$  is, in general, zero unless  $(k, \ell)$  is  $(i - 1, j)$ ,  $(i, j - 1)$ , or  $(i, j)$ ] so that it would be feasible to take  $M = 3$  or  $4$  or  $5$  in (11) to produce more accurate approximations to the true expectations. Moreover, this scheme of approximation is valid for all the processes we have derived. The principal difficulty would be computation of powers of  $Q$  but since  $Q$  consists mostly of zeroes it might well be feasible to "diagonalize"  $Q$ ; that is, to find a representation of the form

$$Q = A^{-1}DA$$

where  $D$  is a diagonal matrix, in which case

$$Q^n = A^{-1}D^nA$$

becomes trivial to compute and approximations in (11) with  $M$  essentially arbitrary would become feasible. Indeed in this case, we can obtain  $P_t$  explicitly as

$$P_t = A^{-1} e^{tD} A.$$

Further research is desirable on such matters.

## V. A COMPENDIUM OF STOCHASTIC LANCHESTER ATTRITION PROCESSES

We present here the family of attrition process as derived in this research effort. The format is to give for each process the set of assumptions from which we have derived it, and a description of its properties as a regular step process, namely, its jump function, transition kernel, and infinitesimal generator, followed by interpretive remarks. We also give specific versions of expression (12) for each process.

We first present some remarks on the general structure of the families of assumptions leading to the processes described in this section. Each family includes assumptions concerning the following:

- 1) Structure of the two opposing forces;
- 2) The rate at which engagements are initiated;
- 3) The evolution and possible outcomes of an engagement (in all processes but L2 an engagement occurs instantaneously and can end only in death of some weapons on the engaged side and loss of contact);
- 4) If necessary, allocation of fire;
- 5) Interaction among weapons on a given side;
- 6) Interaction of a weapon on one side with weapons on the opposing side, other than engagement.

In all models, 5) and 6) are independence assumptions stipulating no interaction.

As general notations we establish  $B_t$  for Blue survivors at time  $t$  and  $R_t$  for Red survivors at time  $t$ . These will sometimes be vectors.

For each process we give the values of jump function, transition kernel, and infinitesimal generator only for nonabsorbing states.

The set of absorbing states is left to the reader to determine (it is the family of states in which one side or the other is exterminated). If  $x$  is absorbing  $\lambda(x) = 0$ ,  $Q(x, y) = 0$  for all  $y$  and  $P(x, \cdot)$  is not defined.

## A1 Homogeneous Square Law Area Fire Process

### Assumptions

1. All weapons on each side are identical.
2. Times between shots fired by a surviving Blue weapon are independent and identically exponentially distributed with mean  $1/r_1$ .
3. If a shot is fired by a Blue weapon when there are  $k$  Reds surviving the shot has the instantaneous effect of killing a number of Red weapons which is binomially distributed with parameters  $(k, p_1)$ .  
This means that the shot has probability  $p_1$  of killing each particular Red weapon and that different Red weapons are killed independently of one another.
4. Red weapons satisfy assumptions 2 and 3 with mean time between shots  $1/r_2$  and individual target kill probability  $p_2$ .
5. Firing processes of all weapons are mutually independent.

### Process Characterization

Under the preceding set of assumptions the stochastic survivor process  $((B_t, R_t))_{t \geq 0}$  is a regular step process with jump function  $\lambda$  given by

$$\lambda(i, j) = ir_1(1 - (1 - p_1)^j) + jr_2(1 - (1 - p_2)^i) ,$$

transition kernel  $P$  given by

$$P((i, j); (i, \ell)) = \frac{ir_1 \binom{j}{\ell} (1 - p_1)^\ell p_1^{j-\ell}}{\lambda(i, j)} , \quad \ell = 0, \dots, j - 1$$

$$P((i, j); (k, j)) = \frac{jr_2 \binom{i}{k} (1 - p_2)^k p_2^{i-k}}{\lambda(i, j)} , \quad k = 0, \dots, i - 1$$

and infinitesimal generator  $Q$  given by

$$Q((i, j); (i, l)) = ir_1 \binom{j}{l} (1 - p_1)^l p_1^{j-l}, \quad l=0, \dots, j-1$$

$$Q((i, j); (i, j)) = -[ir_1(1 - (1 - p_1)^j) + jr_2(1 - (1 - p_2)^i)]$$

$$Q((i, j); (k, j)) = jr_2 \binom{i}{k} (1 - p_2)^k p_2^{i-k}, \quad k=0, \dots, i-1.$$

The binomial distribution of assumption 2 can be replaced by any probability distribution on  $0, \dots, k$ , so this process is in fact a whole class of similar processes. In particular, the distribution which places mass  $p_1$  on 1 and mass  $1 - p_1$  on 0, yields the Process S1 described below.

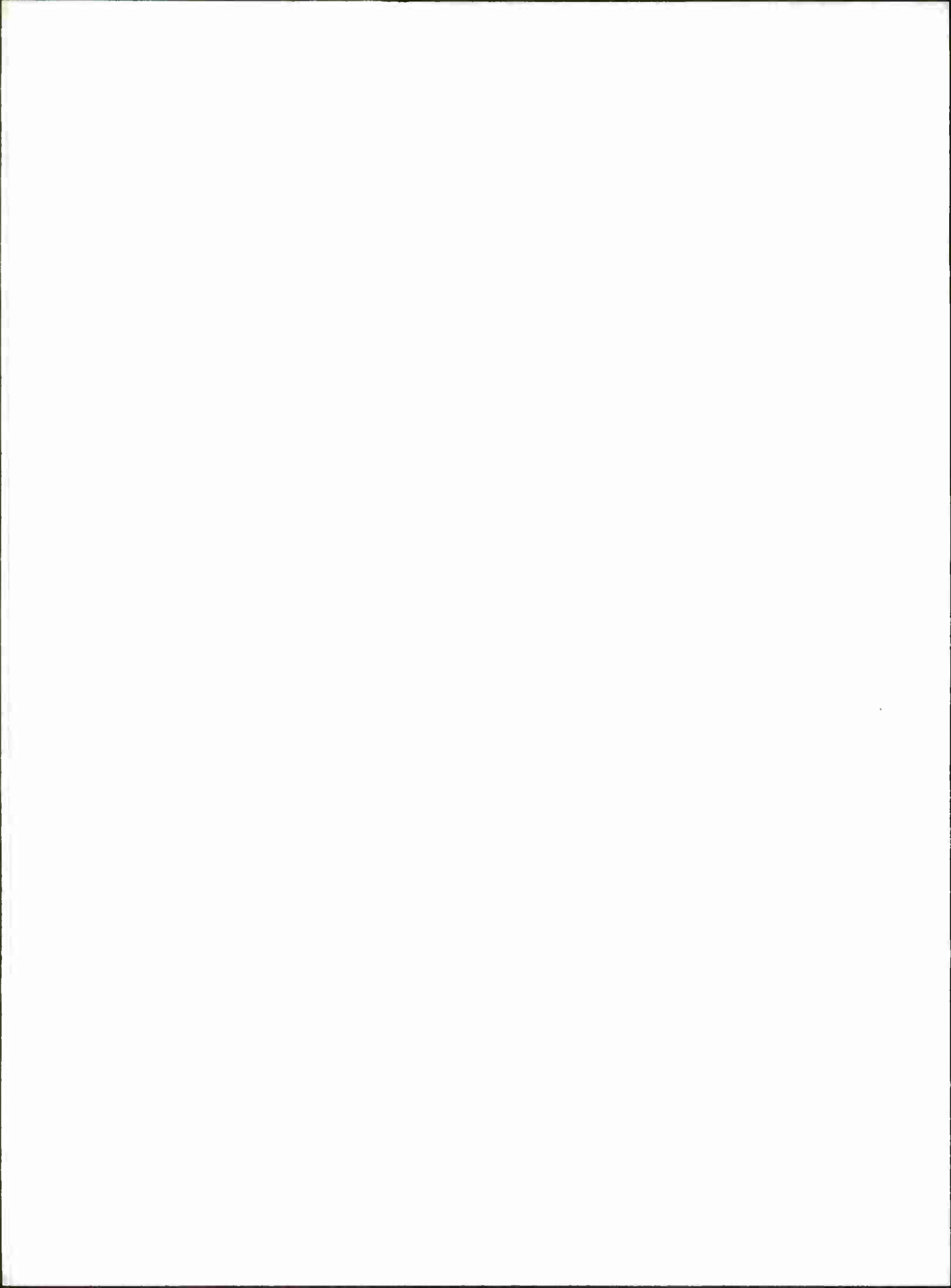
This process is in many ways the most interesting of those we have derived, because of the number of intriguing questions it raises. The interpretation of its assumptions, especially that concern binomially distributed multiple kills, seems to be unequivocally that of area fire. (Indeed, the possibility of multiple kills may well be the distinguishing feature of area fire combat processes). What is at issue is whether this is a square law process or a linear law process. Consider its infinitesimal generator  $Q$ . Examining the term  $-Q((i,j),(i,j))$ , which is the rate at which the process leaves state  $(i,j)$ , we have what appears to be of square law form (cf. the corresponding term for Process S1). But on the other hand

$$Q((i,j), (i,j - 1)) = ir_1 j (1 - p_1)^{j-1} p_1,$$

which can be thought of as a representation of the instantaneous rate at which Red combatants are being destroyed when the current state of the process is  $(i,j)$ , can be argued to be of linear law form (but not entirely, because of the presence of the factor  $(1 - p_1)^{j-1}$ ).

One way to clarify the situation is to abandon the square law-linear law dichotomy for attrition entirely and to simply regard the processes we have derived, in terms of their underlying

assumptions. What distinguishes Process A1 from all the other processes is the provision for multiple kills, so it might even be argued that this is the only true "area fire" process presented here. The whole situation is complicated by the fact that if the binomial distribution of assumption (2) is replaced by the distribution with mass  $p$  on 1 and mass  $1 - p$  on 0 the Process S1, which seems clearly to be a square law process, is obtained.



Assumptions

1. All weapons on each side are identical.
2. Times between shots fired by a surviving Blue weapon are independent and identically exponentially distributed with mean  $1/r_1$ .
3. When a shot occurs it kills exactly one Red weapon with probability  $p_1$  and no Red weapons with probability  $1 - p_1$ , instantaneously and independent of past history of the process.
4. Red weapons satisfy assumptions 2 and 3 with mean time between shots of  $1/r_2$  and kill probability  $p_2$ .
5. The firing processes of all weapons are mutually independent.

Process Characterization

For  $i = 1, 2$ , let  $c_i = r_i p_i$ . Then subject to Assumptions 1 to 5 above,  $((B_t, R_t))_{t \geq 0}$  is a regular step process with state space  $\mathbb{N} \times \mathbb{N}$ , jump function  $\lambda$  given by

$$\lambda(i, j) = ic_1 + jc_2 ,$$

transition kernel P given by

$$P((i, j); (i, j - 1)) = \frac{ic_1}{ic_1 + jc_2}$$

$$P((i, j); (i - 1, j)) = \frac{jc_2}{ic_1 + jc_2} ,$$

and infinitesimal generator Q given by

$$Q((i, j), (i, j - 1)) = ic_1$$

$$Q((i, j), (i, j)) = - (ic_1 + jc_2)$$

$$Q((i, j), (i - 1, j)) = jc_2 .$$

For this process the approximation (12) is given by

$$E[B_t | (B_0, R_0) = (i, j)] \sim i - jc_2t ;$$

similarly

$$E[R_t | (B_0, R_0) = (i, j)] \sim j - ic_1t .$$

Clearly the larger  $t$  is, the worse the approximation.

By the form of  $Q$  this process is the stochastic version of the deterministic square law (1).  $1/c_1$  is the mean time between fatal shots fired by a Blue weapon, so  $c_1 = r_1p_1$  can be interpreted as the mean rate at which a Blue weapon kills Red weapons (which is independent of the strength of the Red force so long as the latter remains nonzero), giving in physical terms what the coefficients  $c_1$  and  $c_2$  in (1) mean.

## S2 Heterogeneous Square Law Process

### Assumptions

1. There are  $M$  Blue weapon types and  $N$  Red weapon types.
2. Consider a Blue type  $i$  weapon. The times between shots it fires at Red type  $j$  weapons are independent and identically exponentially distributed with mean  $1/r_1(j, i)$ .
3. These  $N$  firing processes occur simultaneously and independently.
4. A shot fired by a Blue type  $i$  weapon at Red type  $j$  weapons is directed at one Red type  $j$  weapon which is chosen according to a uniform distribution and is fatal with probability  $p_1(j, i)$ .
5. Red weapons also satisfy 2 - 4, with parameters  $r_2(i, j)$ ,  $p_2(i, j)$  describing the fire of a Red type  $j$  weapon against Blue type  $i$  weapons.
6. Firing processes of all weapons are mutually independent.

### Process Characterization

For  $q = 1, 2$  and appropriate  $k$  and  $l$ , let  $c_q(k, l) = r_q(k, l)p_q(k, l)$ . Note that  $\underline{B}_t$  and  $\underline{R}_t$  are now vectors of dimension  $M$  and  $N$ , respectively. Then subject to assumptions 1 through 6 above, the vector-valued process  $((\underline{B}_t, \underline{R}_t))_{t \geq 0}$  is a regular step process with state space  $\underline{N} \times \dots \times \underline{N}$  ( $M + N$  times) and, for states  $(x; y) = (x_1, \dots, x_M; y_1, \dots, y_N)$ , jump function  $\lambda$  given by

$$\lambda(x; y) = \sum_{i: x_i > 0} \sum_{j: y_j > 0} (c_2(i, j)y_j + c_1(j, i)x_i) ,$$

transition kernel  $P$  given for  $j$  with  $y_j > 0$  by

$$P((x, y); (x; y_1, \dots, y_j^{-1}, \dots, y_N)) = \frac{\sum_{i=1}^M c_1(j, i)x_i}{\lambda(x, y)} ,$$

and for  $i$  with  $x_i > 0$  by

$$P((x,y); (x_1, \dots, x_{i-1}, \dots, x_M; y)) = \frac{\sum_{j=1}^N c_2(i,j)y_j}{\lambda(x,y)},$$

and infinitesimal generator  $Q$  given by

$$Q((x,y); (x; y_1, \dots, y_{j-1}, \dots, y_N)) = \sum_{i=1}^M c_1(j,i)x_i \quad \text{if } y_j > 0$$

$$Q((x,y); (x,y)) = - \sum_{i:x_i > 0} \sum_{j:y_j > 0} (c_2(i,j)y_j + c_1(j,i)x_i)$$

$$Q((x,y); (x_1, \dots, x_{i-1}, \dots, x_M; y)) = \sum_{j=1}^N c_2(i,j)y_j \quad \text{if } x_i > 0.$$

State  $(x,y)$  is absorbing if and only if  $x \equiv 0$  or  $y \equiv 0$ .

Corresponding to the approximation (12) we have for this process

$$E[B_t^i | (B_0, R_0) = (x,y)] \sim x_i - \left( \sum_{j=1}^N c_2(i,j)y_j \right) t$$

for  $i = 1, \dots, M$  while for  $j = 1, \dots, N$

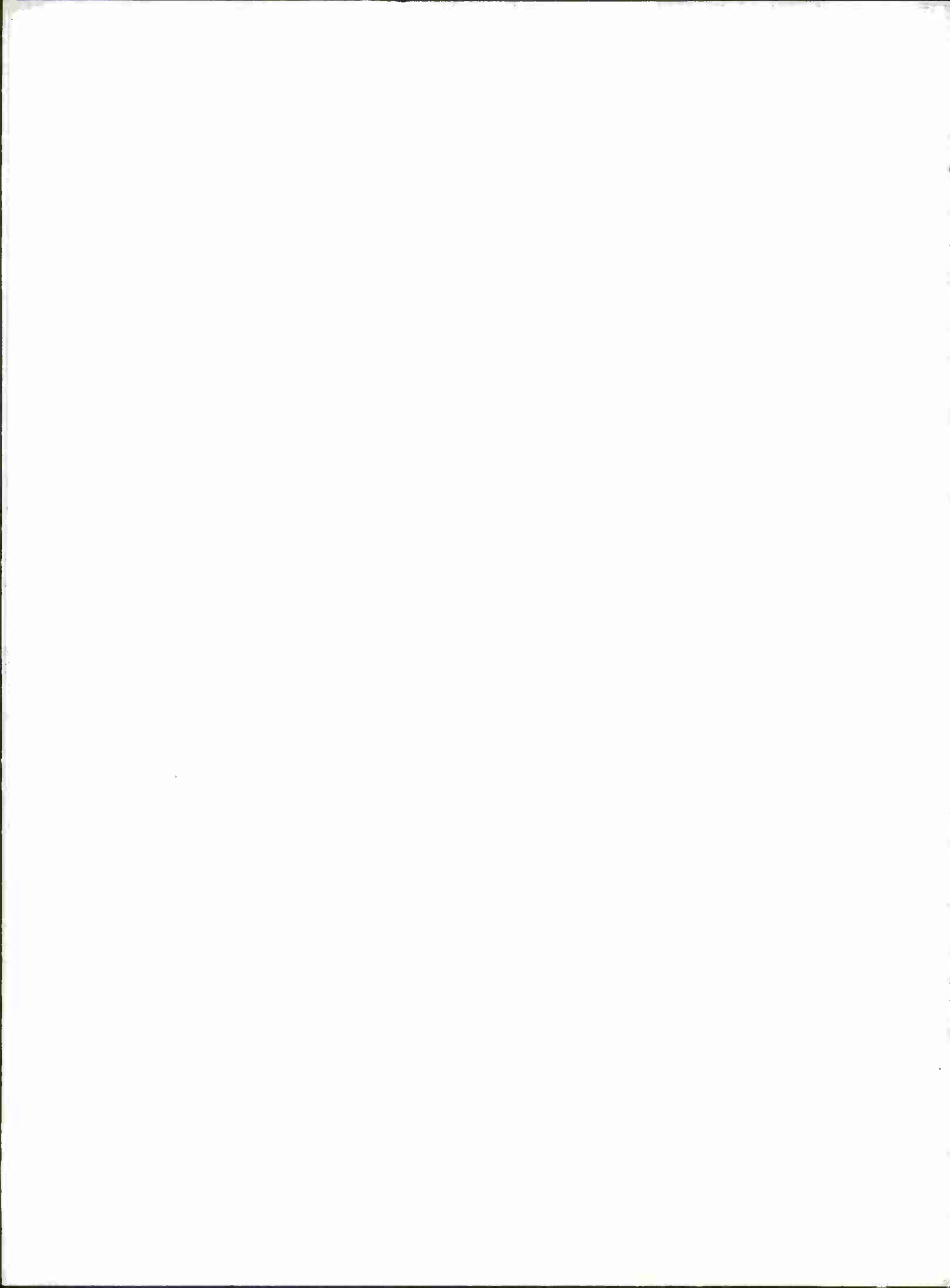
$$E[R_t^j | (B_0, R_0) = (x,y)] \sim y_j - \left( \sum_{i=1}^M c_1(j,i)x_i \right) t.$$

It seems rather natural to call this process "heterogeneous Lanchester square" because of resemblance of  $Q$  to the system (5) of differential equations, and this process provides a physical interpretation of the matrices in (5). But the physical interpretation of the assumptions is quite unappealing. In particular, assumptions 2 and 3 are unpalatable. They state that each weapon carries out one firing process for each type of opposition weapon and that

all these processes are evolving concurrently and independently.

On the contrary, the spirit of assumption 2 of processes A1 and S1 is that a weapon's firing behavior is a function only of its type and of whether there are a positive number of targets present. The next process we present is in that spirit.

It is, in any case, the present process S2 which is commonly referred to as "stochastic heterogeneous Lanchester square". Based on the structure of the underlying families of assumptions, this designation seems to be inappropriate.



### S3a Heterogeneous Square Law Process

#### Assumptions

1. There are  $M$  Blue weapon types and  $N$  Red weapon types.
2. Times between shots fired by a surviving Blue type  $i$  weapon are independent and identically exponentially distributed with mean  $1/r_1(i)$ .
3. When a Blue type  $i$  weapon fires a shot it is directed at a Red weapon chosen from all Red weapons currently surviving according to a uniform distribution.
4. A shot fired by a Blue type  $i$  weapon at a Red type  $j$  weapon is fatal with probability  $p_1(j,i)$ .
5. Assumptions 2 - 4 hold for Red with mean firing rates  $r_2(j)$  and kill probabilities  $p_2(i,j)$ .
6. The firing processes of all weapons initially present are mutually independent.

#### Process Characterization

Under the preceding family of assumptions, the survivor process  $((\underline{B}_t, \underline{R}_t))_{t \geq 0}$  is a regular step process with state space  $\mathbb{N} \times \dots \times \mathbb{N}$  ( $M + N$  times), jump function  $\lambda$  given by

$$\lambda(x,y) = \sum_{i=1}^M \sum_{j=1}^N \left[ \frac{x_i}{x \cdot 1} p_2(i,j)r_2(j)y_j + \frac{y_j}{y \cdot 1} p_1(j,i)r_1(i)x_i \right]$$

where  $x \cdot 1 = \sum_{k=1}^M x_k$ ,  $y \cdot 1 = \sum_{\ell=1}^N y_\ell$ , transition kernel  $P$  given by

$$P((x,y); (x; y_1, \dots, y_j - 1, \dots, y_M)) = \frac{y_j}{y \cdot 1} \frac{\sum_{i=1}^M p_1(j,i)r_1(i)x_i}{\lambda(x,y)}$$

$$P((x,y); (x_1, \dots, x_i - 1, \dots, x_M; y)) = \frac{x_i}{x \cdot 1} \frac{\sum_{j=1}^N p_2(i,j)r_2(j)y_j}{\lambda(x,y)}$$

and infinitesimal generator  $Q$  given by

$$Q((x,y); (x; y_1, \dots, y_j^{-1}, \dots, y_M)) = \frac{y_j}{y \cdot 1} \sum_{i=1}^M p_1(j,i)r_1(i)x_i$$

$$Q((x,y); (x,y)) = - \sum_{i=1}^M \sum_{j=1}^N \left[ \frac{x_i}{x \cdot 1} p_2(i,j)r_2(j)y_j + \frac{y_j}{y \cdot 1} p_1(j,i)r_1(i)x_i \right]$$

$$Q((x,y); (x_1, \dots, x_i^{-1}, \dots, x_M; y)) = \frac{x_i}{x \cdot 1} \sum_{j=1}^N p_2(i,j)r_2(j)y_j .$$

For this process the first-order approximation (12) is given by

$$E[\underline{B}_t^i | (\underline{B}_0, \underline{R}_0) = (x, y)] \sim x_i - \left( \frac{x_i}{x \cdot 1} \sum_{j=1}^N p_2(i,j)r_2(j)y_j \right) t$$

provided  $x_i > 0$ , while if  $y_j > 0$

$$E[\underline{R}_t^j | (\underline{B}_0, \underline{R}_0) = (x, y)] \sim y_j - \left( \frac{y_j}{y \cdot 1} \sum_{i=1}^M p_1(j,i)r_1(i)x_i \right) t;$$

here  $i = 1, \dots, M$  and  $j = 1, \dots, N$ .

The reader may well protest that  $Q$  is of linear-law rather than square-law form, but this really isn't so; the factors  $(x_i)/(x \cdot 1)$  and  $(y_j)/(y \cdot 1)$  are simply normalizing constants. In view of assumption 3,  $(y_j)/(y \cdot 1)$  is the probability that a Blue weapon, if it fires at a Red target force  $y$ , will direct the shot at a Red type  $j$  weapon.

Implicit (but not really necessary) in the statement of assumption 3 of the Homogeneous Square Law Process S1 is the notion that the one Red weapon attacked each time a shot is fired is uniformly chosen from among all Red weapons then present. Therefore the assumptions of Process S3a, and not those of the commonly studied and applied Process S2, are the

proper generalization of the assumptions of Process S1 to the heterogeneous case. This generalization preserves dependence of mean firing rate on only the type of the shooting weapon and the uniform fire allocation. Hence, we believe, it is more appropriate to call process S3a a stochastic model of heterogeneous Lanchester-square attrition that it is to so call Process S2.

The system of differential equations analogous to this process is given by

$$r'_j = \frac{-r_j}{\sum_{\ell=1}^N r_\ell} \sum_{i=1}^M k_1(j, i) b_i$$

$$b'_i = \frac{-b_i}{\sum_{k=1}^M b_k} \sum_{j=1}^N k_2(i, j) r_j$$

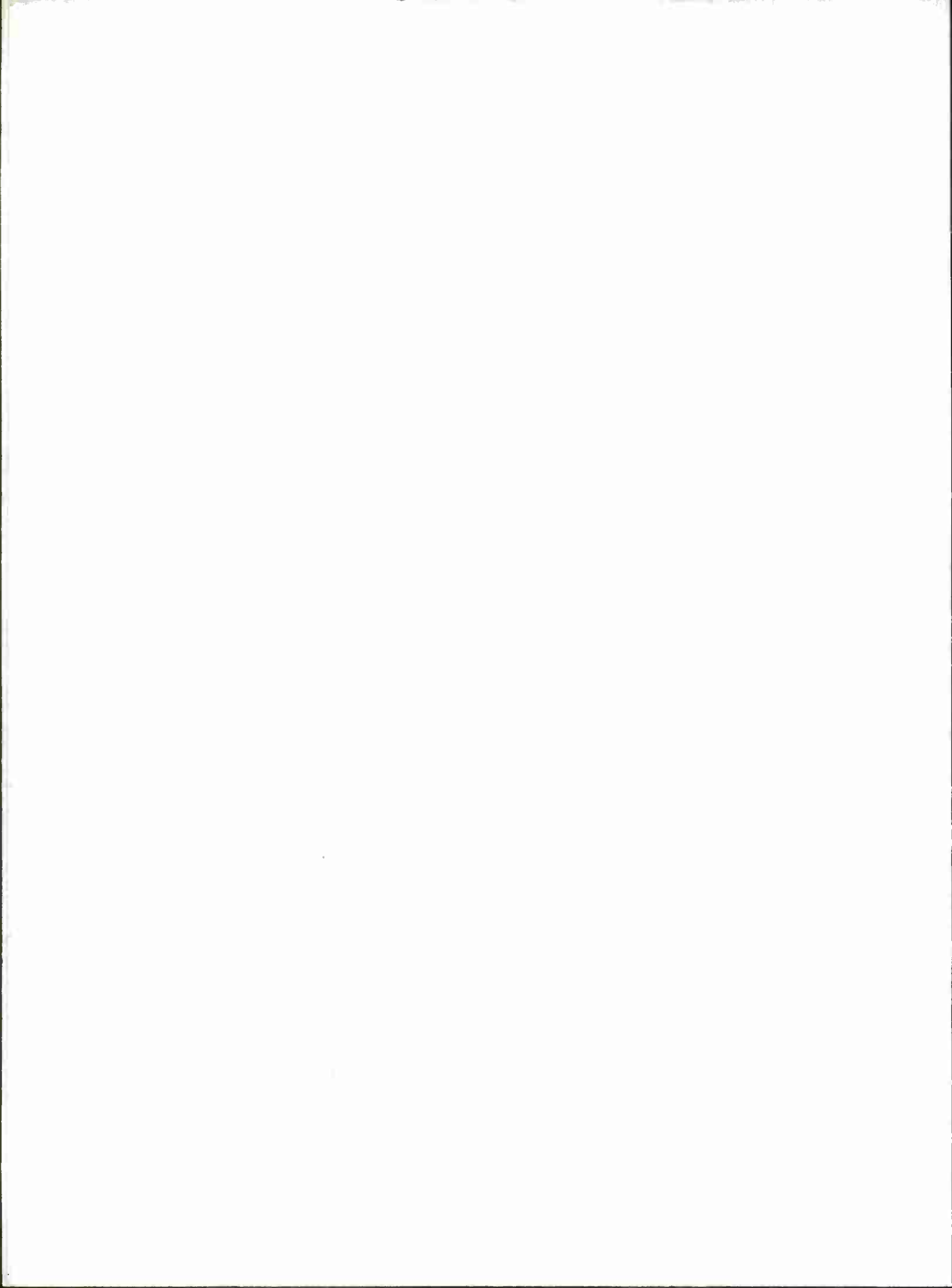
where

$$k_1(j, i) = p_1(j, i) r_1(i)$$

and

$$k_2(i, j) = p_2(i, j) r_2(j) .$$

In particular this provides an interpretation of the matrices  $k_1$ ,  $k_2$ .



### S3 Heterogeneous Square Law Process

#### Assumptions

The assumptions here are those of the Process S3a but with assumption 3 there modified as follows.

3. If a Blue type  $i$  weapon fires a shot at a time when the Red force is (the  $N$ -vector)  $y$  then the probability that the one target attacked is of type  $j$  is  $\psi_i(y, j)$ ,  $j = 1, \dots, N$ ; within the class of type  $j$  targets the specific target attacked is chosen uniformly. For each  $i$  and  $y$ ,  $\psi_i(y, \cdot)$  is a probability distribution over the set  $\{1, \dots, N\}$  of Red weapon type indices.

Similarly, assumption 5 is changed to include fire allocation distributions  $\eta_j(x, \cdot)$  for Red type  $j$  weapons firing at a Blue force of composition  $x$ .

#### Process Characterization

Under the family of assumptions given above,  $((\tilde{B}_t, \tilde{R}_t))_{t \geq 0}$  is a regular step process with jump function  $\lambda$  given by

$$\lambda(x, y) = \sum_{i=1}^M \sum_{j=1}^N \{ \psi_i(y, j) p_1(j, i) r_1(i) x_i + \eta_j(x, i) p_2(i, j) r_2(j) y_j \},$$

transition kernel  $P$  given by

$$P((x, y); (x; y_1, \dots, y_j - 1, \dots, y_N)) = \frac{\sum_{i=1}^M \psi_i(y, j) p_1(j, i) r_1(i) x_i}{\lambda(x, y)}$$

$$P((x, y); x_1, \dots, x_i - 1, \dots, x_M; y) = \frac{\sum_{j=1}^N \eta_j(x, i) p_2(i, j) r_2(j) y_j}{\lambda(x, y)},$$

and infinitesimal generator  $Q$  given by

$$Q((x,y); (x; y_1, \dots, y_j - 1, \dots, y_N)) = \sum_{i=1}^M \psi_i(y, j) p_1(j, i) r_1(i) x_i$$

$$Q((x,y); (x,y)) = - \sum_{i=1}^M \sum_{j=1}^N \{ \psi_i(y, j) p_1(j, i) r_1(i) x_i + \eta_j(x, i) p_2(i, j) r_2(j) y_j \}$$

$$Q((x,y); (x_1, \dots, x_i - 1, \dots, x_M; y)) = \sum_{j=1}^N \eta_j(x, i) p_2(i, j) r_2(j) y_j .$$

For this process the approximation (12) has the form

$$E[\underline{B}_t^i | (\underline{B}_0, \underline{R}_0) = (x, y)] \sim x_i - \left[ \sum_{j=1}^N \eta_j(x, i) p_2(i, j) r_2(j) y_j \right] t$$

for  $i$  such that  $x_i > 0$ . Similarly, for  $j$  such that  $y_j > 0$  we have

$$E[\underline{R}_t^j | (\underline{B}_0, \underline{R}_0) = (x, y)] \sim y_j - \left[ \sum_{i=1}^M \psi_i(y, j) p_1(j, i) r_1(i) x_i \right] t .$$

Process S3a is obtained as a special case of Process S3 by taking

$$\psi_i(y, j) = \frac{y_j}{y \cdot 1}$$

and

$$\eta_j(x, i) = \frac{x_i}{x \cdot 1}$$

for all  $i$  and  $j$ . The generator  $Q$  of Process S3 is clearly of Lanchester square form, in the sense of resembling (5). Thus, Process S3a is a square law process. This similarity, incidentally, conveys additional information about what the coefficients in (5) should mean.

Effects such as target priority and axiomatically derived allocations are included within the large class of S3 processes. See the Appendix for details.

We note that the system of differential equations analogous to this stochastic process is given by

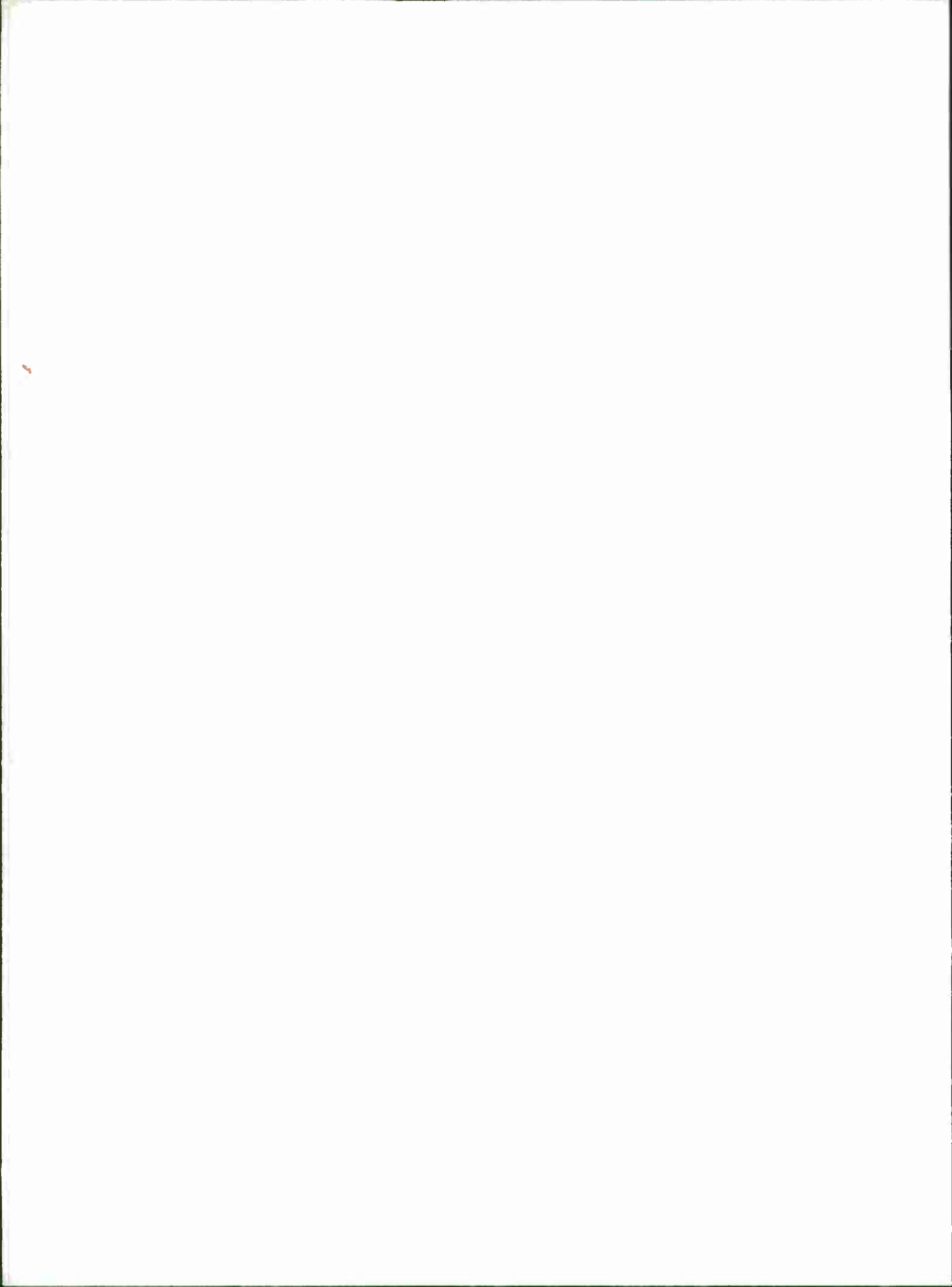
$$r'_j(t) = - \sum_{i=1}^M \psi_i(r(t), j) p_1(j, i) r_1(i) b_i(t)$$

for  $j = 1, \dots, N$  and by

$$b'_i(t) = - \sum_{j=1}^N \psi_j(b(t), i) p_1(i, j) r_2(j) r_j(t)$$

for  $i = 1, \dots, M$ . The reader should be aware of the potentially confusing notation: the  $r_1(i)$  and  $r_2(j)$  are mean firing rates while the  $r_j(t)$  are surviving Red weapons.

The assumptions for Processes S1, S2, S3 are all clearly similar. To interpret any of these as an area fire process requires the following reasoning: since a combatant's mean rate of fire is independent of the size of the opposing force, he can be envisioned as simply firing into an enemy-containing region according to a Poisson process determined by his own attributes (e.g., weapon, location, ...), which seems plausible. But it must then be assumed that a shot kills either exactly one enemy combatant or none, which doesn't seem entirely reasonable. Indeed, we have previously suggested that area fire attrition processes are distinguished by the possibility of multiple kills arising from one shot. Hence it can be argued that of all the processes presented here only the Process A1 is an area fire process, and that all the others are point fire processes which are grouped into two classes (square-law and linear-law) according to the form of mean engagement rates. For square-law processes the engagement rate depends only on the strength of the side initiating engagements, while for linear-law processes each engagement rate is proportional to the product of the strengths of the two sides. These, of course, correspond to rather different physical situations.



## L1 Homogeneous Linear Law Process

### Assumptions

1. All weapons on each side are identical
2. The time required for a Blue weapon to detect a particular Red weapon is exponentially distributed with mean  $1/s_1$ . Each Blue weapon detects different Red weapons independently.
3. A Blue weapon attacks every Red weapon it detects; the conditional probability of kill given attack is  $q_1$ . The attack occurs instantaneously and contact is immediately lost. An attack cannot occur without a detection.
4. Red weapons satisfy the same assumptions with parameters  $s_2, q_2$ .
5. The detection and attack processes of all weapons initially present are mutually independent.

### Process Characterization

Let  $k_l = s_l q_l$  for  $l = 1, 2$ .

Under assumptions 1 - 5,  $((B_t, R_t))$  is a regular step process with state space  $\underline{N} \times \underline{N}$ , jump function  $\lambda$  given by

$$\lambda(i,j) = ij(k_1 + k_2)$$

transition kernel  $P$  given by

$$P((i,j), (i,j - 1)) = \frac{k_1}{k_1 + k_2}$$
$$P((i,j), (i - 1,j)) = \frac{k_2}{k_1 + k_2},$$

and infinitesimal generator  $Q$  given by

$$Q((i,j), (i,j - 1)) = k_1 ij$$

$$Q((i,j), (i,j)) = - ij(k_1 + k_2)$$

$$Q((i,j), (i - 1, j)) = k_2 ij \quad .$$

As first order approximations for this process we have

$$\begin{aligned} E[B_t | (B_0, R_0) = (i, j)] &\sim i - (k_1 ij)t \\ &= i(1 - k_1 jt) \end{aligned}$$

and

$$\begin{aligned} E[R_t | (B_0, R_0) = (i, j)] &\sim j - (k_2 ij)t \\ &= j(1 - k_2 it) \quad . \end{aligned}$$

The similarity of  $Q$  and the deterministic homogeneous Lanchester linear equations (4) is evident, and our process provides a physical interpretation (namely, inverse of mean time required to kill a particular opponent) for the coefficients in (4).

The reader should note the presence of an assumption concerning detection, rather than firing, times in this model. Indeed, square- and linear-law models are distinguished, in our structure, in that models of the former type include assumptions on firing times independent of the strength of the opposing force while the latter include assumptions on one-on-one detection times.

## L2 Homogeneous Linear Law Process With Engagements

### Assumptions

1. All weapons on each side are identical.
2. The time required for a Blue weapon to detect a prescribed Red weapon is exponentially distributed with mean  $1/s_1$ . Each Blue weapon detects different Red weapons independently.
3. A Blue weapon engages every Red weapon it attacks.
4. All engagements are binary (i.e., one-on-one). During an engagement both combatants are invulnerable to other weapons and unable to make further detections.
5. The length of an engagement is exponentially distributed with mean  $1/u$ . An engagement terminates in exactly one of the following outcomes with the probabilities indicated, independent of past history.

<u>Outcome</u>	<u>Probability</u>
Blue killed, but not Red	$P_1$
Red killed, but not Blue	$P_2$
Mutual survival	$P_3 = 1 - P_1 - P_2$

These parameters are independent of the initiator of the engagement.

6. Red weapons satisfy assumptions 2 and 3, with mean detection time  $1/s_2$ .

7. All detection processes, engagement lengths, and engagement outcomes are mutually independent.

### Process Characterization

To study this model we define some further notation. Let  $B_t^*$  denote the number of unengaged Blue weapons at time  $t$ ,  $R_t^*$  the number of unengaged Red weapons at time  $t$  and  $D_t$  the number of binary engagements in progress at time  $t$ , which is also the number of Blue weapons and Red weapons engaged at time  $t$ . We first consider the three component process  $((B_t^*, R_t^*, D_t))_{t \geq 0}$ .

Under assumptions 1 through 7 above, the process  $((B_t^*, R_t^*, D_t))_{t \geq 0}$  is a regular step process with state space  $\underline{N} \times \underline{N} \times \underline{N}$ , jump function  $\lambda$  given by

$$\lambda(i, j, k) = ku + ij(s_1 + s_2) ,$$

transition kernel  $P$  given by

$$P((i, j, k), (i + 1, j, k - 1)) = \frac{kp_2 u}{ku + ij(s_1 + s_2)}$$

$$P((i, j, k), (i, j + 1, k - 1)) = \frac{kp_1 u}{ku + ij(s_1 + s_2)}$$

$$P((i, j, k), (i + 1, j + 1, k - 1)) = \frac{kp_3 u}{ku + ij(s_1 + s_2)}$$

$$P((i, j, k), (i - 1, j - 1, k + 1)) = \frac{ij(s_1 + s_2)}{ku + ij(s_1 + s_2)} ,$$

and infinitesimal generator  $Q$  given by

$$Q((i, j, k), (i + 1, j, k - 1)) = kp_2 u$$

$$Q((i, j, k), (i, j + 1, k - 1)) = kp_1 u$$

$$Q((i, j, k), (i + 1, j + 1, k - 1)) = kp_3 u$$

$$Q((i, j, k), (i - 1, j - 1, k + 1)) = ij(s_1 + s_2)$$

$$Q((i, j, k), (i, j, k)) = - [ku + ij(s_1 + s_2)] .$$

From the state of  $i$  unengaged Blue weapons,  $j$  unengaged Red weapons, and  $k$  engagements in progress, four states may be reached, corresponding to the events "termination of one engagement with kill of a Red weapon", "termination of one engagement with kill of a Blue weapon", "termination of one engagement with mutual survival", and "initiation of one additional engagement". These occur at the rates indicated by the expressions for  $Q$  (which, as usual, assume that  $i > 0$ ,  $j > 0$ ,  $k > 0$  and must be suitably adjusted for other states).

Based on its assumptions and the term in  $Q$  expressing engagement initiation, this process is of linear law type. But our

principal interest is in the survivor process  $((B_t, R_t))$  defined by

$$B_t = B_t^* + D_t$$

and

$$R_t = R_t^* + D_t ,$$

since the number of Blue weapons surviving at time  $t$  is the sum of the numbers of unengaged and engaged Blue weapons surviving at time  $t$ , and similarly for Red. The process  $((B_t, R_t))$  is not, unfortunately, a regular step process and we have not yet characterized it in a useful manner. However, expectations of  $B_t$  and  $R_t$  are easy to compute from those of  $B_t^*$ ,  $R_t^*$ , and  $D_t$ , because

$$E[B_t] = E[B_t^*] + E[D_t]$$

and

$$E[R_t] = E[R_t^*] + E[D_t] .$$

The latter expectations might be computed, since  $((B_t^*, R_t^*, D_t))_{t \geq 0}$  is a regular step process, using the methods described in Section IV.

To compute first-order approximations to  $E[B_t]$  and  $E[R_t]$  we first note that (12) and the form of the generator  $Q$  of this process imply that

$$E[B_t^* | (B_0^*, R_0^*, D_0) = (i, j, k)] \sim i - [ij(s_1 + s_2) + ku(p_2 + p_3)]t ,$$

that

$$E[R_t^* | (B_0^*, R_0^*, D_0) = (i, j, k)] \sim j - [ij(s_1 + s_2) - ku(p_1 + p_3)]t ,$$

and also that

$$E[D_t | (B_0^*, R_0^*, D_0) = (i, j, k)] \sim k - [ku - ij(s_1 + s_2)]t .$$

Thus

$$\begin{aligned} E[B_t | (B_0^*, R_0^*, D_0) = (i, j, k)] \\ &= E[B_t^* | (B_0^*, R_0^*, D_0) = (i, j, k)] + E[D_t | (B_0^*, R_0^*, D_0) = (i, j, k)] \\ &\sim i - ij(s_1 + s_2)t + ku(p_2 + p_3)t + k - kut + ij(s_1 + s_2)t \\ &= i + k - (p_1 ku)t \quad ; \end{aligned}$$

we similarly obtain

$$E[R_t | (B_0^*, R_0^*, D_0) = (i, j, k)] = j + k - (p_2 ku)t \quad .$$

The assumption that the two weapons in a binary engagement be invulnerable to other opposition weapons is questionable (an engaged weapon would seem to be relatively more vulnerable than an unengaged weapon) and alternatives should be sought.

### L3 Heterogeneous Linear Law Process

#### Assumptions

1. There are  $M$  Blue weapon types and  $N$  Red weapon types.
2. The time required for a Blue type  $i$  weapon to detect a particular Red type  $j$  weapon is exponentially distributed with mean  $1/s_1(j,i)$ . Each Blue weapon detects different Red weapons independently.
3. A Blue weapon attacks every Red weapon it detects. The conditional probability that a Blue type  $i$  weapon kills a Red type  $j$  weapon, given detection and attack, is  $p_1(j,i)$ .
4. Red weapons satisfy assumptions 2 and 3 with mean detection times  $1/s_2(i,j)$  and kill probabilities  $p_2(i,j)$  describing Red type  $j$  weapons opposing Blue type  $i$  weapons.
5. Detection and attack processes of all weapons present are mutually independent.

#### Process Characterization

For  $q = 1, 2$  and appropriate  $k$  and  $\ell$  define  $k_q(k,\ell) = s_q(k,\ell)p_q(k,\ell)$ . Then under assumptions 1 through 5 above, the vector-valued survivor process  $((\tilde{B}_t, \tilde{R}_t))_{t \geq 0}$  is a regular step process with state space  $\mathbb{N} \times \dots \times \mathbb{N}$  ( $M + N$  times), jump function  $\lambda$  given by

$$\lambda(x,y) = \sum_{i=1}^M \sum_{j=1}^N x_i y_j [k_1(j,i) + k_2(i,j)] ,$$

transition kernel  $P$  given by

$$P((x,y); (x; y_1, \dots, y_j - 1, \dots, y_N)) = \frac{y_j \sum_{i=1}^M k_1(j,i) x_i}{\lambda(x,y)}$$

$$P((x,y); (x_1, \dots, x_i - 1, \dots, x_M; y)) = \frac{x_i \sum_{j=1}^N k_2(i,j) y_j}{\lambda(x,y)} ,$$

and infinitesimal generator  $Q$  given by

$$Q((x,y); (x; y_1, \dots, y_j - 1, \dots, y_N)) = y_j \sum_{i=1}^M k_1(j,i)x_i$$

$$Q((x,y); (x,y)) = - \sum_{i=1}^M \sum_{j=1}^N x_i y_j [k_1(j,i) + k_2(i,j)]$$

$$Q((x,y); (x_1, \dots, x_i - 1, \dots, x_M; y)) = x_i \sum_{j=1}^N k_2(i,j)y_j .$$

Expectations of numbers of survivors in this process may be approximated using (12) in the following manner:

$$\begin{aligned} E[B_t^i | (B_0, R_0) = (x,y)] &\sim x_i - [x_i \sum_{j=1}^N k_2(i,j)y_j]t \\ &= x_i [1 - t \sum_{j=1}^N k_2(i,j)y_j] \end{aligned}$$

and

$$\begin{aligned} E[R_t^j | (B_0, R_0) = (x,y)] &\sim y_j - [y_j \sum_{i=1}^M k_1(j,i)x_i]t \\ &= y_j [1 - t \sum_{i=1}^M k_1(j,i)x_i] , \end{aligned}$$

for  $i = 1, \dots, M$  and  $j = 1, \dots, N$ .

Based on its underlying assumptions this process is clearly the heterogeneous analog of the stochastic homogeneous linear law Process L1. Its infinitesimal generator  $Q$  is similar to the system (6), which confirms that (6) is an appropriate deterministic heterogeneous linear model and provides the proper interpretations of the coefficients in (6); namely, the inverse of the mean time required to detect and kill a particular opposition weapon of a given type.

## M1 Heterogeneous Mixed Law Process

### Assumptions

1. Each side possesses exactly two types of weapons.
2. Times between shots fired by a surviving type 1 Blue weapon are independent and identically exponentially distributed with mean  $1/r_1$ .
3. Each such shot is directed at a target chosen according to a uniform distribution on the set of all Red weapons surviving at that time. If the target is a Red type  $j$  weapon it is killed with probability  $p_1(j)$ .
4. Red type 1 weapons satisfy assumptions 2 and 3 with mean firing rate  $r_2$  and kill probability  $p_2(i)$  against a Blue target of type  $i$ .
5. The time required for a Blue type 2 weapon to detect a particular Red weapon of type  $j$  is exponentially distributed with mean  $1/s_1(j)$ . A fixed Blue type 2 weapon detects different Red weapons independently of one another.
6. A Blue type 2 weapon attacks every Red weapon it detects. The conditional probability, given detection and attack, that a Red weapon of type  $j$  is killed, is  $q_1(j)$ . The attack occurs instantaneously and contact is lost.
7. Red type 2 weapons satisfy assumptions 5 and 6 with mean detection times  $1/s_2(i)$  and kill probabilities  $q_2(i)$  against type  $i$  weapons.
8. All weapons detect and attack independently of one another.

### Process Characterization

Under assumptions 1 through 8 above, the survivor process  $((B_t^1, B_t^2, R_t^1, R_t^2))_{t \geq 0}$  is a regular step process with state space

$\tilde{N} \times \tilde{N} \times \tilde{N} \times \tilde{N}$ , jump function  $\lambda$  given by

$$\begin{aligned} \lambda(i, j, k, \ell) = & ir_1 \left( \frac{p_1(1)k + p_1(2)\ell}{k + \ell} \right) \\ & + j [ks_1(1)q_1(1) + \ell s_1(2)q_1(2)] \\ & + kr_2 \left( \frac{p_2(1)i + p_2(2)j}{i + j} \right) \\ & + \ell [is_2(1)q_2(1) + js_2(2)q_2(2)] \quad , \end{aligned}$$

transition kernel  $P$  given by

$$\begin{aligned} P((i, j, k, \ell); (i, j, k - 1, \ell)) &= \frac{ir_1 p_1(1) \frac{k}{k + \ell} + jks_1(1)q_1(1)}{\lambda(i, j, k, \ell)} \\ P((i, j, k, \ell); (i, j, k, \ell - 1)) &= \frac{ir_1 p_1(2) \frac{\ell}{k + \ell} + j\ell s_1(2)q_1(2)}{\lambda(i, j, k, \ell)} \\ P((i, j, k, \ell); (i - 1, j, k, \ell)) &= \frac{kr_2 p_2(1) \frac{i}{i + j} + \ell is_2(1)q_2(1)}{\lambda(i, j, k, \ell)} \\ P((i, j, k, \ell); (i, j - 1, k, \ell)) &= \frac{kr_2 p_2(2) \frac{j}{i + j} + \ell js_2(2)q_2(2)}{\lambda(i, j, k, \ell)} \end{aligned}$$

and infinitesimal generator  $Q$  given by

$$Q((i,j,k,\ell); (i,j,k-1,\ell)) = ir_1 p_1(1) \frac{k}{k+\ell} + jks_1(1)q_1(1)$$

$$Q((i,j,k,\ell); (i,j,k,\ell-1)) = ir_1 p_1(2) \frac{\ell}{k+\ell} + j\ell s_1(2)q_1(2)$$

$$\begin{aligned} Q((i,j,k,\ell); (i,j,k,\ell)) = & - \left\{ ir_1 \left( \frac{p_1(1)k + p_1(2)\ell}{k+\ell} \right) \right. \\ & + j[ks_1(1)q_1(1) + \ell s_1(2)q_1(2)] \\ & + kr_2 \left( \frac{p_2(1)i + p_2(2)j}{i+j} \right) \\ & \left. + \ell[is_2(1)q_2(1) + js_2(2)q_2(2)] \right\} \end{aligned}$$

$$Q((i,j,k,\ell); (i-1,j,k,\ell)) = kr_2 p_2(1) \frac{i}{i+j} + \ell is_2(1)q_2(1)$$

$$Q((i,j,k,\ell); (i,j-1,k,\ell)) = kr_2 p_2(2) \frac{j}{i+j} + \ell js_2(2)q_2(2) .$$

For the approximation (12) we have

$$E[B_t^1 | (B_0^1, B_0^2, R_0^1, R_0^2) = (i,j,k,\ell)]$$

$$\begin{aligned} & \sim i - [kr_2 p_2(1) \frac{i}{i+j} \ell is_2(1)q_2(1)]t \\ & = i(1 - t[\frac{kr_2 p_2(1)}{i+j} \ell s_2(1)q_2(1)]) , \end{aligned}$$

$$E[B_t^2 | (B_0^1, B_0^2, R_0^1, R_0^2) = (i,j,k,\ell)]$$

$$\sim j(1 - t[\frac{kr_2 p_2(2)}{i+j} + \ell s_2(2)q_2(2)]) ,$$

$$E[R_t^1 | (B_0^1, B_0^2, R_0^1, R_0^2) = (i, j, k, \ell)] \\ \sim k(1 - \tau[\frac{ir_1p_1(1)}{k + \ell} + js_1(1)q_1(1)]) ,$$

and

$$E[R_t^2 | (B_0^1, B_0^2, R_0^1, R_0^2) = (i, j, k, \ell)] \\ \sim \ell(1 - \tau[\frac{ir_1p_1(2)}{k + \ell} + js_1(2)q_1(2)]) ,$$

respectively.

The four states which can be reached from state  $(i, j, k, \ell)$  correspond to deaths of the four different weapon types and are entered at the "rates" indicated in the expression for  $Q$ . Comparing  $Q$  here with the generators of the Processes S1 and L3 yields the "mixed law" interpretation of this process, as does a comparison of the families of assumptions underlying the three processes.

No heterogeneous mixed laws have been discovered before. Without families of assumptions to use in making generalizations, it was not known what appropriate process would be. Moreover, the Process M1 can be extended to allow an arbitrary number of weapon types on each side.

## M1a Homogeneous Mixed Law Process

### Assumptions

1. All weapons on each side are identical.
2. Red weapons satisfy assumptions 2 and 3 of process M1 with parameters  $r, p$ .
3. Blue weapons satisfy assumptions 5 and 6 of process M1 with parameters  $s, q$ .
4. Detection and attack processes of all weapons are mutually independent.

### Process Characterization

Under assumptions 1 to 4, the process  $((B_t, R_t))_{t \geq 0}$  is a regular step process with state space  $\mathbb{N} \times \mathbb{N}$ , jump function  $\lambda$  given by

$$\lambda(i, j) = irp + ijsq ,$$

transition kernel  $P$  given by

$$P((i, j), (i, j - 1)) = \frac{rp}{rp + jsq}$$

$$P((i, j), (i - 1, j)) = \frac{jsq}{rp + jsq}$$

and infinitesimal generator  $Q$  given by

$$Q((i, j), (i, j - 1)) = irp$$

$$Q((i, j), (i, j)) = - (irp + ijsq)$$

$$Q((i, j), (i - 1, j)) = ijsq .$$

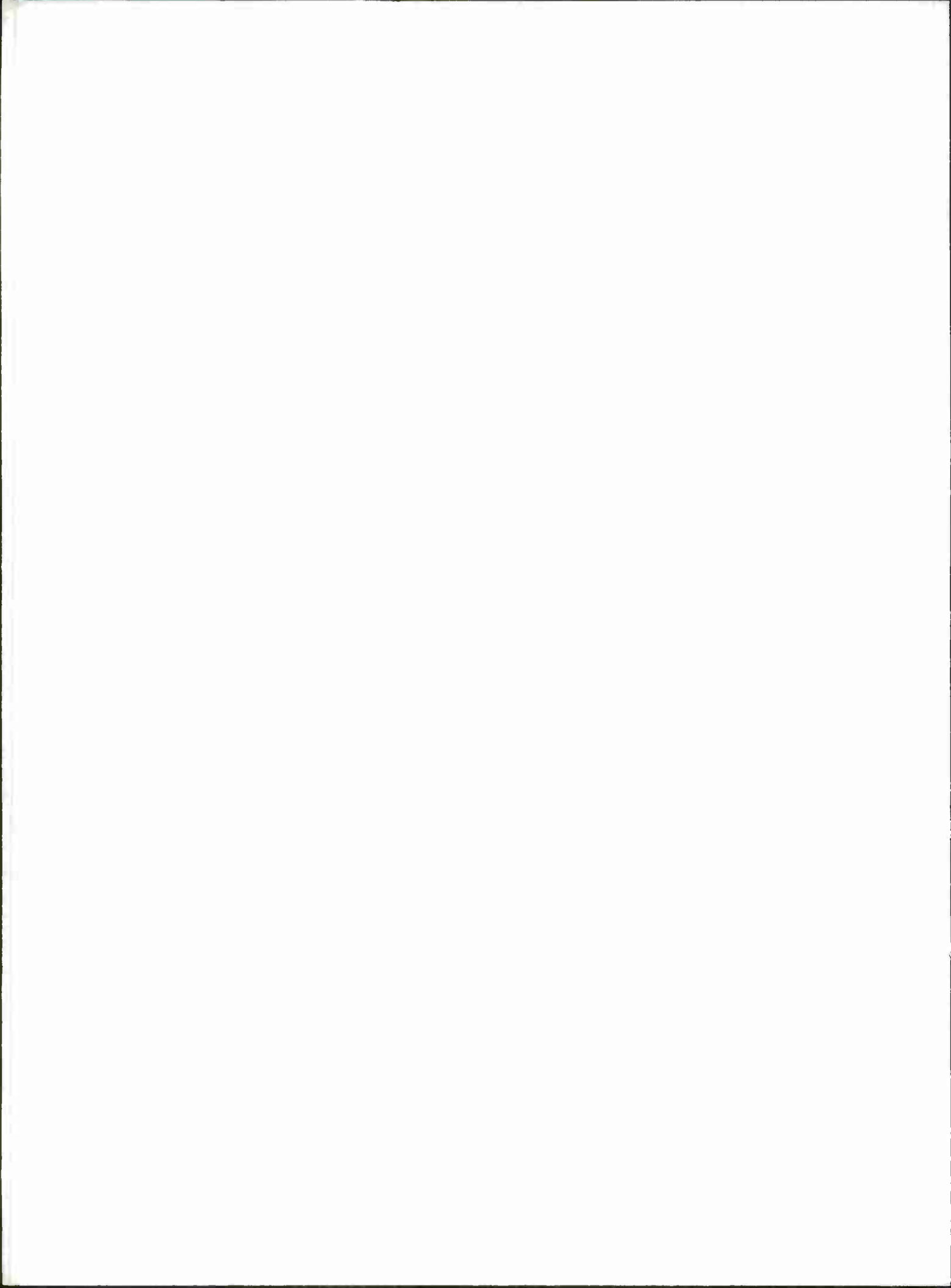
First-order approximations in this case are given by

$$E[B_t | (B_0, R_0) = (i, j)] = i - jrpt$$

and

$$E[R_t | (B_0, R_0) = (i, j)] = j(1 - t[isq]),$$

respectively.



## APPENDIX

### PROBABILISTIC TECHNICALITIES AND PROOFS OF RESULTS

We give here the proofs of the characterization theorems in the main body of the paper. First, however, some probabilistic concepts related to regular step processes will be discussed in further detail.

Let  $E$  be a countable set with discrete  $\sigma$ -algebra  $\underline{E}$ . A Markov kernel on the measurable space  $(E, \underline{E})$  is a mapping  $P$  of  $E \times \underline{E}$  into  $[0, 1]$  such that  $A \rightarrow P(i, A)$  is a probability measure on  $\underline{E}$  for each  $i \in E$ . Since  $\underline{E}$  is discrete the probability  $P(i, \cdot)$  is determined by its values on singleton sets and  $P$  may hence be considered as the matrix  $P$  defined by

$$P(i, j) = P(i, \{j\}) .$$

If  $f$  is a bounded or nonnegative function on  $E$ ,  $Pf$  denotes the function defined by

$$Pf(i) = \sum_{j \in E} P(i, j)f(j) .$$

Given a function  $\lambda: E \rightarrow [0, \infty)$  and a Markov kernel  $P$  on  $(E, \underline{E})$  such that  $P(i, i) = 0$  for all  $i \in E$  such that  $\lambda(i) > 0$ , there exists a continuous parameter Markov process  $X = (\Omega, \underline{E}, X_t, P^i)$  with state space  $(E, \underline{E})$  satisfying the following intuitive description. If the process enters a state  $i$  with  $\lambda(i) = 0$  it remains there forever after (such states are said to be absorbing). When the process enters a state  $i$  with  $\lambda(i) > 0$  its sojourn time there is exponentially distributed with mean  $1/\lambda(i)$  and independent of the past history of the process. At the end of this time, the process jumps (instantaneously) to a new state of  $E$  according to the probability distribution

$P(i, \cdot)$ , independent of the present time, the length of its sojourn in state  $i$  and all previous jumps. Hence, successive states entered form a Markov chain with transition matrix  $P$ .

DEFINITION.  $X$  is the regular step process with jump function  $\lambda$  and transition kernel  $P$ .

$P^i$  denotes the probability law of the process given that  $X_0 = i$ ; expectation with respect to this measure is denoted by  $E^i$ , so

$$E^i[Y] = \int_{\Omega} Y(\omega) P^i(d\omega)$$

for suitable  $\underline{E}$ -measurable random variables  $Y$ .

Let  $b\underline{E}$  denote the set of all bounded (and necessarily  $\underline{E}$ -measurable) functions on  $E$ . Then the relations

$$P_t f(i) = E^i[f(X_t)]$$

define a semigroup  $(P_t)_{t \geq 0}$  of bounded linear operators on  $b\underline{E}$ , called the transition function of  $X$ . If  $f$  is the indicator function  $1_{\{j\}}$  of the singleton set  $\{j\}$  then

$$P_t f(i) = P^i\{X_t = j\} \equiv P_t(i, j)$$

and for any  $g$ ,

$$P_t g(i) = \sum_j P_t(i, j) g(j) .$$

Hence  $(P_t)$  is represented by a family of stochastic matrices; we do not notationally distinguish the operator  $P_t$  and the corresponding matrix.  $P_0$  is the identity matrix.

The transition function  $(P_t)$  is uniquely determined by its infinitesimal generator, which is the (possibly unbounded) linear operator  $Q$  defined by

$$Qf(i) = \lim_{t \downarrow 0} \frac{P_t f(i) - f(i)}{t} .$$

Q is represented by the matrix Q given by

$$Q(i, j) = \lim_{t \downarrow 0} \frac{P_t(i, j) - \delta(i, j)}{t} ,$$

since  $P_0$  is the identity matrix.

Stated alternatively, the matrix-valued function  $t \rightarrow P_t$  is differentiable (componentwise) and

$$P'_t = QP_t$$

for all  $t$  ; in particular

$$Q = P'_0 .$$

The transition semigroup is given in terms of the infinitesimal generator Q by formal solution of the system  $P' = QP$  of linear differential equations by exponentials; thus

$$P_t = e^{tQ}$$

where

$$e^{tQ} = \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!} .$$

For regular step process, Q is given explicitly in terms of the jump function  $\lambda$  and transition kernel P by the relations

$$Q(i, j) = \begin{cases} -\lambda(i) & \text{if } j = i \\ \lambda(i)P(i, j) & \text{if } j \neq i , \end{cases}$$

which may be proved computationally.

Fundamental to our claim that the stochastic processes characterized in the following theorems are analogs of Lanchester's differential equation models of combat is the "differential" interpretation of the infinitesimal generator  $Q$ . From the representation  $P_t = e^{tQ}$  we see that

$$P_h(i, j) = \delta(i, j) + hQ(i, j) + o(h)$$

as  $h \rightarrow 0$ . In particular for  $j \neq i$

$$P_h(i, j) = hQ(i, j) + o(h) \quad , \quad h \rightarrow 0 \quad .$$

That is, the probability of a jump from  $i$  to  $j$  in the interval  $(0, h]$  is approximately  $h \cdot Q(i, j)$  so that one can interpret  $Q(i, j)$  as the infinitesimal rate at which the process moves from state  $i$  to state  $j$ .

We next present proofs of the characterization theorems for each of our stochastic attrition processes.

## A1 Homogeneous Square Law Area Fire Process

We construct this process in the following manner. Let  $E = \underline{\mathbb{N}} \times \underline{\mathbb{N}}$ , where  $\underline{\mathbb{N}} = \{0, 1, 2, \dots\}$ , the set of natural numbers. As the sample space  $\Omega$  we take the family of all functions  $\omega = (\omega_1, \omega_2)$  mapping  $[0, \infty)$  into  $E$  with the properties that

a)  $\omega$  is right continuous on  $[0, \infty)$ : for each time  $t \geq 0$ ,

$$\lim_{u \downarrow t} \omega(u) = \omega(t);$$

all limits in  $E$  are taken in the product of the discrete topologies on the factor spaces;

b)  $\omega$  possesses left-hand limits on  $(0, \infty)$ ; that is, for each  $t > 0$

$$\lim_{u \uparrow t} f(u)$$

exists.

We define two families  $(B_t)_{t \geq 0}$  and  $(R_t)_{t \geq 0}$  of coordinate random variables on  $\Omega$  by

$$B_t(\omega) = \omega_1(t) \quad , \quad t \geq 0$$

and

$$R_t(\omega) = \omega_2(t) \quad , \quad t \geq 0 \quad ,$$

respectively. The interpretation is that  $B_t$  is the number of surviving Blue combatants at time  $t$  and  $R_t$  the number of Red combatants surviving at time  $t$ .

Further, we let  $\underline{F}_t$  be the history generated by the random variables,  $\{(B_s, R_s) : 0 \leq s \leq t\}$ , which can be interpreted as the history of the attrition process up until time  $t$ , and let

$$\underline{F} = \sigma((B_u, R_u) : u \geq 0),$$

which is the entire history of the process.

For each  $(i, j) \in E$ , let  $P^{(i,j)}$  denote the probability law of the attrition process governed by assumptions 1-5 on page 23, conditioned on the event  $\{(B_0, R_0) = (i, j)\}$ . While one can give the details of the construction of these measures (beginning with probabilities assigned to appropriate cylinder sets and proceeding through an application of the Kolmogorov Extension Theorem) such an approach is not appropriate in this exposition. So, instead, we take for granted the existence of such probabilities and derive in our Theorems characterizations of the resultant stochastic attrition process.

(1) THEOREM. Subject to assumptions 1-5 of the family A1, the process

$$(\Omega, \underline{F}, \underline{F}_t, (B_t, R_t), P^{(i,j)})$$

is a regular step process with

a) state space  $(E, \underline{E})$ ; we remind that  $E = \mathbb{N} \times \mathbb{N}$  and  $\underline{E}$  is the discrete  $\sigma$ -algebra;

b) jump function  $\lambda$  given by

$$(2) \quad \lambda(i,j) = \begin{cases} ir_1[1 - (1-p_1)^j] + jr_2[1 - (1-p_2)^i] & \text{if } i > 0, j > 0 \\ 0 & \text{if } i = 0 \text{ or } j = 0; \end{cases}$$

c) transition kernel  $P$  given for states  $(i,j)$  with  $i > 0$  and  $j > 0$  by

$$P((i,j); (i,\ell)) = \frac{ir_1 \binom{j}{\ell} (1-p_1)^\ell p_1^{\ell-j}}{\lambda(i,j)}, \quad 0 \leq \ell < j$$

(3)

$$P((i,j); (k,j)) = \frac{jr_2 \binom{i}{k} (1-p_2)^k p_2^{i-k}}{\lambda(i,j)}, \quad 0 \leq k < i ;$$

d) infinitesimal generator  $Q$  given for states  $(i,j)$  with  $i > 0$  and  $j > 0$  by

$$Q((i,j); (i,\ell)) = ir_1 \binom{j}{\ell} (1-p_1)^\ell p_1^{j-\ell}, \quad 0 \leq \ell < j$$

$$Q((i,j); (i,j)) = - (ir_1 [1 - (1-p_1)^j] + jr_2 [1 - (1-p_2)^i])$$

$$Q((i,j); (k,j)) = jr_2 \binom{i}{k} (1-p_1)^k p_1^{i-k}, \quad 0 \leq k < i .$$

If  $i = 0$  or  $j = 0$ ,  $Q((i,j); x) = 0$  for all  $x \in E$ .

PROOF. For detailed arguments we refer to the proof of Theorem (6).

First of all, note that if some Blue combatant fires a shot at an instant when there are  $j$  surviving Reds, the probability that  $\ell$  Reds survive is the binomial probability

$$\binom{j}{\ell} (1-p_1)^\ell p_1^{j-\ell},$$

in particular, the probability that the shot causes one or more fatalities is

$$q_1 = 1 - (1-p_1)^j .$$

Thus if we denote by  $S_B$  and  $S_R$  the times of the first fatality-causing shots fired by Blue and Red, respectively, then with respect to the probability  $P^{(i,j)}$ ,  $S_B$  and  $S_R$  are, each conditioned on non-occurrence of the other, independent and exponentially distributed with means  $1/ir_1 q_1$  and  $1/jr_2 q_2$ , respectively. It follows (see the

proof of Theorem (6)) that the time

$$V = \inf\{t: (B_t, R_t) \neq (B_0, R_0)\}$$

of the first change of state of the process is exponentially distributed with mean  $1/(ir_1q_1 + jr_2q_2)$  (with respect to  $P^{(i,j)}$ , of course; note also that  $q_1$  is a function of  $j$  and  $q_2$  a function of  $i$ , even though we have suppressed this dependence in our notation) and that, moreover

$$\begin{aligned} P^{(i,j)}\{(B_{V+}, R_{V+}) \in \{(i, \ell): 0 \leq \ell < j\}\} \\ &= P^{(i,j)}\{S_B < S_R\} \\ &= \frac{ir_1q_1}{ir_1q_1 + jr_2q_2} \end{aligned}$$

while

$$\begin{aligned} P^{(i,j)}\{(B_{V+}, R_{V+}) \in \{(k, j): 0 \leq k < i\}\} \\ &= P^{(i,j)}\{S_R < S_B\} \\ &= \frac{jr_2q_2}{ir_1q_1 + jr_2q_2} . \end{aligned}$$

Let  $K_B$  be the number of fatalities caused by the first fatality-causing shot, if there is one, fired by a Blue combatant. Then conditioned on the event  $\{S_B < S_R\}$ ,  $K_B$  is "binomially" distributed on  $\{0, \dots, j-1\}$ , with respect to  $P^{(i,j)}$ , in the sense that

$$\begin{aligned} P^{(i,j)}\{K_B = \ell | S_B < S_R\} \\ &= \frac{\binom{j}{\ell} p_1^\ell (1-p_1)^{j-\ell}}{1 - (1-p_1)^j} , \quad \ell = 0, \dots, j-1 ; \end{aligned}$$

the normalization is required in order that

$$\sum_{\ell=0}^{j-1} P^{(i,j)} \{K_B = \ell | S_B < S_R\} = 1 .$$

We then have for  $0 \leq \ell < j$

$$\begin{aligned} P^{(i,j)} \{ (B_{V+}, R_{V+}) = (i, \ell) \} &= P^{(i,j)} \{ S_B < S_R, K_B = j - \ell \} \\ &= P^{(i,j)} \{ S_B < S_R \} P^{(i,j)} \{ K_B = j - \ell | S_B < S_R \} \\ &= \frac{ir_1 q_1}{ir_1 q_1 + jr_2 q_2} \frac{\binom{j}{j-\ell} (1-p_1)^\ell p_1^{j-\ell}}{1 - (1-p_1)^j} \\ &= \frac{ir_1 q_1}{ir_1 q_1 + jr_2 q_2} \frac{\binom{j}{\ell} (1-p_1)^\ell p_1^{j-\ell}}{q_1} \\ &= \frac{ir_1 \binom{j}{\ell} (1-p_1)^\ell p_1^{j-\ell}}{\lambda(i,j)} \\ &= P((i,j), (i,\ell)) \end{aligned}$$

where  $P$  is defined by (3) .

We leave to the reader, should he desire the details, the entirely analogous proof that

$$\begin{aligned} P^{(i,j)} \{ (B_{V+}, R_{V+}) = (k,j) \} &= \frac{jr_2 \binom{i}{k} (1-p_2)^k p_2^{i-k}}{\lambda(i,j)} \\ &= P((i,j); (k,j)) \end{aligned}$$

for  $0 \leq k < i$  .

Hence with respect to  $P^{(i,j)}$ , the random variable  $V$  is exponentially distributed with mean  $1/\lambda(i,j)$  where  $\lambda$  is defined by (2) and  $(B_{V+}, R_{V+})$  is distributed as  $P((i,j); \cdot)$ . The extension to arbitrary fixed times is omitted.  $\square$

The binomial distributions postulated in assumptions 3 and 4 have the (perhaps) unrealistic effect of permitting each side with positive probability to annihilate the other with a single shot.

It is, from a theoretical standpoint, quite acceptable to replace the family of binomial distributions by two families  $\{\varphi_j(\cdot): j \geq 1\}$  and  $\{\psi_i(\cdot): i \geq 1\}$  of probabilities on the nonnegative integers with the property that for each  $j$ ,  $\varphi_j$  is a probability distribution on  $\{0, \dots, j\}$  and for each  $i$ ,  $\psi_i$  is a probability distribution on  $\{0, \dots, i\}$ . The interpretation is then that  $\varphi_j(\ell)$  is the probability that a shot fired by a Blue combatant at an instant when there are  $j$  surviving Reds, kills exactly  $j - \ell$  of those  $j$  Reds, and similarly for the  $\psi_i$ . We may then, with only notational changes, obtain the following generalization of Theorem (1).

(4) THEOREM. The process

$$(\Omega, \underline{E}, \underline{E}_t, (B_t, R_t), P^{(i,j)})$$

is a regular step process with

- a) state space  $(E, \underline{E})$ ;
- b) jump function  $\lambda$  given by

$$\lambda(i, j) = \begin{cases} ir_1(1 - \varphi_j(j)) + jr_2(1 - \psi_i(i)) & \text{if } i > 0, j > 0 \\ 0 & \text{if } i = 0 \text{ or } j = 0; \end{cases}$$

c) transition kernel  $P$  given for states  $(i,j)$  with  $i > 0$  and  $j > 0$  by

$$P((i,j); (i,\ell)) = \frac{ir_1 \varphi_j(\ell)}{\lambda(i,j)} \quad 0 \leq \ell < j$$

$$P((i,j); (k,j)) = \frac{jr_2 \psi_i(k)}{\lambda(i,j)} \quad 0 \leq k < i \quad ;$$

d) infinitesimal generator  $Q$  given for states  $(i,j)$  with  $i > 0$ ,  $j > 0$  by

$$Q((i,j), (i,\ell)) = ir_1 \varphi_j(\ell) \quad , \quad 0 \leq \ell < j$$

$$Q((i,j), (i,j)) = -[ir_1(1 - \varphi_j(j)) + jr_2(1 - \psi_i(i))]$$

$$Q((i,j), (k,j)) = jr_2 \psi_i(k) \quad , \quad 0 \leq k < i \quad .$$

If  $i = 0$  or  $j = 0$ ,  $Q((i,j),x) = 0$  for all  $x \in E$ .

There is no need to repeat the proof of (1); it goes through here with only notational changes.

One pays a price, however, for such generality and that price is that most such distributions  $\{\varphi_j\}$  and  $\{\psi_i\}$  do not arise from a set of physical assumptions concerning the attrition process being modeled. The binomial distributions, for all their other unpalatable characteristics, at least do have a physical basis in the assumption that each shot fired, for example, at a Red target force of  $j$  survivors kills each Red with probability  $p_1$  and different Reds independently of one another, for then the number of Reds surviving the shot is binomially distributed with parameters  $(j, 1 - p_1)$ .

If the distributions  $\{\varphi_j\}$  and  $\{\psi_i\}$  are all of the form

$$\varphi_j(j-1) = p_1$$

$$\varphi_j(j) = 1-p_1$$

(5)

$$\Psi_i(i-1) = p_2$$

$$\Psi_i(i) = 1-p_2 \quad ,$$

where  $0 \leq p_1 \leq 1$ ,  $0 \leq p_2 \leq 1$ , then the Process S1 below is obtained.

## S1 Homogeneous Square Law Process

The canonical stochastic process

$$(\Omega, \underline{F}, \underline{F}_t, (B_t, R_t))$$

remains unchanged. The relevant assumptions are the family of assumptions 1 through 5 on page 27. At the risk of possible confusion, but in order to prevent a hopeless proliferation of notations for various families of probability measures on  $(\Omega, \underline{F})$  we again denote by  $P^{(i,j)}$  the probability law of the attrition process governed by this family of assumptions, subject to the initial conditions  $B_0 = i, R_0 = j$ . Thus, while the family of assumptions has changed, the notation has not; the reader should keep this in mind here and throughout the exposition.

(6) THEOREM. Under assumptions 1-5 of the family S1, the process

$$(\Omega, \underline{F}, \underline{F}_t, (B_t, R_t), P^{(i,j)})$$

is a regular step Markov process with

a) state space  $(E, \underline{E})$ , where  $\underline{E}$  is the  $\sigma$ -algebra of all subsets of  $E$ ;

b) jump function  $\lambda$  given by

$$(7) \quad \lambda(i, j) = \begin{cases} ic_1 + jc_2 & \text{if } i > 0 \text{ and } j > 0 \\ 0 & \text{if } i = 0 \text{ or } j = 0 \end{cases} ;$$

c) transition kernel  $P$  given for states  $(i, j) \in E$  with  $i > 0$  and  $j > 0$  by

$$(8) \quad \begin{aligned} P((i, j), (i, j - 1)) &= \frac{ic_1}{ic_1 + jc_2} \\ P((i, j), (i - 1, j)) &= \frac{j c_2}{ic_1 + jc_2} ; \end{aligned}$$

d) infinitesimal generator  $Q$  given for states  $(i, j)$  with  $i > 0$  and  $j > 0$  by

$$\begin{aligned} Q((i, j), (i, j - 1)) &= ic_1 \\ (9) \quad Q((i, j), (i, j)) &= - (ic_1 + jc_2) \\ Q((i, j), (i - 1, j)) &= jc_2 \end{aligned}$$

If  $i = 0$  or  $j = 0$ ,  $Q((i, j), x) = 0$  for all  $x \in E$ .

PROOF. Let  $E_0 = \{(i, j) : i > 0, j > 0\}$ . Since there clearly is no further attrition once one side is annihilated, every state in  $E - E_0$  is absorbing. Also, with probability one the state  $(0, 0)$  is never entered unless the process begins there.

We begin by showing that if  $(i, j) \in E_0$  and  $t \geq 0$ , then

$$\begin{aligned} P^{(i,j)}\{(B_u, R_u) = (i, j) \text{ for all } u \in [0, t]\} \\ &= P^{(i,j)}\{(B_t, R_t) = (i, j)\} \\ &= \exp [ - (ic_1 + jc_2)t ] \end{aligned}$$

Let  $V$  denote the time at which the process first changes state, which in this case is also the first time at which a fatal shot is fired. We are trying to compute  $P^{(i,j)}\{V > t\}$ .

Fix attention on a particular Blue combatant, say the  $k^{\text{th}}$  one and consider the times  $B_0^{(k)} \equiv 0, B_1^{(k)}, B_2^{(k)}, \dots$  at which he would fire shots if he survived and continued firing forever. By assumptions 2) and 3) the sequence  $(B_n^{(k)})_{n \geq 0}$  is a simple Poisson process with rate  $r_1$ . Suppose in addition that all shots fired by all combatants have only "hypothetical" effects in the sense that no combatants are killed but a record is kept of shots which would have been fatal without this restriction. This converts the attrition process into a non-terminating process in which each combatant continues firing shots

forever, some of which cause "hypothetical" kills of opponents. The hypothetical process not only has nicer behavior than the original attrition process but also serves to describe the attrition process because of the following two properties:

(a) The time of the first hypothetical kill is the time at which the attrition process first changes state;

(b) The side suffering the first hypothetical casualty also suffers the first casualty in the attrition process.

Thus consideration of the process of hypothetical kills suffices by (a) to compute the jump function of the attrition process and by (b) to compute its transition kernel, as well as by (a) and (b) together to establish the Markov property. Computations involving the process of hypothetical kills are easy to do, so we shall use throughout the approach of considering processes of hypothetical kills.

By the Random Sampling Theorem for Poisson processes, the times  $(B_n^{S(k)})_{n \geq 0}$  of hypothetically fatal shots fired by the  $k^{\text{th}}$  Blue combatant form a Poisson process with rate  $c_1 = r_1 p_1$ .

Similarly, for each  $l$ , the set  $(R_n^{S(l)})$  of times of hypothetically fatal shots fired by the  $l^{\text{th}}$  Red combatant is a simple Poisson process with rate  $c_2$ .

By assumption 5) the random variables  $B_1^{S(1)}, \dots, B_1^{S(i)}, R_1^{S(1)}, \dots, R_1^{S(j)}$  are mutually independent with respect to  $P^{(i,j)}$ . Furthermore,  $V > t$  if and only if  $B_1^{S(k)} > t$  for  $k = 1, \dots, i$  and  $R_1^{S(l)} > t$  for  $l = 1, \dots, j$ , since

$$V = \min \{ B_1^{S(1)}, \dots, B_1^{S(i)}, R_1^{S(1)}, \dots, R_1^{S(j)} \}.$$

Therefore

$$\begin{aligned}
 P^{(i,j)}\{V > t\} &= P^{(i,j)}\left\{B_1^{(k)} > t, k = 1, \dots, i; R_1^{(\ell)} > t, \ell = 1, \dots, j\right\} \\
 &= \left[ \prod_{k=1}^i P^{(i,j)}\{B_1^{(k)} > t\} \right] \left[ \prod_{\ell=1}^j P^{(i,j)}\{R_1^{(\ell)} > t\} \right] \\
 &= \left[ \prod_{k=1}^i e^{-c_1 t} \right] \left[ \prod_{\ell=1}^j e^{-c_2 t} \right] \\
 &= \exp [ - (ic_1 + jc_2)t ] .
 \end{aligned}$$

The same reasoning and "memoryless" properties of the exponential distribution apply to show that for each  $t$  and  $s$

$$\begin{aligned}
 P^{(i,j)}\{(B_u, R_u) = (B_s, R_s) \text{ for all } u \in [s, s+t] | \underline{E}_s\} \\
 = \exp [ - (c_1 B_s + c_2 R_s)t ] .
 \end{aligned}$$

This proves that the process  $((B_t, R_t))$  has jump function  $\lambda$  defined in (7) above.

We next show that for  $(i, j) \in E_0$ ,

$$P^{(i,j)}\{(B_{V+}, R_{V+}) = (i-1, j)\} = \frac{jc_2}{ic_1 + jc_2}$$

and

$$P^{(i,j)}\{(B_{V+}, R_{V+}) = (i, j-1)\} = \frac{ic_1}{ic_1 + jc_2} ,$$

where

$$(B_{V+}, R_{V+}) = \lim_{t \downarrow 0} (B_{V+t}, R_{V+t})$$

is the state entered by the process at the time  $V$  of the first fatality. It seems plausible that that state must be either  $(i-1, j)$ , if the first fatality is suffered by Blue, or  $(i, j-1)$ , if the first fatality is suffered by Red.

Let

$$S_B = \min\{B_1^{S(1)}, \dots, B_1^{S(i)}\}$$

and

$$S_R = \min\{R_1^{S(1)}, \dots, R_1^{S(j)}\} .$$

Then

$$\begin{aligned} P^{(i,j)}\{(B_{V^+}, R_{V^+}) = (i-1, j)\} &= P^{(i,j)}\{V = S_R\} \\ &= P^{(i,j)}\{S_B > S_R\} \end{aligned}$$

and

$$\begin{aligned} P^{(i,j)}\{(B_{V^+}, R_{V^+}) = (i, j-1)\} &= P^{(i,j)}\{V = S_B\} \\ &= P^{(i,j)}\{S_B < S_R\} . \end{aligned}$$

Since  $S_B$  is exponentially distributed with parameter  $ic_1$  and  $S_R$  is exponentially distributed with parameter  $jc_2$  and  $S_B, S_R$  are independent (all these assertions hold with respect to the probability measure  $P^{(i,j)}$ , of course), it follows that

$$P^{(i,j)}\{S_B = S_R\} = 0$$

and therefore

$$P^{(i,j)}\{(B_{V^+}, R_{V^+}) \in \{(i-1, j), (i, j-1)\}\} = 1 .$$

We next compute that

$$\begin{aligned} P^{(i,j)}\{S_R < S_B\} &= E^{(i,j)}[P^{(i,j)}\{S_R < S_B | S_R\}] \\ &= jc_2 \int_0^\infty \exp(-jc_2 v) \exp(-ic_1 v) dv \\ &= \frac{jc_2}{ic_1 + jc_2} , \end{aligned}$$

where  $E^{(i,j)}$  denotes expectation with respect to the probability  $p^{(i,j)}$ . Similarly,

$$P^{(i,j)} \{S_B < S_R\} = \frac{ic_1}{ic_1 + jc_2} .$$

For each  $t$  let  $V_t$  denote the time of the first fatal shot fired after time  $t$ . Then again using the "memoryless" property of the exponential distribution, one obtains

$$\begin{aligned} P^{(i,j)} \{(B_{V_{t+}}, R_{V_{t+}}) = (k, \ell) | \underline{E}_t\} \\ = P^{(B_t, R_t)} \{(B_{V_{t+}}, R_{V_{t+}}) = (k, \ell)\} \\ = P((B_t, R_t), (k, \ell)), \end{aligned}$$

where  $P$  is defined by (8), for each  $t$  and  $(k, \ell)$ .

Hence  $((B_t, R_t))_{t \geq 0}$  has the Markov property, the given jump function, and the given transition kernel. That this process then has the infinitesimal generator  $Q$  given in (9) follows from the theory of regular step processes, cf p. 57. □

Here are some additional properties of this process which are of physical interest.

COROLLARY. The following properties hold.

- 1) With probability one, the functions  $t \rightarrow B_t$  and  $t \rightarrow R_t$  are nondecreasing;
- 2) If  $i \neq 0, j \neq 0$ , then

$$P^{(i,j)} \{(B_t, R_t) = (0, 0) \text{ for some } t\} = 0 .$$

Property 1) certainly makes sense: if there are no reinforcements, surviving force strengths can only decrease. The second assertion, which is true even if  $c_1 = c_2$  and initial force strengths are numerically equal, states the "mutual annihilation at time  $t = \infty$ " which occurs in the deterministic Lanchester square model is almost surely impossible in this model.

We wish to emphasize that although the vector process  $((B_t, R_t))_{t \geq 0}$  is a Markov process, neither of the individual survivor processes  $(B_t)_{t \geq 0}$  and  $(R_t)_{t \geq 0}$  is Markovian, at least with respect to the family of histories  $(\underline{E}_t)_{t \geq 0}$ . If  $t$  is fixed, then the history  $\underline{E}_t$  contains information other than the current state  $B_t$  which is helpful in predicting the future course of the process  $(B_u)$ ; that is, in predicting  $(B_s)_{s > t}$ . That information is, of course, the current number  $R_t$  of Red survivors. (It couldn't be anything else, for otherwise  $((B_u, R_u))$  wouldn't have the Markov property.)

This model can be extended to allow for reinforcements which arrive in a particular manner, namely, according to compound Poisson processes, as we now indicate.

Feller (1966) contains sufficient background. For our purposes a compound Poisson process is defined by a positive number  $\alpha$  and a probability distribution  $\varphi$  on  $\mathbb{N}^* = \{1, 2, \dots\}$ . Let  $T_0 = 0$ ,  $T_1, T_2, \dots$  be the times of arrivals of a simple Poisson process with rate  $\alpha$  and let  $\chi_1, \chi_2, \dots$  be independent and identically distributed as  $\varphi$ . Assume that  $(T_n)$  and  $(\chi_k)$  are independent and define

$$N_t = \sum_{n: T_n \leq t} \chi_n, \quad t \geq 0.$$

One interprets  $(N_t)$  is an arrival counting process; that is,  $N_t$  is the number of objects which have arrived up until time  $t$ . At each of the times  $T_1, T_2, \dots$  a random number of objects arrives, which is distributed as  $\varphi$ .

Consider now the following additional assumptions.

6. Reinforcements to the Blue side arrive according to a compound Poisson process with parameters  $(\alpha_1, \varphi_1)$ . Reinforcements to the Red side arrive according to a compound Poisson process with parameters  $(\alpha_2, \varphi_2)$ . All reinforcements immediately enter the combat.

7. The two reinforcement processes are independent of each other and of all firing processes.

For  $(i, j) \in E$ , let  $P^{(i, j)}$  denote the probability law of the homogeneous square law process with reinforcements, governed by assumptions 1-7 above and conditioned on the event  $\{(B_0, R_0) = (i, j)\}$ .

(10) THEOREM. The process

$$(\Omega, \underline{E}, \underline{E}_t, (B_t, R_t), P^{(i, j)})$$

is a regular step process with

- a) state space  $(E, \underline{E})$  as defined for Theorem (6);
- b) jump function  $\lambda$  given

$$(11) \quad \lambda(i, j) = \begin{cases} ic_1 + jc_2 + \alpha_1 + \alpha_2 & \text{if } i > 0 \text{ and } j > 0 \\ \alpha_1 + \alpha_2 & \text{if } i = 0 \text{ or } j = 0 \end{cases} ;$$

c) transition kernel  $P$  given for states  $(i, j)$  with  $i > 0$  and  $j > 0$  by

$$(12a) \quad \begin{aligned} P((i, j); (i, j - 1)) &= \frac{ic_1}{ic_1 + jc_2 + \alpha_1 + \alpha_2} \\ P((i, j); (i - 1, j)) &= \frac{jc_2}{ic_1 + jc_2 + \alpha_1 + \alpha_2} \\ P((i, j); (i + k, j)) &= \varphi_1(k) \frac{\alpha_1}{ic_1 + jc_2 + \alpha_1 + \alpha_2}, \quad k \geq 1 \\ P((i, j); (i, j + l)) &= \varphi_2(l) \frac{\alpha_2}{ic_1 + jc_2 + \alpha_1 + \alpha_2}, \quad l \geq 1 \end{aligned}$$

and for states  $(i, j)$  with  $i = 0$  or  $j = 0$  by

$$(12b) \quad \begin{aligned} P((i, j); (i + k, j)) &= \varphi_1(k) \frac{\alpha_1}{\alpha_1 + \alpha_2}, \quad k \geq 1 \\ P((i, j); (i, j + l)) &= \varphi_2(l) \frac{\alpha_2}{\alpha_1 + \alpha_2}, \quad l \geq 1 ; \end{aligned}$$

d) infinitesimal generator  $Q$  given for states  $(i, j)$  with  $i > 0, j > 0$  by

$$\begin{aligned}
 Q((i, j); (i, j - 1)) &= ic_1 \\
 Q((i, j); (i - 1, j)) &= jc_2 \\
 (13a) \quad Q((i, j); (i + k, j)) &= \alpha_1 \varphi_1(k) \\
 Q((i, j); (i, j + l)) &= \alpha_2 \varphi_2(l) \\
 Q((i, j); (i, j)) &= - (ic_1 + jc_2 + \alpha_1 + \alpha_2)
 \end{aligned}$$

and for states  $(i, j)$  with  $i = 0$  or  $j = 0$  by

$$\begin{aligned}
 Q((i, j); (i + k, j)) &= \alpha_1 \varphi_1(k) \\
 (13b) \quad Q((i, j); (i, j + l)) &= \alpha_2 \varphi_2(l) \\
 Q((i, j); (i, j)) &= - (\alpha_1 + \alpha_2)
 \end{aligned}$$

PROOF. We employ the same notation as in the proof of the preceding Theorem, with the following additions. Let  $A_B$  denote the time of the arrival of the first group of Blue reinforcements, let  $A_R$  denote the time at which the first contingent of Red reinforcements arrives. Let  $\chi_B, \chi_R$  denote the respective sizes of these two groups of reinforcements.

The time

$$V = \inf\{t: (B_t, R_t) \neq (B_0, R_0)\}$$

at which the process first changes state is now given by

$$V = \min\{S_B, S_R, A_B, A_R\}$$

where  $S_B, S_R$  are as defined in the proof of Theorem (6).

That is, the first change of state occurs in exactly one of the following ways:

i) A fatal shot is fired by Blue before Red fires a fatal shot and before any reinforcements arrive; that is

$$S_B < S_R, \quad S_B < A_B, \quad S_B < A_R$$

ii) Red fires a fatal shot before Blue does and before any reinforcements arrive:

$$S_R < S_B, \quad S_R < A_B, \quad S_R < A_R$$

iii) Some Blue reinforcements arrive before any Red reinforcements do and before any fatal shots are fired:

$$A_B < S_B, \quad A_B < S_R, \quad A_B < A_R$$

v) Some Red reinforcements arrive before the arrival of any Blue reinforcements and before any fatalities:

$$A_R < S_B, \quad A_R < S_R, \quad A_R < A_B .$$

By assumptions 2, 3, 4, 5, 6, and 7, the random variables  $S_B$ ,  $S_R$ ,  $A_B$ ,  $A_R$  are independent and exponentially distributed with means  $1/ic_1$ ,  $1/jc_2$ ,  $1/\alpha_1$  and  $1/\alpha_2$ , respectively. Therefore  $V$  is exponentially distributed with mean  $1/(ic_1 + jc_2 + \alpha_1 + \alpha_2) = \lambda(i, j)$ ; in other words

$$(14) \quad P^{(i, j)} \{ (B_u, R_u) = (i, j) \text{ for } 0 \leq u \leq t \} = \exp[ - \lambda(i, j)t ]$$

for all  $(i, j)$  and  $t$ , where  $\lambda$  is given by (11).

Moreover, since

$$P^{(i, j)} \{ S_B = S_R \text{ or } S_B = A_B \text{ or } S_B = A_R \text{ or } S_R = A_B \text{ or } S_R = A_R \text{ or } A_B = A_R \} = 0$$

it follows that

$$P^{(i, j)} \{ (B_{V+}, R_{V+}) \in E_{i, j} \} = 1$$

where

$$E_{ij} = \{ (i - 1, j), (i, j - 1) \} \cup \{ (k, j) : k > i \} \\ \cup \{ (i, l) : l > j \}.$$

That is, two or more simultaneous fatalities (on either the same or opposite sides) or a simultaneous fatality and arrival of reinforcements, fail to occur almost surely.

Proceeding as in the proof of Theorem (6) we see that

$$\begin{aligned}
 P^{(i,j)} \{ (B_{V+}, R_{V+}) = (i, j - 1) \} \\
 &= P^{(i,j)} \{ S_B < S_R, S_B < A_B, S_B < A_R \} \\
 &= \frac{ic_1}{ic_1 + jc_2 + \alpha_1 + \alpha_2},
 \end{aligned}$$

that

$$\begin{aligned}
 P^{(i,j)} \{ (B_{V+}, R_{V+}) = (i - 1, j) \} \\
 &= P^{(i,j)} \{ S_R < S_B, S_R < A_B, S_R < A_R \} \\
 &= \frac{jc_2}{ic_1 + jc_2 + \alpha_1 + \alpha_2},
 \end{aligned}$$

that

$$\begin{aligned}
 P^{(i,j)} \{ (B_{V+}, R_{V+}) \in \{(k, j): k > i\} \} \\
 &= P^{(i,j)} \{ A_B < S_B, A_B < S_R, A_B < A_R \} \\
 &= \frac{\alpha_1}{ic_1 + jc_2 + \alpha_1 + \alpha_2}
 \end{aligned}$$

and, finally, that

$$\begin{aligned}
 P^{(i,j)} \{ (B_{V+}, R_{V+}) \in \{(i, l): l > j\} \} \\
 &= P^{(i,j)} \{ A_R < S_B, A_R < S_R, A_R < A_B \} \\
 &= \frac{\alpha_2}{ic_1 + jc_2 + \alpha_1 + \alpha_2}.
 \end{aligned}$$

Let us consider the latter two expressions in more detail. Since  $(B_{V+}, R_{V+}) = (k + i, j)$  with  $k > 0$  if and only if the first event causing a change of state is the arrival of a group of Blue reinforcements

which number exactly  $k$ , we have

$$\begin{aligned} P^{(i,j)}\{(B_{V+}, R_{V+}) = (i+k, j)\} \\ = P^{(i,j)}\{A_B < S_B, A_B < S_R, A_B < A_R, \chi_B = k\} . \end{aligned}$$

By properties of the compound Poisson process of Blue reinforcements and assumption 7,  $\chi_B$  is, with respect to the probability  $P^{(i,j)}$ , independent of  $A_B, S_B, S_R,$  and  $A_R,$  and therefore

$$\begin{aligned} (15a) \quad & P^{(i,j)}\{A_B < S_B, A_B < S_R, A_B < S_R, \chi_B = k\} \\ & = P^{(i,j)}\{A_B < S_B, A_B < S_R, A_B < S_R\} P^{(i,j)}\{\chi_B = k\} \\ & = \frac{\alpha_1}{ic_1 + jc_2 + \alpha_1 + \alpha_2} \varphi_1(k) . \end{aligned}$$

Through entirely analogous reasoning we are able to obtain

$$\begin{aligned} (15b) \quad & P^{(i,j)}\{(B_{V+}, R_{V+}) = (i, j+l)\} \\ & = \frac{\alpha_2}{ic_1 + jc_2 + \alpha_1 + \alpha_2} \varphi_2(k) \end{aligned}$$

for each  $l > 0$ .

We may, on the basis of (12), summarize (15) in the form

$$(16) \quad P^{(i,j)}\{(B_{V+}, R_{V+}) = (k, l)\} = P((i,j), (k, l))$$

for all  $(i, j), (k, l) \in E$ .

Using standard arguments concerning the exponential distribution, one extends (14) to

$$\begin{aligned}
 P^{(i,j)}\{(B_u, R_u) = (B_s, R_s) \text{ for } s \leq u \leq s + t | \underline{E}_s\} \\
 = \exp[-\lambda(B_s, R_s)t]
 \end{aligned}$$

and (16) to

$$\begin{aligned}
 P^{(i,j)}\{(B_{V_{s^+}}, R_{V_{s^+}}) = (k, l) | \underline{E}_s\} \\
 = P((B_s, R_s); (k, l)),
 \end{aligned}$$

where

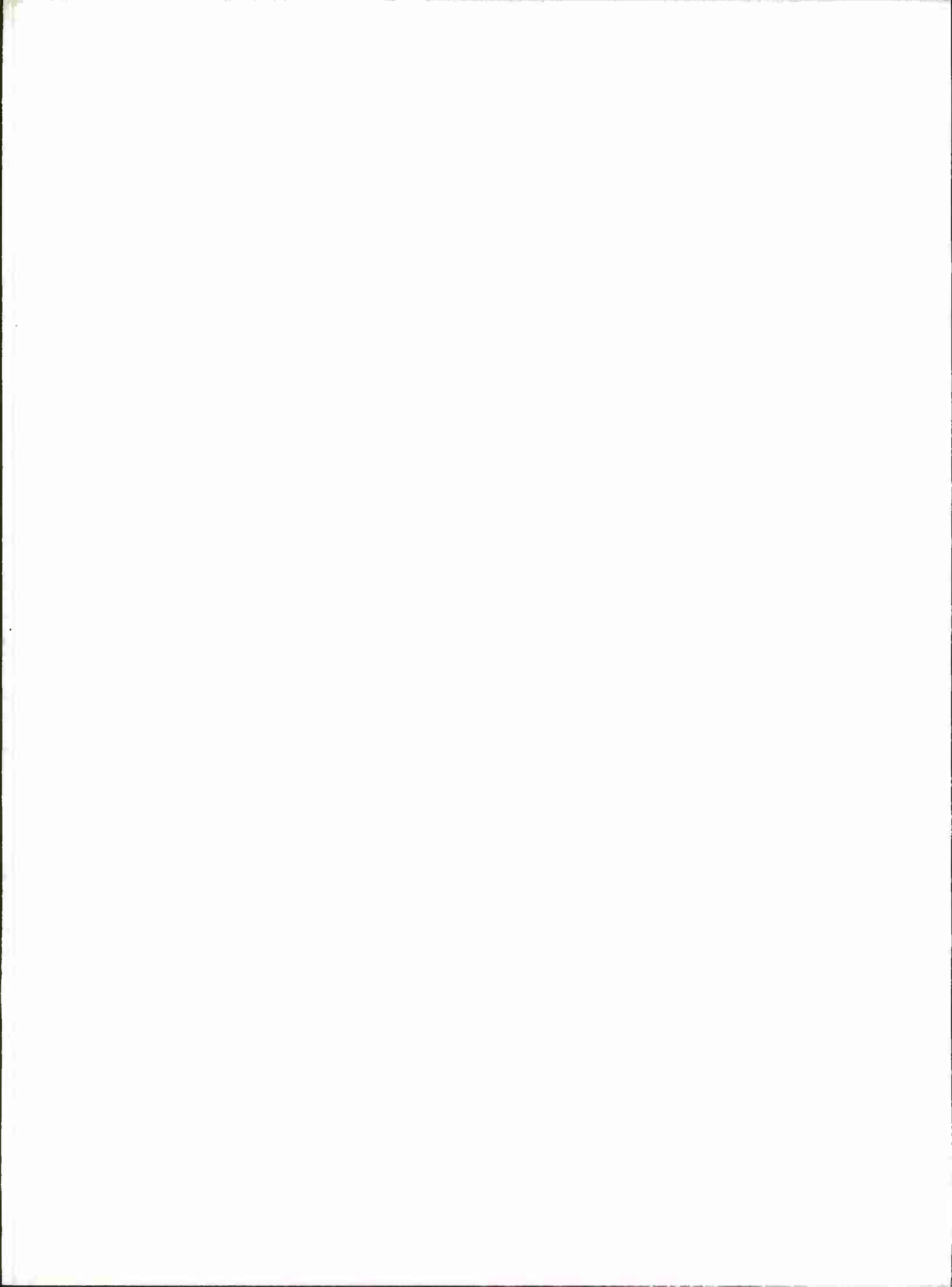
$$V_s = \inf\{u > s: (B_u, R_u) \neq (B_s, R_s)\}$$

is the time of the first change of state of the process after time  $t$ .

These last two statements combine to complete the proof of the Theorem. II

Note that the process with reinforcements has no absorbing states.

Other types of assumptions concerning reinforcements are possible, but would be more difficult to handle and are almost certain to destroy the Markovian property of the process, although if the resultant model were a semi-Markov process it might still be tractable.



## S2 Heterogeneous Square Law Process

For heterogeneous processes we need to construct a new sample space. Throughout we assume that there are  $M$  Blue weapon types (the well-established terminology "weapon types", as opposed to "types of combatants", will be used in all our discussions of heterogeneous processes) and  $N$  Red weapon types. There is no assumption that at least one of each type of weapon be present.

We take as our state space, therefore, the set

$$E = \underline{N} \times \underline{N} \times \dots \times \underline{N} \quad (M + N \text{ times})$$

together with the discrete  $\sigma$ -algebra  $\underline{E}$ .

As sample space  $\Omega$  we again have the family of all mappings  $\omega = (\omega_1, \dots, \omega_M, \omega'_1, \dots, \omega'_N)$  of mappings of  $[0, \infty)$  into  $\Omega$  which are right continuous and have left-hand limits everywhere. We define coordinate vector-valued random processes  $(\underline{B}_t)_{t \geq 0}$  and  $(\underline{R}_t)_{t \geq 0}$  by

$$\underline{B}_t(\omega) = (\omega_1(t), \dots, \omega_M(t))$$

$$\underline{R}_t(\omega) = (\omega'_1(t), \dots, \omega'_N(t)) \quad ,$$

and coordinate processes  $(B_t^1)_{t \geq 0}, \dots, (B_t^M)_{t \geq 0}, (R_t^1)_{t \geq 0}, \dots, (R_t^N)_{t \geq 0}$  by

$$B_t^k(\omega) = \omega_k(t) \quad , \quad 1 \leq k \leq M$$

$$R_t^\ell(\omega) = \omega'_\ell(t) \quad , \quad 1 \leq \ell \leq N$$

so that, in particular,

$$\underline{B}_t = (B_t^1, \dots, B_t^M)$$

and

$$\underline{R}_t = (R_t^1, \dots, R_t^N)$$

for each  $t$ .  $B_t^k$  is the number of Blue type  $k$  weapons surviving at time  $t$ ,  $\underline{B}_t$  the Blue force surviving at time  $t$ ,  $R_t^l$  the number of type  $l$  Red weapons surviving at time  $t$ , and  $\underline{R}_t$  the surviving Red force at time  $t$ .

As before, we put

$$\underline{E}_t = \sigma\{(\underline{B}_s, \underline{R}_s) : 0 \leq s \leq t\}$$

and

$$\underline{E} = \sigma\{(\underline{B}_s, \underline{R}_s) : s \geq 0\} .$$

$\underline{E}_t$  is the history of the attrition process up until time  $t$  and  $\underline{E}$  its entire history

We denote points in  $E$  in the form  $(x, y)$  where  $x \in \underline{N}^M$  and  $y \in \underline{N}^N$ ;  $x$  thus is a state which the vector of Blue surviving weapons may assume and similarly for  $y$  and Red.  $x_i$  corresponds to surviving Blue type  $i$  weapons and  $y_j$  to surviving Red type  $j$  weapons.

For each  $(x, y) \in E$ , let  $P^{(x, y)}$  denote the probability law on  $(\Omega, \underline{E})$  of the attrition process governed by assumptions 1-6 of the Process S2, when  $(\underline{B}_0, \underline{R}_0) = (x, y)$ .

(17) THEOREM. Under the S2 assumptions,

$$(\Omega, \underline{E}, \underline{E}_t, (\underline{B}_t, \underline{R}_t), P^{(x, y)})$$

is a regular step process with

- a) state space  $(E, \underline{E})$  as defined above;
- b) jump function  $\lambda$  given by

$$\lambda(x, y) = \begin{cases} \sum_{i: x_i > 0} \sum_{j: y_j > 0} [c_2(i, j)y_j + c_1(j, i)x_i] & \text{if } x \neq 0, y \neq 0 \\ 0 & \text{if } x = 0 \text{ or } y = 0; \end{cases}$$

c) transition kernel  $P$  defined for states  $(x,y)$  with  $x \neq 0$  and  $y \neq 0$  by

$$P((x,y); (x; y_1, \dots, y_j^{-1}, \dots, y_N)) = \frac{\sum_{i=1}^M c_1(j,i)x_i}{\lambda(x,y)}$$

for  $j = 1, \dots, N$  such that  $y_j > 0$  and by

$$P((x,y); (x_1, \dots, x_i^{-1}, \dots, x_M; y)) = \frac{\sum_{j=1}^N c_2(i,j)y_j}{\lambda(x,y)}$$

for all  $i$  with  $1 \leq i \leq M$  and  $x_i > 0$ ;

d) infinitesimal generator  $Q$  given for states  $(x,y)$  with  $x \neq 0$  and  $y \neq 0$  by

$$Q((x,y); (x; y_1, \dots, y_j^{-1}, \dots, y_N)) = \begin{cases} \sum_{i=1}^M c_1(j,i)x_i & \text{if } y_j > 0 \\ 0 & \text{if } y_j = 0 \end{cases}$$

$$Q((x,y); (x,y)) = - \sum_{i: x_i > 0} \sum_{j: y_j > 0} [c_2(i,j)y_j + c_1(j,i)x_i]$$

$$Q((x,y); (x_1, \dots, x_i^{-1}, \dots, x_M; y)) = \begin{cases} \sum_{j=1}^N c_2(i,j)y_j & \text{if } x_i > 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

and for states  $(x,y)$  with  $x = 0$  or  $y = 0$  by  $Q((x,y); \zeta) = 0$  for all  $\zeta \in E$ .

PROOF. We recall that for  $q = 1, 2$  and appropriate  $k, l$  we have defined

$$c_q(k, l) = r_q(k, l)p_q(k, l) \quad .$$

For  $i = 1, \dots, M$  let

$$S_R(i) = \inf\{t: B_t^i \neq B_0^i\}$$

be the time at which the first fatality to a Blue type  $i$  weapon occurs. If there is no such time, and in particular if  $B_0^i = 0$ , we put  $S_R(i) = +\infty$ . Similarly, let

$$S_B(j) = \inf\{t: R_t^j \neq R_0^j\} \quad , \quad j = 1, \dots, N$$

be the time of the first fatality, if there is one, to a Red type  $j$  weapon.

We claim that

$$\begin{aligned} & P^{(x, y)}\{S_R(i) > t, i = 1, \dots, M, S_B(j) > t, j = 1, \dots, N\} \\ (18) \quad & = \exp\left(-t \sum_{i: x_i > 0} \sum_{j: y_j > 0} [c_2(i, j)y_j + c_1(j, i)x_i]\right) \end{aligned}$$

from which it follows that, if we define

$$V = \inf\{t: (B_t, R_t) \neq (B_0, R_0)\}$$

then since

$$V = \min\{S_B(1), \dots, S_B(N), S_R(1), \dots, S_R(M)\} \quad ,$$

one has

$$P^{(x, y)}\{V > t\} = \exp[-\lambda(x, y)t] \quad .$$

Now, how are we to prove (18)? If all kills were hypothetical only, then for a given  $j$  and  $i$ , the times of "hypothetical" kills of type  $j$  Red weapons by type  $i$  Blue weapons would, by assumptions 2 and 4 form a Poisson process with rate  $c_1(j,i)x_i$ , so long as  $y_j > 0$ . Hence by hypothesis 6, "kills" of type  $j$  Red weapons occur hypothetically at the arrival times of a Poisson process with rate

$$\sum_i c_1(j,i)x_i ;$$

in particular the time  $S_B^*(j)$  of the first such hypothetical kill is exponentially distributed with mean  $(\sum_i c_1(j,i)x_i)^{-1}$  if  $y_j > 0$  and

mean  $+\infty$  if  $y_j = 0$ . Moreover, by hypothesis 3,  $S_B^*(1), \dots, S_B^*(N)$  are independent with respect to  $P^{(x,y)}$ .

If  $S_R^*(i)$  similarly stands for the time of the first hypothetical kill of a Blue type  $i$  weapon, then  $S_R^*(i)$  is exponentially distributed, with respect to the probability  $P^{(x,y)}$ , with mean

$$\begin{cases} (\sum_j c_2(i,j)y_j)^{-1} & \text{if } x_i > 0 \\ + \infty & \text{if } x_i = 0 \end{cases} .$$

Assumption 3 assures us that  $S_R^*(1), \dots, S_R^*(M)$  are independent and assumption 6 implies that  $\{S_B^*(1), \dots, S_B^*(N)\}$  and  $\{S_R^*(1), \dots, S_R^*(M)\}$  are independent.

Furthermore, even though by definition

$$V = \min\{S_B(1), \dots, S_B(N), S_R(1), \dots, S_R(M)\}$$

it is also true that

$$V = \min\{S_B^*(1), \dots, S_B^*(N), S_R^*(1), \dots, S_R^*(M)\}$$

since the times of the first actual and hypothetical kills necessarily coincide.

Thus

$$\begin{aligned}
P^{(x,y)}\{V > t\} &= P^{(x,y)}\{S_B^*(1) > t, \dots, S_B^*(N) > t, \\
&\quad S_R^*(1) > t, \dots, S_R^*(M) > t\} \\
&= P^{(x,y)}\{S_B^*(1) > t\} \dots P^{(x,y)}\{S_B^*(N) > t\} \\
&\quad \times P^{(x,y)}\{S_R^*(1) > t\} \dots P^{(x,y)}\{S_R^*(M) > t\} \\
&= \exp \left[ -t \sum_{j:y_j > 0} \sum_i c_i(j,i)x_i \right] \\
&\quad \times \exp \left[ -t \sum_{i:x_i > 0} \sum_j c_2(i,j)y_j \right]
\end{aligned}$$

which completes the proof of (18).

We note that if  $j \neq l$ ,  $i \neq k$

$$P^{(x,y)}\{S_B^*(j) = S_B^*(l), S_B^*(j) < \infty, S_B^*(l) < \infty\} = 0$$

and

$$P^{(x,y)}\{S_R^*(i) = S_R^*(k), S_R^*(i) < \infty, S_R^*(k) < \infty\} = 0$$

and that

$$P^{(x,y)}\{S_B^*(j) = S_R^*(i), S_B^*(j) < \infty, S_R^*(i) < \infty\} = 0$$

for all  $i, j$ , by independence and exponential distributions of the random variables involved.

Since

$$\begin{aligned}
P^{(x,y)}\{(B_{V^+}, R_{V^+}) = (x; y_1, \dots, y_{j-1}, \dots, y_N)\} \\
= P^{(x,y)}\{S_B^*(j) < S_B^*(l), l \neq j; S_B^*(j) < S_R^*(i) \text{ for all } i\}
\end{aligned}$$

it follows that

$$P^{(x,y)} \left\{ (\underline{B}_{V_+}, \underline{R}_{V_+}) \in \{(w,z): w_i = x_i - 1 \text{ for exactly one } i \text{ and otherwise } w = x \text{ and } z = y \text{ or } z_j = y_j - 1 \text{ for exactly one } j \text{ and otherwise, } z = y \text{ and } w = x\} \right\} = 1.$$

Again by independence and exponential distributions

$$P^{(x,y)} \{ S_B^*(j) < S_B^*(l), l \neq j; S_B^*(j) < S_R^*(i) \text{ for all } i \}$$

$$= \begin{cases} 0 & \text{if } y_j = 0 \\ \frac{\sum_i c_1(j,i)x_i}{\sum_{j:y_j>0} \sum_i c_1(j,i)x_i + \sum_{i:x_i>0} \sum_j c_1(i,j)x_j} & \text{if } y_j > 0 \end{cases}$$

and similarly,

$$P^{(x,y)} \{ (\underline{B}_{V_+}, \underline{R}_{V_+}) = (x_1, \dots, x_i - 1, \dots, x_M; y) \}$$

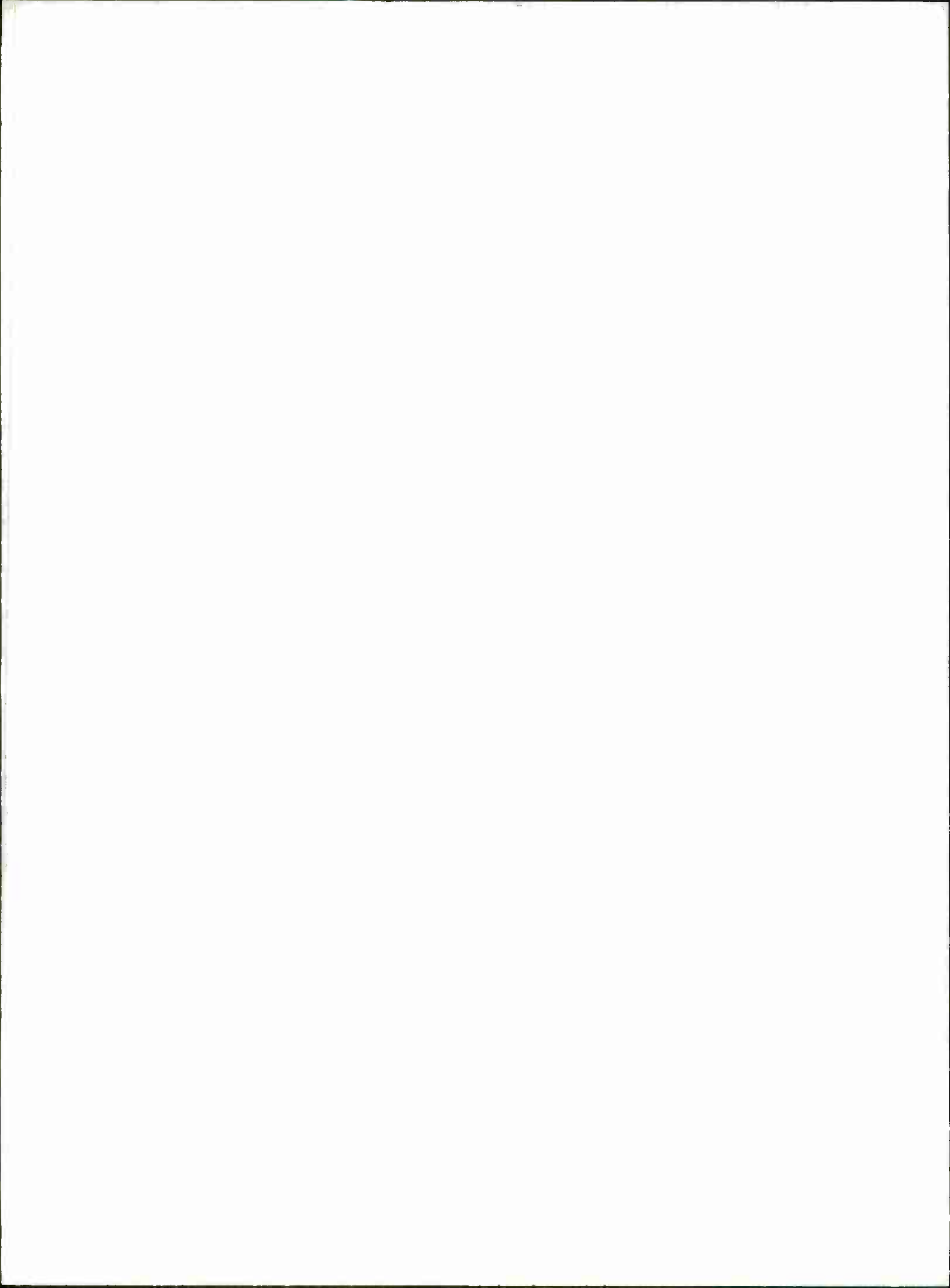
$$= P^{(x,y)} \{ S_R^*(i) < S_R^*(k), k \neq i; S_R^*(i) < S_B^*(j) \text{ for all } j \}$$

$$= \begin{cases} 0 & \text{if } x_i = 0 \\ \frac{\sum_j c_2(i,j)y_j}{\sum_{j:y_j>0} \sum_i c_1(j,i)x_i + \sum_{i:x_i>0} \sum_j c_1(i,j)y_j} & \text{if } x_i > 0 \end{cases}.$$

Hence we have shown that

$$P^{(x,y)} \{ (\underline{B}_{V_+}, \underline{R}_{V_+}) = \xi \} = P((x,y); \xi)$$

for all  $\xi \in E$ , where  $P$  is the transition kernel defined in the statement of the Theorem. □



S3a Heterogeneous Square Law Process

We continue to assume that there are  $M$  Blue weapon types and  $N$  Red weapon types, and that the canonical stochastic process

$$(\Omega, \underline{E}, \underline{E}_t, (\underline{B}_t, \underline{R}_t))$$

is as described in the preceding section.  $P^{(x,y)}$  will now denote the probability law of the attrition process governed by the family of five assumptions for the Process S3a, subject to the initial conditions

$$\begin{aligned} \underline{B}_0 &= x \in \underline{N}^M \\ \underline{R}_0 &= y \in \underline{N}^N . \end{aligned}$$

(19) THEOREM. Under the assumptions of Process S3a, the collection

$$(\Omega, \underline{E}, \underline{E}_t, (\underline{B}_t, \underline{R}_t), P^{(x,y)})$$

is a regular step process with

- a) state space  $(E, \underline{E})$  as defined in the description of Process S2;
- b) jump function  $\lambda$  given by

$$\lambda(x,y) = \begin{cases} \sum_{i=1}^M \sum_{j=1}^N [\tilde{x}_i p_2(i,j) r_2(j) y_j + \tilde{y}_j p_1(j,i) r_1(i) x_i] & \text{if } x \neq 0, y \neq 0 \\ 0 & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

where

$$\tilde{x}_i = x_i / \sum_{k=1}^M x_k, \quad i = 1, \dots, M$$

and

$$\tilde{y}_j = y_j / \sum_{\ell=1}^N y_\ell, \quad j = 1, \dots, N ;$$

these quantities are defined only if the denominator is positive;

c) transition kernel  $P$  given for states  $(x,y) \in E$  such that  $x \neq 0$  and  $y \neq 0$  by

$$P((x,y); (x; y_1, \dots, y_j - 1, \dots, y_N)) \\ = \tilde{y}_j \frac{\sum_{i=1}^M p_1(j,i) r_1(i) x_i}{\lambda(x,y)}, \quad j = 1, \dots, N$$

$$P((x,y); (x_1, \dots, x_{i-1}, \dots, x_M; y)) \\ = \tilde{x}_i \frac{\sum_{j=1}^N p_2(i,j) r_2(j) y_j}{\lambda(x,y)}, \quad i = 1, \dots, M;$$

d) infinitesimal generator  $Q$  given for states  $(x,y)$  with  $x \neq 0$  and  $y \neq 0$  by

$$Q((x,y), (x; y_1, \dots, y_j - 1, \dots, y_N)) = \tilde{y}_j \sum_{i=1}^M p_1(j,i) r_1(i) x_i$$

$$Q((x,y), (x,y)) = - \sum_{i=1}^M \sum_{j=1}^N [\tilde{x}_i p_2(i,j) r_2(j) y_j + \tilde{y}_j p_1(j,i) r_1(i) x_i]$$

$$Q((x,y); (x_1, \dots, x_{i-1}, \dots, x_M; y)) = \tilde{x}_i \sum_{j=1}^N p_2(i,j) r_2(j) y_j.$$

If  $x = 0$  or  $y = 0$ ,  $Q((x,y), \alpha) = 0$  for all  $\alpha \in E$ .

PROOF. As usual consider the hypothetical situation in which no weapons are actually killed, but only "potential" kills are recorded, in terms of the times at which they occur. Following the notation of the preceding proof we let  $S_B^*(j)$  denote the time of the first hypothetical kill of a Red type  $j$  weapon,  $j = 1, \dots, N$  and for  $i = 1, \dots, M$  we denote by  $S_R^*(i)$  the time at which the first hypothetical kill occurs of a Blue type  $i$  weapon. Then the time

$$V^* = \min\{S_B^*(1), \dots, S_B^*(N), S_R^*(1), \dots, S_R^*(M)\}$$

of the first hypothetical kill coincides with the time

$$V = \inf\{t: (\underline{B}_t, \underline{R}_t) \neq (\underline{B}_0, \underline{R}_0)\}$$

of the first actual kill, which is also the time at which the attrition process first changes state.

For  $i = 1, \dots, M$  and  $j = 1, \dots, N$  let  $K_R^*(i,j)$  be the time of the first hypothetical shot which is fatal to a Blue type  $i$  weapon and was fired by a Red type  $j$  weapon. Similarly, let  $K_B^*(j,i)$  be the first time at which a Red type  $j$  weapon is hypothetically killed by a Blue type  $i$  weapon. Then

$$S_R^*(i) = \min\{K_R^*(i,1), \dots, K_R^*(i,N)\}$$

and

$$S_B^*(j) = \min\{K_B^*(j,1), \dots, K_B^*(j,M)\}$$

for all appropriate  $i$  and  $j$ .

Consider now a fixed Blue weapon of type  $i$ . The times of hypothetical shots which it fires form a Poisson process with rate  $r_1(i)$ . Further, by assumption 3

$$\begin{aligned} p^{(x,y)} & \left\{ \begin{array}{l} \text{a hypothetical shot fired by a Blue type } i \text{ weapon} \\ \text{is directed at some Red weapon of type } j \end{array} \right\} \\ (20) \quad & = \tilde{y}_j = y_j / \sum_{\ell=1}^N y_\ell ; \end{aligned}$$

this is also the probability that an actual shot fired at a Red force of composition  $y$  is directed at a target of type  $j$ . A similar expression holds for shots and hypothetical shots fired by Red weapons.

Hence hypothetical shots fired by a particular Blue type  $i$  which are directed at Red weapons of type  $j$  form a Poisson process with rate  $r_1(i)\tilde{y}_j$

with respect, of course, to the probability  $P^{(x,y)}$ , which is assumed fixed for the purposes of this discussion. The fatal shots fired by this weapon at Red type  $j$  weapons thus constitute a Poisson process with rate  $r_1(i)\tilde{y}_j p_1(j,i)$ . By assumption 6, therefore, hypothetically fatal shots fired at Red type  $j$  weapons by all type  $i$  Blue weapons form a Poisson process with rate  $x_i r_1(i)\tilde{y}_j p_1(j,i)$ .

Another application of assumption 6 shows that the times of all hypothetically fatal shots fired at Red type  $j$  weapons form a Poisson process with rate

$$\sum_i x_i r_1(i)\tilde{y}_j p_1(j,i)$$

and therefore

$$(21) \quad P^{(x,y)}\{S_B^*(j) > t\} = \exp[-t\tilde{y}_j \sum_i p_1(j,i)r_1(i)x_i] .$$

Similarly, since

$$P^{(x,y)}\{K_R^*(i,j) > t\} = \exp[-t\tilde{x}_i p_2(i,j)r_2(j)y_j]$$

for each  $i$  and  $j$  yet another application of the independence assumption 6 yields

$$(22) \quad P^{(x,y)}\{S_R^*(i) > t\} = \exp[-t\tilde{x}_i \sum_j p_2(i,j)r_2(j)y_j] .$$

From these two expressions

$$P^{(x,y)}\{V^* > t\} = \exp[-\lambda(x,y)t]$$

follows and hence

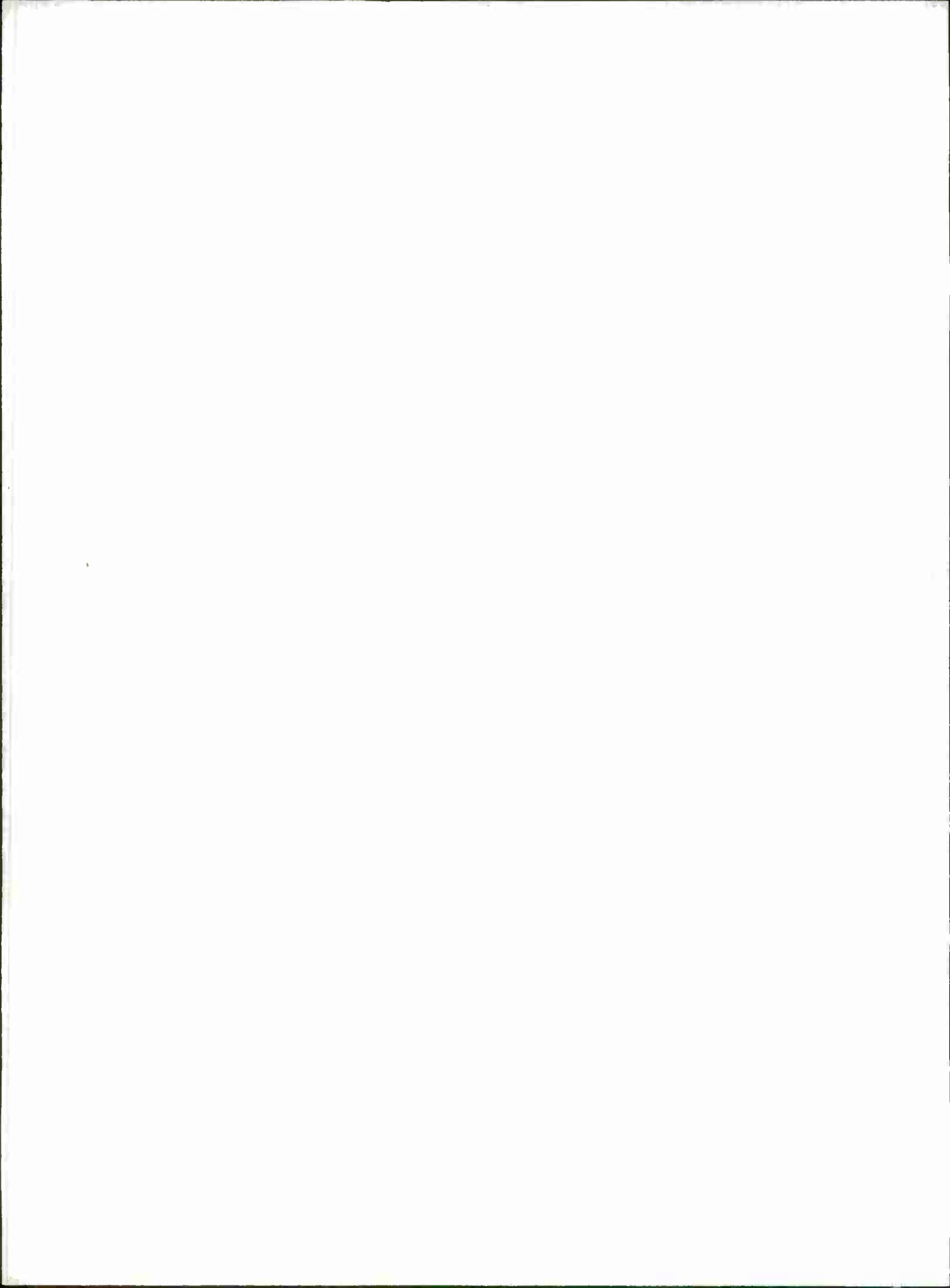
$$P^{(x,y)}\{V > t\} = \exp[-\lambda(x,y)t] .$$

Proceeding exactly as in the proof of Theorem (17) we see that

$$\begin{aligned} P^{(x,y)}\{(B_{V+}, R_{V+}) = (x; y_1, \dots, y_{j-1}, \dots, y_N)\} \\ &= P^{(x,y)}\{V^* = S_B^*(j)\} \\ (23) \quad &= P^{(x,y)}\{S_B^*(j) < S_B^*(l) \text{ for } l \neq j, S_B^*(j) < S_R^*(i) \text{ for all } i\} \\ &= \frac{\tilde{y}_j \sum_i p_1(j,i) r_1(i) x_i}{\lambda(x,y)} \end{aligned}$$

and, analogously, that

$$\begin{aligned} P^{(x,y)}\{(B_{V+}, R_{V+}) = (x_1, \dots, x_{i-1}, \dots, x_M; y)\} \\ &= P^{(x,y)}\{V^* = S_R^*(i)\} \\ (24) \quad &= P^{(x,y)}\{S_R^*(i) < S_R^*(k) \text{ for } k \neq i, S_R^*(i) < S_B^*(j) \text{ for all } j\} \\ &= \frac{\tilde{x}_i \sum_j p_2(i,j) r_2(j) y_j}{\lambda(x,y)} . \end{aligned} \quad \square$$



### S3 Heterogeneous Square Law Process

The notation is exactly that of the previous two processes.  $P^{(x,y)}$  now denotes the probability law of the attrition process governed by the family of assumptions of Process S3, subject to the initial conditions  $(\underline{B}_0, \underline{R}_0) = (x,y)$ .

(25) THEOREM. Under the assumptions for Process S3, the collection

$$(\Omega, \underline{E}, \underline{E}_t, (\underline{B}_t, \underline{R}_t), P^{(x,y)})$$

is a regular step process with

- a) state space  $(E, \underline{E})$  as defined above;
- b) jump function  $\lambda$  given by

$$\lambda(x,y) = \sum_{i=1}^M \sum_{j=1}^N [\Psi_i(y,j)p_1(j,i)r_1(i)x_i + \eta_j(x,i)p_2(i,j)r_2(j)y_j]$$

(we assume that  $y_j = 0$  implies  $\Psi_i(y,j) = 0$  for all  $i$ );

- c) transition kernel  $P$  given for states  $(x,y)$  with  $x \neq 0$  and  $y \neq 0$  by

$$P((x,y);(x;y_1, \dots, y_{j-1}, \dots, y_N)) = \frac{\sum_{i=1}^M \Psi_i(y,j)p_1(j,i)r_1(i)x_i}{\lambda(x,y)}$$

$$P((x,y);(x_1, \dots, x_{i-1}, \dots, x_M;y)) = \frac{\sum_j \eta_j(x,i)p_2(i,j)r_2(j)y_j}{\lambda(x,y)} ;$$

- d) infinitesimal generator  $Q$  given for states  $(x,y)$  with  $x \neq 0$  and  $y \neq 0$  by

$$Q((x,y);(x;y_1, \dots, y_{j-1}, \dots, y_N)) = \sum_{i=1}^M \Psi_i(y,j)p_1(j,i)r_1(i)x_i$$

$$Q((x,y);(x,y)) = - \sum_{i=1}^M \sum_{j=1}^N [\Psi_i(y,j)p_1(j,i)r_1(i)x_i + \\ + \eta_j(x,i)p_2(i,j)r_2(j)y_j]$$

$$Q((x,y);(x_1, \dots, x_{i-1}, \dots, x_M; y)) = \sum_{j=1}^N \eta_j(x,i)p_2(i,j)r_2(j)y_j .$$

If  $x = 0$  or  $y = 0$ ,  $Q((x,y),\alpha) = 0$  for all  $\alpha \in E$ .

PROOF. The proof of Theorem (19) requires only notational changes. By the new assumption 3, (20) becomes

$$(20^*) \quad P^{(x,y)} \left\{ \begin{array}{l} \text{A hypothetical shot fired by a Blue type } i \text{ weapon} \\ \text{is directed at some Red weapon of type } j \end{array} \right\} \\ = \Psi_i(y,j)$$

so that hypothetical shots fired by a given Blue type  $i$  weapon at Red type  $j$  weapons form a Poisson process with rate  $r_1(i)\Psi_i(y,j)$  and hypothetical fatal shots fired by this type  $i$  weapon at Red type  $j$  weapons form a Poisson process with rate  $r_1(i)\Psi_i(y,j)p_1(j,i)$ , in which case for (21) one obtains

$$(21^*) \quad P^{(x,y)} \left\{ S_B^*(j) > t \right\} \\ = \exp \left[ - t \sum_i \Psi_i(y,j)p_1(j,i)r_1(i)x_i \right] ;$$

analogously, (22) becomes

$$(22^*) \quad P^{(x,y)} \left\{ S_R^*(i) > t \right\} \\ = \exp \left[ - t \sum_j \eta_j(x,i)p_2(i,j)r_2(j)y_j \right] .$$

In the same manner one obtains as analogs of (23) and (24) the following:

$$(23^*) \quad P^{(x,y)} \{S_B^*(j) < S_B^*(\ell) \text{ for } \ell \neq j, S_B^*(j) < S_R^*(i) \text{ for all } i\}$$

$$= \frac{\sum_i \psi_i(y,j) p_1(j,i) r_1(i) x_i}{\lambda(x,y)}$$

for  $j = 1, \dots, N$  and

$$(24^*) \quad P^{(x,y)} \{S_R^*(i) < S_R^*(k) \text{ for } k \neq i, S_R^*(i) < S_B^*(j) \text{ for all } j\}$$

$$= \frac{\sum_j \eta_j(x,i) p_2(i,j) r_2(j) y_j}{\lambda(x,y)},$$

respectively.

The Theorem now follows. ↓

EXAMPLES. The fire allocation scheme of this process is quite general, but also entirely abstract. Here are three specific cases.

1. Uniform Fire Allocation occurs when

$$\psi_i(y,j) = \tilde{y}_j = y_j / \sum_{\ell=1}^N y_\ell$$

and

$$\varphi_j(x,i) = \tilde{x}_i = x_i / \sum_{k=1}^M x_k$$

for all  $i$  and  $j$  and in this case the Process S3 reduces to the process S3a. It becomes clear, therefore, that Process S3a is a Lanchester square process.

2. Priority Fire Allocation can occur in the following manner. Consider a fixed type of Blue weapon, say type  $i$ , and suppose that

the Red weapon type 1, ..., N are ranked in some order  $j_1, \dots, j_N$  (here  $\{j_1, \dots, j_N\}$  is simply a permutation of the set  $\{1, \dots, N\}$ ) in such a manner that a Red weapon of type  $j_\ell$  will never be fired upon when a weapon of type  $j_{\ell-1}$  is present. One models this by putting

$$\Psi_i(y, j_{\ell^*}) = 1$$

where

$$\ell^* = \inf\{\ell: y_{j_\ell} > 0\}.$$

Other more flexible and possibly more realistic priority schemes can certainly be developed.

3. One may also derive fire allocations from physically meaningful axioms. For example, consider the following hypotheses on the family  $\{\Psi_i\}$  of fire allocation distributions

- i)  $\Psi_i(y, j) = 0$  if and only if  $y_j = 0$ ;
- ii) For all  $y$  and  $z$

$$\begin{aligned} \Psi_i(y + z, j) = & \Psi_i(y, j) \left[ \sum_{\ell=1}^N \frac{y_\ell}{y_\ell + z_\ell} \Psi_i(y + z)(\ell) \right] \\ & + \Psi_i(z, j) \left[ \sum_{\ell=1}^N \frac{z_\ell}{y_\ell + z_\ell} \Psi_i(y + z)(\ell) \right] \end{aligned}$$

for all  $j = 1, \dots, N$ .

The interpretations of all these assumptions are physically meaningful. The first states that a weapon type not present receives no fire but that every weapon type present receives

a positive fraction of the fire. The second hypothesis is best explained step-by-step. Consider the effect of combining two target forces  $y$  and  $z$  into the single force  $w = y + z$ .  $\Psi_l(y + z, l)$  is the proportion of fire directed at the combined force  $w$  which is allocated to type  $l$  weapons. If this is further allocated among the type  $l$  weapons from the two component forces ( $y$  and  $z$ ) in proportion to the relative numbers of such weapons present then

$$\frac{y_l}{y_l + z_l} \Psi(y + z, l)$$

is the fraction of fire directed at the combined force that is allocated to type  $l$  weapons originally part of the  $y$ -force. Thus

$$\alpha_y = \sum_{l=1}^N \frac{y_l}{y_l + z_l} \Psi(y + z, l)$$

is the proportion of fire allocated to weapons of types originally and the  $y$ -force and

$$\alpha_z = \sum_{l=1}^N \frac{z_l}{y_l + z_l} \Psi(y + z, l)$$

the fire allocated to weapons originally in the  $z$ -force. But the fire represented by  $\alpha_y$  should be allocated among weapon types according to the distribution  $\Psi(y, \cdot)$  and similarly for  $\alpha_z$  so that one should have

$$\Psi(y + z, \cdot) = \alpha_y \Psi(y, \cdot) + \alpha_z \Psi(z, \cdot)$$

which is exactly the second assumption. If an assumption like this did not hold, consistency problems would arise, with fire allocation dependent on names given targets rather than only numbers of targets.

(For example, arbitrarily splitting a class of  $n$  indistinguishable weapons into two subclasses of  $n_1$  "Type A" and  $n - n_1$  "Type B" weapons would change the fire allocation).

We have conjectured, and J. Blankenship (1973) has proved, using the notion of invariant measures for Markov matrices, that any family  $\{\psi\}$  of allocation distributions satisfying i), ii), iii) above is necessarily of the form

$$\psi(y, j) = \frac{a_j y_j}{\sum_{\ell=1}^N a_\ell y_\ell}$$

where

$$a_j = \psi(\underline{1}, j) \quad , \quad j = 1, \dots, N$$

and  $\underline{1} = (1, \dots, 1)$ . It is possible, moreover, to express the  $\psi$  and terms of  $\psi(\tilde{y}, \cdot)$  for any fixed  $\tilde{y}$  such that  $\tilde{y}_j > 0$  for all  $j$ . Thus if one accepts the hypotheses i), ii), iii) then the fire allocation for every target force can be computed from that for a "base" force.

We remark in conclusion that even if different weapons may employ qualitatively different means of allocations, the model remains valid.

## L1 Homogeneous Linear Law Process

We return to the sample space and canonical stochastic process previously established for the homogeneous case. Thus, the state space is

$$E = \mathbb{N} \times \mathbb{N}$$

together with the discrete  $\sigma$ -algebra  $\underline{E}$ ,  $\Omega$  is the set of mappings  $\omega = (\omega_1, \omega_2)$  from  $[0, \infty)$  into  $E$  which are right-continuous with left-hand limits everywhere,

$$B_t(\omega) = \omega_1(t) \in E$$

$$R_t(\omega) = \omega_2(t) \in E$$

for each  $t \geq 0$ ,

$$\underline{E}_t = \sigma((B_u, R_u) : 0 \leq u \leq t)$$

is the history of the attrition process up until time  $t$  and

$$\underline{E} = \sigma((B_u, R_u) : u \geq 0)$$

the entire history.

From each  $(i, j) \in E$  we denote by  $P^{(i, j)}$ , as usual, the probability of the attrition process conditioned on the event  $\{(B_0, R_0) = (i, j)\}$  and this time governed by the five assumptions listed in the main text for the Process L1.

(26) **THEOREM.** Under the assumptions of Process L1, the collection

$$(\Omega, \underline{E}, \underline{E}_t, (B_t, R_t), P^{(i, j)})$$

is a regular step process with

a) state space  $E = \mathbb{N}^2$ ;

b) jump function  $\lambda$  given by

$$\lambda(i,j) = ij(k_1 + k_2);$$

c) transition kernel  $P$  given for states  $(i,j)$  with  $i > 0$  and  $j > 0$  by

$$P((i,j), (i,j - 1)) = \frac{k_1}{k_1 + k_2}$$
$$P((i,j), (i - 1, j)) = \frac{k_2}{k_1 + k_2} ;$$

d) infinitesimal generator  $Q$  given by

$$Q((i,j), (i,j - 1)) = k_1 ij$$
$$Q((i,j), (i,j)) = - (k_1 + k_2) ij$$
$$Q((i,j), (i - 1, j)) = k_2 ij .$$

We remind that for  $\ell = 1, 2$

$$k_\ell = s_\ell q_\ell$$

where  $1/s_1$  is the mean time required for a particular Blue weapon to detect a particular Red weapon and  $q_1$  the conditional probability that a Blue weapon destroys a Red weapon (in a one-on-one engagement, the only sort of engagement permitted in this model, which is also assumed to occur instantaneously) given detection and attack.  $s_2$  and  $q_2$  are analogous descriptions of Red weapons.

PROOF. We again use the device of hypothetical kills, which is useful because it preserves independence and stationarity properties which are not valid for the actual attrition process, yet allows us to make statements about the actual process because the time of the first hypothetical kill is also the time of the first actual kill

(namely, the first time at which the attrition process changes state) and the weapon killed at the first hypothetical kill is also the first weapon actually killed. Thereafter the two processes diverge from one other and the hypothetical process is no longer a useful means of dealing with the actual process.

We therefore denote by  $T_B^*(k, \ell)$  the time at which the  $k^{\text{th}}$  Blue weapon (first) hypothetically detects and kills the  $\ell^{\text{th}}$  Red weapon (not every detection results in a kill unless  $q_1 = 1$ ) and let  $T_R^*(\ell, k)$  be the time at which the  $\ell^{\text{th}}$  Red weapon first hypothetically kills the  $k^{\text{th}}$  Blue weapon. Then according to assumptions 2, 3, 4, 5 and the Random Sampling Theorem for Poisson processes, the random variables  $T_B^*(1,1), T_B^*(1,2), \dots, T_B^*(1,j), T_B^*(2,1), T_B^*(2,2), \dots, T_B^*(2,j), \dots, T_B^*(i,1), T_B^*(i,2), \dots, T_B^*(i,j)$  are independent and identically exponentially distributed with mean  $1/k_1$ . Symmetrically,  $T_R^*(1,1), \dots, T_R^*(1,i), \dots, T_R^*(j,1), \dots, T_R^*(j,i)$  are independent and identically exponentially distributed with mean  $1/k_2$ . Furthermore, these two families of random variables are independent. All of these statements hold with respect to the probability  $P^{(i,j)}$ .

Also with respect to  $P^{(i,j)}$ , the time

$$V^* = \min_{\substack{k < i \\ \ell \leq j}} \min \{ T_B^*(k, \ell), T_R^*(\ell, k) \}$$

of the first hypothetical fatality and the time

$$V = \inf \{ t : (B_t, R_t) \neq (B_0, R_0) \}$$

of the first actual fatality, coincide almost surely.

By independence, therefore,

$$\begin{aligned}
 P^{(i,j)}\{V^* > t\} &= P^{(i,j)}\{T_B^*(k,l) > t, T_R^*(l,k) > t \\
 &\qquad\qquad\qquad \text{for } 1 \leq k \leq i, i \leq l \leq j\} \\
 &= P^{(i,j)}\{T_B^*(k,l) > t \text{ for } 1 \leq k \leq i, 1 \leq l \leq j\} \\
 &\quad \times P^{(i,j)}\{T_R^*(l,k) > t \text{ for } 1 \leq k \leq i, 1 \leq l \leq j\} \\
 &= (P^{(i,j)}\{T_B^*(1,1) > t\})^{ij} \times (P^{(i,j)}\{T_R^*(1,1) > t\})^{ij} \\
 &= \exp [ - t(k_1 + k_2)ij ] .
 \end{aligned}$$

Since the side which suffers the first hypothetical casualty also suffers the first actual casualty (at, as noted above, the time of the first hypothetical casualty) and since by independence and non-atomicity of the exponential distribution

$$(27) \quad P^{(i,j)}\{\text{the } T_B^*(k,l) \text{ and } T_R^*(l,k) \text{ are all distinct from one another}\} = 1,$$

it follows that

$$(28) \quad P^{(i,j)}\{(B_{V^+}, R_{V^+}) \in \{(i,j-1), (i-1,j)\}\} = 1$$

and, moreover, that

$$\begin{aligned}
 P^{(i,j)}\{(B_{V^+}, R_{V^+}) = (i,j-1)\} \\
 &= P^{(i,j)}\{T_B^*(k,l) = V^* \text{ for some } k \text{ and } l\} \\
 &= \sum_{k=1}^i \sum_{l=1}^j P^{(i,j)}\{T_B^*(k,l) = V^*\}
 \end{aligned}$$

(we have used (27) here)

$$= ij P^{(i,j)} \{T_B^*(1,1) = v^*\}$$

by identical distributions of the  $T_B^*(k,l)$ .

Finally, we compute that

$$P^{(i,j)} \{T_B^*(1,1) = v^*\}$$

$$= P^{(i,j)} \{T_B^*(1,1) < T_R^*(l,k) \text{ for } k \leq i, l \leq j ;$$

$$T_B^*(1,1) < T_B^*(k,l) \text{ for } 2 \leq k \leq i, 2 \leq l \leq j;$$

$$T_B^*(1,1) < T_B^*(1,l) \text{ for } 2 \leq l \leq j;$$

$$T_B^*(1,1) < T_B^*(k,1) \text{ for } 2 \leq k \leq i\}$$

$$= \frac{k_1}{[ijk_2 + (i-1)(j-1)k_1 + (j-1)k_1 + (i-1)k_1] + k_1}$$

$$= \frac{k_1}{ijk_2 + k_1[(i-1)(j-1) + (j-1) + (i-1) + 1]}$$

$$= \frac{k_1}{ij(k_1 + k_2)} .$$

Thus

$$P^{(i,j)} \{(B_{V+}, R_{V+}) = (i, j-1)\} = ij \left( \frac{k_1}{ij(k_1 + k_2)} \right)$$

$$= \frac{k_1}{k_1 + k_2} ,$$

and by (28) the complementary event  $\{(B_{V+}, R_{V+}) = (i-1, j)\}$  has  $P^{(i,j)}$ -probability given as follows:

$$\begin{aligned}
P^{(i,j)}\{B_{V+}, R_{V+} = (i-1, j)\} \\
&= 1 - P^{(i,j)}\{B_{V+}, R_{V+} = (i, j-1)\} \\
&= 1 - \frac{k_1}{k_1 + k_2} \\
&= \frac{k_2}{k_1 + k_2} \cdot
\end{aligned}$$

□

## L2 Homogeneous Linear Law Process With Engagements

To study this process we must introduce further notation, which will be used only here. Let

$$E = \mathbb{N} \times \mathbb{N} \times \mathbb{N} ,$$

the state space of this process, be endowed with the discrete  $\sigma$ -algebra  $\underline{E}$  and the discrete topology. We take as the sample space  $\Omega$  the set of mappings  $\omega = (\omega_1, \omega_2, \omega_3)$  of  $[0, \infty)$  into the state space  $E$  which are right continuous and possess left-hand limits everywhere (with respect to the discrete topology on  $E$ ).

For  $t \geq 0$  define

$$B_t^*(\omega) = \omega_1(t)$$

$$R_t^*(\omega) = \omega_2(t)$$

and

$$D_t(\omega) = \omega_3(t) .$$

We begin discussion of the process with binary engagements with a model of a three-component process in which  $B_t^*$  denotes the number of unengaged Blue weapons surviving at time  $t$ ,  $R_t^*$  the number of Red weapons surviving and unengaged at time  $t$  and  $D_t$  the number of binary engagements (duels) going on at time  $t$ . Since only binary engagements are permitted,  $D_t$  is then also the number of surviving and engaged Blue weapons at time  $t$ , which necessarily is also the number of Red weapons surviving at time  $t$  but engaged at that instant.

The history of the process up until time  $t$  is again given by

$$\underline{E}_t = \sigma((B_u^*, R_u^*, D_u): 0 \leq u \leq t)$$

and the entire history by

$$\underline{E} = \sigma((B_u^*, R_u^*, D_u): u \geq 0) .$$

Let  $P^{(i,j,k)}$  denote the probability law on  $(\Omega, \underline{F})$  of the three component process  $((B_t^*, R_t^*, D_t))_{t \geq 0}$  governed by the assumptions of Process L2, subject to the initial conditions

$$\begin{aligned} B_0^* &= i && \text{initially unengaged Blue survivors} \\ R_0^* &= j && \text{initially unengaged Red survivors} \\ D_0 &= k && \text{engagements initially in progress .} \end{aligned}$$

For most applications one would have  $k = 0$ .

(29) THEOREM. Under the family of assumptions for Process L2, the collection

$$(\Omega, \underline{F}, \underline{F}_t, (B_t^*, R_t^*, D_t), P^{(i,j,k)})$$

is a regular step process with

- a) state space  $E = \mathbb{N}^3$  ;
- b) jump function  $\lambda$  given by

$$\lambda(i,j,k) = ku + ij(s_1 + s_2) ;$$

- c) transition kernel  $P$  given by

$$P((i,j,k), (i + 1, j, k - 1)) = \frac{kp_2u}{ku + ij(s_1 + s_2)}$$

$$P((i,j,k), (i, j + 1, k - 1)) = \frac{kp_1u}{ku + ij(s_1 + s_2)}$$

$$P((i,j,k), (i + 1, j + 1, k - 1)) = \frac{kp_3u}{ku + ij(s_1 + s_2)}$$

$$P((i,j,k), (i - 1, j - 1, k + 1)) = \frac{ij(s_1 + s_2)}{ku + ij(s_1 + s_2)} ;$$

d) infinitesimal generator  $Q$  given by

$$Q((i,j,k), (i+1, j, k-1)) = kp_2u$$

$$Q((i,j,k), (i, j+1, k-1)) = kp_1u$$

$$Q((i,j,k), (i+1, j+1, k-1)) = kp_3u$$

$$Q((i,j,k), (i-1, j-1, k+1)) = ij(s_1 + s_2)$$

$$Q((i,j,k), (i,j,k)) = - [ku + ij(s_1 + s_2)] .$$

PROOF. Denote by  $T_1$  the time at which a duel is first initiated and let  $T_2$  be the time at which the first initially ongoing duel terminates ( $= +\infty$  if there is no such duel). Then the first time at which the process changes state is given by

$$V = \min\{T_1, T_2\} .$$

From the proof of Theorem (26) and assumption 7 we see that with respect to the probability  $P^{(i,j,k)}$ ,  $T_1$  and  $T_2$  are independent and exponentially distributed with means  $1/ij(s_1 + s_2)$  and  $1/ku$ , respectively. Therefore

$$\begin{aligned} P^{(i,j,k)}\{V > t\} &= P^{(i,j,k)}\{T_1 > t, T_2 > t\} \\ &= P^{(i,j,k)}\{T_1 > t\}P^{(i,j,k)}\{T_2 > t\} \\ &= \exp[-ij(s_1 + s_2)t] \exp[-(ku)t] \\ &= \exp[-\lambda(i,j,k)t] . \end{aligned}$$

At time  $V$  exactly one of four events occurs:

- 1) One engagement terminates with the destruction of a Red weapon;
- 2) One engagement terminates with the destruction of a Blue weapon;
- 3) One engagement terminates with the destruction of neither combatant;
- 4) One new engagement is initiated.

If the current state is  $(i, j, k)$  then the new states reached in these four cases are

- 1)  $(i + 1, j, k - 1)$
- 2)  $(i, j + 1, k - 1)$
- 3)  $(i + 1, j + 1, k - 1)$
- 4)  $(i - 1, j - 1, k + 1)$ ,

respectively.

By now familiar computations,

$$\begin{aligned}
 P^{(i,j,k)}\{(B_{V+}^*, R_{V+}^*, D_{V+}^*) = (i + 1, j, k - 1)\} \\
 &= P^{(i,j,k)}\{T_2 < T_1, \text{ outcome of terminated duel is} \\
 &\quad \text{destroyed Red weapon}\} \\
 &= P^{(i,j,k)}\{T_2 < T_1\} \times \\
 &\quad P^{(i,j,k)}\{\text{outcome of terminated duel is destroyed} \\
 &\quad \text{Red weapon}\} \\
 &= \frac{ku}{ku + ij(s_1 + s_2)} P_2 \cdot
 \end{aligned}$$

Similarly,

$$P^{(i,j,k)}\{(B_{V+}^*, R_{V+}^*, D_{V+}^*) = (i, j + 1, k - 1)\}$$

$$\begin{aligned}
&= P^{(i,j,k)} \{T_2 < T_1, \text{outcome of terminated duel is destroyed} \\
&\quad \text{Blue weapon}\} \\
&= P^{(i,j,k)} \{T_2 < T_1\} \times \\
&\quad P^{(i,j,k)} \{\text{outcome is destroyed Blue weapon}\} \\
&= \frac{ku}{ku + ij(s_1 + s_2)} P_1,
\end{aligned}$$

and

$$\begin{aligned}
&P^{(i,j,k)} \{(B_{V+}^*, R_{V+}^*, D_{V+}) = (i + 1, j + 1, k - 1)\} \\
&= P^{(i,j,k)} \{T_2 < T_1, \text{outcome of terminated duel is destruction} \\
&\quad \text{of neither weapon}\} \\
&= P^{(i,j,k)} \{T_2 < T_1\} \times \\
&\quad P^{(i,j,k)} \{\text{outcome is mutual survival}\} \\
&= \frac{ku}{ku + ij(s_1 + s_2)} P_3.
\end{aligned}$$

Finally,

$$\begin{aligned}
&P^{(i,j,k)} \{(B_{V+}^*, R_{V+}^*, D_{V+}) = (i - 1, j - 1, k + 1)\} \\
&= P^{(i,j,k)} \{T_1 < T_2\} \\
&= \frac{ij(s_1 + s_2)}{ku + ij(s_1 + s_2)}.
\end{aligned}$$

Standard arguments now complete the proof. □

Consider now the two-dimensional process  $((B_t, R_t))_{t \geq 0}$  of total numbers of survivors defined by

$$B_t = B_t^* + D_t$$

and

$$R_t = R_t^* + D_t .$$

Unfortunately,  $((B_t, R_t))_{t \geq 0}$  is not a Markov process with respect to the  $\sigma$ -algebras  $(\underline{F}_t)_{t \geq 0}$  because for each  $t$  the history  $\underline{F}_t$  of the process up until time  $t$  contains information about the process, namely the components  $B_t^*$ ,  $R_t^*$  and  $D_t$ , which is not contained in the pair  $(B_t, R_t)$ .

Functions of Markov process have been studied but mostly in terms of limiting behavior (e.g., Laws of Large Numbers and Central Limit Theorems, cf. Chung (1967)) and identifiability problems (cf. Rosenblatt (1971)) and hardly at all in terms of transient behavior and computation of distributions. Further research is desirable.

We can make some simple remarks about computations. If for each  $t$  we know

$$P_t((i,j,k), (l,m,n)) = P^{(i,j,k)}\{(B_t^*, R_t^*, D_t) = (l,m,n)\}$$

for all  $(i, j, k)$  and  $(l, m, n)$ , then we can compute

$$\begin{aligned} P^{(i,j,k)}\{B_t = \mu\} \\ = \sum_m \sum_{n+l = \mu} P_t((i,j,k), (l,m,n)) . \end{aligned}$$

This computation is not so hard as appears at first glance, since we know from the theory of regular step processes that

$$P_t((i,j,k), (l,m,n)) = [\exp (tQ)] ((i,j,k), (l,m,n)),$$

and hence can be approximated as

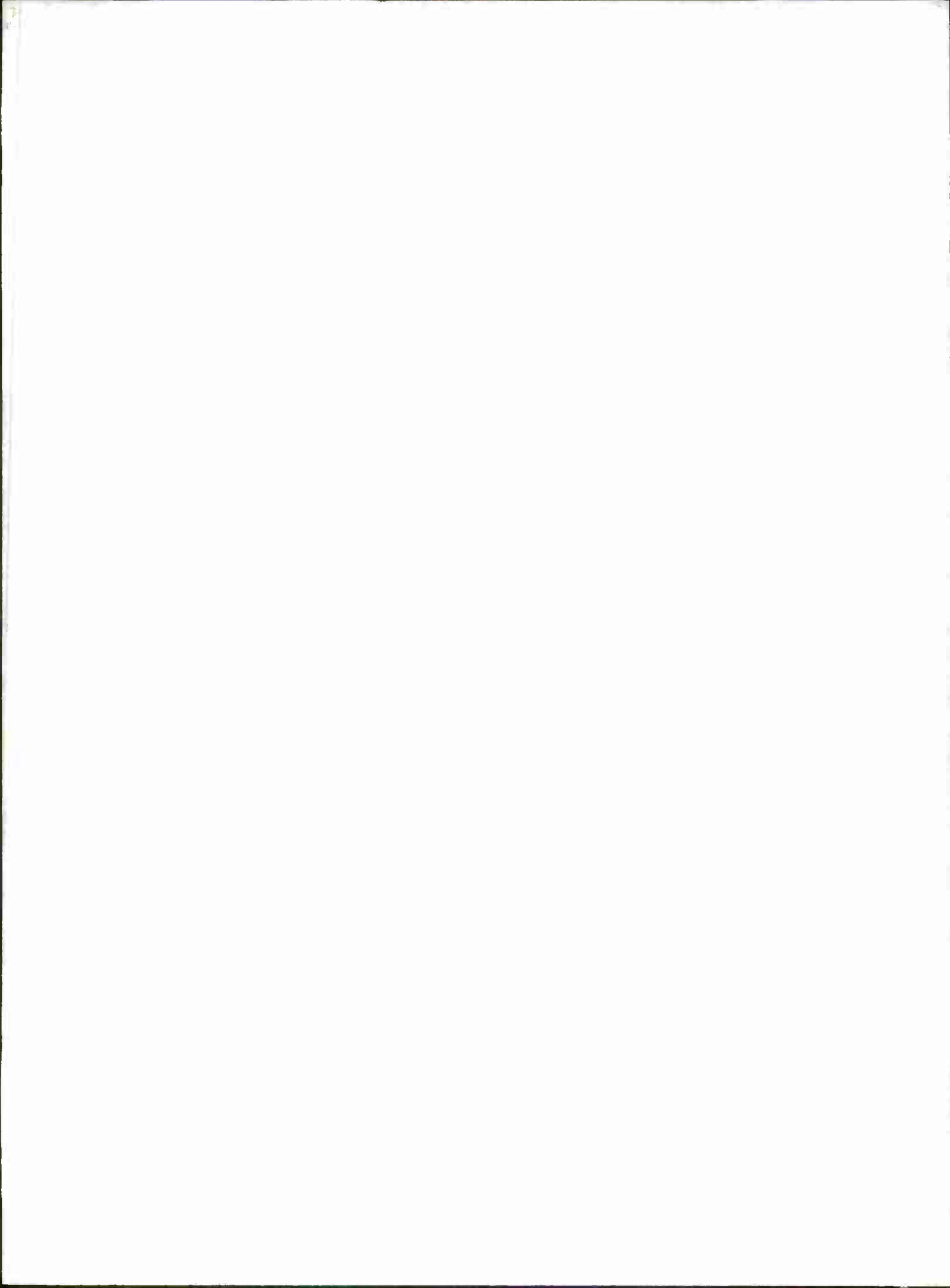
$$P_t((i,j,k), (l,m,n)) \sim \sum_{p=0}^M \frac{t^p}{p!} Q^p((i,j,k), (l,m,n)),$$

Since by linearity of expectation we have for each  $(i,j,k) \in E$

$$E^{(i,j,k)} [B_t] = E^{(i,j,k)} [B_t^*] + E^{(i,j,k)} [D_t]$$

any scheme (in particular, that discussed in Section IV of the paper) for approximating expectations of a regular step process also approximates expected numbers of survivors in this process.

To produce a good probabilistic characterization of the process  $((B_t, R_t))$  will take further research into the behavior and characterization of functions of Markov processes.



### L3 Heterogeneous Linear Law Process

Assume that there are  $M$  Blue weapon types and  $N$  Red weapon types. The state space  $E$  for the heterogeneous linear law process is then given by

$$E = \mathbb{N} \times \dots \times \mathbb{N} \text{ (} M + N \text{ times) ,}$$

along with the discrete  $\sigma$ -algebra  $\underline{E}$ . As sample space  $\Omega$  we have the family of functions from  $[0, \infty)$  into  $E$  which are right-continuous and have left-hand limits everywhere. For  $\omega = (\omega_1, \dots, \omega_M, \omega'_1, \dots, \omega'_N)$ , where  $\omega_1, \dots, \omega_M, \omega'_1, \dots, \omega'_N$  are mappings of  $[0, \infty)$  into  $\mathbb{N}$  which are right-continuous and have left-hand limits, with respect to the discrete topology on  $\mathbb{N}$ , and  $t \geq 0$ , we define

$$\begin{aligned} \underline{B}_t(\omega) &= (\omega_1(t), \dots, \omega_M(t)) \\ \underline{R}_t(\omega) &= (\omega'_1(t), \dots, \omega'_N(t)) \\ B_t^i(\omega) &= \omega_i(t) , \quad i = 1, \dots, M , \\ R_t^j(\omega) &= \omega'_j(t) , \quad j = 1, \dots, N . \end{aligned}$$

$B_t^i$  is the number of type  $i$  Blue weapons surviving at time  $t$ ,  $\underline{B}_t = (B_t^1, \dots, B_t^M)$  the Blue force surviving at time  $t$ ,  $R_t^j$  the number of type  $j$  Red weapons surviving at time  $t$  and  $\underline{R}_t$  the surviving Red force at time  $t$ .

The history of the attrition process up until time  $t$  is

$$\underline{F}_t = \sigma((\underline{B}_s, \underline{R}_s) : 0 \leq s \leq t)$$

and the entire history is

$$\underline{F} = \sigma((\underline{B}_s, \underline{R}_s) : s \geq 0) .$$

Points in  $E$  are written in the form  $(x,y)$  with  $x \in \underline{N}^M$  and  $x_i$  denoting the Blue type  $i$  weapons and  $y \in \underline{N}^N$  with  $y_j$  the number of type  $j$  Red weapons.

For each  $(x,y) \in E$  let  $P^{(x,y)}$  be the probability law on  $(\Omega, \underline{E})$  of the attrition process governed by the family of hypotheses of Process L3, subject to the initial conditions

$$\underline{B}_0 = x$$

and

$$\underline{R}_0 = y .$$

(30) THEOREM. Under the assumptions of Process L2, the process

$$(\Omega, \underline{E}, \underline{E}_t, (\underline{B}_t, \underline{R}_t), P^{(x,y)})$$

is a regular step process with

- a) state space  $(E, \underline{E})$  as defined above;
- b) jump function  $\lambda$  given by

$$(31) \quad \lambda(x,y) = \sum_{i=1}^M \sum_{j=1}^N x_i y_j [k_1(j,i) + k_2(i,j)]$$

where  $k_q(\mu, \nu) = s_q(\mu, \nu) p_q(\mu, \nu)$  for  $q = 1, 2$  and appropriate values of  $\mu$  and  $\nu$ ;

- c) transition kernel  $P$  given by

$$(32) \quad P((x,y); (x; y_1, \dots, y_{j-1}, \dots, y_N)) = \frac{y_j \sum_{i=1}^M k_1(j,i) x_i}{\lambda(x,y)}$$

$$P((x,y); (x_1, \dots, x_{i-1}, \dots, x_M; y)) = \frac{x_i \sum_{j=1}^N k_2(i,j) y_j}{\lambda(x,y)} ;$$

d) infinitesimal generator  $Q$  given by

$$Q((x,y), (x; y_1, \dots, y_j - 1, \dots, y_N)) = y_j \sum_{i=1}^M k_1(j,i)x_i$$

$$Q((x,y), (x,y)) = - \sum_{i=1}^M \sum_{j=1}^N x_i y_j (k_1(j,i) + k_2(i,j))$$

$$Q((x,y), (x_1, \dots, x_i - 1, \dots, x_M; y)) = x_i \sum_{j=1}^N k_2(i,j)y_j .$$

PROOF. We follow the pattern established in the proof of Theorem (26), only now the notation is more complicated.

Denote by  $T_B^*(i,j; k, \ell)$  the time at which the  $k^{\text{th}}$  Blue weapon of type  $i$  first hypothetically detects and kills the  $\ell^{\text{th}}$  Red weapon of type  $j$  and by  $T_R^*(j,i; \ell, k)$  the time at which the first hypothetical kill of that specific Blue weapon by the same particular Red weapon occurs. With respect to the probability law  $P^{(x,y)}$ , for each  $i$  and  $j$  the random variables  $T_B^*(i,j; 1,1), T_B^*(i,j; 1,2), \dots, T_B^*(i,j; 1,y_j), T_B^*(i,j; 2,1), \dots, T_B^*(i,j; 2,y_j), \dots, T_B^*(i,j; x_i,1), \dots, T_B^*(i,j; x_i,y_j)$  are independent and identically exponentially distributed with mean  $1/k_1(j,i)$ . Similarly,  $T_R^*(j,i; 1,1), \dots, T_R^*(j,i; 1,x_i), \dots, T_R^*(j,i; y_j,1), \dots, T_R^*(j,i; y_j, x_i)$  are independent and identically exponentially distributed with mean  $1/k_2(i,j)$ . These two families of random variables are independent with respect to  $P^{(x,y)}$  and as the weapon types  $i$  and  $j$  vary all families of random variables so obtained are independent.

With respect to  $P^{(x,y)}$  the time

$$V^* = \min_{\substack{i=1, \dots, M \\ j=1, \dots, N}} \min_{\substack{k \leq x_i \\ \ell \leq y_j}} \min \{ T_B^*(i,j; k, \ell); T_R^*(j,i; \ell, k) \}$$

of the first hypothetical fatality and the time

$$V = \inf\{t: (\underline{B}_t, \underline{R}_t) \neq (\underline{B}_0, \underline{R}_0)\}$$

at which the first actual fatality occurs are equal almost surely.

By the independence hypothesis 5, we have

$$\begin{aligned} & P^{(x,y)}\{V^* > t\} \\ &= P^{(x,y)}\{T_B^*(i,j; k, \ell) > t, T_R^*(j,i; \ell, k) > t \text{ for all} \\ &\quad \text{appropriate } i, j, k, \ell\} \\ &= \prod_{i=1}^M \prod_{j=1}^N P^{(x,y)}\{T_B^*(i,j; k, \ell) > t \text{ for all } k \leq x_i, \ell \leq y_j\} \\ &\quad \times \prod_{j=1}^N \prod_{i=1}^M P^{(x,y)}\{T_R^*(j,i; \ell, k) > t \text{ for } k \leq x_i, \ell \leq y_j\} \\ &= \prod_{i=1}^M \prod_{j=1}^N \prod_{k=1}^{x_i} \prod_{\ell=1}^{y_j} P^{(x,y)}\{T_B^*(i,j; k, \ell) > t\} \\ &\quad \times \prod_{j=1}^N \prod_{i=1}^M \prod_{\ell=1}^{y_j} \prod_{k=1}^{x_i} P^{(x,y)}\{T_R^*(j,i; \ell, k) > t\} \\ &= \prod_{i=1}^M \prod_{j=1}^N \prod_{k=1}^{x_i} \prod_{\ell=1}^{y_j} \exp[-k_1(j,i)t] \\ &\quad \times \prod_{j=1}^N \prod_{i=1}^M \prod_{\ell=1}^{y_j} \prod_{k=1}^{x_i} \exp[-k_2(i,j)t] \\ &= \prod_{i=1}^M \prod_{j=1}^N \exp[-x_i y_j k_1(j,i)t] \\ &\quad \times \prod_{j=1}^N \prod_{i=1}^M \exp[-y_j x_i k_2(i,j)t] \end{aligned}$$

$$= \exp \left[ -t \sum_{i=1}^M \sum_{j=1}^N x_i y_j (k_1(j,i) + k_2(i,j)) \right]$$

$$= \exp \left[ -\lambda(x,y)t \right],$$

where  $\lambda$  is defined by (31), so that

$$P^{(x,y)}\{V > t\} = \exp \left[ -\lambda(x,y)t \right]$$

as required.

Since almost surely with respect to  $P^{(x,y)}$  the  $T_B^*(i,j; k, \ell)$  and  $T_R^*(j,i; \ell, k)$  are all distinct from one another, it follows first of all that

$$P^{(x,y)}\left\{ \left( \underline{B}_{V^+}, \underline{R}_{V^+} \right) \in \left\{ (x; y_1^{-1}, y_2, \dots, y_N), \dots, \right. \right. \\ \left. \left. (x; y_1, \dots, y_N^{-1}), (x_1^{-1}, \dots, y_M; y), \right. \right. \\ \left. \left. \dots, (x_1, \dots, x_M^{-1}; y) \right\} \right\} = 1.$$

For a fixed  $j$  we have

$$P^{(x,y)}\left\{ \left( \underline{B}_{V^+}, \underline{R}_{V^+} \right) = (x; y_1, \dots, y_j^{-1}, \dots, y_N) \right\} \\ = P^{(x,y)}\{V^* = T_B^*(i,j; k, \ell) \text{ for some } i, k \leq x_i; \text{ and } \ell \leq y_j\} \\ = \sum_{i=1}^M \sum_{k=1}^{x_i} \sum_{\ell=1}^{y_j} P^{(x,y)}\{V^* = T_B^*(i,j; k, \ell)\} \\ = y_j \sum_{i=1}^M x_i P^{(x,y)}\{V^* = T_B^*(i,j; 1, 1)\}.$$

But

$$P^{(x,y)}\{V^* = T_B^*(i,j; 1, 1)\}$$

$$\begin{aligned}
&= P^{(x,y)}\{T_B^*(i,j; 1,1) < T_R^*(j,i; \ell,k) \text{ for all } k, \ell; \\
&\quad T_B^*(i,j; 1,1) < T_R^*(n,m; \ell,k) \text{ for } m \neq i, n \neq j, \text{ all } k, \ell; \\
&\quad T_B^*(i,j; 1,1) < T_B^*(m,n; k,\ell) \text{ for } m \neq i, n \neq j, \text{ all } k, \ell; \\
&\quad T_B^*(i,j; 1,1) < T_B^*(i,j; p,q) \text{ for } p > 1 \text{ or } q > 1\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{k_1(j,i)}{x_i y_j k_2(i,j) + \sum_{m \neq i} \sum_{n \neq j} x_m y_n [k_2(m,n) + k_1(n,m)] + x_i y_j k_1(j,i)} \\
&= P((x,y); (x; y_1, \dots, y_{j-1}, \dots, y_N))
\end{aligned}$$

where  $P(\cdot, \cdot)$  is defined by (32). The proof that

$$\begin{aligned}
&P^{(x,y)}\{(B_{V_t^+}, R_{V_t^+}) = (x_1, \dots, x_{i-1}, \dots, x_M; y)\} \\
&= P((x,y), (x_1, \dots, x_{i-1}, \dots, x_M; y))
\end{aligned}$$

for each  $i$  is entirely analogous and is hence omitted.

The proof is complete upon application of standard arguments based on the "memoryless" property of the exponential distribution which extend the preceding computations to show that if one defines for each  $t$  the time

$$V_t = \inf\{s > t: (B_s, R_s) \neq (B_t, R_t)\}$$

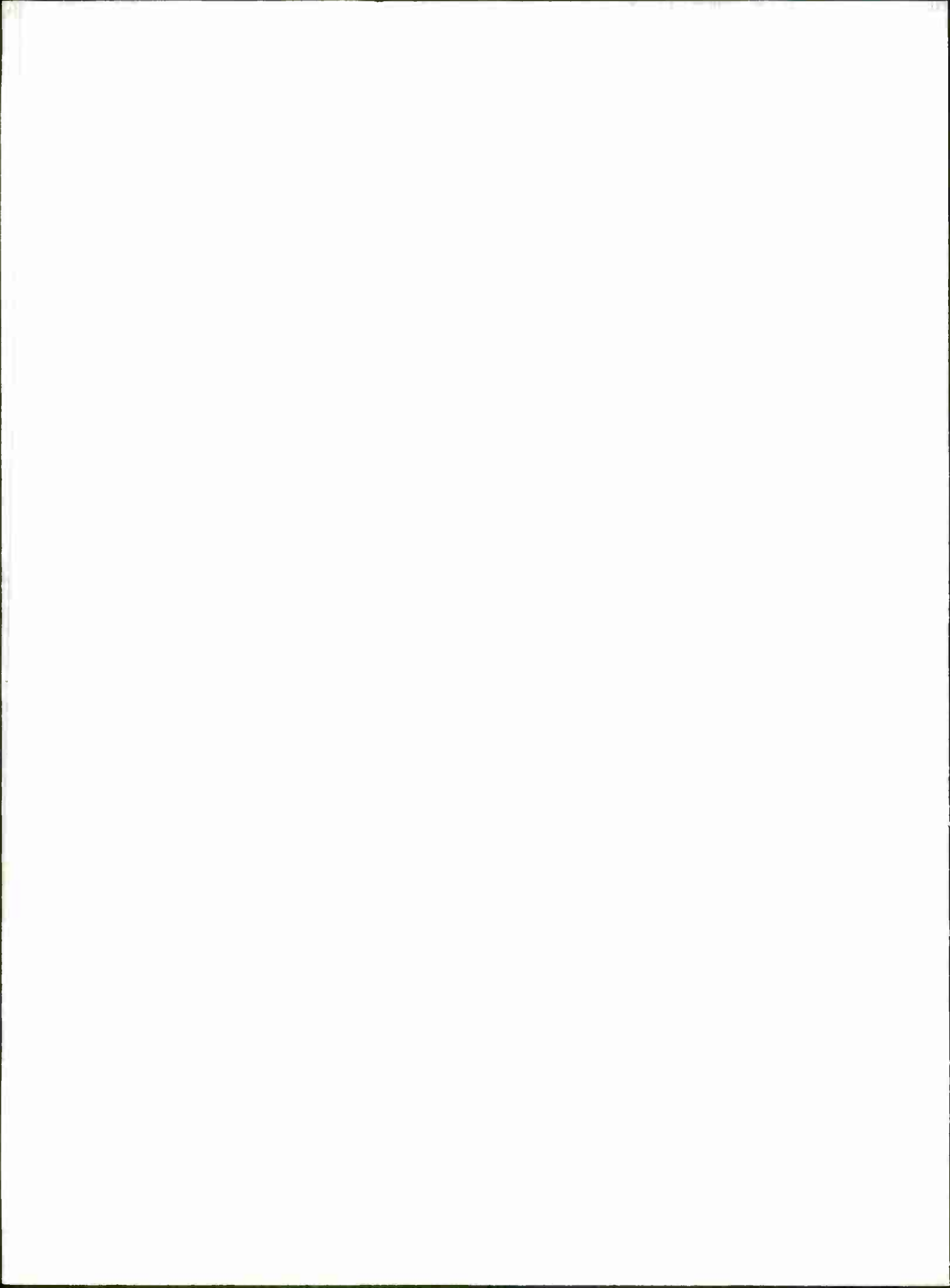
of the first transition of the process after time  $t$  then

$$P^{(x,y)}\{V_t - t > u \mid \underline{E}_t\} = \exp[-\lambda(B_t, R_t)u]$$

and

$$P^{(x,y)}\{(B_{V_t^+}, R_{V_t^+}) = \alpha\} = P((B_t, R_t); \alpha)$$

for all  $(x,y)$  and  $\alpha \in E$ . From these two statements one concludes that  $(\Omega, \underline{E}, \underline{E}_t, (\underline{B}_t, \underline{B}_t), P^{(x,y)})$  is indeed a regular step process with jump function  $\lambda$  and transition kernel  $P$ . □



## M1 Heterogeneous Mixed Law Process

For simplicity and clarity we assume that each side possesses exactly two types of weapons, one of which obeys assumptions of weapons in square law models and the other of which behaves according to the assumptions of the linear law models. Our convention is that Type 1 weapons satisfy square law assumptions and Type 2 weapons linear law assumptions.

The state space becomes  $E = \mathbb{N}^4$  with, as usual, the discrete  $\sigma$ -algebra  $\underline{E}$ . As sample space  $\Omega$  we have the family of right-continuous functions from  $[0, \infty)$  into  $E$  which everywhere have left-hand limits. For  $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega$  we define

$$\begin{aligned} B_t^1(\omega) &= \omega_1(t) \\ B_t^2(\omega) &= \omega_2(t) \\ R_t^1(\omega) &= \omega_3(t) \\ R_t^2(\omega) &= \omega_4(t) . \end{aligned}$$

For  $q = 1, 2$ ,  $B_t^q$  (resp.,  $R_t^q$ ) is the number of type  $q$  Blue (resp., Red) weapons surviving at time  $t$ . We further define

$$\underline{B}_t = (B_t^1, B_t^2)$$

and

$$\underline{R}_t = (R_t^1, R_t^2)$$

which are the surviving Blue and Red forces, respectively, at time  $t$ .

As usual the history of the process until time  $t$  is

$$\underline{E}_t = \sigma(\underline{B}_s, \underline{R}_s): 0 \leq s \leq t)$$

and the entire history is

$$\underline{E} = \sigma((B_s, R_s): s \geq 0).$$

Denote by  $P^{(i,j,k,\ell)}$  the probability law on the measurable space  $(\Omega, \underline{E})$  of the attrition process governed by the family of eight assumptions of the Process M1, conditioned on the event

$$\{B_0^1 = i, B_0^2 = j, R_0^1 = k, R_0^2 = \ell\}.$$

(33) THEOREM. Under the assumptions of Process M1,

$$(\Omega, \underline{E}, \underline{E}_t, (B_t, R_t), P^{(i,j,k,\ell)})$$

is a regular step process with

a) state space  $(E, \underline{E})$ , as defined above;

b) jump function  $\lambda$  given for nonabsorbing states [that is, states  $(m,n,p,q)$  such that  $m + n > 0$  and  $p + q > 0$ ] by

$$\begin{aligned} \lambda(i,j,k,\ell) = & ir_1 \left( \frac{kp_1(1) + \ell p_1(2)}{k + \ell} \right) \\ & + j(ks_1(1)q_1(1) + \ell s_1(2)q_1(2)) \\ & + kr_2 \left( \frac{ip_2(1) + jp_2(2)}{i + j} \right) \\ & + \ell(is_2(1)q_2(1) + js_2(2)q_2(2)); \end{aligned}$$

c) transition kernel  $P$  given by

$$P((i,j,k,\ell); (i,j, k-1, \ell)) = \frac{ir_1 p_1(1) \frac{k}{k+\ell} + jk \cdot s_1(1)q_1(1)}{\lambda(i,j,k,\ell)}$$

$$P((i,j,k,\ell); (i,j; k, \ell-1)) = \frac{ir_1 p_1(2) \frac{\ell}{\ell+k} + j\ell \cdot s_1(2)q_1(2)}{\lambda(i,j,k,\ell)}$$

$$P((i,j,k,l); (i-1,j,k,l)) = \frac{kr_2 p_2(1) \frac{i}{i+j} + li \cdot s_2(1)q_2(1)}{\lambda(i,j,k,l)}$$

$$P((i,j,k,l); (i,j-1,k,l)) = \frac{lr_2 p_2(2) \frac{j}{i+j} + lj \cdot s_2(2)q_2(2)}{\lambda(i,j,k,l)} ;$$

d) infinitesimal generator  $Q$  given for nonabsorbing states  $(i,j,k,l)$  by

$$Q((i,j,k,l); (i,j,k-1,l)) = ir_1 p_1(1) \frac{k}{k+l} + jk \cdot s_1(1)q_1(1)$$

$$Q((i,j,k,l); (i,j,k,l-1)) = ir_1 p_1(2) \frac{l}{k+l} + jl \cdot s_1(2)q_1(2)$$

$$Q((i,j,k,l); (i,j,k,l)) = - \left\{ ir_1 \left( \frac{kp_1(1) + lp_1(2)}{k+l} \right) \right. \\ \left. + j(ks_1(1)q_1(1) + ls_1(2)q_1(2)) \right. \\ \left. + kr_2 \left( \frac{ip_2(1) + jp_2(2)}{i+j} \right) \right. \\ \left. + l(is_2(1)q_2(1) + js_2(2)q_2(2)) \right\}$$

$$Q((i,j,k,l); (i-1,j,k,l)) = kr_2 p_2(1) \frac{i}{i+j} + li \cdot s_2(1)q_2(1)$$

$$Q((i,j,k,l); (i,j-1,k,l)) = lr_2 p_2(2) \frac{j}{i+j} + lj \cdot s_2(2)q_2(2) .$$

If state  $(i,j,k,l)$  is absorbing then  $Q((i,j,k,l), \alpha) = 0$  for all  $\alpha \in E$ .

PROOF. For one last time we resort to the fiction of hypothetical kills.

Denote by  $T_B^*(m,n)$  the time at which some Red type  $n$  weapon is first hypothetically killed by some Blue type  $m$  weapon and by

$T_R^*(n,m)$  the time at which some Blue type  $m$  weapon is first hypothetically killed by some type  $n$  Red weapon, for  $m, n = 1, 2$ .

Hypothetical shots fired by a type 1 Blue weapon form a Poisson process with rate  $r_1$ , so such shots directed at type 1 and type 2 Red weapons form independent Poisson processes with rates  $r_1 \frac{k}{k+l}$  and  $r_1 \frac{l}{k+l}$ , respectively, relative, of course, to the probability measure  $P^{(i,j,k,l)}$ . Hypothetical fatal shots fired by this particular Blue type 1 weapon against type 1 and type 2 Red weapons thus form independent Poisson processes with respective rates  $r_1 p_1(1) \frac{k}{k+l}$  and  $r_1 p_1(2) \frac{l}{k+l}$  and hence by assumption 8 and the Superposition Theorem for Poisson processes it follows that

$$(34) \quad \begin{aligned} P^{(i,j,k,l)}\{T_B^*(1,1) > t\} &= \exp\left[-t\left(r_1 p_1(1) \frac{k}{k+l}\right)\right] \\ P^{(i,j,k,l)}\{T_B^*(1,2) > t\} &= \exp\left[-t\left(r_1 p_1(2) \frac{l}{k+l}\right)\right]. \end{aligned}$$

Entirely analogously we have

$$(35) \quad \begin{aligned} P^{(i,j,k,l)}\{T_R^*(1,1) > t\} &= \exp\left[-t\left(k r_2 p_2(1) \frac{i}{i+j}\right)\right] \\ P^{(i,j,k,l)}\{T_R^*(1,2) > t\} &= \exp\left[-t\left(k r_2 p_2(2) \frac{j}{i+j}\right)\right]. \end{aligned}$$

Hypothetical detections of a particular type  $n$  Red weapon by a particular type 2 Blue weapon form a Poisson process with rate  $s_1(n)$ , and hypothetical fatal shots by the same Blue weapon at the same Red weapon a Poisson process with rate  $s_1(n)q_1(n)$  from which one obtains by the independence assumption 8 that

$$(36) \quad \begin{aligned} P^{(i,j,k,l)}\{T_B^*(2,1) > t\} &= \exp\left[-t\left(j k s_1(1) q_1(1)\right)\right] \\ P^{(i,j,k,l)}\{T_B^*(2,2) > t\} &= \exp\left[-t\left(j l s_1(2) q_1(2)\right)\right]. \end{aligned}$$

Similarly

$$P^{(i,j,k,l)}\{T_R^*(2,1) > t\} = \exp[-t(\ell i s_2(1)q_2(1))] \quad (37)$$

$$P^{(i,j,k,l)}\{T_R^*(2,2) > t\} = \exp[-t(\ell j s_2(2)q_2(2))].$$

But the independence assumption 8 implies that with respect to  $P^{(i,j,k,l)}$  the random variables  $T_B^*(m,n)$ ,  $m, n = 1, 2$  and  $T_R^*(n,m)$ ,  $n, m = 1, 2$  are mutually independent. The minimum  $V^*$  of these eight random variables is, moreover, equal with probability one to the time

$$V = \inf\{t: (\underline{B}_t, \underline{R}_t) \neq (\underline{B}_0, \underline{R}_0)\}.$$

Therefore, by (34), (35), (36), and (37)

$$\begin{aligned} P^{(i,j,k,l)}\{V > t\} &= \prod_{m=1}^2 \prod_{n=1}^2 P^{(i,j,k,l)}\{T_B^*(m,n) > t\} P^{(i,j,k,l)}\{T_R^*(n,m) > t\} \\ &= \exp \left[ -t(ir_1 p_1(1) \frac{k}{k+l} + ir_2 p_1(2) \frac{l}{l+k} \right. \\ &\quad \left. + kr_2 p_2(1) \frac{i}{i+j} + kr_2 p_2(2) \frac{j}{i+j} \right. \\ &\quad \left. + jk \cdot s_1(1)q_1(1) + jl \cdot s_1(2)q_1(2) \right. \\ &\quad \left. + \ell i \cdot s_2(1)q_2(1) + \ell j \cdot s_2(2)q_2(2) \right] \\ &= \exp [-t\lambda(i,j,k,l)] \end{aligned}$$

by a simple rearrangement of terms.

With probability one the eight aforementioned random variables are all distinct from one another. We then have, for example,

$$\begin{aligned}
& P^{(i,j,k,\ell)} \{ (B_{V+}, R_{V+}) = (i,j,k-1,\ell) \} \\
&= P^{(i,j,k,\ell)} \{ v^* = T_B^*(1,1) \text{ or } v^* = T_B^*(2,1) \} \\
&= P^{(i,j,k,\ell)} \{ v^* = T_B^*(1,1) \} + P^{(i,j,k,\ell)} \{ v^* = T_B^*(2,1) \} \\
&= P^{(i,j,k,\ell)} \{ T_B^*(1,1) < \min \{ T_B^*(1,2), T_B^*(2,1), \\
&\quad T_B^*(2,2), T_R^*(1,1), \\
&\quad T_R^*(1,2), T_R^*(2,1), \\
&\quad T_R^*(2,2) \} \} \\
&\quad + P^{(i,j,k,\ell)} \{ T_B^*(2,1) < \min \{ T_B^*(1,1), T_B^*(1,2), \\
&\quad T_B^*(2,2), T_R^*(1,1), \\
&\quad T_R^*(1,2), T_R^*(2,1), \\
&\quad T_R^*(2,2) \} \} \\
&= \frac{i r_1 p_1(1) \frac{k}{k+\ell}}{\lambda(i,j,k,\ell)} + \frac{j k \cdot s_i(1) q_1(1)}{\lambda(i,j,k,\ell)} ;
\end{aligned}$$

here we have omitted some long and unenlightening expressions occurring in the course of the computation.

One similarly checks that

$$\begin{aligned}
& P^{(i,j,k,\ell)} \{ (B_{V+}, R_{V+}) = (i,j,k,\ell-1) \} \\
&= \frac{i r_1 p_1(2) \frac{\ell}{\ell+k} + j \ell \cdot s_1(2) q_1(2)}{\lambda(i,j,k,\ell)} ,
\end{aligned}$$

that

$$\begin{aligned}
 p^{(i,j,k,\ell)} \{(\underline{B}_{V+}, \underline{R}_{V+}) = (i-1, j, k, \ell)\} \\
 = \frac{kr_2 p_2(1) \frac{i}{i+j} + i\ell \cdot s_2(1) q_2(1)}{\lambda(i,j,k,\ell)}
 \end{aligned}$$

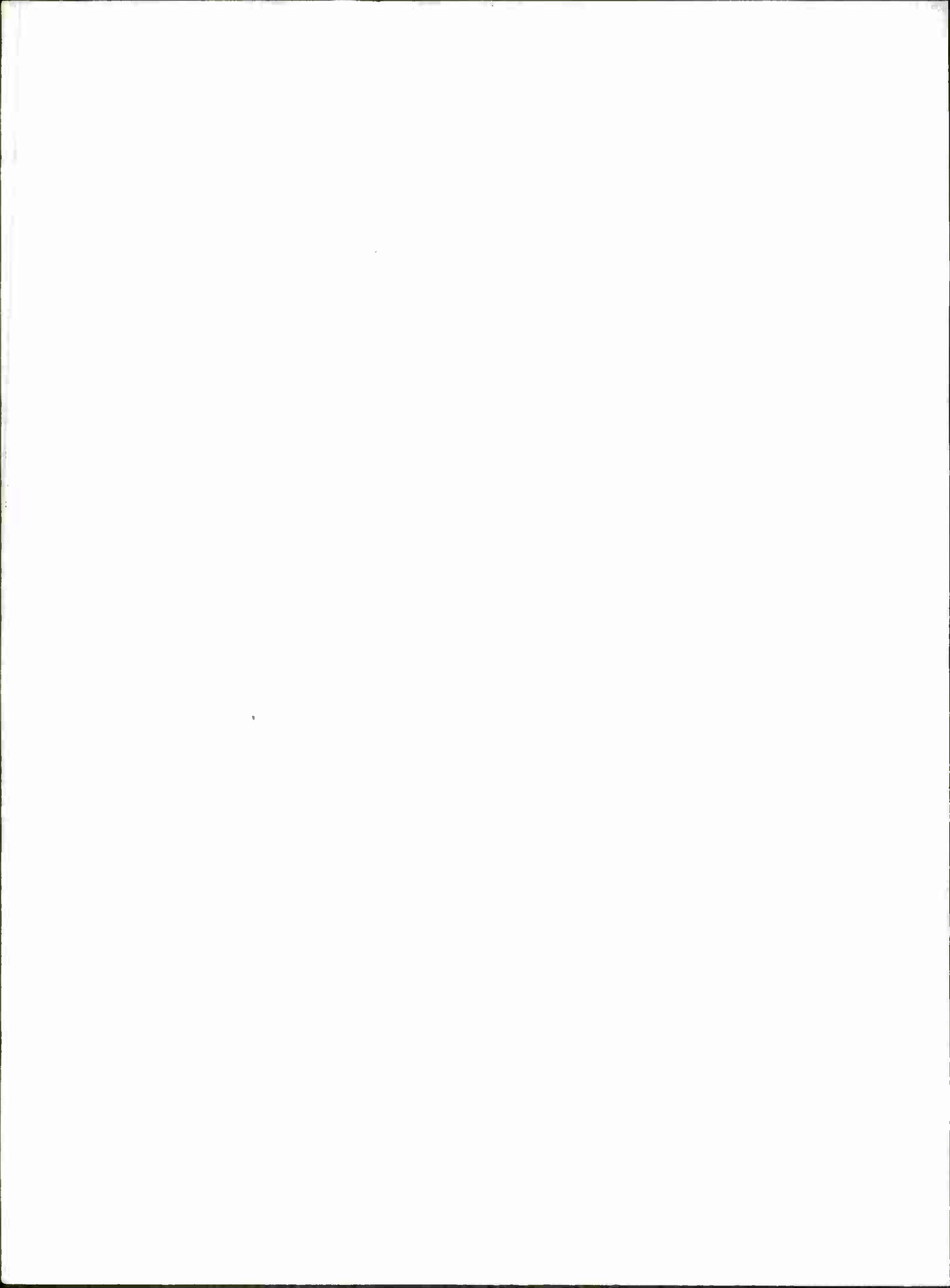
and, finally, that

$$\begin{aligned}
 p^{(i,j,k,\ell)} \{(\underline{B}_{V+}, \underline{R}_{V+}) = (i, j-1, k, \ell)\} \\
 = \frac{kr_2 p_2(2) \frac{j}{i+j} + j\ell \cdot s_2(2) q_2(2)}{\lambda(i,j,k,\ell)}.
 \end{aligned}$$

Hence in all cases we have

$$p^{(i,j,k,\ell)} \{(\underline{B}_{V+}, \underline{R}_{V+}) = \alpha\} = P((i,j,k,\ell); \alpha)$$

and the proof of the Theorem is complete following standard extensions based on the "memoryless" property of the exponential distribution.



M1a Homogeneous Mixed Law Process

We assume that the Blue side has only type 1 weapons and the Red side only type 2 weapons.

Consider the 4-dimensional Heterogeneous Mixed Law Process M1 derived above. With respect to a probability of the form  $P^{(i,0,0,l)}$ ,  $B_t^2$  and  $R_t^1$  are identically zero almost surely and can be disregarded. Now

$$\lambda(i,0,0,l) = ir_1 p_1(2) + il \cdot s_2(1)q_2(1)$$

while

$$P((i,0,0,l), (i,0,0,l-1)) = \frac{ir_1 p_1(2)}{\lambda(i,0,0,l)}$$

and

$$P((i,0,0,l), (i-1,0,0,l)) = \frac{il \cdot s_2(1)q_2(1)}{\lambda(i,0,0,l)} .$$

Hence we have the following result.

(38) THEOREM. Let the assumptions of Process M1 be satisfied. Then the stochastic process

$$(\Omega, \underline{F}, \underline{F}_t, (B_t^1, R_t^2), P^{(i,0,0,l)})$$

is equivalent--in the sense of being a Markov process with the same infinitesimal generator (and thus a regular step process with the same jump function and transition kernel)--to any regular step process with state space  $\underline{N} \times \underline{N}$ , jump function  $\hat{\lambda}$  given by

$$\hat{\lambda}(i,l) = ir_1 p_1(2) + il \cdot s_2(1)q_2(1)$$

and transition kernel  $\hat{P}$  given by

$$\hat{P}((i, \ell); (i, \ell-1)) = \frac{ir_1 p_1(2)}{\lambda(i, \ell)}$$

$$\hat{P}((i, \ell); (i-1, \ell)) = \frac{i\ell \cdot s_2(1)q_2(1)}{\lambda(i, \ell)} .$$

From Theorem (38) one trivially obtains our final Theorem.

(39) THEOREM. The attrition process governed by the family of assumptions of Process M1a is a regular step process with

a) state space  $\underline{N} \times \underline{N}$ ;

b) jump function  $\lambda$  given by

$$\lambda(i, j) = \begin{cases} irp + ijsq & \text{if } i > 0, j > 0 \\ 0 & \text{if } i = 0 \text{ or } j = 0 ; \end{cases}$$

c) transition kernel  $P$  given for states  $(i, j)$  with  $i > 0$  and  $j > 0$  by

$$P((i, j), (i, j-1)) = \frac{rp}{rp + jsq}$$

$$P((i, j), (i, j+1)) = \frac{jsq}{rp + jsq} ;$$

d) infinitesimal generator  $Q$  given for states  $(i, j)$  with  $i > 0$  and  $j > 0$  by

$$Q((i, j), (i, j-1)) = irp$$

$$Q((i, j), (i, j)) = - (irp + ijsq)$$

$$Q((i, j), (i-1, j)) = ijsq .$$

If  $i = 0$  or  $j = 0$ ,  $Q((i, j); (k, \ell)) = 0$  for all  $(k, \ell) \in E$ . □

The particularly simple form of the transition kernel  $P$  is noteworthy. Each column is constant; perhaps this structure can be exploited in computational applications.

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