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ON THE OMEGA-VALUE OF A MATRIX

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The omega-value of a matrix is a function of a parameter σ and is defined as a limit of a sequence of successive min and max operations applied to convex combinations of entries of the matrix. It arises naturally from a game-theoretical model. In this paper it is shown that the omega-value always exists† and that it can be obtained from certain systems of nonlinear equations. Some of its properties are also investigated.

†See Remark on page 24

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1. Introduction.

The omega-value of a matrix $[a_{ij}]$, the formal definition of which is given in Section 3, is a function of a parameter σ and is obtained, roughly speaking, by applying an infinite chain of min and max operations to convex combinations of the entries. As the already well established Von Neumann's value

$$v = \max_x \min_y \sum_i \sum_j a_{ij} x_i y_j = \min_y \max_x \sum_i \sum_j a_{ij} x_i y_j ,$$

the omega-value seems to be a mathematical concept interesting enough to be studied on its own merit. However, like the Von Neumann's value before it, the omega-value has been born by game theory, and it was in that framework the author was first introduced to the omega-value by its father John M. Danskin. We, therefore, include a section (Section 2) containing a brief description of the game-theoretical motivation leading to this concept. The reader may, if he so wishes, skip that section without any loss of continuity.

The main purpose of this paper is to show that the omega-value, as defined in Sec. 3, exists† for any matrix and can in fact be obtained explicitly from some systems of nonlinear equations. These results are contained in Sec. 6. In the preceding two sections we study certain transformations which serve as the main tool for the subsequent results. Some general properties of the omega-value and a formula for the two-by-two case are also established.

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† See Remark on page 24.

2. A matrix-differential game.

Consider a two-person, zero-sum game with a payoff matrix $[a_{ij}]$, where, as customary, Player 1 controls the row index and Player 2 the column index. Unlike the usual matrix game, however, this game is to be played continuously for one unit of time and we wish to allow each player to react "instantaneously" to his opponent's previous choice. The entries a_{ij} are supposed to represent instantaneous payoffs in the sense that if during some small time interval Δt the players play i and j respectively the payoff to Player 1 is $a_{ij}\Delta t$. Payoffs are assumed to be accumulated during the course of play and thus one would be tempted to write the total payoff as an integral

$$\int_0^1 a_{ij}dt.$$

Of course, since the players are supposed to be able to change i and j instantaneously the above symbol is meaningless. Clearly, we have to find some other way to express the total payoff.

Such a problem, in a more general context of differential games, has been encountered before. In his book on differential games [2] Rufus Isaacs suggested an idea (attributed by him to Samuel Karlin) of first permitting each player to change their control variables only at a fixed sequence of time instances, and then letting the lengths of the resulting time intervals shrink to zero in some orderly way. Many sophisticated variants of this idea have been introduced since, most of which can be found in recent Danskin's book [1]. It was also John Danskin who proposed a specific approach to the matrix game described above, which he calls a matrix-differential game.

The unit time interval $[0,1]$ is first divided into n equal subintervals $\left[\frac{k-1}{n}, \frac{k}{n}\right)$, $k = 1,\ldots,n$. Each of these subintervals is further subdivided into two parts in a fixed ratio $\sigma/(1-\sigma)$, where $0 \le \sigma \le 1$ is a parameter. One of the players, say Player 2, is then allowed to change the column index only at time instants $t_k = \frac{k-1}{n}$ while Player 1 may change the row index only at $t_k + \frac{\sigma}{n}$. Notice that the fraction $\sigma/(1-\sigma)$ can be interpreted as the ratio of reaction times of the two players. The trouble-some integral then becomes a finite sum

$$\frac{\sigma}{n} a_{i_0 j_1} + \frac{(1-\sigma)}{n} a_{i_1 j_1} + \frac{\sigma}{n} a_{i_1 j_2} + \frac{(1-\sigma)}{n} a_{i_2 j_2} + \cdots$$

$$+ \frac{\sigma}{n} a_{i_{n-1} j_n} + \frac{(1-\sigma)}{n} a_{i_n j_n},$$

and the game itself is now a well defined finite game with perfect information and with alternate moves. Hence it has a pure Von Neumann's value

$$\Omega_n(\sigma, i_0) = \min_{j_1} \max_{i_1} \ldots \min_{j_n} \max_{i_n} \frac{1}{n} \sum_{k=1}^{n} \sigma a_{i_{k-1} j_k} + (1-\sigma) a_{i_k j_k}$$

depending, of course, on n and the initial row index i_0. The next step is to let n increase to infinity. If the sequence $\Omega_1(\sigma,i_0), \Omega_2(\sigma,i_0),\ldots$ converges and its limit is independent of i_0 then this limit $\Omega(\sigma)$ is the omega-value and can be regarded as a solution of the matrix-differential game $[a_{ij}]$.

Alternate approach is to assign the time instants t_k to Player 1 and $t_k + \frac{1-\sigma}{n}$ to Player 2. Here σ must be replaced by $1-\sigma$ to preserve the reaction time ratio. Clearly, if the whole idea is sound this ought to lead to the same $\Omega(\sigma)$. As will be shown later it indeed does.

3. Definition of the omega-value.

Throughout this paper let $A = [a_{ij}]$ be an $N \times M$ matrix with real entries a_{ij} and with I and J being reserved for the sets of its row and column indices. The letter σ will always denote a real number in the interval $[0,1]$. To simplify the notation we abbreviate

$$A_{ij}^{k}(\sigma) = \sigma a_{kj} + (1-\sigma)a_{ij}, \tag{3.1}$$

and

$$B_{ij}^{\ell}(\sigma) = (1-\sigma)a_{i\ell} + \sigma a_{ij}, \tag{3.2}$$

dropping the variable σ whenever possible.

Definition: Let for every $n = 1,2,\ldots,$ $i_0 \in I$, $j_0 \in J$ and $\sigma \in [0,1]$

$$\Omega_n^{(1)}(\sigma,i_0) = \min_{j_1 \in J}\ \max_{i_1 \in I} \ldots \min_{j_n \in J}\ \max_{i_n \in I} \frac{1}{n}\sum_{k=1}^{n} A_{i_k,j_k}^{i_{k-1}}(\sigma),$$

$$\Omega_n^{(2)}(\sigma,j_0) = \max_{i_1 \in I}\ \min_{j_1 \in J} \ldots \max_{i_n \in I}\ \min_{j_n \in J} \frac{1}{n}\sum_{k=1}^{n} B_{i_k,j_k}^{j_{k-1}}(\sigma).$$

If
$$\lim_{n\to\infty} \Omega_n^{(1)}(\sigma,i_0) \quad \text{and} \quad \lim_{n\to\infty} \Omega_n^{(2)}(\sigma,j_0)$$

both exist, are independent of i_0 and j_0, and are equal then their common value $\Omega(\sigma)$ is called the omega-value of the matrix A.

4. The min-max transformation.

Let R^N be the space of all N-tuples $x = (x_1, \ldots, x_N)$ of real numbers endowed with the maximum norm

$$\|x\| = \max_{i \in I} | x_i |.$$

Also let X be a subset of R^N defined by

$$X = \{x \in R^N : \|x\| \le \|A\|\},$$

where

$$\|A\| = \max_{i \in I} \max_{j \in J} | a_{ij} |.$$

For every $\lambda \in (0,1]$ we define a transformation

$$T^{(1)}(\lambda;x) = (T_1^{(1)}(\lambda;x), \ldots, T_N^{(1)}(\lambda;x))$$

from X into R^N by

$$T_k^{(1)}(\lambda;x) = \min_{j \in J} \max_{i \in I} [\lambda A_{ij}^k + (1-\lambda)x_i], \qquad k \in I, \tag{4.1}$$

where A_{ij}^k is the expression (3.1).

We are now ready to investigate some properties of this transformation.

Lemma 1: For any $\lambda \in (0,1]$

$$x \in X \Rightarrow T^{(1)}(\lambda;x) \in X.$$

Proof: $$\|T^{(1)}(\lambda;x)\| = \max_{k \in I} | T^{(1)}(\lambda;x) | \le \max_{k \in I} \max_{j \in J} \max_{i \in J} | \lambda A_{ij}^k + (1-\lambda)x_i |$$

$$\le \lambda\|A\| + (1-\lambda)\|x\| \le \|A\| .$$

Lemma 2: For any $\lambda \in (0,1]$, $x \in X$, $x' \in X$,

$$\|T^{(1)}(\lambda;x) - T^{(1)}(\lambda;x')\| \le (1-\lambda)\|x-x'\|$$

Proof: We apply the well-known inequalities

$$\left| \min_{j\in J} \alpha_j - \min_{j\in J} \beta_j \right| \le \max_{j\in J} \left| \alpha_j - \beta_j \right| , \tag{4.2}$$

$$\left| \max_{i\in I} \alpha_i - \max_{i\in I} \beta_i \right| \le \max_{i\in I} \left| \alpha_i - \beta_i \right| , \tag{4.3}$$

which hold for any real α_j's and β_j's. We obtain

$$\|T^{(1)}(\lambda;x) - T^{(1)}(\lambda;x')\| = \max_{k\in I} \Big| \min_{j\in J} \max_{i\in I} [\lambda A^k_{ij} + (1-\lambda)x_i]$$
$$- \min_{j\in J} \max_{i\in I} [\lambda A^k_{ij} + (1-\lambda)x'_i] \Big| \le \max_{k\in I} \max_{j\in J} \Big| \max_{i\in I} [\lambda A^k_{ij} + (1-\lambda)x_i]$$
$$- \max_{i\in I} [\lambda A^k_{ij} + (1-\lambda)x'_i] \Big| \le \max_{k\in I} \max_{j\in J} \max_{i\in I} \left| (1-\lambda)(x_i-x'_i) \right| = (1-\lambda)\|x-x'\|.$$

Theorem 1: For every $\lambda \in (0,1]$ there is a unique fixed point

$$\tilde{x}(\lambda) = T^{(1)}(\lambda;\tilde{x}(\lambda)).$$

Proof: By Lemma 2 the map $T^{(1)}$ is for any fixed $\lambda \in (0,1]$ a contraction and by Lemma 1 it maps a closed, bounded set X into itself. Hence, the statement follows by the classical Banach's fixed point theorem.

To find out how the fixed point $\tilde{x}(\lambda)$ changes with λ we first need the following lemma.

Lemma 3: There exists $\lambda_0 > 0$ and $p = (p(1),\ldots,p(N))$, $q = (q(1),\ldots,q(N))$, $p(k) \in I$, $q(k) \in J$, $k \in I$, such that for every $\lambda \in (0,\lambda_0]$, $\tilde{x}(\lambda)$ is the unique solution of the linear system

$$\tilde{x}_k(\lambda) = \lambda A^k_{p(k),q(k)} + (1-\lambda)\tilde{x}_{p(k)}(\lambda), \qquad k \in I. \tag{4.4}$$

<u>Proof</u>: Let P denote the set of all N-tuples $p = (p(1),\ldots,p(N))$ of row indices $p(k) \in I$, let Q be the set of all N-tuples $q = (q(1),\ldots, q(N))$ of column indices $q(k) \in J$.

Since by Theorem 1 the system of equations

$$x_k = \min_{j\in J} \max_{i\in I} [\lambda A^k_{ij} + (1-\lambda)x_i], \qquad k \in I \tag{4.5}$$

has a unique solution $x = \tilde{x}(\lambda)$ for every $\lambda \in (0,1]$, there is, for every such λ, a $p \in P$ and a $q \in Q$ such that

$$\min_{j\in J} \max_{i\in I} [\lambda A^k_{ij} + (1-\lambda)\tilde{x}_i(\lambda)] = \lambda A^k_{p(k),q(k)} + (1-\lambda)\tilde{x}_{p(k)}(\lambda) = \tilde{x}_k(\lambda), \qquad k \in I. \tag{4.6}$$

Consider now the system of linear equations

$$x_k + (1-\lambda)x_{p(k)} = \lambda A^k_{p(k),q(k)}, \qquad k \in I, \tag{4.7}$$

where $p \in P$, $q \in Q$, and let $V_p(\lambda)$ be the determinant of such a system. Since clearly $V_p(1) = 1$ the determinant cannot be identically zero. Hence, it can only have a finite number of zeros in the interval $\lambda \in (0,1]$. Since the set P is also finite this implies that there must be $\lambda_1 > 0$ such that

$$\lambda \in (0,\lambda_1]\ ,\ p \in P \Rightarrow V_p(\lambda) \neq 0. \tag{4.8}$$

Consequently if $\lambda \in (0,\lambda_1]$ and $\tilde{x}(\lambda)$ satisfies (4.6) with some $p \in P$, $q \in Q$ then it is also a unique solution of the linear system (4.7) (with the same p, q). Hence as long as $0 < \lambda \le \lambda_1$, any $\tilde{x}(\lambda)$ is of the form

$$\tilde{x}_k(\lambda) = \lambda V_p^{-1}(\lambda) W_{p,q}^k(\lambda), \qquad k \in I, \tag{4.9}$$

where $W_{p,q}^k(\lambda)$ are polynomials in λ of degree not exceeding N.

Substitute now (4.9) for x_k into (4.5). After multiplying both sides by $V_p(\lambda)$ and cancelling λ we obtain

$$W_{p,q}^k(\lambda) = \min_{j\in J} \max_{i\in I} [A_{ij}^k V_p(\lambda) + (1-\lambda) W_{p,q}^{p(i)}(\lambda)], \qquad k \in I. \tag{4.10}$$

Now, since $\tilde{x}(\lambda)$ is a unique solution of (4.5) and any $\tilde{x}(\lambda)$ is of the form (4.9) it is given by (4.9) for some particular $p \in P$, $q \in Q$ and $\lambda \in (0,\lambda_1]$ if and only if (4.10) is satisfied with these p, q and λ.

Let for every $p \in P$, $q \in Q$

$$\Lambda_{p,q} = \{\lambda \in (0,\lambda_1] : (4.10) \text{ holds with these } p, \; q.\}$$

Writing the equations (4.10) as the system of $2N$ inequalities

$$\min_{j\in J} \max_{i\in I} f_{ij}^k(\lambda) \geq 0, \qquad k \in I,$$

$$\min_{j\in J} \max_{i\in J} f_{ij}^k(\lambda) \leq 0, \qquad k \in I,$$

where

$$f_{ij}^k(\lambda) = A_{ij}^k V_p(\lambda) + (1-\lambda) W_{p,q}^{p(i)}(\lambda) - W_{p,q}^k(\lambda),$$

it is easily seen that

$$\Lambda_{p,q} = \bigcap_{k\in I} [(\bigcap_{j\in J} \bigcup_{i\in I} \{\lambda : f_{ij}^k(\lambda) \geq 0\}) \cap (\bigcup_{j\in J} \bigcap_{i\in I} \{\lambda : f_{ij}^k(\lambda) \leq 0\})].$$

Now, since for every $\lambda \in (0,\lambda_1]$ there is a pair $p \in P$, $q \in Q$ such that (4.10) holds the finite union

$$\bigcup_{p\in P} \bigcup_{q\in Q} \Lambda_{p,q}$$

covers the entire interval $(0,\lambda_1]$. On the other hand the functions $f^k_{ij}(\lambda)$ are all polynomials in λ. Hence, each $\Lambda_{p,q}$ must be a finite union of closed intervals, some of them possibly vacuous. Therefore, there must be a pair $p \in P$, $q \in Q$ and $\Lambda_0 > 0$ such that

$$(0,\lambda_0] \subset \Lambda_{p,q},$$

i.e. (4.10) and consequently (4.6) must be satisfied with this p, q for every $\lambda \in (0,\lambda_0]$.

The lemma is proved.

<u>Theorem 2</u>: There is a number ω such that for every $k \in I$

$$\lim_{\lambda\to 0+} \tilde{x}_k(\lambda) = \omega. \tag{T2.1}$$

Furthermore, there is a nonempty interval $(0,\lambda_0]$ and a constant C such that the functions $\tilde{x}_k(\lambda)$, $k \in I$ are infinitely differentiable on $(0,\lambda_0]$ and

$$\left| \tilde{x}_k(\lambda) - \tilde{x}_k(\lambda') \right| \leq C \left| \lambda - \lambda' \right| \tag{T2.2}$$

for all $\lambda \in (0,\lambda_0]$, $\lambda' \in (0,\lambda_0]$.

<u>Proof</u>: By Lemma 1 $\left| \tilde{x}_k(\lambda) \right| \leq \|A\|$ for every $\lambda \in (0,1]$. By Lemma 3 there is a $\lambda_0 > 0$ such that for $\lambda \in (0,\lambda_0]$, $\tilde{x}_k(\lambda)$, $k \in I$ is a unique solution of the linear system (4.4). Hence, it can be written as a ratio of two polynomials

$$\tilde{x}_k(\lambda) = \frac{F_k(\lambda)}{G_k(\lambda)},$$

where F_k and G_k have no roots in common. Next since Lemma 1 implies that $F_k(\lambda)/G_k(\lambda)$ is bounded on $(0,\lambda_0]$ there is an $\varepsilon > 0$ such that $G_k(\lambda)$ can have no root in the interval $(-\varepsilon,\lambda_0+\varepsilon)$. Hence $\tilde{x}_k(\lambda)$ is infinitely differentiable and $\frac{d}{d\lambda}\tilde{x}_k(\lambda)$ is uniformly bounded on $(0,\lambda_0]$. This proves (T2.1) and establishes the existence of a finite limit of $\tilde{x}_k(\lambda)$ as $\lambda \to 0+$.

To show that for $k \in I$ all these limits are equal consider again the equations for $\tilde{x}(\lambda)$:

$$\tilde{x}_k(\lambda) = \min_{j\in J} \max_{i\in I} [\lambda A^k_{ij} + (1-\lambda)\tilde{x}_i(\lambda)], \qquad k \in I.$$

By Lemma 3 there is an index $q(k) \in J$ such that for every $\lambda \in (0,\lambda_0]$

$$\min_{j\in J} \max_{i\in I} [\lambda A^k_{ij} + (1-\lambda)\tilde{x}_i(\lambda)] = \max_{i\in I} [\lambda A^k_{i,q(k)} + (1-\lambda)\tilde{x}_i(\lambda)].$$

Hence, for every $i \in I$ and $k \in I$

$$\lambda A^k_{i,q(k)} + (1-\lambda)\tilde{x}_i(\lambda) \le \tilde{x}_k(\lambda),$$

from which by letting $\lambda \to 0+$ we obtain

$$\lim_{\lambda\to 0+} \tilde{x}_i(\lambda) \le \lim_{\lambda\to 0+} \tilde{x}_k(\lambda).$$

Interchanging i and k completes the proof.

<u>Theorem 3</u>: Let $x(0), x(1), \ldots$ be a sequence of points in X defined by

$$x(0) \in X, \; x(n+1) = T^{(1)}\left(\tfrac{1}{n+1}; x(n)\right), \qquad n = 0,1,\ldots$$

Then the sequence converges and for every $k \in I$

$$\lim_{n\to\infty} x_k(n) = \lim_{\lambda\to 0+} \tilde{x}_k(\lambda).$$

Proof: Let n_0 be a positive integer such that

$$\frac{1}{n_0} \le \lambda_0,$$

and let n be an arbitrary nonnegative integer. By Lemma 2

$$\|x(n_0+n) - \tilde{x}(\tfrac{1}{n_0+n})\| = \|T^{(1)}(\tfrac{1}{n_0+n};x(n_0+n-1)) - T^{(1)}(\tfrac{1}{n_0+n};\tilde{x}(\tfrac{1}{n_0+n}))\|$$

$$\le \frac{n_0+n-1}{n_0+n}\|x(n_0+n-1) - \tilde{x}(\tfrac{1}{n_0+n})\| .$$

By triangular inequality and (T2.2) of Theorem 2 we have

$$\|x(n_0+n-1) - \tilde{x}(\tfrac{1}{n_0+n})\| \le \|x(n_0+n-1) - \tilde{x}(\tfrac{1}{n_0+n-1})\| + \|x(\tfrac{1}{n_0+n-1})$$

$$- \tilde{x}(\tfrac{1}{n_0+n})\| \le \|x(n_0+n-1) - \tilde{x}(\tfrac{1}{n_0+n-1})\| + \frac{C}{(n_0+n)(n_0+n-1)} .$$

Calling $\Delta_n = \|x(n_0+n) - \tilde{x}(\frac{1}{n_0+n})\|$ we obtain the inequality

$$\Delta_n \le \frac{n_0+n-1}{n_0+n}\Delta_{n-1} + \frac{C}{(n_0+n)^2} ,$$

from which by iteration

$$\Delta_n \le \frac{n_0}{n_0+n}\Delta_0 + \frac{C}{n_0+n}\sum_{k=0}^{n}\frac{1}{n_0+k} .$$

Hence

$$\lim_{n\to\infty}\Delta_n = 0$$

since the harmonic series is Cesaro summable to zero. The statement now follows from the convergence of the sequence $\tilde{x}(\frac{1}{n_0})$, $\tilde{x}(\frac{1}{n_0+1})$,... implied by (T2.1) of Theorem 2.

5. The max-min transformation.

Let this time R^M be the space of all real M-tuples $x = (x_1,\dots,x_M)$ again with the maximum norm $\|x\|$. For every $\lambda \in (0,1]$ define the transformation

$$T^{(2)}(\lambda;x) = (T_1^{(2)}(\lambda;x),\dots,T_M^{(2)}(\lambda;x))$$

from the set

$$X = \{x \in R^M : \|x\| \le \|A\|\} \quad \text{into} \quad R^M \text{ by}$$

$$T_\ell^{(2)}(\lambda;x) = \max_{i\in I} \min_{j\in J} [\lambda B_{ij}^\ell + (1-\lambda)x_j], \qquad \ell \in J, \tag{5.1}$$

where B_{ij}^ℓ is the expression (3.2).

Consider now the matrix $-A^t$, the negative transpose of the matrix A, and let $T^{(1)}$ be the min-max transformation (4.1) for this matrix. The expression (3.1) now becomes

$$A_{ij}^k(\sigma) = -\sigma a_{jk} - (1-\sigma)a_{ji}, \tag{5.2}$$

where $j \in J$ is now a row index and $i \in I$, $k \in I$ are column indices of the matrix $-A^t$. Substituting (5.2) into (4.1) and replacing σ by 1 - and x by $-x$ it is easily seen that

$$T^{(2)}(\lambda;x) = -T^{(1)}(\lambda;-x),$$

where $T^{(2)}$ is the max-min transformation (5.1) for the matrix A with some $\sigma \in [0,1]$ while $T^{(1)}$ is the min-max transformation (4.1) for the matrix $-A^t$ with $\sigma' = 1 - \sigma$. Thus $\tilde{x}(\lambda)$ is a fixed point of $T^{(1)}(\lambda;\cdot)$ if and only if $-\tilde{x}(\lambda)$ is a fixed point of $T^{(2)}(\lambda;\cdot)$. Since the matrix A in the previous section was an arbitrary matrix all the results obtained there for the min-max transformation are also valid for the max-min transformation. We thus have:

<u>Theorem 4</u>: The statements of Theorem 1, 2 and 3 remain true if the min-max transformation $T^{(1)}$ is replaced by the max-min transformation $T^{(2)}$ and the set of row indices I by the set of column indices J.

6. Existence and further properties of the omega-value.

The results obtained in the previous two sections will now be used to prove the main theorem of this paper.

Theorem 5: The omega-value $\Omega(\sigma)$, $\sigma \in [0,1]$ exists[†] for any $N \times M$ matrix $A = [a_{ij}]$. Further, if $\tilde{x}(\lambda)$, $\lambda \in (0,1]$, is a solution of the system of N equations

$$\tilde{x}_k(\lambda) = \min_{j\in J} \max_{i\in I}\{\lambda[\sigma a_{kj} + (1-\sigma)a_{ij}] + (1-\lambda)\tilde{x}_i(\lambda)\}, \qquad k \in I, \tag{T5.1}$$

or the system of M equations

$$\tilde{x}_\ell(\lambda) = \max_{i\in I} \min_{j\in J}\{\lambda[(1-\sigma)a_{i\ell} + \sigma a_{ij}] + (1-\lambda)\tilde{x}_j(\lambda)\}, \qquad \ell \in J, \tag{T5.2}$$

then

$$\Omega(\sigma) = \lim_{\lambda\to 0+} \tilde{x}_k(\lambda) \quad \text{for all } k \in I,$$

or

$$\Omega(\sigma) = \lim_{\lambda\to 0+} \tilde{x}_\ell(\lambda) \quad \text{for all } \ell \in J.$$

Proof: According to the definition $\Omega(\sigma)$ is the common limit of

$$\{\Omega_n^{(1)}(\sigma;i_0)\} \text{ and } \{\Omega_n^{(2)}(\sigma;j_0)\}, \tag{6.1}$$

provided both these sequences converge to the same limit for any $i_0 \in I$, $j_0 \in J$.

Upon writing

$$\Omega_n^{(1)}(\sigma,i_0) = \min_{j_1\in J}\{\frac{\sigma}{n} a_{i_0 j_1} + \max_{i_1\in I}\min_{j_2\in J} \cdots \max_{i_{n-1}\in I}\min_{j_n\in J}\{\frac{1}{n}[(1-\sigma)a_{i_1,j_1} +$$

$$\sigma a_{i_1 j_2} + \ldots + (1-\sigma)a_{i_{n-1},j_{n-1}} + \sigma a_{i_{n-1},j_n}] + \max_{i_n\in I}\frac{(1-\sigma)}{n} a_{i_n,j_n}\}\}$$

[†]See Remark on page 24.

$$\leq \frac{\sigma}{n} a_{i_0 j_1} + \Omega^{(2)}_{n-1}(\sigma, j_1) + \frac{1-\sigma}{n} \max_{i \in I} \max_{j \in J} a_{ij} \leq \frac{1}{n} \|A\| + \Omega^{(2)}_{n-1}(\sigma; j),$$

$$j \in J,$$

and similarly beginning with $\Omega^{(2)}_n(\sigma; j_0)$,

$$\Omega^{(2)}_n(\sigma, j_0) \leq \frac{1}{n} \|A\| + \Omega^{(1)}_{n-1}(\sigma, i), \qquad i \in I,$$

we see that if either of the two sequences (6.1) converges the other must also converge to the same limit.

Next since for any $n = 0, 1, \ldots$

$$\Omega^{(1)}_{n+1}(\sigma, i_0) = \min_{j_1 \in J} \max_{i_1 \in I} \left\{ \frac{1}{n+1}[\sigma a_{i_0, j_1} + (1-\sigma) a_{i_1, j_1}] + \frac{n}{n+1} \Omega^{(1)}_n(\sigma, i_1) \right\}$$

with $\Omega^{(1)}_0$ arbitrary we see that

$$\Omega^{(1)}_{n+1}(\sigma, k) = T^{(1)}_k\left(\frac{1}{n+1}; \Omega^{(1)}_n(\sigma, \cdot)\right), \qquad k \in I,$$

where $T^{(1)}$ is the min-max transformation (4.1). Similarly

$$\Omega^{(2)}_{n+1}(\sigma, \ell) = T^{(2)}_\ell\left(\frac{1}{n+1}; \Omega^{(2)}_n(\sigma, \cdot)\right), \qquad \ell \in J.$$

Application of theorems 2, 3 and 4 completes the proof.

Although the equations (T5.1) or (T5.2) can be used to compute $\Omega(\sigma)$ (by finding $\tilde{x}(\lambda)$ for all λ is some positive neighborhood of zero and computing the limit $\lambda \to 0$) there is a related pair of systems of equations yielding the omega-value directly.

<u>Theorem 6</u>: The system of N equations

$$\omega = \min_{j \in J} \max_{i \in I} \{\sigma a_{kj} + (1-\sigma) a_{ij} + y_i - y_k\}, \qquad k \in I, \qquad \text{(T6.1)}$$

in $N + 1$ unknowns $\omega, y_1, \ldots, y_N$ has always a solution. Further, if $(\omega, y_1, \ldots, y_N)$ is any solution of this system then

$$\omega = \Omega(\sigma).$$

Proof: Let $\tilde{x}_k(\lambda)$, $k \in I$ be the solution of (T5.1), and let $\tilde{y}_k(\lambda)$ be defined by

$$\tilde{x}_k(\lambda) = \omega + \lambda \tilde{y}_k(\lambda), \qquad k \in I.$$

Substituting into (T5.1) we obtain

$$\omega + \lambda \tilde{y}_k(\lambda) = \min_{j \in J} \max_{i \in I} [\lambda A^k_{ij} + (1-\lambda)\omega + \lambda(1-\lambda)\tilde{y}_i(\lambda)], \qquad k \in I,$$

where again

$$A^k_{ij} = \sigma a_{ki} + (1-\sigma) a_{ij}.$$

Subtracting ω and dividing by $\lambda > 0$ this becomes

$$\tilde{y}_k(\lambda) = \min_{j \in J} \max_{i \in I} [A^k_{ij} - \omega + (1-\lambda)\tilde{y}_i(\lambda)]. \qquad (6.2)$$

By (T2.2) of Theorem 2 the functions $\tilde{x}_k(\lambda)$ are uniformly Lipschitzian in $\lambda \in (0, \lambda_0]$. Hence

$$\lim_{\lambda \to 0+} \tilde{y}_k(\lambda) = y_k, \qquad k \in I$$

exists and (6.2) yields

$$y_k = \min_{j \in J} \max_{i \in I} [A^k_{ij} - \omega + y_i], \qquad k \in I,$$

which is the same system as (T6.1). Since by Theorem 5

$$\lim_{\lambda \to 0+} \tilde{x}_k(\lambda) = \Omega(\sigma), \qquad k \in I,$$

the system (T6.1) has at least one solution $(\omega, y_1, \ldots, y_N)$ with $\omega = \Omega(\sigma)$.

To show that the ω-component of any solution of (T6.1) is the omega-value we must prove that if $(\omega, y_1, \ldots, y_N)$ and $(\omega', y_1', \ldots, y_N')$ are any two solutions of this system then necessarily $\omega = \omega'$.

Let for any real $y_1, \ldots, y_k$ and $k \in I$

$$F_k(y_1, \ldots, y_N) = \min_{j \in J} \max_{i \in I} [A_{ij}^k + y_i - y_k]. \tag{6.3}$$

By assumption

$$F_k(y_1, \ldots, y_N) = \omega \quad \text{and} \quad F_k(y_1', \ldots, y_N') = \omega' \quad \text{for all} \quad k \in I.$$

Write $y_i' = y_i + z_i$, $i \in I$ and let $r \in I$ and $s \in I$ be such that

$$z_r = \min\{z_1, \ldots, z_N\}, \quad z_s = \max\{z_1, \ldots, z_N\}.$$

Define $y_i^{(r)} = y_i + z_i - z_r$, $y_i^{(s)} = y_i + z_i - z_s$, $i \in I$. Since adding a constant tc all variables does not change the value of the function F_k we must have

$$F_k(y_1^{(r)}, \ldots, y_N^{(r)}) = F_k(y_1^{(s)}, \ldots, y_N^{(s)}) = \omega' \quad \text{for all} \quad k \in I.$$

Notice that $y_r^{(r)} = y_r$, $y_s^{(s)} = y_s$ and

$$y_i^{(r)} \geq y_i \geq y_i^{(s)} \quad \text{for all} \quad i \in I. \tag{6.4}$$

Now consider the identity

$$\omega - \omega' = \sum_{i=1}^{n} [F_r(y_1^{(r)}, \ldots, y_{i-1}^{(r)}, y_i, \ldots, y_N) - F_r(y_1^{(r)}, \ldots, y_i^{(r)}, y_{i+1}, \ldots, y_N)].$$

Notice that since $y_r^{(r)} = y_r$ the $r^{\underline{th}}$ bracket is zero while from (6.3) and (6.4) it is easy to see that all the other terms in brackets are non-positive. Hence

$$\omega - \omega' \leq 0.$$

On the other hand in the identity

$$\omega - \omega' = \sum_{i=1}^{n} [F_s(y_1^{(s)},\ldots,y_{i-1}^{(s)},y_i,\ldots,y_N) - F_s(y_1^{(s)},\ldots,y_i^{(s)},y_{i+1},\ldots,y_N)]$$

the $s^{\underline{th}}$ bracket is zero while by (6.3) and (6.4) all others are nonnegative. Thus

$$\omega - \omega' \geq 0$$

and the proof is complete.

Starting with the max-min transformation and repeating the above argument we obtain a dual theorem.

Theorem 7: The system of M equations

$$\omega = \max_{i\in I} \min_{j\in J} \{(1-\sigma)a_{i\ell} + \sigma a_{ij} + y_i - y_\ell\}, \qquad \ell \in J, \tag{T7.1}$$

in $M + 1$ unknowns $\omega, y_1, \ldots, y_M$ has always a solution. Further, if $(\omega, y_1, \ldots, y_M)$ is any solution of this system then

$$\omega = \Omega(\sigma).$$

The systems (T6.1) or (T7.1) are particulary easy to solve if $N = 2$ or $M = 2$ respectively. Just replace the difference $y_1 - y_2$ by a single variable $z = y_1 - y_2$, plot the two functions of z

$$F_1(z) = \min_{j \in J} \max \{a_{1j}, \sigma a_{1j} + (1-\sigma)a_{2j} - z\},$$

$$F_2(z) = \min_{j \in J} \max \{\sigma a_{2j} + (1-\sigma)a_{1j} + z, a_{2j}\},$$

and compute the ordinate ω of their intersection. This way we can arrive at a general expression for the omega-value of a two-by-two matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

namely

$$\Omega(\sigma) = \begin{cases} V_1 & \text{if } w(\sigma) \le V_1, \\ w(\sigma) & \text{if } V_2 < w(\sigma) < V_1, \\ V_2 & \text{if } V_2 \le w(\sigma), \end{cases}$$

where $V_1 = \min_i \max_j a_{ij}$, $V_2 = \max_i \min_j a_{ij}$, and

$$w(\sigma) = \frac{1}{2}[\sigma(V_2 + \min_{i,j} a_{ij}) + (1-\sigma)(V_1 + \max_{i,j} a_{ij})].$$

This is the same formula established in [3] directly from the system (T5.1).

Of course, if $N > 2$ and $M > 2$ finding the solution of (T6.1) or (T7.1) is no longer such a simple matter. We would like to mention, however, that Alan Washburn [4] has recently discovered quite an efficient algorithm for solving these systems.

To the end we present some general properties of the omega-value as a function of σ. Few of them follow directly from the definition, for instance

$$\Omega(0) = \min_{j\in J}\max_{i\in I} a_{ij}, \quad \Omega(1) = \max_{i\in I}\min_{j\in J} a_{ij},$$

or

$$\alpha > 0, -\infty < \beta < +\infty \Rightarrow \Omega_{\alpha A+\beta I}(\sigma) = \alpha\Omega_A(\sigma) + \beta,$$

where the subscript denotes the matrix in question. Similarly from the relation between the min-max and max-min transformations discussed in Section 5 we have

$$\Omega_A(\sigma) = -\Omega_{-A^t}(1-\sigma),$$

which in particular implies that if the matrix A is skew-symmetric then $\Omega_A(\sigma) = -\Omega_A(1-\sigma)$.

Some of the less obvious but perhaps not surprising properties are states as the following last theorem of this paper.

Theorem 8: The omega value $\Omega(\sigma)$, $\sigma \in [0,1]$ is a continuous, nonincreasing, piece-wise linear function of σ.

Proof. To prove continuity we employ the min-max transformation. Let $\lambda > 0$, $\sigma_1 \in [0,1]$, $\sigma_2 \in [0,1]$. Using again the inequalities (4.2) and (4.3) we have

$$\|\tilde{x}(\lambda,\sigma_1) - \tilde{x}(\lambda,\sigma_2)\| \le \max_{k\in I}\max_{j\in J}\max_{i\in I} | \lambda[A^k_{ij}(\sigma_1) - A^k_{ij}(\lambda_2)]$$

$$+ (1-\lambda)[\tilde{x}_i(\lambda,\sigma_1) - \tilde{x}_i(\lambda,\sigma_2)] \quad \le \quad \lambda\max_{k\in I}\max_{j\in J}\max_{i\in I} | A_{ij}(\sigma_1) - A_{ij}(\sigma_2) |$$

$$+ (1-\lambda)\|\tilde{x}(\lambda,\sigma_1) - \tilde{x}(\lambda,\sigma_2)\| \qquad (6.5)$$

However,

$$\max_{k \in I} \max_{j \in J} \max_{i \in I} \mid A^k_{ij}(\sigma_1) - A^k_{ij}(\sigma_2) \mid \le 2\|A\| \mid \sigma_1 - \sigma_2 \mid$$

so that (6.5) becomes

$$0 \le 2\lambda\|A\| \mid \sigma_1 - \sigma_2 \mid - \lambda\|\tilde{x}(\lambda,\sigma_1) - \tilde{x}(\lambda,\sigma_2)\|.$$

Dividing by λ and applying Theorem 5 we obtain

$$\mid \Omega(\sigma_1) - \Omega(\sigma_2) \mid \le 2\|A\| \mid \sigma_1 - \sigma_2 \mid,$$

which implies continuity.

The remaining two properties of $\Omega(\sigma)$ are proved with the aid of Theorem 6. Fix $\sigma \in [0,1]$ and consider the system of equation (T6.1). Let for each $k \in I$, $p(k) \in I$ and $q(k) \in J$ be such that

$$\omega = A^k_{p(k),q(k)} + y_{p(k)} - y_k. \tag{6.6}$$

Next let $K = p(I)$ be the image of I under the mapping $p(\cdot)$, and let m be the number of elements in K. Since p restricted to K is a bijection by adding (6.6) over $k \in K$ we obtain

$$m\omega = \sum_{k \in K} A^k_{p(k),q(k)} = \sigma \sum_{k \in K} a_{k,q(k)} + (1-\sigma) \sum_{k \in K} a_{p(k),q(k)}. \tag{6.7}$$

By Theorem 6 for every $\sigma \in [0,1]$ there exist $p(\cdot)$ and $q(\cdot)$ such that $\Omega(\sigma) = \omega$ satisfies (6.7). However, since there is only a finite number of mappings $p(\cdot)$ and $q(\cdot)$ and since $\Omega(\sigma)$ is continuous it must in fact consist of a finite number of linear segments of the form

$$\Omega(\sigma) = \frac{\sigma}{m} \sum_{k \in K} a_{k,q(k)} + \frac{1-\sigma}{m} \sum_{k \in K} a_{p(k),q(k)}, \tag{6.8}$$

where $\sigma_1 \le \sigma \le \sigma_2$.

Finally to prove that $\Omega(\sigma)$ is nonincreasing notice that from (T6.1) and by the definition of $q(k)$ we have

$$\omega \geq A^k_{i,q(k)} + y_i - y_k$$

for every $i \in I$, $k \in I$. Hence, taking $i = k$ and summing over the set K we obtain

$$n\omega \geq \sum_{k \in K} A^k_{k,q(k)} = \sigma \sum_{k \in K} a_{k,q(k)} + (1-\sigma) \sum_{k \in K} a_{k,q(k)}.$$

Comparison with (6.7) yields

$$\sum_{k \in K} a_{k,q(k)} \leq \sum_{k \in K} a_{p(k),q(k)}$$

for every $\sigma \in [0,1)$, which together with (6.8) completes the proof.

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[4] Washburn, Alan R., (paper in preparation).

Remark (Added in the proof.)

After this report had been completed the author's attention was drawn to the paper by R. J. Elliott, A. Friedman and N. J. Kalton, "Alternate Play in Differential Games" (Journal of Differential Equations, 15, May 1974, p. 694), which contains a simple proof of the existence of the omega-value. The proof consists of showing that the sequence Ω_n, of which the omega-value is the limit, is a Cauchy sequence. However, this method of proving the existence does not yield the equations (T5.1-2) nor further results contained in this report.