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COMPLETE CLASSIFICATION OF (24,12)  
AND (22,11) SELF-DUAL CODES

Vera Pless, et al

Massachusetts Institute of Technology

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COMPLETE CLASSIFICATION OF (24,12) AND (22,11) SELF-DUAL CODES

by

Vera Pless\*

Project MAC, MIT, Cambridge, Massachusetts

and

N. J. A. Sloane

Bell Laboratories, Murray Hill, N. J.

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ABSTRACT

A complete classification is given of all  $[22, 11]$  and  $[24, 12]$  self-dual codes. For each code we give the order of its group, the number of codes equivalent to it, and its weight distribution. There is a unique  $[24, 12, 6]$  self-dual code. Several theorems on the enumeration of self-orthogonal codes are used, including formulas for the number of such codes with minimum distance  $\geq 4$ , and for the sum of the weight enumerators of all self-dual codes.

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1. Introduction

In spite of 25 years of research ([2], [31]), even the codes of only moderate length, up to 50 say, are a long way from being understood. Slepian [38] used Pólya's counting theorem to find the number of inequivalent codes of length  $n$  and dimension  $k$ . But the enumeration by length, dimension and minimum distance seems much more difficult. Some results on the enumeration of self-dual codes ( $C = C^\perp$ ) have been given in [24], [32], [33], [35]; and in [34] Pless has classified and enumerated all self-dual codes of length  $n \leq 20$ . In the present paper we first give several new general theorems (§3-§6) including a canonical form for self-orthogonal codes generated by codewords of weight 4 (Th. 7.5). We then apply these theorems to enumerate all self-dual codes of length 22 and 24 (§7, §8). For each code we give the order of its group, the number of codes equivalent to it, and its weight distribution. These codes provide 22 and 24 dimensional representations over  $GF(2)$  of their groups. There is a

unique self-dual code of length 24 and minimum distance 6; its group is a maximal subgroup of  $M_{24}$ .

The numbers of inequivalent codes are as follows.

|                      |   |   |   |   |    |    |    |    |    |    |    |    |
|----------------------|---|---|---|---|----|----|----|----|----|----|----|----|
| Length n             | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| Indecomposable codes | 1 | 0 | 0 | 1 | 0  | 1  | 1  | 2  | 2  | 6  | 8  | 26 |
| All Codes            | 1 | 1 | 1 | 2 | 2  | 3  | 4  | 7  | 9  | 16 | 25 | 55 |

If we require that the weights of codewords be divisible by 4, the corresponding numbers are:

|                      |   |    |    |
|----------------------|---|----|----|
| Length n             | 8 | 16 | 24 |
| Indecomposable codes | 1 | 1  | 7  |
| All Codes            | 1 | 2  | 9  |

The 9 codes of length 24 with weights divisible by 4 were first found by J. H. Conway (unpublished). Niemeier ([29], see also [28]) has found that there are  $2^4$  inequivalent even unimodular lattices in dimension 24, of which 9 correspond to these codes.

[34] also classifies  $[n, \frac{1}{2}(n-1)]$  self-orthogonal codes ( $C \subset C^\perp$ ) for  $n = 1, 3, \dots, 19$ . Although we have not classified the  $[21, 10]$  or  $[23, 11]$  self-orthogonal codes, Tables I, II would be of considerable help in doing so.

## §2. Terms from Coding Theory

For standard coding theory terms see [2], [31].

All codes are binary and linear. An  $[n, k, d]$  (or  $[n, k]$  for short) code has length  $n$ , dimension  $k$ , and (minimum) distance exactly  $d$ , and is a subspace of  $F^n$ , where  $F = \{0, 1\}$ .  $|u|$



The (symmetry) group  $\mathcal{G}(C)$  of  $C$  consists of all permutations of the coordinates which send codewords into codewords (i.e. fix  $C$  setwise).  $\mathcal{G}(C)$  is a subgroup of the symmetric group  $\mathcal{S}_n$ . E.g.  $\mathcal{G}(C_2)$  is  $Z_2$ , the cyclic group of order 2;  $\mathcal{G}(E_8)$  is the general affine group  $\mathcal{G}A_3(2)$  of order  $1344$  (all transformations  $\underline{y} \rightarrow \underline{x}A + \underline{b}$  where  $A$  is an invertible  $3 \times 3$  matrix); and  $\mathcal{G}(G_{24})$  is the Mathieu group  $M_{24}$  of order  $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ . There is an extensive literature on  $G_{24}$ ,  $M_{24}$ , and the associated Steiner system and Leech lattice - see references 1,3,7-10,15,16,19,21, 22, 29, 33, 39, 40, 42, 43.

Two codes  $C, C'$  are equivalent if there exists a permutation in  $\mathcal{S}_n$  sending  $C$  into  $C'$ . The size of the equivalence class containing  $C$  is  $n! / \text{order of } \mathcal{G}(C)$ .

The direct sum of codes  $C[n, k, d]$  and  $C'[n', k', d']$  is the  $[n+n', k+k', \min(d, d')]$  code  $C \oplus C' = \{(u_1 \dots u_n v_1 \dots v_{n'}) : (u_1 \dots u_n) \in C, (v_1 \dots v_{n'}) \in C'\}$ .  $C \oplus C$  will be written  $2C$ , etc. If  $D$  can be written  $C \oplus C'$  it is called decomposable, otherwise indecomposable ([38]).

If  $\mathcal{G}, \mathcal{H}$  are groups we write  $\mathcal{G} \times \mathcal{H}$  for their direct product,  $\mathcal{G}^k$  for  $\mathcal{G} \times \dots \times \mathcal{G}$  ( $k$  factors), and  $\mathcal{G} \rtimes \mathcal{H}$  for a semidirect product.

Lemma 2.3 If  $C = C_1 \oplus \dots \oplus C_k$  where the  $C_i$  are indecomposable and equivalent then  $\mathcal{G}(C) = \mathcal{G}(C_1)^k \cdot \mathcal{S}_k$

Lemma 2.4 Let  $C = D_1 \oplus \dots \oplus D_l$  where each  $D_i$  is a direct sum of equivalent codes, and for  $i \neq j$  no summand of  $D_i$  is equivalent to a summand of  $D_j$ . Then

$$G(C) = \prod_{i=1}^l G(D_i).$$

Let us say that a self-orthogonal code has property  $P(d, \delta)$  if it has minimum distance  $\geq d$  and all weights are divisible by  $\delta$ . Then it is worth mentioning that the number of indecomposable codes with property  $P(d, \delta)$  and the total number of all such codes are related by exactly the same Riddell-Gilbert formula ([6], [11], [12], [36 p. 147]) which relates the numbers of connected graphs and all graphs.

The weight distribution of  $C$  consists of the numbers  $\alpha_0, \dots, \alpha_n$  where  $\alpha_i$  is the number of codewords of weight  $i$ . The weight enumerator of  $C$  is the polynomial

$$\omega(C) = \omega(C; x) = \sum_{i=0}^n \alpha_i x^i. \quad \text{E.g. } \omega(C_2) = 1 + x^2, \quad \omega(E_8) = 1 + 14x^4 + x^8, \quad \omega(G_{24}) = 1 + 759x^8 + 2576x^{12} + 759x^{16} + x^{24}.$$

Theorem 2.5 (Gleason [13]; see also [4], [14], [23], [25])

(a) The weight enumerator of a self dual code is a polynomial in  $\omega(C_2)$  and  $\omega(E_8)$ . (b) If in addition the weight of every codeword is multiple of 4, then the weight enumerator is a polynomial in  $\omega(E_8)$  and  $\omega(G_{24})$ .

Notation Usually capital Latin letters

( $A_{24}, \dots$ ) denote codes, the subscript giving

the length.  $d_n, e_n$  are special codes, &  $\underline{1}, a, a', b, c$  are special vectors (see §6).  $y_{22}$  and  $y_{24}$  are special integers. Capital script letters ( $\mathbb{M}_{24}, \dots$ ) denote groups.

### §3 General Enumeration Theorems

Define, for  $0 \leq k \leq \frac{1}{2} n$ ,

$\Phi_{n,k}$  = the class of self-orthogonal  $[n,k]$  codes,

$\Phi'_{n,k}$  = subclass of  $\Phi_{n,k}$  of codes which contain  $\underline{1}$ ,

$\Psi_{n,k}$  = subclass of  $\Phi_{n,k}$  of codes in which every codeword has weight divisible by 4,

$\Psi'_{n,k}$  = subclass of  $\Psi_{n,k}$  of codes which contain  $\underline{1}$ .

Then  $\Phi_{n, \frac{1}{2}n} = \Phi'_{n, \frac{1}{2}n}$  is the class of self dual codes of length  $n$ .

The following results are useful for enumerating self dual codes. Some of these results appeared in [24], [32], [33]. They are all proved by the methods of [24], [32], i.e. by induction on  $k$ . An empty product is equal to 1.

Theorem 3.1 Let  $n$  be even and  $C \in \Phi'_{n,s}$ . The number of codes in  $\Phi'_{n,k}$  ( $k \geq s$ ) which contain  $C$  is

$$\prod_{j=0}^{k-s-1} \frac{2^{n-2s-2j} - 1}{2^{j+1} - 1}.$$

Cor. 3.2 [24] Let  $n$  be even and  $C \in \Phi'_{n,s}$ . The number of codes in  $\Phi'_{n, \frac{1}{2}n}$  which contain  $C$  is

$$\prod_{j=1}^{\frac{1}{2}n-s} (2^j+1).$$

Cor. 3.3 [32] The total number of codes in  $\Phi'_{n, \frac{1}{2}n}$  is

$$\prod_{j=1}^{\frac{1}{2}n-1} (2^j+1)$$

Cor. 3.4 The total number of codes in  $\Phi'_{n,k}$  is

$$\prod_{j=1}^{k-1} \frac{2^{n-2j}-1}{2^j-1} \text{ if } n \text{ even,} \quad 0 \text{ if } n \text{ odd.}$$

Theorem 3.5 Let  $C \in \Phi_{n,s} - \Phi'_{n,s}$ . The number of codes in  $\Phi_{n,k} - \Phi'_{n,k}$  ( $k \geq s$ ) which contain  $C$  is

$$2^{k-s} \prod_{j=1}^{k-s} \frac{2^{n-2s-2j}-1}{2^j-1} \text{ (n even),} \quad \prod_{j=1}^{k-s} \frac{2^{n-2s-2j+1}-1}{2^j-1} \text{ (n odd).}$$

Cor. 3.6 The total number of codes in  $\Phi_{n,k} - \Phi'_{n,k}$  is

$$2^k \prod_{j=1}^k \frac{2^{n-2j}-1}{2^j-1} \text{ (n even),} \quad \prod_{j=1}^k \frac{2^{n-2j+1}-1}{2^j-1} \text{ (n odd).}$$

Cor. 3.7 Let  $n$  be even and  $C \in \Phi_{n,s} - \Phi'_{n,s}$ . The number of codes in  $\Phi_{n,k}$  ( $k > s$ ) which contain  $C$  is

$$(2^{n-k-s}-1) \prod_{j=1}^{k-s-1} (2^{n-2s-2j}-1) / \prod_{j=1}^{k-s} (2^j-1).$$

Cor. 3.8 [32] If  $n$  is even, the total number of codes in  $\Phi_{n,k}$  is

$$(2^{n-k-1}) \prod_{j=1}^{k-1} (2^{n-2j-1}) / \prod_{j=1}^k (2^j - 1).$$

For codes with weights divisible by 4 we do not give as much detail.

Theorem 3.9 Let  $n$  be a multiple of 8, and  $C \in \Psi'_{n,s}$ . The number of codes in  $\Phi'_{n,k} = \Psi'_{n,k}$  ( $k > s$ ) which contain  $C$  is

$$(2^{n-s-k-2^{\frac{1}{2}n-k}})^{k-s-1} \prod_{j=1}^{k-s-1} \frac{2^{n-2s-2j} - 1}{2^j - 1}$$

Cor. 3.10 Same hypothesis as Th. 3.9. Then the number of codes in  $\Psi'_{n,k}$  ( $k > s$ ) which contain  $C$  is

$$(2^{\frac{1}{2}n-s-1})(2^{\frac{1}{2}n-k+1})^{k-s-1} \prod_{j=1}^{k-s-1} (2^{n-2s-2j-1}) / \prod_{j=1}^{k-s} (2^j - 1)$$

Cor. 3.11 [24] Same hypothesis as Th. 3.9. The number of codes in  $\Psi'_{n, \frac{1}{2}n}$  which contain  $C$  is

$$\prod_{j=0}^{\frac{1}{2}n-s-1} (2^{j+1}).$$

Cor. 3.12 [24] If  $n$  is a multiple of 8, the total number of codes in  $\Psi'_{n, \frac{1}{2}n}$  is

$$\prod_{j=0}^{\frac{1}{2}n-2} (2^{j+1}).$$

§4. The Sum of all Weight Enumerators

Let

$$\sigma_n(x) = \sum_{C \in \Phi_{n, \frac{1}{2}n}} \omega(C) \text{ and } \tau_n(x) = \sum_{C \in \Psi_{n, \frac{1}{2}n}} \omega(C),$$

giving the sum of the weight enumerators of all self dual codes of length  $n$ , and the corresponding sum when the weights are divisible by 4.

Theorem 4.1 (a) For  $n$  even,

$$\sigma_n(x) = \prod_{j=1}^{\frac{n}{2}-2} (2^{j+1}) \cdot \left[ 2^{\frac{1}{2}n-1} (1+x^n) + \sum_{2 \nmid i} \binom{n}{i} x^i \right]$$

$$\tau_n(x) = \prod_{j=0}^{\frac{n}{2}-3} (2^{j+1}) \cdot \left[ 2^{\frac{1}{2}n-2} (1+x^n) + \sum_{4 \nmid i} \binom{n}{i} x^i \right]$$

Proof (a). Write

$$\sigma_n(x) = \sum_{C \in \Phi_{n, \frac{1}{2}n}} \sum_{u \in C} x^{|u|}$$

and use Cors. 3.2, 3.3. Similarly (b) follows from Cors. 3.11, 3.12.

### Examples

$$\sigma_8(x) = 15(9+28x^2+70x^4+28x^6+9x^8),$$

$$\tau_8(x) = 30(1+14x^4+x^8),$$

$$\sigma_{24}(x) = \frac{305,836,524}{1127} y_{24} (2049+276x^2+10626x^4+134,596x^6+735,471x^8 \\ +1,961,256x^{10}+2,704,156x^{12}+1,961,256x^{14} \\ +\dots+x^{24}),$$

$$\tau_{24}(x) = \frac{596,754}{1127} y_{24} (1025+10626x^4+735,471x^6+2,704,156x^{12} \\ +735,471x^{16}+\dots+x^{24}),$$

where

$$y_{24} = 1.3.5.7. \dots .21.23 = 316,234,143,225. \quad (4.2)$$

### §5. Codes with Minimum Distance at least 4

Let  $C$  be a s.o. code of length  $n$  with minimum distance 2.

Lemma 5.1  $C$  is decomposable if  $n > 2$ .

Proof. Let  $u = (u_1, \dots, u_n) \in C$  have weight 2. If  $v \in C$ , since  $u \cdot v = 0$ ,  $|v \cap u| = 0$  or 2. Let  $D = \{v \in C: |v \cap u| = 0\}$ . Then  $C = D \cup (u+D)$ . Let  $D'$  be obtained from  $D$  by deleting the two coordinates  $i$  for which  $u_i = 1$ . Then  $C = D \oplus C_2$ ,  $C_2 = \{00, 11\}$ .

Lemma 5.2 All codewords of weight 2 in  $C$  are nonzero on disjoint sets of coordinates.

Theorem 5.3 Let  $n$  be even. The number of s.o.  $[n, n-r]$  codes with minimum distance  $\geq 4$  is

$$\sum_{i=0}^{n/2} \frac{(-1)^i n!}{2^i i! (n-2i)!} a(n, r)$$

where

$$a(n, r) = (2^r - 1) \prod_{j=1}^{n-r-1} (2^{n-2j-1}) / \prod_{j=1}^{n-r} (2^j - 1).$$

Proof. Let  $c(n, r, i)$  be the number of s.o.  $[n, n-r]$  codes containing  $i$  codewords of weight 2. From Cor. 3.8,

$$\sum_{i=0}^{n/2} c(n, r, i) = a(n, r).$$

From Lemmas 5.1, 5.2,

$$c(n, r, i) = \frac{n!}{2^i i! (n-2i)!} c(n-2i, r, 0),$$

therefore

$$\frac{n!}{2^{\frac{1}{2}n}} \sum_{j=0}^{n/2} \frac{2^j}{(2j)! (\frac{1}{2}n-j)!} c(j, r, 0) = a(n, r)$$

The coefficients on the left are those of the Hermite polynomial  $H_n(-x)$  [20]. The desired result follows from the orthogonality of these polynomials.

§6. Codes With Minimum Distance Exactly 4

For  $n = 4, 6, 8, \dots$  let  $d_n$  be the s.o.  $[n, \frac{1}{2}n-1]$  code with generator matrix

$$d_n: \begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ & 1 & 1 & 1 & 1 & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 & 1 \end{bmatrix}$$

$d_n$  may also be obtained from the  $[\frac{1}{2}n, \frac{1}{2}n-1]$  code consisting of all vectors of even weight, upon replacing 0 by 00 and 1 by 11.  $d_n$  has deficiency 1, weight enumerator  $\frac{1}{2}[(1+x^2)^{n/2} + (1-x^2)^{n/2}]$ , and dual code

$$d_n^\perp = d_n \cup (a+d_n) \cup (b+d_n) \cup (a'+d_n) \quad (6.1)$$

where

$$a = 101010\dots 10,$$

$$b = 110000\dots 00,$$

$$a' = a + b = 011010\dots 10. \quad (6.2)$$

The group of  $d_n$  is:  $G(d_4) = S_4$ ,  $G(d_n) = Z_2^{n/2} \cdot S_{n/2}$  if  $n > 4$  ([34]).

For  $n = 7, 11, 15, \dots$  let  $e_n$  be the s.o.  $[n, \frac{1}{2}(n-1)]$  code with generator matrix

$$e_n: \begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ & & 1 & 1 & 1 & 1 & \\ & & & \cdot & \cdot & \cdot & \\ & & & 1 & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 & 1 \\ 1 & & 1 & & 1 & & 1 & & 1 \end{bmatrix}$$

$e_n$  has deficiency  $\frac{1}{2}$ , weight enumerator  $\frac{1}{2}[(1+x^2)^{(n-1)/2} + (1-x^2)^{(n-1)/2}] + 2^{(n-3)/2}x^{(n+1)/2}$ , and dual code

$$e_n^1 = e_n \cup (c+e_n), \quad (6.1)'$$

where  $c = \underline{1} = 11\dots 1$ . The group is:  $G(e_7) = G_3(2) \simeq \text{PSL}_2(7)$ , of order 168;  $G(e_n) = Z_2^{(n-3)/2} \cdot S_{\frac{1}{2}(n-1)}$  if  $n > 7$  ([34]).

For  $n = 8, 12, 16, \dots$  let  $E_n$  be the  $[n, \frac{1}{2}n]$  self-dual code  $d_n \cup (a+d_n)$ , i.e. with generator matrix

$$E_n: \begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ & & 1 & 1 & 1 & 1 & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 & 1 \\ 1 & & 1 & & 1 & & 1 & & 1 \end{bmatrix}$$

For  $E_8$  see (2.1). The weight enumerator is  $\frac{1}{2}[(1+x^2)^{n/2} + (1-x^2)^{n/2}] + 2^{\frac{1}{2}n-1}x^{n/2}$ . The group is:  $\mathcal{G}(E_8) = \mathcal{CG}_3(2)$ , of order 1344;  $\mathcal{G}(E_n) = \mathbb{Z}_2^{\frac{1}{2}n-1} \cdot \mathcal{S}_{\frac{1}{2}n}$  if  $n > 8$  ([34]).

Note: In [34],  $E_8, E_{12}, E_{16}, E_{20}$  were called  $A_8, B_{12}, E_{16}, J_{20}$  respectively. From (6.1), (6.1)' and the fact that  $E_n$  is self-dual, we have:

Lemma 6.3 Any codeword of  $d_n^1$  is equal to one of 0, a, b, or a' (modulo  $d_n$ ); any codeword of  $e_n^1$  is equal to 0 or c (modulo  $e_n$ ); and any codeword of  $E_n^1$  is equal to 0 (modulo  $E_n$ ).

Cor. 6.4 If C is a s.o. code containing  $E_n$  as a subcode, then C is decomposable.

These codes are important because they provide a canonical form for codes generated by codewords of weight 4, given in Th. 6.5. This result is the basis of the classification in [34] and is used again in §§7,8. The result was derived independently by J. H. Conway (unpublished).

Theorem 6.5 An indecomposable, self-orthogonal code C of length n which is generated by codewords of weight 4 is either  $d_n$  ( $n = 4, 6, 8, \dots$ ),  $e_7$  or  $E_8$ .

Proof: Let I be the subset of the n coordinate indices with the property that there exists at least one vector in C with 1 on an index in I. We say that C is of type H if I can be partitioned into pairs in such a way that every vector in  $F^n$  of weight 4 with ones on any 2 of these pairs is in C. If C is of type H, |I| must be even. Note that a code is of type H iff it is a  $d_n$  with  $n \geq 4$ .

Consider any  $C$ . If  $\dim C = 1$ ,  $C$  is equivalent to  $d_4$ . If  $\dim C = 2$ ,  $C$  is equivalent to  $d_6$ . If  $\dim C \geq 3$ ,  $C$  contains a  $d_6$  and hence must contain a  $d_n$  of maximal dimension. Denote this subcode by  $\bar{C}$ . If  $C = \bar{C}$ , we are finished. So suppose  $C \neq \bar{C}$ . Then there is a vector  $v$  of weight 4 in  $C - \bar{C}$ . Since  $v$  is orthogonal to all vectors in  $C$  we have the following four possibilities.

- a)  $v$  has no coordinate indices in  $I$ .
- b)  $v$  has 2 coordinate indices in a pair of  $I$ .
- c)  $v$  has 3 coordinate indices in  $I$ , no two being in a pair of  $I$ .
- d)  $v$  has all 4 coordinate indices in  $I$ , no two being in a pair of  $I$ .

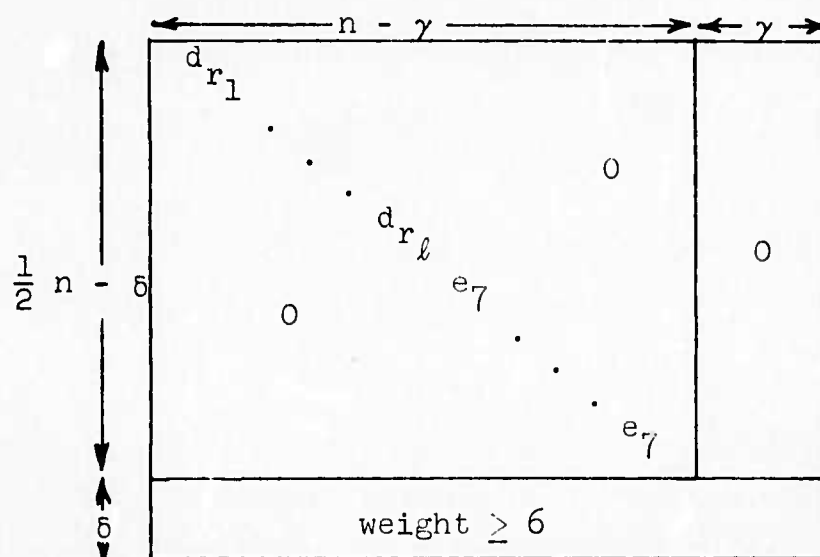
Since  $C$  is indecomposable, case c) implies that  $C = e_7$  and case d) implies that  $C = E_8$ . Case b) is not possible since  $v$  could then be added to  $\bar{C}$  contradicting its maximal dimension. Case a) is not possible since  $\bar{C}$  would then be a direct summand.

Cor. 6.6 The only self-dual codes which are generated by codewords of weight 4 are  $E_8 \oplus \dots \oplus E_8$ .

Our notation for describing the generator matrix of an indecomposable self-dual code  $C$  with minimum distance equal to 4 is as follows. We take the maximum number of linearly independent codewords of weight 4 as the top left-hand corner of the generator matrix. By Th. 6.5 and Cor. 6.4

this has the form  $d_{r_1} \oplus \dots \oplus d_{r_\ell} \oplus e_7 \oplus \dots \oplus e_7$

(with  $m$  copies of  $e_7$ ), or  $d_{r_1} \dots d_{r_\ell} e_7^m$  for short, for suitable  $r_1, \dots, r_\ell, m$ . The generator matrix is



It is convenient to use the same symbol ( $d_r$ ,  $e_7$ , etc.) both for the code and its generator matrix. Here  $\gamma$  is called the gap of  $C$ , and  $\delta = \ell + \frac{1}{2}m + \frac{1}{2}\gamma$  is the deficiency of the subcode generated by codewords of weight 4. The last  $\delta$  rows have weight  $\geq 6$ . If  $u$  is one of the last  $\delta$  rows, by Lemma 6.3 we may assume that under each  $d_r$ ,  $u$  is one of 0,  $a$ ,  $b$ , or  $a'$  (see(6.2)), and under each  $e_7$ ,  $u$  is either 0 or  $c$ .

To avoid writing the generator matrix in full we adopt a shorthand notation, best explained by two examples. The code  $A_{24}$  of §8, with generator matrix given in (6.8)



will be written  $d_8 e_7^2 + 2/bc010/boc01/ao^2 1^2$ . The explicit form of the generator matrices for indecomposable self-dual codes of length  $\leq 20$  can be found in [34].

It seems difficult to find a formula for the number of self-dual codes of length  $n$  and minimum distance 4. However, the next theorem does provide a useful check on the enumeration of some of these codes.

For  $n = 4m$ , let  $\Omega_n$  denote the class of self-dual codes of length  $n$  with the property that the codeword  $\underline{1}$  is the sum of  $m$  disjoint codewords of weight 4. For  $C \in \Omega_n$  let  $h(C)$  be the number of ways of writing  $\underline{1}$  as a sum of  $m$  codewords of weight 4, and let

$$\theta_n = \sum_{C \in \Omega_n} h(C),$$

$$\varphi_n = \theta_n / \binom{n}{4} \binom{n-4}{4} \cdots \binom{4}{4}.$$

Theorem 6.10 An explicit formula for  $\varphi_n$  is

$$\varphi_n = \sum_{i=0}^m (-3)^{m-i} \binom{m}{i} \psi_i,$$

where

$$\psi_0 = 1, \quad \psi_i = \prod_{j=1}^i (2^{j+1}).$$

In particular  $\varphi_8 = 6$ ,  $\varphi_{24} = 3,811,050$ .

Proof By Cor. 3.2, the total number of self-dual codes containing the  $m$  codewords

$$\left[ \begin{array}{cccccccc} 1 & 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & 1 & & & \\ & & 1 & 1 & 1 & 1 & & \\ & & & 1 & 1 & 1 & 1 & \\ & & & & 1 & 1 & 1 & 1 \end{array} \right]$$

is  $\psi_m = \prod_{j=1}^m (2^j + 1)$ . Each of these codes contains a certain number  $2i$ , where  $i = 0, 1, \dots, m$ , of codewords of weight 2.

These codewords come in pairs, as each block of 4 coordinates contains 0 or 2 codewords of weight 2. If one of these blocks contains 2 such codewords they can be chosen in 3 ways:

1100 & 0011, 1010 & 0101, or 1001 & 0110. Therefore

$$\psi_m = \sum_{i=0}^m 3^i \binom{m}{i} \varphi_{n-4i}, \quad \text{with } \varphi_0 = 1.$$

Inversion of this recurrence (cf [36,p49]) gives the desired result.

To calculate  $h(C)$ , it is sufficient to look at the subcode of  $C$  generated by codewords of weight 4. It is easily seen that:

$$h(d_n) = \begin{cases} (\frac{1}{2}n-1)(\frac{1}{2}n-3)\dots 5.3.1 & \text{if } 4|n \\ 0 & \text{otherwise} \end{cases}$$

$$h(e_7) = 0, \quad h(E_8) = 7,$$

$$h(d_{r_1} \oplus d_{r_2} \oplus \dots) = h(d_{r_1})h(d_{r_2})\dots$$

As an example of Th. 6.10, for  $n = 8$  there is one code  $E_8$  in  $\mathcal{C}_8$ , the number of codes equivalent to  $E_8$  is 30 ([34]), and so  $\theta_8 = 7.30$ ,  $\phi_8 = 6$ , which agrees with Th. 6.10. For  $n = 24$ , 15 codes from Table II are in  $\mathcal{C}_{24}$ , namely  $3E_8$ ,  $2E_{12}$ ,  $E_8 \oplus E_{16}$ ,  $E_8 \oplus F_{16}$ ,  $A_{24}$ ,  $C_{24}$ ,  $E_{24}$ ,  $F_{24}$ ,  $H_{24}$ ,  $I_{24}$ ,  $L_{24}$ ,  $M_{24}$ ,  $O_{24}$ ,  $T_{24}$  and  $V_{24}$ . Again the result agrees with Th. 6.10.

## §7. Self Dual Codes of Length 22

Theorem 7.1 There are 25 inequivalent self-dual codes of length 22, 17 of which are decomposable and 8 indecomposable.

These codes are shown in Table I, where for each code  $C$  we give:

- (i) either its direct sum decomposition if  $C$  is decomposable, or a generator matrix in the notation of §6 if  $C$  is indecomposable; (ii) the order of the group  $\mathcal{G}(C)$ ; (iii) the number of codes equivalent to  $C$ , written as a multiple of

$$y_{22} = 1.3.5.7. \dots .19.21 = 13,749,310,575;$$

- (iv) the weight distribution  $\alpha_i = \alpha_{22-i}$  ( $i=2,4,\dots,10$ ), omitting  $\alpha_0 = \alpha_{22} = 1$ .

For codes of length  $\leq 20$  appearing in Tables I, II we use the notation of [34]. Table I also gives the number of codes with minimum distance  $\geq 4$ , and the total number.

These are in agreement with Th. 5.3 and Cor. 3.3. Furthermore the sum of the weight enumerators agrees with Th. 4.1.

Theorem 7.1 is proved by the same method as Theorem 8.1, except that 7.1 is simpler. We omit the details.

Notes on Table I  $G_{22}$  is obtained from the Golay code  $G_{24}$  by writing that code as

$$G_{24} = G^{(00)} \cup G^{(01)} \cup G^{(10)} \cup G^{(11)},$$

according to the values of the first two coordinates. Then  $G_{22}$  is  $G^{(00)} \cup G^{(11)}$  with the first two coordinates deleted. The weight distribution of  $G_{22}$  is uniquely determined (given that its minimum distance is 6) from Th. 2.5, or can be obtained from the tables on page 80 of [8]. The group of  $G_{22}$  is twice  $M_{22}$ .

$U_{22}$  has generator matrix enclosed by the double line in (7.2).

$U_{22}$  and  $Z_{24}$ :

|                                                                 |                                                                 |                                                                 |                                                                 |                                                                 |                                                                 |
|-----------------------------------------------------------------|-----------------------------------------------------------------|-----------------------------------------------------------------|-----------------------------------------------------------------|-----------------------------------------------------------------|-----------------------------------------------------------------|
| $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ |
| $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ |
| $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ |
| $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ |
| $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ |
| $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ | $\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array}$ |

(7.2)

Table I

- 22 -

## Self Dual Codes of Length 22

| Code                           | Order of Group<br>(I) Decomposable Codes | Number $\div y_{22}$  | $\alpha_2$ | $\alpha_4$ | Weight Distribution |            |               |
|--------------------------------|------------------------------------------|-----------------------|------------|------------|---------------------|------------|---------------|
|                                |                                          |                       |            |            | $\alpha_6$          | $\alpha_8$ | $\alpha_{10}$ |
| $11C_2$                        | $2^{11}.11!$                             | 1                     | 11         | 55         | 165                 | 330        | 462           |
| $7C_2 \oplus E_8$              | $2^7.7!.1344$                            | $94 \frac{2}{7}$      | 7          | 35         | 133                 | 330        | 518           |
| $5C_2 \oplus E_{12}$           | $2^5.5!.2^5.6!$                          | 924                   | 5          | 25         | 117                 | 330        | 546           |
| $4C_2 \oplus D_{14}$           | $2^4.4!.168^2.2$                         | $3,771 \frac{3}{7}$   | 4          | 20         | 109                 | 330        | 560           |
| $3C_2 \oplus 2E_8$             | $2^3.3!.1344^2.2$                        | $471 \frac{3}{7}$     | 3          | 31         | 85                  | 282        | 622           |
| $3C_2 \oplus E_{16}$           | $2^3.3!.2^7.8!$                          | 330                   | 3          | 31         | 85                  | 282        | 622           |
| $3C_2 \oplus F_{16}$           | $2^3.3!.192^2.2$                         | 23,100                | 3          | 15         | 101                 | 330        | 574           |
| $2C_2 \oplus H_{18}$           | $2^2.2!.24^3.6!$                         | 123,200               | 2          | 10         | 93                  | 330        | 588           |
| $2C_2 \oplus I_{18}$           | $2^2.2!.168.2^4.5!$                      | 31,680                | 2          | 18         | 85                  | 306        | 612           |
| $C_2 \oplus E_8 \oplus E_{12}$ | $2.1344.2^5.6!$                          | 1,320                 | 1          | 29         | 61                  | 258        | 674           |
| $C_2 \oplus E_{20}$            | $2.2^9.10!$                              | 22                    | 1          | 45         | 45                  | 210        | 722           |
| $C_2 \oplus K_{20}$            | $2.2^3.4!.2^5.6!$                        | 9,240                 | 1          | 21         | 69                  | 282        | 650           |
| $C_2 \oplus L_{20}$            | $2.48.168^2$                             | $30,171 \frac{3}{7}$  | 1          | 17         | 73                  | 294        | 638           |
| $C_2 \oplus S_{20}$            | $2.8.192^2$                              | 138,600               | 1          | 13         | 77                  | 306        | 626           |
| $C_2 \oplus R_{20}$            | $2.6.24^3$                               | 492,800               | 1          | 9          | 81                  | 318        | 614           |
| $C_2 \oplus M_{20}$            | $2.4^5.5!$                               | 332,640               | 1          | 5          | 85                  | 330        | 602           |
| $E_8 \oplus D_{14}$            | $1344.168^2.2$                           | $1,077 \frac{27}{49}$ | 0          | 28         | 49                  | 246        | 700           |

Table I  
Self Dual Codes of Length 22 (cont.)

| Code                                                                                      | Generator matrix<br>Order of Group                                                                                   | Number $\div y_{22}$ | Weight Distribution                           |            |            |            |               |
|-------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|----------------------|-----------------------------------------------|------------|------------|------------|---------------|
|                                                                                           |                                                                                                                      |                      | $\alpha_2$                                    | $\alpha_4$ | $\alpha_6$ | $\alpha_8$ | $\alpha_{10}$ |
| (II) Indecomposable Codes                                                                 |                                                                                                                      |                      |                                               |            |            |            |               |
| $G_{22}$                                                                                  | Shortened Golay code<br>$2^8 3^2 5 \cdot 7 11$                                                                       | 92,160               | 0                                             | 0          | 77         | 330        | 616           |
| $N_{22}$                                                                                  | $d_{14} e_7 + 1/bcl/aol$<br>$2^6 \cdot 7 \cdot 168$                                                                  | 1,508 $\frac{4}{7}$  | 0                                             | 28         | 49         | 246        | 700           |
| $P_{22}$                                                                                  | $d_{10}^2 + 2/b^2 l^2/ao0l/oa10$<br>$(2^4 \cdot 5!)^2 \cdot 2$                                                       | 11,088               | 0                                             | 20         | 57         | 270        | 676           |
| $Q_{22}$                                                                                  | $d_6^2 d_{10}/b^3/a^2 o/a'oa$<br>$(2^2 \cdot 3!)^2 \cdot 2^4 \cdot 5! \cdot 2$                                       | 36,960               | 0                                             | 16         | 61         | 282        | 664           |
| $R_{22}$                                                                                  | $d_6 d_8 e_7 + 1/bocl/abol/bao0$<br>$2^2 \cdot 3! \cdot 2^3 \cdot 4! \cdot 168$                                      | 105,600              | 0                                             | 16         | 61         | 282        | 664           |
| $S_{22}$                                                                                  | $d_6^2 d_8 + 2/aobl0/o^2 al^2/b^2 ol^2/a^2 ol^2$<br>$(2^2 \cdot 3!)^2 2^3 \cdot 4! \cdot 2$                          | 369,600              | 0                                             | 12         | 65         | 294        | 652           |
| $T_{22}$                                                                                  | $d_4^2 d_6^2 + 2/aa'bo00/aaol0/aa'obl^2/oa'oa10/b^2 o^2 l^2$<br>$4^2 \cdot (2^2 \cdot 3!)^2 \cdot 2 \cdot 2 \cdot 2$ | 2,217,600            | 0                                             | 8          | 69         | 306        | 640           |
| $U_{22}$                                                                                  | $d_4^4 + 6/\dots$ (see (7.2))<br>$4^4 \cdot 4! \cdot 6$                                                              | 2,217,600            | 0                                             | 4          | 73         | 318        | 628           |
| Subtotal with min $\frac{m}{4}$ distance $\geq 4$ : 5,053,194 $\frac{6}{49} \cdot y_{22}$ |                                                                                                                      |                      | Total: 6,241,559 $\frac{34}{49} \cdot y_{22}$ |            |            |            |               |

## §8. Self Dual Codes of Length 24

Theorem 8.1 There are 55 inequivalent self dual codes of length 24, 29 of which are decomposable and 26 indecomposable (Table II; for  $y_{24}$  see Eq. (4.2)).

Proof. First we find the decomposable codes as direct sums of shorter codes. The groups of these codes are obtained from Lemma 2.4, [34], and Table I. The indecomposable codes are then classified according to minimum distance. By lemma 5.1 there is no indecomposable code with minimum distance 2. It is known [33], [39] that the Golay code  $G_{24}$  is the unique code of length 24 and distance 8.

Now suppose the minimum distance is 4. Let  $C$  be an indecomposable self dual code of length 24 and distance 4, and let

$$C' = d_{r_1} \oplus \dots \oplus d_{r_\ell} \oplus e_7 \oplus \dots \oplus e_7 = d_{r_1} \dots d_{r_\ell} e_7^m \quad (8.2)$$

be the maximal subcode generated by codewords of weight 4 (§6).

$C'$  has gap  $\gamma = 24 - r_1 - \dots - r_\ell - 7m$ , and deficiency  $\delta = \ell + \frac{1}{2}m + \frac{1}{2}\gamma$ .

Our method is to consider each possible form (8.2) for  $C'$ , and to find all ways of adding  $\delta$  linearly independent generators to  $C'$  so as to obtain an indecomposable self dual code  $C$  of distance 4. We call such a code  $C$  (indecomposable, self dual, minimum distance 4, and with all codewords of weight 4 contained in the subcode  $C'$ ) an extension of  $C'$ .  $C$  must

Table II

- 24a -

## Self Dual Codes of Length 24 (Page 1)

| Code                            | Order of Group<br>(I) Decomposable Codes | Number $\div y_{24}$   | $\alpha_2$ | $\alpha_4$ | $\alpha_6$ | $\alpha_8$ | $\alpha_{10}$ | $\alpha_{12}$ |
|---------------------------------|------------------------------------------|------------------------|------------|------------|------------|------------|---------------|---------------|
| $12 C_2$                        | $2^{12}.12!$                             | 1                      | 12         | 66         | 220        | 495        | 792           | 924           |
| $8C_2 \oplus E_8$               | $2^8.8!.1344$                            | $141 \frac{3}{7}$      | 8          | 42         | 168        | 463        | 848           | 1036          |
| $6C_2 \oplus E_{12}$            | $2^6.6!.2^5.6!$                          | 1,848                  | 6          | 30         | 142        | 447        | 876           | 1092          |
| $5C_2 \oplus D_{14}$            | $2^5.5!.168^2.2$                         | $9,051 \frac{3}{7}$    | 5          | 24         | 129        | 439        | 890           | 1120          |
| $4C_2 \oplus 2E_8$              | $2^4.4!.1344^2.2$                        | $1,414 \frac{2}{7}$    | 4          | 34         | 116        | 367        | 904           | 1244          |
| $4C_2 \oplus E_{16}$            | $2^4.4!.2^7.8!$                          | 990                    | 4          | 34         | 116        | 367        | 904           | 1244          |
| $4C_2 \oplus F_{16}$            | $2^4.4!.192^2.2$                         | 69,300                 | 4          | 18         | 116        | 431        | 904           | 1148          |
| $3C_2 \oplus H_{18}$            | $2^3.3!.24^3.6$                          | 492,800                | 3          | 12         | 103        | 423        | 918           | 1176          |
| $3C_2 \oplus I_{18}$            | $2^3.3!.168.2^4.5!$                      | 126,720                | 3          | 20         | 103        | 391        | 918           | 1224          |
| $2C_2 \oplus E_8 \oplus E_{12}$ | $2^2.2!.1344.2^5.6!$                     | 7,920                  | 2          | 30         | 90         | 319        | 932           | 1348          |
| $2C_2 \oplus E_{20}$            | $2^2.2!.2^9.10!$                         | 132                    | 2          | 46         | 90         | 255        | 932           | 1444          |
| $2C_2 \oplus K_{20}$            | $2^2.2!.2^3.4!.2^6.6!$                   | 55,440                 | 2          | 22         | 90         | 351        | 932           | 1300          |
| $2C_2 \oplus L_{20}$            | $2^2.2!.48.168^2$                        | $181,028 \frac{4}{7}$  | 2          | 18         | 90         | 367        | 932           | 1276          |
| $2C_2 \oplus S_{20}$            | $2^2.2!.8.192^2$                         | 831,600                | 2          | 14         | 90         | 383        | 932           | 1252          |
| $2C_2 \oplus R_{20}$            | $2^2.2!.6.24^3$                          | 2,956,800              | 2          | 10         | 90         | 399        | 932           | 1228          |
| $2C_2 \oplus M_{20}$            | $2^2.2!.4^5.5!$                          | 1,995,840              | 2          | 6          | 90         | 415        | 932           | 1204          |
| $C_2 \oplus E_8 \oplus D_{14}$  | $2.1344.168^2.2$                         | $12,930 \frac{30}{19}$ | 1          | 28         | 77         | 295        | 946           | 1400          |
| $C_2 \oplus G_{22}$             | $2^9.3^2.5.7.11$                         | 1,105,920              | 1          | 0          | 77         | 407        | 946           | 1232          |

Table II

Self Dual Codes of Length 24 (Page 2)

| Code                      | Order of Group                                                                                   | Number $\div y_{24}$ | $\alpha_2$ | $\alpha_4$ | $\alpha_6$ | $\alpha_8$ | $\alpha_{10}$ | $\alpha_{12}$ |
|---------------------------|--------------------------------------------------------------------------------------------------|----------------------|------------|------------|------------|------------|---------------|---------------|
| (I) Decomposable Codes    |                                                                                                  |                      |            |            |            |            |               |               |
| $C_2 \oplus N_{22}$       | $2.2^6.7!.168$                                                                                   | 18,102 $\frac{6}{7}$ | 1          | 28         | 77         | 295        | 946           | 1400          |
| $C_2 \oplus P_{22}$       | $2.(2^4.5!)^2.2$                                                                                 | 133,056              | 1          | 20         | 77         | 327        | 946           | 1352          |
| $C_2 \oplus Q_{22}$       | $2^2.(2^2.3!)^2.2^4.5!$                                                                          | 443,520              | 1          | 16         | 77         | 343        | 946           | 1328          |
| $C_2 \oplus R_{22}$       | $2.2^2.3!2^3.4!.168$                                                                             | 1,267,200            | 1          | 16         | 77         | 343        | 946           | 1328          |
| $C_2 \oplus S_{22}$       | $2^2.(2^2.3!)^2.2^3.4!$                                                                          | 4,435,200            | 1          | 12         | 77         | 359        | 946           | 1304          |
| $C_2 \oplus T_{22}$       | $2^7.(2^2.3!)^2$                                                                                 | 26,611,200           | 1          | 8          | 77         | 375        | 946           | 1280          |
| $C_2 \oplus U_{22}$       | $2.4^4.4!.6$                                                                                     | 26,611,200           | 1          | 4          | 77         | 391        | 946           | 1256          |
| $3E_8$                    | $* 1344^3.3!$                                                                                    | 134 $\frac{34}{49}$  | 0          | 42         | 0          | 591        | 0             | 2828          |
| $E_8 \oplus E_{16}$       | $* 1344.2^7.8!$                                                                                  | 282 $\frac{6}{7}$    | 0          | 42         | 0          | 591        | 0             | 2828          |
| $E_8 \oplus F_{16}$       | $1344.192^2.2$                                                                                   | 19,800               | 0          | 26         | 64         | 271        | 960           | 1452          |
| $2E_{12}$                 | $(2^5.6!)^2.2$                                                                                   | 1,848                | 0          | 30         | 64         | 255        | 960           | 1476          |
| (II) Indecomposable Codes |                                                                                                  |                      |            |            |            |            |               |               |
| $A_{24}$                  | $* \left\{ \begin{array}{l} d_{12}^2/ab/ba(\text{see}(6.8)) \\ (2^5.6!)^2.2 \end{array} \right.$ | 1,848                | 0          | 30         | 0          | 639        | 0             | 2756          |
| $B_{24}$                  | $* \left\{ \begin{array}{l} d_{10}e_7^2/bcc/aoc \\ 2^4.5!168^2.2 \end{array} \right.$            | 18,102 $\frac{6}{7}$ | 0          | 24         | 0          | 663        | 0             | 2720          |
| $C_{24}$                  | $* \left\{ \begin{array}{l} d_8^3(a)/abb/bab/bba \\ (2^3.4!)^3.3! \end{array} \right.$           | 46,200               | 0          | 18         | 0          | 687        | 0             | 2684          |

Table II

- 24c -

## Self Dual Codes of Length 24 (Page 3)

| Code                              | Generator Matrix<br>Order of Group                                                                                                     | Number $\div y_{24}$  | $\alpha_2$ | $\alpha_4$ | $\alpha_6$ | $\alpha_8$ | $\alpha_{10}$ | $\alpha_{12}$ |
|-----------------------------------|----------------------------------------------------------------------------------------------------------------------------------------|-----------------------|------------|------------|------------|------------|---------------|---------------|
| (II) Indecomposable Codes (Cont.) |                                                                                                                                        |                       |            |            |            |            |               |               |
| $D_{24}$                          | $\left\{ \begin{array}{l} * d_6^4(a)/baao/oba\bar{a}/aoba/aaob \\ (2^2.3!)^4 4! \end{array} \right\}$                                  | 246,400               | 0          | 12         | 0          | 711        | 0             | 2648          |
| $E_{24}$                          | $\left\{ \begin{array}{l} * d_{24}^{11}/a \\ 2^{11}.12 \end{array} \right\}$                                                           | 2                     | 0          | 66         | 0          | 495        | 0             | 2972          |
| $F_{24}$                          | $\left\{ \begin{array}{l} * d_4^6(a)/ba\bar{a}^3/o\bar{b}o\bar{a}^3/aob\bar{a}^2/a^2oba/a^3obo/oa^3ob \\ 4^6.5:3 \end{array} \right\}$ | 221,760               | 0          | 6          | 0          | 735        | 0             | 2612          |
| $G_{24}$                          | $\left\{ \begin{array}{l} * \text{Golay code (see (2.2))} \\ 2^{10}.3^3.5.7.11.23 \end{array} \right\}$                                | 8,013 $\frac{21}{23}$ | 0          | 0          | 0          | 759        | 0             | 2576          |
| $H_{24}$                          | $\left\{ \begin{array}{l} d_8 d_{16}/ab/ba \\ 2^3.4!2^7.8! \end{array} \right\}$                                                       | 1,980                 | 0          | 34         | 64         | 239        | 960           | 1500          |
| $I_{24}$                          | $\left\{ \begin{array}{l} d_4 d_8 d_{12}/b^3/a^2/o/oa^2 \\ 2.2!2^3.4!2^5.6! \end{array} \right\}$                                      | 110,880               | 0          | 22         | 64         | 287        | 960           | 1428          |
| $J_{24}$                          | $\left\{ \begin{array}{l} d_8 e_7^2/bco10/boc01/ao^2 l^2 \text{ (see (6.9))} \\ 2^3.4!168^2.2 \end{array} \right\}$                    | 181,028 $\frac{4}{7}$ | 0          | 20         | 64         | 295        | 960           | 1416          |

Table II

Self Dual Codes of Length 24 (Page 4)

- 24d -

| Code                              | Generator matrix<br>Order of Group                                                                                                                            | Number $\div y_{24}$ | $\alpha_2$ | $\alpha_4$ | $\alpha_6$ | $\alpha_8$ | $\alpha_{10}$ | $\alpha_{12}$ |
|-----------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------|------------|------------|------------|------------|---------------|---------------|
| (II) Indecomposable Codes (Cont.) |                                                                                                                                                               |                      |            |            |            |            |               |               |
| $K_{24}$                          | $d_6 d_{10} e_7 + 1/b^2 c l / o a o l / a b o l$                                                                                                              |                      |            |            |            |            |               |               |
|                                   | $\left\{ \begin{array}{l} 2^2 \cdot 3! 2^4 \cdot 5! 168 \\ d_8^3 (h) / b^3 / a^2 o / o a^2 \\ (2^3 \cdot 4!)^3 \cdot 3! \end{array} \right\}$                 | 253,440              | 0          | 20         | 64         | 295        | 960           | 1416          |
| $L_{24}$                          | $\left\{ \begin{array}{l} d_8^3 (c) / a^3 / b a^3 o / b o a^3 \\ (2^3 \cdot 4!)^3 \cdot 2 \end{array} \right\}$                                               | 46,200               | 0          | 18         | 64         | 303        | 960           | 1404          |
| $M_{24}$                          | $\left\{ \begin{array}{l} d_6^2 d_{10} + 2/b^3 l l / o a^2 l l / a b o l / b a o l o \\ (2^3 \cdot 4!)^3 \cdot 2 \end{array} \right\}$                        | 138,600              | 0          | 18         | 64         | 303        | 960           | 1404          |
| $O_{24}$                          | $\left\{ \begin{array}{l} (2^2 \cdot 3!)^2 2^4 \cdot 5! 2 \\ d_4^2 d_8^2 / a b^2 o / b o a o / o b o a / b a o b \end{array} \right\}$                        | 887,040              | 0          | 16         | 64         | 311        | 960           | 1392          |
| $P_{24}$                          | $\left\{ \begin{array}{l} (2 \cdot 2!)^2 (2^3 \cdot 4!)^2 \cdot 2 \\ d_4 d_6^2 e_7 + 1 / o b^2 c l / a b^2 o o / o a a' o o / b o a o l \end{array} \right\}$ | 1,663,200            | 0          | 14         | 64         | 319        | 960           | 1380          |
| $Q_{24}$                          | $\left\{ \begin{array}{l} 2 \cdot 2! (2^2 \cdot 3!)^2 168 \cdot 2 \\ d_6^4 (b) / a o a o / b o a^2 / o a o a' / o b a' a \end{array} \right\}$                | 2,534,400            | 0          | 14         | 64         | 319        | 960           | 1380          |
|                                   | $\left\{ (2^2 \cdot 3!)^4 \cdot 8 \right\}$                                                                                                                   | 739,200              | 0          | 12         | 64         | 327        | 960           | 1368          |

Table II

- 24e -

## Self Dual Codes of Length 24 (Page 5)

| Code                              | Generator matrix<br>Order of Group                                                                                  | Number $\div y_{24}$ | $\alpha_2$ | $\alpha_4$ | $\alpha_6$ | $\alpha_8$ | $\alpha_{10}$ | $\alpha_{12}$ |
|-----------------------------------|---------------------------------------------------------------------------------------------------------------------|----------------------|------------|------------|------------|------------|---------------|---------------|
| (II) Indecomposable Codes (Cont.) |                                                                                                                     |                      |            |            |            |            |               |               |
| $R_{24}$                          | $d_6^2 d_8 + 4/b^2 o l^4 / b o b l^2 o^2 / o^2 a o l^2 o / a o^2 o l^3 / o a o l^3 o$                               | 8,870,400            | 0          | 12         | 64         | 327        | 960           | 1368          |
| $S_{24}$                          | $(2^2 . 3!)^2 2^3 . 4! . 2$<br>$d_4 d_6^3 + 2/a b o^2 l^2 / o a o b l o / a o b^2 o^2 / b o a o o l / b o^2 a l o$  | 17,740,800           | 0          | 10         | 64         | 335        | 960           | 1356          |
| $T_{24}$                          | $2 . 2! (2^2 . 3!)^3 . 2$<br>$d_4^4 d_8 / b a b a b / b a^2 o a / o a b^2 a' / a o b a^2 / b^2 o a a'$              | 4,989,600            | 0          | 10         | 64         | 335        | 960           | 1356          |
| $U_{24}$                          | $d_4^2 d_6^2 + 4/o b^2 o l^2 o^2 / o a^2 o o^3 l / o b o b o^2 l^2 / o a o a o l o^2 / b^2 o l^4 / a^2 o^2 l o l o$ | 53,222,400           | 0          | 8          | 64         | 343        | 960           | 1344          |
| $V_{24}$                          | $4^2 (2^2 . 3!)^2 . 4$<br>$d_4^6 (b) / b a b o^3 / o b a b o^2 / o^2 b a b o / o^3 b a b / b o^3 b a / a b o^3 b$   | 9,979,200            | 0          | 6          | 64         | 351        | 960           | 1332          |

Table II

- 24f -

## Self Dual Codes of Length 24 (Page 6)

| Code                                         | Generator matrix<br>Order of Group                                                                                                    | Number $\div y_{24}$ | $\alpha_2$                      | $\alpha_4$ | $\alpha_6$ | $\alpha_8$ | $\alpha_{10}$ | $\alpha_{12}$ |
|----------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------|----------------------|---------------------------------|------------|------------|------------|---------------|---------------|
| (II) Indecomposable Codes (Cont.)            |                                                                                                                                       |                      |                                 |            |            |            |               |               |
| $W_{24}$                                     | $\left\{ \begin{array}{l} d_4^3 d_6 + 6 / \dots (\text{see } (8.10)) \\ 4^3 \cdot 2^2 \cdot 3! \cdot 3! \cdot 2 \end{array} \right\}$ | 106,444,800          | 0                               | 6          | 64         | 351        | 960           | 1332          |
| $X_{24}$                                     | $\left\{ \begin{array}{l} d_4^4 + 8 / \dots (\text{see } (8.11)) \\ 4^4 \cdot 4! \cdot 2 \end{array} \right\}$                        | 159,667,200          | 0                               | 4          | 64         | 359        | 960           | 1320          |
| $Y_{24}$                                     | $\left\{ \begin{array}{l} d_4^2 + 16 / \dots (\text{see } (8.9)) \\ 2^{11} \cdot 3^2 \end{array} \right\}$                            | 106,444,800          | 0                               | 2          | 64         | 367        | 960           | 1308          |
| $Z_{24}$                                     | $\left\{ \begin{array}{l} \text{see } (8.12) \\ 2^{10} \cdot 3^3 \cdot 5 \end{array} \right\}$                                        | 14,192,640           | 0                               | 0          | 64         | 375        | 960           | 1296          |
| Subtotal with $\min \frac{m}{2}$ distance 2: |                                                                                                                                       | 67,369,356           | $\frac{9}{49} \cdot y_{24}$     |            |            |            |               |               |
| * Subtotal with weights divisible by 4:      |                                                                                                                                       | 542,744              | $\frac{362}{1127} \cdot y_{24}$ |            |            |            |               |               |
| Total:                                       |                                                                                                                                       | 556,041,557          | $\frac{86}{1127} \cdot y_{24}$  |            |            |            |               |               |

Subtotal with  $\min \frac{m}{2}$  distance 2:

\* Subtotal with weights divisible by 4:

Total:

contain the vector  $\underline{1}$ . So for each  $C'$  we must find all its extensions  $C$ . Lemma 6.3 is our chief weapon. Having found a  $C$ , we compute its group  $\mathcal{G}(C)$ , and then the number of codes equivalent to  $C$  is  $24!/\text{order of } \mathcal{G}(C)$ .

Lemma 8.3  $C' = d_{24}$  (with  $\gamma = 0$ ,  $\delta = 1$ ) has a unique extension  $C = E_{24} = d_{24}/a$  (in the notation of §7).

Proof. We must add 1 vector,  $u$  say, to  $C'$ . By 6.3 we may assume  $u$  is  $a = 1010\dots 10$ ,  $b = 1100\dots 00$ , or  $a' = 0110\dots 10$ . But  $a'$  is equivalent to  $a$ , and  $b$  has weight 2, so we may take  $u = a$ .

The group of  $E_{24}$  is  $Z_2^{11} \cdot S_{12}$ .

Lemma 8.4  $C' = d_r$  ( $4 \leq r \leq 22$ ) has no extension  $C$ .

Proof. By 6.3, the generator matrix of  $C$  has the form

$$\begin{array}{l} u = \\ v = \end{array} \begin{array}{|c|c|} \hline \begin{array}{c} r \\ d_r \end{array} & \begin{array}{c} \gamma \\ 0 \end{array} \\ \hline a & \dots\dots \\ \hline b & \dots\dots \\ \hline 0 & Q \\ \hline \end{array},$$

where  $u$  and  $v$  may be absent. If both are absent  $C$  is decomposable. If one is absent,  $Q$  has deficiency 0, length  $\underline{20}$ , and distance 6, which is impossible by Table III. If both  $u$ ,  $v$  are present,  $Q$  has deficiency 1. By Table III there is a  $[20, 9, 6]$  code  $Q$ . But the next lemma shows that this  $Q$ , and hence  $C$ , ~~does~~ not contain  $\underline{1}$ , a contradiction.

Table III, which is frequently used in the proof of Th. 8.1, shows, for each dimension  $k$ , the length  $n_0$  of the shortest s.o.  $[n_0, k, 6]$  code.

Table III

|       |    |     |     |    |    |     |    |    |    |    |     |     |
|-------|----|-----|-----|----|----|-----|----|----|----|----|-----|-----|
| $k$   | 1  | 2   | 3   | 4  | 5  | 6   | 7  | 8  | 9  | 10 | 11  | 12  |
| $n_0$ | 6* | 10* | 12* | 14 | 15 | 16* | 18 | 19 | 20 | 21 | 22* | 24* |

\*: code is unique.

This table was constructed by direct search, with the help of [18]. We omit the details. An asterisk indicates that the code is unique. The asterisk for  $k = 6$  follows using the known list of  $[16, 8, 4]$  self dual codes [34]. The asterisk for  $k = 11$  is from Th. 7.1.

Lemma 8.5 There is no s.o.  $[20, 9, 6]$  code containing  $\underline{1}$ .

Proof. Suppose such a code  $D'$  exists. By Cor. 3.2 there is a self dual  $[20, 10, d]$  code  $D$  containing  $D'$ . If  $d = 4$ ,  $D$  must be one of the codes  $E_{20}, K_{20}, L_{20}, M_{20}, R_{20}, S_{20}$  of [34]. Suppose  $D = M_{20}$ . Let  $v_1, \dots, v_5$  be the 5 vectors of weight 4 in  $M_{20}$ . Then we may assume  $M_{20}$  is generated by  $D'$  and  $v_1$ . Therefore the following vectors are in  $D'$ :  $v_1 + v_2, v_1 + v_3, v_1 + v_4$ , hence  $v_1 + v_2 + v_3 + v_4 = \underline{1} + v_5$ , hence  $v_5$ . But  $v_5$  has weight 4, a contradiction. The other possibilities for  $D$ , and the case  $d = 2$ , are similar.

Lemma 8.6  $d_r d_{24-r}$  (with  $\gamma = 0, \delta = 2$ ) has a unique extension  $d_r d_{n-r}/ab/ba$  provided  $r = 8, 12$ . (This gives the entries  $A_{24}, H_{24}$  of Table IV).

Lemma 8.7  $d_r d_s$  with  $8 < r + s < 24$  has no extension.

Lemma 8.8  $d_4^2$  has a unique extension  $C = Y_{24}$  shown in (8.9).

[illegible]

$$(8.9)$$

Proof. The generator matrix for  $C$  must have the form

|         |         |     |           |
|---------|---------|-----|-----------|
| 1 1 1 1 | 0       | 0   |           |
| 0       | 1 1 1 1 | 0   |           |
| a       | 0       | ... | q         |
| b       | 0       | ... | r         |
| 0       | a       | ... | s         |
| 0       | b       | ... | t         |
| 0       | 0       | Q   | u...<br>z |

where  $Q$  is the unique  $[16, 6, 6]$  code mentioned in Table III. To describe  $Q$ , let  $x_1, \dots, x_4$  be binary variables. As in describing Reed-Muller codes, we identify each of the  $2^{16}$  polynomials  $f(x_1, \dots, x_4)$  over  $GF(2)$  with the corresponding vector of length 16. The first order Reed Muller  $[16, 5, 8]$

code R consists of all linear functions  $\sum_{i=1}^4 \alpha_i x_i + \beta$ , where  $\alpha_i, \beta = 0$  or  $1$  ([31]§5.5). Then  $Q \stackrel{\text{def}}{=} R \cup (x_1 x_2 + x_3 x_4 + R)$ , so we may take as generators for  $Q$ :  $u=1, v=x_1, w=x_2, x=x_3, y=x_4, z=x_1 x_2 + x_3 x_4$ . The group of R is the general affine group  $\mathcal{GA}_1(2)$  consisting of all transformations  $(x_1, x_2, x_3, x_4) \rightarrow (x_1, x_2, x_3, x_4) A + b$ , where A is an invertible  $4 \times 4$  binary matrix and b is a binary 4-tuple.

It is now straightforward to calculate the group of Q, and to show that there is essentially only one way to choose q, r, s, t, namely  $q = x_1 x_3, r = x_2 x_4, s = x_1 x_4, t = x_2 x_3$ , as shown in (8.9).

The group of  $Y_{24}$  is as follows. To every permutation  $\pi$  of the first 4 coordinates there corresponds a permutation  $g \in \mathcal{G}(Q)$  such that  $\pi \circ g$  fixes  $Y_{24}$ . Similarly on the second set of 4. Also the two sets of 4 may be exchanged. Finally there are the 16 permutations generated by  $x_i \rightarrow x_i + 1$  ( $i = 1, \dots, 4$ ). Thus  $|\mathcal{G}(Y_{24})| = 2^4 \cdot 2 \cdot 2^4$ .

The remaining codes in Table II with minimum distance 4 are found in the same way (although none are as complicated as  $Y_{24}$ ). It is worth pointing out that  $d_8^3$  has three inequivalent extensions:  $C_{24}, L_{24}, M_{24}$ ; and  $d_6^4, d_4^6$  each have two.

$d_4^3 d_6$  has a unique extension  $W_{24}$  shown in (8.10),

$$W_{21}^2:$$

(8.10)

[illegible]

(i) For each  $d_r$  block, those permutations in  $Z_2^{\frac{1}{2}r-1} \cdot S_r$  which act inside the block, possibly followed by a permutation of the gap (and similarly for each  $e_r$  block, if present). Thus  $\mathcal{G}(W_{24})$  contains a Klein 4-group  $Z_2 \cdot S_2$  acting on each  $d_4$  block, e.g. (13)(24) and (12)(34) fix the code and generate a Klein 4-group on block 1. Again (13 15)(14 16), (13 17) (14 18), (13 14) (15 16), (13 14) (17 18) generate a  $Z_2^2 \cdot S_3$  on block 4.

(ii) Permutations of the blocks, possibly followed by permutations inside the blocks and inside the gap. Thus in  $W_{24}$  a group  $S_3$  acts on blocks 1,2,3 as follows. Convention:  $\pi \circ \rho$  means first apply  $\pi$ , then  $\rho$ . Let  $\pi_{12} = (\text{block 1, block 2}) = (15)(26)(37) (48)$ , etc. Then

$$\pi_{12} \circ (23)(67)(9\ 11)(19\ 21)(22\ 24)$$

$$\pi_{123} \circ (123)(67)(13\ 14)(19\ 23\ 21\ 22\ 20\ 24)$$

fix the code and generate an  $S_3$  on the blocks.

(iii) Exceptional permutations, not of class (i), which act inside each block, possibly followed by a permutation of the gap. Thus  $\mathcal{G}(W_{24})$  contains the exceptional permutation (1 2)(5 7)(9 11)(13 14)(19 22)(20 23)(21 24) of order 2. No other permutations of  $W_{24}$  are possible, and the order of  $\mathcal{G}(W_{24})$  is  $4^3 \cdot (2^2 \cdot 3!) \cdot 3! \cdot 2$ .

The only codes containing exceptional permutations are  $F_{24}$ ,  $W_{24}$ ,  $X_{24}$  (8.11) and  $Y_{24}$ .

$$X_{24}: \left[ \begin{array}{cccccc} \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} \\ \hline \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \\ | \end{array} \end{array} \quad (8.11)$$

Finally it remains to consider the case of minimum distance 6. Let  $C$  be a  $[24,12,6]$  self dual code. By deleting 2 coordinates from  $C$  we obtain a  $[22,11,4]$  **self** dual code  $D$ , which must be in Table I. It is straightforward to show that the only possibility is  $D = U_{22}$ , and further that there is a unique way to add two columns and one row to the generator matrix of  $U_{22}$  to obtain  $C$ , as shown in (7.2). Therefore  $C$  is unique, and is denoted by  $Z_{24}$ .

To simplify calculation of the group of  $Z_{24}$ , we give an alternative construction for this code based on the Golay code  $G_{24}$ , using the notation of Todd's paper [42].

Let  $\Omega = \{\infty, 0, 1, \dots, 22\}$  be the coordinates of  $G_{24}$ . A subset of  $\Omega$  giving the location of the 1's in a codeword of  $G_{24}$  of weight 8 is called an octad. A list of the 759 octads is given in [42].  $\Omega$  may be partitioned into 6 sets of 4 (called mutually complementary tetrads) such that the union of any two tetrads is an octad, for example (using

Todds notation for the octads).

$\infty$  0 1 2 3 5 14 17, 4 13 16 22, 6 7 19 21, 9 10 15 20, 8 11 12 18.

(\*)

Associated with any set of mutually complementary tetrads is a set of 64 non-special hexads (i.e. 6-sets of  $\mathcal{C}$ ) with the properties: (i) A non-special hexad is not contained in any octad; and (ii) let  $H = (a_1 a_2 a_3 a_4 a_5 a_6)$  be any non-special hexad, choose any point, say  $a_1$ , of  $H$ , and find the unique octad  $a_2 a_3 a_4 a_5 a_6 b_2 b_3 b_4$  containing the other 5 points of  $H$ . Then  $a_1 b_2 b_3 b_4$  must be one of the tetrads.

A method of constructing the non-special hexads is given in [42]. A set of 12 non-special hexads associated with the tetrads(\*) form the rows of (8.12). These rows do indeed generate a  $[24, 12, 6]$  code, which therefore must be  $Z_{24}$ . The group of this code is that subgroup of  $M_{24}$  which fixes the set of mutually complementary tetrads. This is the group  $G_5$  described in [42], of order  $2^{10} \cdot 3^3 \cdot 5$  and index 1771 in  $M_{24}$ . The permutations and character table are given in Table VII of [42].

This completes the enumeration of the codes and the proof of Theorem 8.1.

As checks on table II we verified the number of codes of minimum distance  $\geq 4$  (5.3), the number of codes with weights divisible by 4 (3.12), the sum of the weight enumerators of the latter codes (4.1), the total number of

$Z_{24}$ :

| $\infty$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|----------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 2        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 3        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 4        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 5        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 6        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 7        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 8        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 9        | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 10       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 11       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 12       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 13       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 14       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 15       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 16       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 17       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 18       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 19       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 20       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 21       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 22       | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |

(8.12)

codes (3.3), the sum of all weight enumerators (4.1), and

$\varphi_{24}$  of Th. 6.10.

Cor. 8.13 There are 9 self dual codes of length 24 with all weights divisible by 4 (denoted by an asterisk\* in Table II).

Cor 8.14 There is a unique self dual code of length 24 and minimum distance 6.

Cor.8.15 Let  $C$  be an indecomposable self dual code of length 24, with weight distribution  $\alpha_i$ . Either  $\alpha_6 = \alpha_{10} = 0$  or  $\alpha_6 = 64$ ,  $\alpha_{10} = 960$ .

Proof. 1. From Table II; or

2. From Th. 2.5 (using the version in [4]),

the weight enumerator of  $C$  is, for suitable  $\ell, m$ ,

$$(1+x^2)^{12} - 12x^2(1+x^2)^8(1-x^2)^2 + \ell x^4(1+x^2)^4(1-x^2)^4 + mx^6(1-x^2)^6 \\ = 1 + (\ell-6)x^4 + (m+64)x^6 + (399-4\ell-6m)x^8 + 15(m+64)x^{10} + \dots,$$

so  $\alpha_{10} = 15\alpha_6$ . But the codewords with weights divisible by 4 form a subcode of  $C$  of dimension 11 or 12, so  $\alpha_6 + \alpha_{10} = 0$  or  $2^{10}$ . This completes the proof.

Remarks (1) The latter proof can be used for lengths 8 and 16 to decide which of the possible weight enumerators given by Th. 2.3 can be realized by codes.

(2) Note that  $N_{22}$ ,  $P_{22}$ ,  $K_{24}$  can also be written  $e_7e_{15}/\dots, e_{11}^2/\dots, d_6e_7e_{11}/\dots$ .

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