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# COLLOCATION AND FINITE ELEMENTS - A COMBINED METHOD

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13. ABSTRACT

Discusses advantages/disadvantages of the finite element techniques and of the modified mapping-collocation (MMC) techniques for analytical solutions of fracture mechanics problems. Reports one of several AMMRC investigations toward development of a procedure aimed at combining the two techniques to take advantage of the best features of each. Presents examples and their solutions using first the MMC formulation and then the finite element formulation; lastly, the examples are addressed using a combined methods formulation. Results and observations relative to the three methods are discussed. (Author)



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## CONTENTS

# COLLOCATION AND FINITE ELEMENTS - A COMBINED METHOD

C. E. FREESE

## INTRODUCTION

The ever-increasing use of numerical methods for the analytical solutions of fracture mechanics problems has led to a need for a general technique capable of solving a wide variety of mechanics problems. A reluctance to abandon the obvious advantages offered by a single numerical technique has often led to a sacrifice of economy and accuracy. In such an instance, the economic factors must not only include the cost of a numerical solution but also the time and effort necessary to extend the capabilities of an existing method into areas beyond which it was initially designed.

In the case of finite elements, an accurate solution to stress concentration and even stress singularity problems has led to much manipulation of the basic finite element formulation. It is equally true that methods such as the modified mapping-collocation techniques (hereafter referred to as the MMC method) [1], designed explicitly to handle areas of high stress gradients and stress singularites, suffers the need of additional cost and formulation when boundary conditions and areas of relatively low stress gradients are remote to the high concentration area. Another factor, most apparent to the novice user, is that terms such as "adequate representation" and "remote conditions" are quite problem-dependent and not well defined. There is, however, an immediate confrontation with many such expressions in the initial phases of any numerical application. More explicitly this means a choice of element types, mesh representation, or perhaps a partitioning scheme [2]. Furthermore, the number of degrees of freedom necessary to achieve a given accuracy in finite elements greatly depends on the representation of that solution in the zones of high stress gradient. The MMC method finds the inverse to be true, the more remote the area the higher the number of degrees of freedom necessary. It is quite evident then, a solution which would combine the advantages of both methods as applied to areas for which they are best suited should lead to a more accurate solution for a given number of degrees of freedom.

Perhaps more important, from the standpoint of application, is the fact that most problems contain a broad band for which either method provides an adequate solution. Thus, considerable freedom exists when a choice of locating the interface boundaries becomes necessary. This useful property first was observed in the successful application of partitioning techniques with the MMC method [2]. The AMMRC team of investigators quickly recognized the possibility of further optimizing the procedure by an appropriate combination of the MMC and finite element techniques. This paper reports one of several investigations by members of this AMMRC group toward the development of a combined procedure.

1. BOWIE, O. L., and NEAL, D. M. *A Modified Mapping-Collocation Technique for Accurate Calculation of Stress Intensity Factors.* Army Materials and Mechanics Research Center, AMMRC TR 69-28, November 1969; also Intern. J. Fracture Mech., v. 6, 1970, p. 199-206.
2. BOWIE, O. L., FREESE, C. E., and NEAL, D. M. *Solution of Plane Problems of Elasticity Utilizing Partitioning Concepts.* J. of Applied Mech., v. 40, 1973, p. 767-772.

## MMC FORMULATION

The successful application of the MMC method to finite, two-dimensional, isotropic and anisotropic problems for a wide variety of discontinuities has demonstrated the versatility of the technique [1] [3] [4] [5]. It is not the intent here to describe in detail the formulation of either the MMC method or the finite element method. However, a general outline and a specific example will be defined in order to demonstrate the successful combination of the two methods.

In particular, the MMC method can be said to differ uniquely from the more common collocation techniques in four separate areas.

1. A very simple mapping function is usually used to transform the discontinuity into a coordinate system more conducive to the application of the arguments offered later with respect to analytic functions.

2. A partial satisfaction of boundary conditions is oftentimes achieved through a reflection argument.

3. The formulation is then reduced to the determination of the coefficients of a Laurent series based on the force and/or displacement conditions as defined by the boundary conditions of the problem. This deviates slightly from the more conventional collocation techniques which attempt to satisfy point-wise stress boundary conditions, often leading to gross off-point stress errors.

4. The final step employs a least-squared solution for the satisfaction of all boundary conditions applied to a truncated version of the Laurent series in order to determine an accurate solution to the dominant terms of the expansion.

The specific examples used in this paper to demonstrate the application of the two methods is the plane stress solution of a noneccentric circular cutout in a square tensile sheet (Figure 1). An accurate solution to this problem is quite within the range of both finite elements and the MMC methods.

The specific application of the MMC method would take the following form:

1. A mapping function of the form

$$Z = \omega(\zeta) = \{(a+b)/2\}\zeta + \{(a-b)/2\}\zeta^{-1} \tag{1}$$

would map the circle in the physical plane Z into the unit circle of the parameter plane $\zeta$. In this particular case a simple scaling would accomplish the same effect, but for purposes that will become clear later a more general function is assumed.

3. BOWIE, O. L., and NEAL, D. M. *A Note on the Central Crack in a Uniformly Stressed Strip.* Eng. Fracture Mech., v. 2, 1970, p. 181-182.

4. BOWIE, O. L., and FREESE, C. E. *Elastic Analysis for a Radial Crack in a Circular Ring.* Army Materials and Mechanics Research Center, AMMRC MS 70-3, September 1970; also Eng. Fracture Mech., v. 4, 1972, p. 315-321.

5. BOWIE, O. L., and FREESE, C. E. *Central Crack in Plane Orthotropic Rectangular Sheet.* Intern. J. Fracture Mech., v. 8, 1972, p. 49-58.

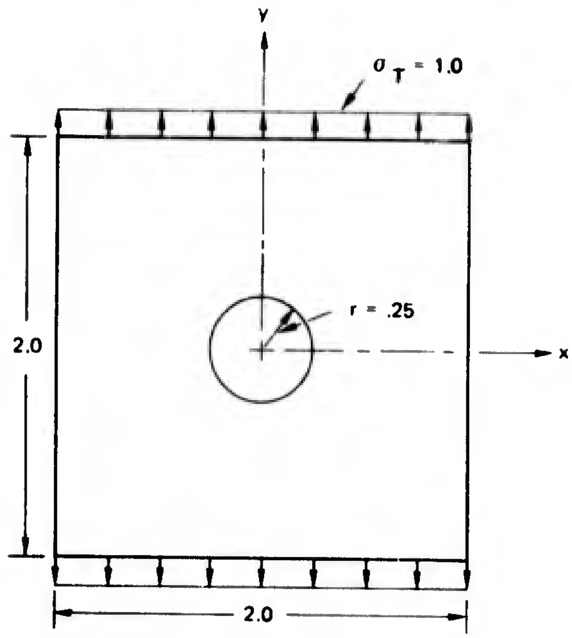


Figure 1. Circular cutout in finite tension plate

2. Using the notation offered by Muskhelishvili [6], we are able to write the force and displacement equations in terms of the two analytic functions $\phi(Z)$ and $\psi(Z)$

$$\phi(Z) + Z\overline{\phi'(Z)} + \overline{\psi(Z)} = i\int (X_n + iY_n)ds = f_1 + if_2 \tag{2}$$

$$(3 - \nu) / (1 + \nu) \quad \phi(Z) - Z\overline{\phi'(Z)} - \overline{\psi(Z)} = 2G\ (u + iv) \tag{3}$$

where bars denote complex conjugates, primes denote derivatives, $G = E/2(1+\nu)$, E is Young's Modulus, $\nu$ is Poisson's Ratio, and $(X_n, Y_n)$, $(u, v)$ are rectangular force and displacement components, respectively.

We can then define

$$\phi(Z) = \phi(w(\zeta)) = \phi(\zeta),\ \phi'(Z) = \phi'(\zeta)/w'(\zeta), \psi(Z) = \psi(\zeta), \text{ etc.}$$

By reflection with respect to the unit circle and the condition that the unit circle be load-free we are able to write the function $\psi(\zeta)$ in terms of the function $\phi(\zeta)$ [7]

$$\psi(\zeta) = -\overline{\phi}(1/\zeta) - \overline{w}(1/\zeta)\phi'(\zeta)/w'(\zeta). \tag{4}$$

Substituting this definition, Equations (2) and (3) become

6. MUSKHELISHVILI, N. I. *Some Basic Problems of Mathematical Theory of Elasticity*. Nordhoff, Groningen, Holland, 1953.
7. KARTZIVADZE, I. N. *Comp. Rend. de l'acad. Sc. de l'U.R.S.S.*, v. 20, 1943, p. 95.

$$\phi(\zeta) - \phi(1/\bar{\zeta}) + [w(\zeta)-w(1/\bar{\zeta})]\ \overline{\phi'(\zeta)}/\overline{w'(\zeta)} = f_1+if_2 \tag{5}$$

$$(3-\nu)/(1+\nu)\phi(\zeta) + \phi(1/\bar{\zeta}) - [w(\zeta)-w(1/\bar{\zeta})]\ \overline{\phi'(\zeta)}/\overline{w'(\zeta)} = 2G(u+iv) \tag{6}$$

Likewise, the stress equations can be written

$$\sigma_x + \sigma_y = 4\mathrm{Re}\{\phi'(z)\} = 4\mathrm{Re}\ \{\phi'(\zeta)/w'(\zeta)\},\ \text{etc.} \tag{7}$$

3. A Laurent series of the form

$$\phi(\zeta) = \sum_{-\infty}^{\infty} \alpha_n\ \zeta^n \tag{8}$$

is chosen to replace the remaining function $\phi(\zeta)$. The validity and completeness of such a substitution are primarily based on its successful application to previous problems.

4. Assuming further a truncated version of the Laurent series together with the obvious stress symmetries of the problem

$$\phi(\zeta) = \sum_{n=m}^{-p} \alpha_n\ \zeta^{2n+1} \tag{9}$$

where $\alpha_n$'s are real.

The boundary conditions for points on the outer boundary are simply stated as

$$f_1 = x,\ f_2 = 0 \tag{10}$$

where $Z = x+iy$ and the number of stations chosen should be greater than $(p + m)$ or more precisely the number of equations generated should be at least twice the number of degrees of freedom.

The results of a solution computed for $m = 8$, $p = 8$ are listed in Table 1.

## FINITE ELEMENT FORMULATION

Finite elements in general require the solution of a set of linear equations of the general form

$$\{S\} = [B]^T [D] [B] \{\delta\}$$

or

$$\{S\} = [K] \{\delta\} \quad (11)$$

where S has force components $S_x$ and $S_y$ and $\delta$ has displacement components u and v [8]. Inherent to the method is an assumption as to the variance of the displacement field within each element.

A solution to demonstrate the predictable difficulties in the use of constant stress triangular elements was computed and the results for the two extreme stress concentration values inserted in Table 1. It is true that not much attention was given to the design of the mesh field (Figure 2) as well as the number of degrees of freedom. However, this is exactly the type of problem which we would like to eliminate when using finite elements.

Additional solutions were determined using the more advanced cubic "serendipity" elements [9] (Figure 3) which seem to converge quite rapidly toward the MMC solution. We will assume, since evidence in the past has shown quite accurate results for this type problem, the MMC solution to have produced the correct results.

An important point which we will take advantage of when formulating the combined solution is that the vertical and horizontal displacement components

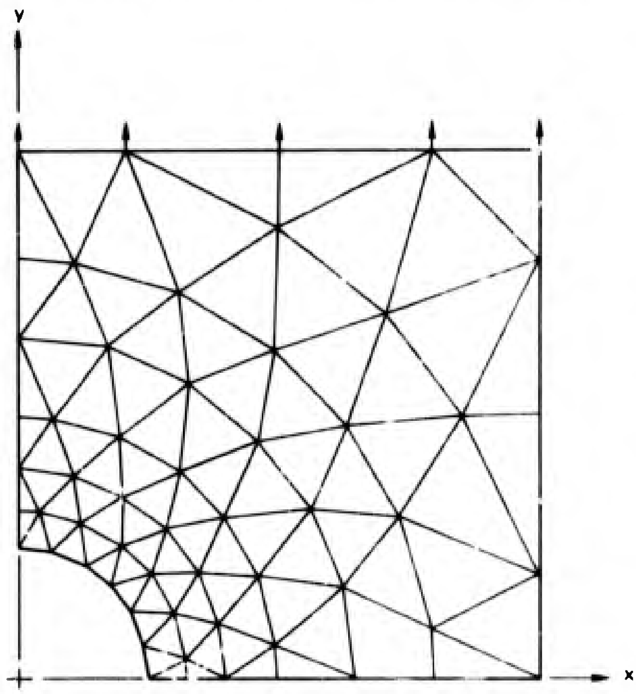


Figure 2. Triangular finite element grid

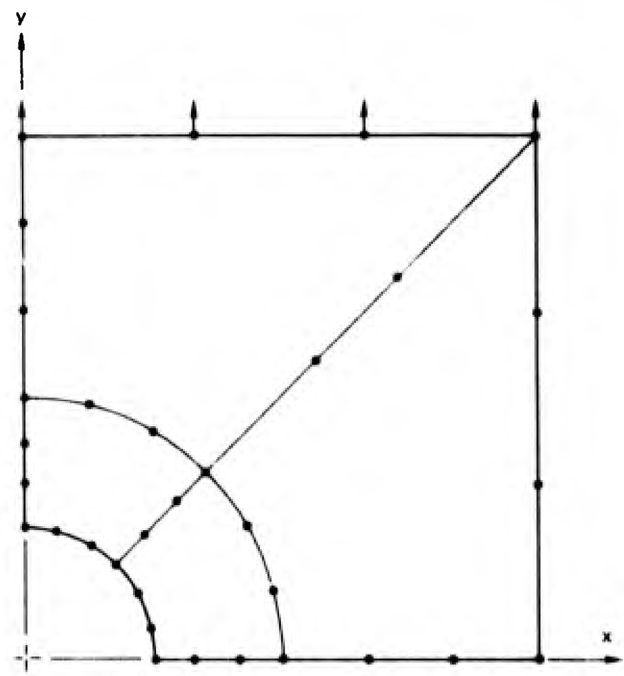


Figure 3. Cubic isoparametric mesh

8. WILSON, E. L. *Finite Element Analysis of Two-Dimensional Structures.* Ph.D. Dissertation, University of California, 1963.
9. ZIENKIEWICZ, O. C. *The Finite Element Method in Engineering Science.* McGraw-Hill, London, 1971.

on an arc of two and three times the radius of the circular cutout are not greatly affected by inaccuracies on the circle itself. This is a point often stressed in the formulation of a finite element solution. In fact, if we were to take the displacement components along an arc equal to twice the radius of the cutout (Figure 3) and use these values as boundary conditions for the MMC formulation the result is identical to the MMC solution for the entire problem. This leads to renewed interest in the possibilities of combining the two methods.

## MMC AND FINITE ELEMENT FORMULATION

In terms of the stiffness matrix formulation of finite elements (referring to Figure 4)

$$\begin{bmatrix} S_1 \\ \vdots \\ S_{13} \\ \hline S_{14} \\ \vdots \\ S_{20} \\ \hline S_{21} \\ \vdots \\ S_{33} \end{bmatrix} = \left[\begin{array}{c:c} K_{1,1} & K_{1,2} \\ \hdashline K_{2,1} & K_{2,2} \\ \hdashline K_{3,1} & K_{3,2} \end{array}\right] \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{13} \\ \hline \delta_{14} \\ \vdots \\ \hline \\ \delta_{33} \end{bmatrix} \tag{12}$$

We can also write

$$\begin{Bmatrix} S_1 \\ \cdot \\ \cdot \\ \cdot \\ S_{13} \end{Bmatrix} = [K_{1,1}] \begin{Bmatrix} \delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta_{13} \end{Bmatrix} + [K_{1,2}] \begin{Bmatrix} \delta_{14} \\ \cdot \\ \cdot \\ \cdot \\ \delta_{33} \end{Bmatrix} \tag{13}$$

and

$$\begin{Bmatrix} S_{14} \\ \cdot \\ \cdot \\ \cdot \\ S_{20} \end{Bmatrix} = [K_{2,1}] \begin{Bmatrix} \delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta_{13} \end{Bmatrix} + [K_{2,2}] \begin{Bmatrix} \delta_{14} \\ \cdot \\ \cdot \\ \cdot \\ \delta_{33} \end{Bmatrix} \tag{14}$$

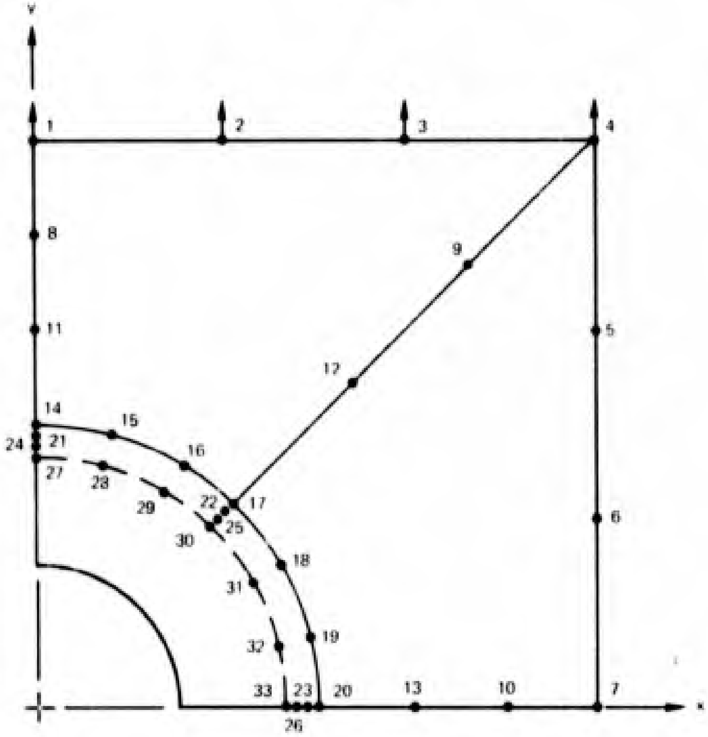


Figure 4. Simple cubic grid for combined solution

If we let $[K_A] = [K_{2,1}]\ [K_{1,1}]^{-1}$ and replace the unknown displacements $\delta_1 \ldots \delta_{13}$ in Equation (14) by their equivalent from Equation (13)

$$\begin{Bmatrix} S_{14} \\ \cdot \\ \cdot \\ \cdot \\ S_{20} \end{Bmatrix} = [K_A] \begin{Bmatrix} S_1 \\ \cdot \\ \cdot \\ \cdot \\ S_{13} \end{Bmatrix} + \left([K_{2,2}] - [K_A]\ [K_{1,2}]\right) \begin{Bmatrix} \delta_{14} \\ \cdot \\ \cdot \\ \cdot \\ \delta_{33} \end{Bmatrix} \tag{15}$$

We are also able to generate the remaining unknown displacement equations in terms of the unknown coefficients of the Laurent series

$$\begin{Bmatrix} \delta_{14} \\ \cdot \\ \cdot \\ \cdot \\ \delta_{33} \end{Bmatrix} = [C] \begin{Bmatrix} \alpha_n \end{Bmatrix} \qquad (16)$$

where the number of coefficients is less than the number of displacement equations. If we let $[K_B] = ([K_A]\ [K_{1,2}] - [K_{2,2}])\ [C]$ the least-squared equivalent performed in the MMC method is achieved by solving the set of equations.

$$[K_B]^T \quad [K_A] \quad \begin{Bmatrix} S_1 \\ \cdot \\ \cdot \\ \cdot \\ S_{13} \end{Bmatrix} = [K_B]^T \ [K_B] \begin{Bmatrix} \alpha_n \end{Bmatrix} . \qquad (17)$$

The vector $S_{14} \ldots S_{20}$ must be zero or replaced by known displacement boundary conditions.

Although this procedure certainly does not exhaust all the possibilities for combining the two techniques, it does provide some advantages. There is no attempt to reduce either method to the equivalent of the other. Because of this, no additional formulation is necessary and the application is quite straightforward.

In general, the joining of two regions, either under the same representation or different representation, requires the satisfaction of both compatibility and equilibrium along their interface. If both regions are represented by a finite element formulation an equilibrium solution is attained by imposing zero resultant force conditions at common nodal points. Compatibility is most often established by assuming the same type of element in both regions. Two regions can be combined with the MMC method by prescribing equality of force and displacement conditions at a discrete number of points along the interface [2]. When combining the discrete finite element method with the continuous MMC method the identity of displacements at common points on the interface seems most obvious. However, an equivalent force relationship, without some additional manipulation of the basic equations, does not seem to exist. This is especially true if the finite element region makes use of some of the higher ordered elements. A band of fictitious elements internal to the region defined by the MMC method completes the force equations along the interface in terms of displacements only. The unknown displacements internal to the region defined by the MMC formulation are then replaced by their MMC equivalent. Thus, no attempt is made to generate equivalent force relationships but equilibrium is achieved by completing the finite element formulation through the use of a fictitious element set.

## RESULTS

The combined methods were first applied to the finite element solution which was carried out using the constant stress triangular elements with an interface zone located at twice the radius of the circular cutout (Figure 5). The results, although much improved, still were quite different from the true values. This should not be too surprising since the displacements along this arc were in error in the original finite element solution. However, when the same method was applied to solutions which take advantage of the higher ordered elements (Figure 4) convergence toward the true solution was quite rapid (Table 1). The small percentage of inaccuracy which remained could easily be attributed to either method as to their exact satisfaction of boundary conditions (Figure 6).

A perhaps more important point is that since the MMC method handles a problem of these dimensions quite easily, the need for finite elements to assume responsibility of the solution is necessary along arcs much more remote than twice the radius of the cutout. This would be especially true for distances four or five times the radius of the circle where the MMC method would find greater difficulty yet finite elements would yield quite accurate results.

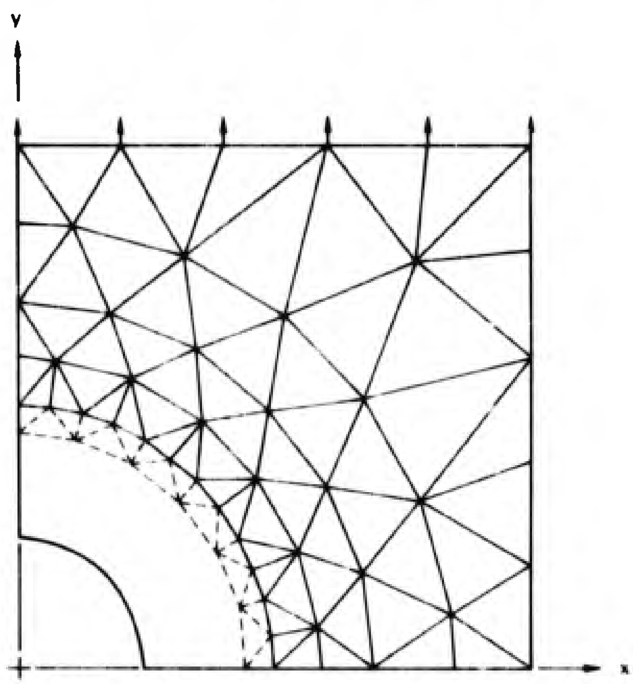


Figure 5. Triangular representation for combined analysis

Table 1. CONVERGENCE WITH DEGREES OF FREEDOM

| Degrees of Freedom | | | $\sigma_\theta(\theta=0^o)$ | $\sigma_\theta(\theta=90^o)$ |
|---|---|---|---|---|
| Finite Element | | MMC | | |
| Linear | Cubic | | | |
| 114 | - | - | 2.905 | - .928 |
| 70 | - | 12 | 3.264 | -1.205 |
| - | 40 | - | 3.394 | -1.036 |
| - | 66 | - | 3.357 | -1.207 |
| - | 26 | 6 | 3.460 | -1.344 |
| - | 48 | 6 | 3.549 | -1.432 |
| - | - | 16 | 3.585 | -1.477 |

E = 30,000 ksi, $\nu$ = .33

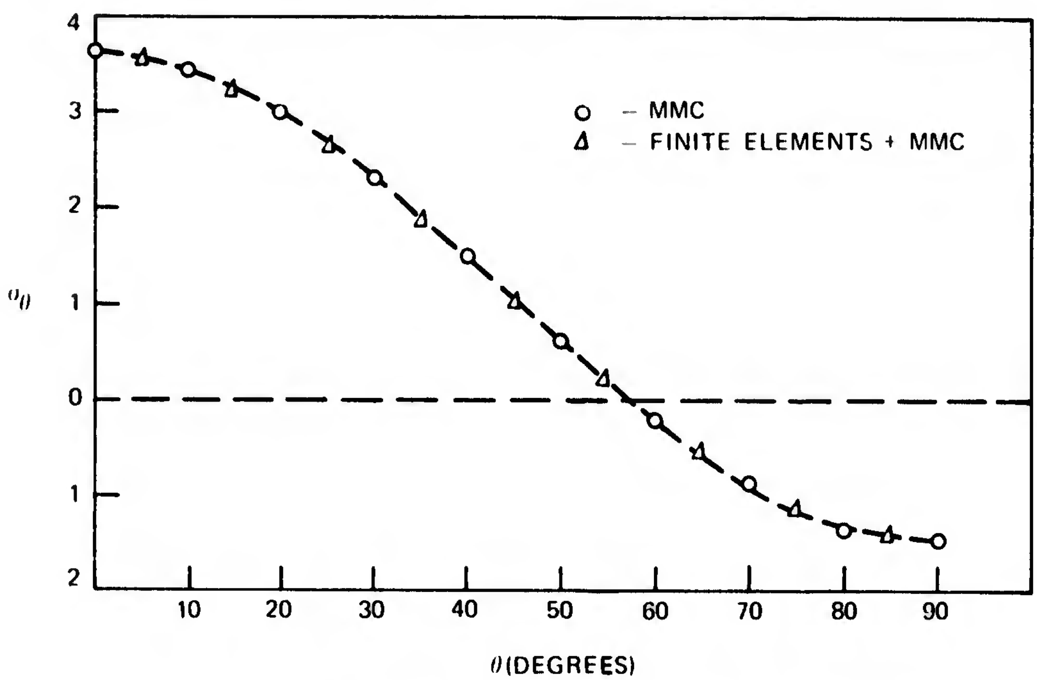


Figure 6. Comparison of cutout stress values for circular hole

In an attempt to prove this statement we can now take advantage of the more general form of the mapping function which we chose in the original formulation of the MMC portion. The variables (a and b) which appear in the mapping function represent the axis of an ellipse. We are therefore able, with a suitable choice of parameters, to generate smaller circular cutouts, ellipses, or even the exact mapping function for the internal crack (b = 0). This is all accomplished without a disturbance of the original finite element mesh field where each subsequent change requires only the regeneration of [C] and the solution of the resulting (6 x 6) system of equations (m = 3, p = 3). The results of solutions carried out on all three types of discontinuities, now more remote from the interface with finite elements, led to values of stress concentrations and stress intensities which were less than one percent inaccurate. Even the constant stress triangular elements gave suitable results (Figure 7).

| a | b | $\sigma_\theta(\theta=0^o)$ | $\sigma_\theta(\theta=90^o)$ | K* |
|---|---|---|---|---|
| .10 | .10 | 3.084 | -1.065 | - |
| .10 | .033 | 7.134 | -1.030 | - |
| .10 | .00 | - | - | 1.014 |
| .20 | .00 | - | - | 1.055 |

*Stress Intensity Factor

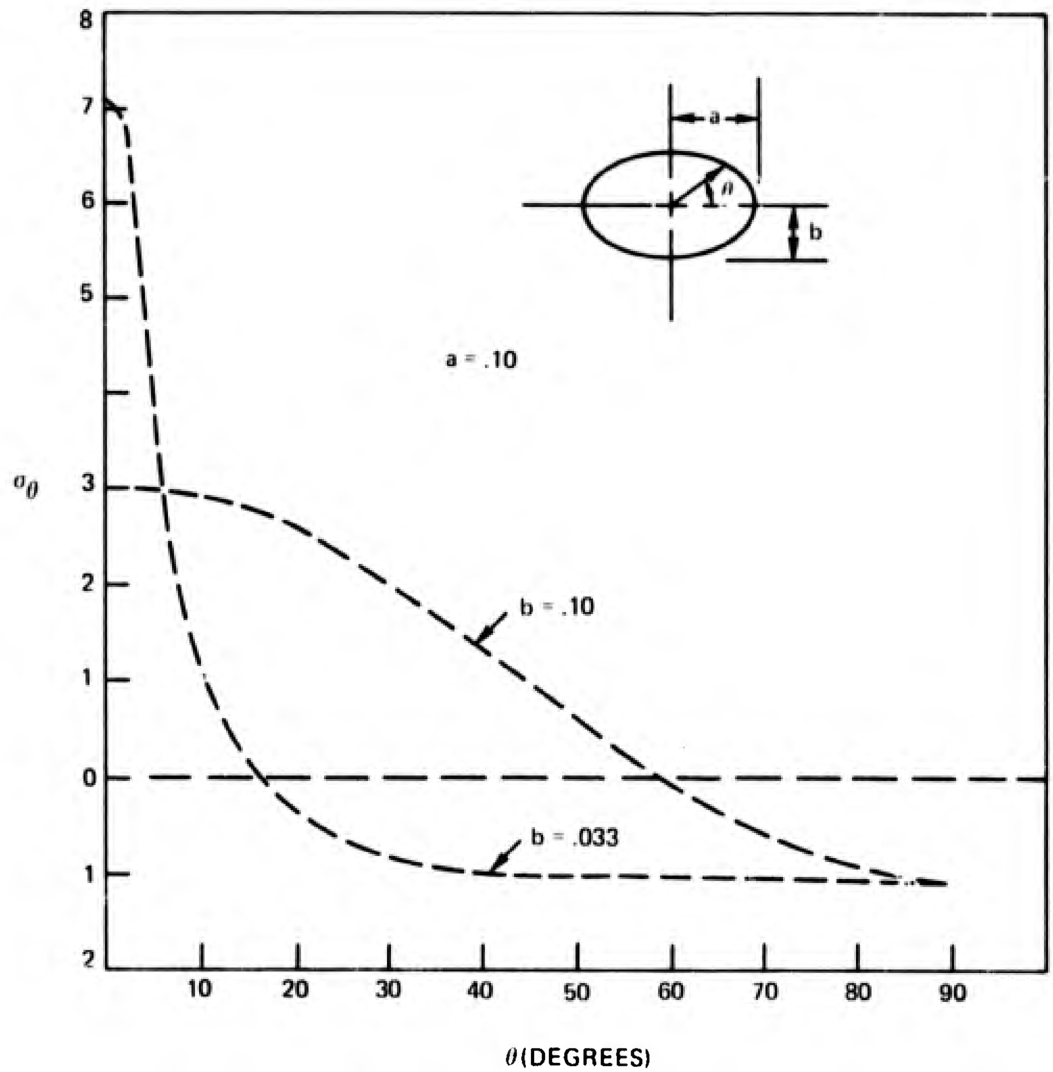


Figure 7. Cutout stress values as a function of aspect ratio

An additional fact that should be obvious is that the MMC method yields a continuous solution and thus the stresses within the zone defined by the MMC solution requires only their evaluation at the points of interest.

## OBSERVATIONS

Although the particular examples chosen here do not demonstrate the areas for which finite elements become necessary to complete an accurate solution, they do demonstrate the value of including the MMC formulation to achieve accuracy in the areas of high stress gradients. Also shown is that when finite elements are necessary for the completion of the formulation (a case that arises often in anisotropic solutions, remote boundary conditions, and discontinuities which are quite close to these boundaries), the choice of representation and the location of the interface will be quite insensitive to the solution.

## ACKNOWLEDGMENT

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