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DERIVATION OF SUFFICIENT CONDITIONS FOR APPLICATION OF THE DISTORTED-WAVE BORN APPROXIMATION TO TURBULENT WAKE BACKSCATTER

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We have derived sufficient conditions under which the distorted-wave Born approximation may be applied to the calculation of the radar cross section of turbulent reentry vehicle wakes. In simplified form the requirements are that $\sin^2\alpha \ll 1$, $(.0752 k_o r_1^2/\gamma L_o)\sin^2\alpha \ll 1$, and $(0.15 k_o^2 L_o r_1/\gamma^{1/2}) \sin^3\alpha \ll 1$, where α is the aspect angle, k_o is the signal wavenumber, r_1 is

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) the wake radius, L_o is the scale size of the turbulent eddies, and $\gamma = -\ln[(n_c^2/n_o^2)\sin^4\alpha]$, where n_c is the critical electron density and n_o is the average electron density on the wake axis.

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Derivation of Sufficient Conditions for Application of the Distorted-Wave Born Approximation to Turbulent Wake Backscatter

1. INTRODUCTION

The distorted-wave Born approximation has been used with some success recently* in predicting the radar cross section of turbulent wakes illuminated at small aspect angles. In this model the actual incident electric field within the turbulent plasma is replaced by the electric field that would be present if the random electron density n_e in the plasma is replaced by the average electron density $\bar{n}_e$. (This field is then used in the right-hand side of the integral equation for the scattered field to obtain the scattering cross section.) This has been termed a "dishonest method" by Keller;[1] and strictly speaking it should not be valid. However, its proponents can point to the fact that, while it should not

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*Among the corporations employing this model, with various modifications, are General Electric, AVCO Everett Research Laboratory, Stanford Research Institute, Aerosyne Corp, and Physical Sciences, Inc.

[1] Keller, J. (1964) Stochastic equations and wave propagation in random media, Proc. of Symposia in Applied Mathematics, Vol XVI (edited by R. Bellman) American Mathematical Society, Providence, R.I.

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work, it does give reasonable results[2-5] for many laboratory experiments and flight test vehicles. Unfortunately, there presently exists some confusion about the conditions under which this model can be employed; therefore, in this report we will carefully derive sufficient conditions for application of the distorted-wave Born approximation, and apply them specifically to the problem of radar backscatter from the turbulent wakes of reentry vehicles.

2. DERIVATION OF SUFFICIENT CONDITIONS FOR APPLICATION OF DISTORTED-WAVE BORN APPROXIMATION

Let us consider the case of a turbulent plasma in which the scale length of both the mean electron density and the fluctuation about the mean is large in comparison with the signal wavelength. In this case if a plane electromagnetic wave is incident on the plasma the incident electric field within the plasma can be approximated by

$$E_i(\underline{x}) \simeq \left\{ \varepsilon^{1/2}(\underline{x}) \frac{d\sigma[\varepsilon(\underline{x})]}{d\sigma_o} \right\}^{-1/2} \exp\left\{ ik_o \int_S \sqrt{\varepsilon}\, ds \right\} , \tag{1}$$

where $\varepsilon = \overline{\varepsilon} + \delta\varepsilon$ is the relative permittivity of the plasma, $d\sigma/d\sigma_o$ is the ratio of the area of the ray tube at position $\underline{x}$ within the plasma to that outside the plasma, k_o is the signal wavenumber in vacuum, and S is the stochastic ray path. The notation $d\sigma[\varepsilon]$ denotes that the area of the ray tube depends on the random variable ε. Equation (1) fails, of course, near the caustic (turning point) where $\varepsilon^{1/2} d\sigma/d\sigma_o = 0$; in this region a different approximation must be used. For the incident field given by Eq. (1), the field scattered by a plasma region of volume V is

$$E_s \simeq \frac{k_o^2 e^{ik_oR}}{4\pi R} \iiint_V d^3x\, E_i(\underline{x})(\varepsilon - 1) e^{-i\underline{k}_s \cdot \underline{x}} , \tag{2}$$

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2. Finson, M., Monsler, M., and Kaplan L. (1970) *Aspect-Dependence Wake Scattering—Final Report*, AVCO Everett Research Laboratory, Document No. 70-945.
3. Finson, M. (1971) *Reentry Experiments Analysis and Predictions—Final Report*, AVCO Everett Research Laboratory, Document 71-520.
4. Finson, M. (1972) *Interpretation of Recent Delco Ballistic Range Observations*, AVCO Everett Research Laboratory, Document 72-385.
5. Guthart, H. and Graf, K. (1974) Radar backscatter by turbulent wakes, *Radio Science* (to be published).

where R is the distance from the center of the plasma volume to the observer and k_o is the wave vector in the direction of the observer. Equation (2) neglects the scattered component of the field within the plasma. That is, in Eq. (2), instead of E_i we should have $E_i + E_M$ where E_M is the multiply scattered field within the plane. We will comment later on the condition necessary for the neglect of E_M in comparison with E_i. From Eqs. (1) and (2) we then have for the ensemble-averaged scattered power

$$\langle E_s E_s^* \rangle = \frac{k_o^4}{(4\pi R)^2} \iiint_V d^3x \iiint_V d^3x' \, e^{i\underline{k}_s \cdot (\underline{x}-\underline{x}')} \Big\langle [\varepsilon(\underline{x})-1] \; [\varepsilon(\underline{x}')-1]$$

$$\times \, \varepsilon^{-1/4}(\underline{x}) \, \varepsilon^{-1/4}(\underline{x}') \left\{ \frac{d\sigma[\varepsilon(x)]}{d\sigma_o} \, \frac{d\sigma[\varepsilon(x')]}{d\sigma_o} \right\}^{-1/2}$$

$$\times \left\{ \exp \; ik_o \int_S \sqrt{\varepsilon(\underline{x}'')} \, ds'' - ik_o \int_S \sqrt{\varepsilon(\underline{x}'')} \, ds'' \right\} \Big\rangle \qquad (3)$$

where S denotes the random path of the ray which terminates at $\underline{x}$, S' is the path of the ray terminating at $\underline{x}'$, and $\langle \quad \rangle$ denotes an ensemble average.

It is clear that in order to study Eq. (3) we shall have to make some approximations. We first assume that the fluctuation $\delta\varepsilon$ in the relative permittivity is small compared with the average relative permittivity $\bar{\varepsilon}$. Of course, this may not be true near turning points but Eq. (3) is not valid there anyway. If $\delta\varepsilon/\bar{\varepsilon} << 1$ we do not expect fluctuations in the amplitude,

$$\varepsilon^{-1/4} \left(\frac{d\sigma[\varepsilon]}{d\sigma_o} \right)^{-1/2} ,$$

about the mean amplitude,

$$(\bar{\varepsilon})^{-1/4} \left(\frac{d\sigma[\bar{\varepsilon}]}{d\sigma_o} \right)^{-1/2} ,$$

to be very large. If we are willing to accept errors of order of several dB in the radar cross section of the plasma we can then approximate Eq. (3) by

$$\langle E_s E_s^* \rangle = \frac{k_o^4}{(4\pi R)^2} \iiint_V d^3x \iiint_V d^3x' \, e^{-i\underline{k}_s \cdot (\underline{x}-\underline{x}')} \left[\bar{\varepsilon}(\underline{x})\bar{\varepsilon}(\underline{x}')\right]^{-1/4}$$

$$\times \left\{ \frac{d\sigma[\bar{\varepsilon}(x)]}{d\sigma_o} \frac{d\sigma[\bar{\varepsilon}(x')]}{d\sigma_o} \right\}^{-1/2} \Big\langle \left[\varepsilon(\underline{x})-1\right]\left[\varepsilon(\underline{x}')-1\right]$$

$$\times \exp\left\{ ik_o \int_S \sqrt{\varepsilon(\underline{x}'')}\, ds'' - ik_o \int_{S'} \sqrt{\varepsilon(\underline{x}'')}\, ds'' \right\} \Big\rangle . \qquad (4)$$

Since we have assumed that $\delta\varepsilon << \bar{\varepsilon}$ we may further approximate

$$\varepsilon^{1/2} = (\bar{\varepsilon} + \delta\varepsilon)^{1/2} \simeq \bar{\varepsilon}^{\,1/2} + \frac{1}{2}\frac{\delta\varepsilon}{\bar{\varepsilon}^{\,1/2}} ,$$

so that

$$\exp\left\{ ik_o \int_S \sqrt{\varepsilon}\, ds'' - ik_o \int_{S'} \sqrt{\varepsilon}\, ds'' \right\} \simeq \exp\left\{ ik_o \int_S \sqrt{\bar{\varepsilon}}\, ds'' - ik_o \int_{S'} \sqrt{\bar{\varepsilon}}\, ds'' \right\}$$

$$\times \exp\left\{ i\frac{k_o}{2} \int_S \frac{\delta\varepsilon(\underline{x}'')}{\bar{\varepsilon}(\underline{x}'')}\, ds'' - i\frac{k_o}{2} \int_{S'} \frac{\delta\varepsilon(\underline{x}'')}{\sqrt{\bar{\varepsilon}(\underline{x}'')}}\, ds'' \right\} . \qquad (5)$$

In order to put Eq. (4) in a form closer to that of the distorted-wave Born approximation we would next like to be able to replace the stochastic path S by the mean ray path $\bar{S}$ ($\bar{S}$ is the path followed by the ray which reaches the point $\underline{x}$ after traversing a a plasma having the relative dielectric constant $\bar{\varepsilon}$). We would like to determine the error involved in replacing S by $\bar{S}$, and then require that this be small. Let us consider successive scatterings of the ray by the turbulent eddies. From Figure 1 it is clear that if $d\xi$ is the element of arc length along the average path S and ds'' is the element along the random ray path, then

$$d\xi = ds'' \cos\theta .$$

Therefore the first phase term in Eq. (5) can be written as

$$\phi \equiv k_o \int_S \sqrt{\bar{\varepsilon}}\, ds'' \simeq k_o \int_{\bar{S}} \sqrt{\bar{\varepsilon}} \frac{d\xi}{\cos\theta(\xi)}$$

$$\simeq k_o \int_{\bar{S}} \sqrt{\bar{\varepsilon}}\, d\xi - \frac{k_o}{2} \int_{\bar{S}} \sqrt{\bar{\varepsilon}}\, \theta^2(\xi)\, d\xi . \qquad (6)$$

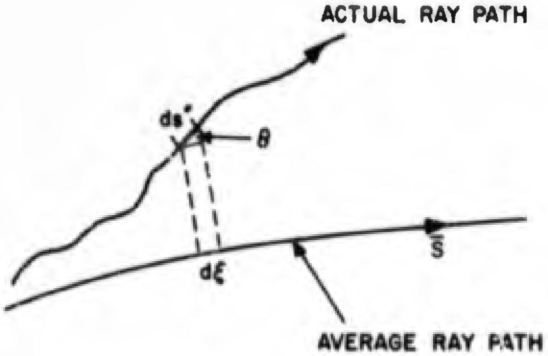

Figure 1. Geometry for Relating the Actual Ray Path to the Average Path

Therefore if ϕ were replaced by $k_o \int_{\bar{S}} \sqrt{\bar{\varepsilon}}\, d\xi$ the ensemble averaged phase error would be

$$\langle \delta\phi \rangle = -\frac{k_o}{2} \int_{\bar{S}} \sqrt{\bar{\varepsilon}}\ \langle \theta^2(\xi) \rangle \ d\xi \ .$$

From Chernov[6] we have that

$$\langle \theta^2 \rangle \simeq \int^{\xi} \frac{\langle \delta\varepsilon^2 \rangle}{L_o}\ d\eta$$

where L_o is the outer scale of the turbulence. Therefore, the ensemble averaged phase error is

$$\langle \delta\phi \rangle = -\frac{k_o}{2L_o} \int_{\bar{S}} \sqrt{\bar{\varepsilon}} \left[\int^{\xi} \langle \delta\varepsilon^2 \rangle \ d\eta \right] d\xi \ , \tag{7}$$

where the integral on $\bar{S}$ is along S with the path terminating at the point ξ.

Requiring the mean phase error to be small results in the condition

$$\frac{1}{2}\left(\frac{k_o}{L_o}\right) \int_{\bar{S}} d\xi \sqrt{\bar{\varepsilon}} \int^{\xi} \langle \delta\varepsilon^2 \rangle \ d\eta << 1. \tag{8}$$

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[6] Chernov, L. (1960) *Wave Propagation in a Random Medium*, McGraw Hill, New York.

When condition (8) holds we can write, correct to terms of order $\langle \delta\varepsilon^2 \rangle$

$$\exp\left\{ ik_o \int_S \sqrt{\varepsilon}\, ds'' - ik_o \int_{S'} \sqrt{\varepsilon}\, ds'' \right\}$$

$$\simeq \exp\left\{ ik_o \int_{\bar{S}} \sqrt{\bar{\varepsilon}}\, ds'' - ik_o \int_{\bar{S}'} \sqrt{\bar{\varepsilon}}\, ds'' \right\} \left[1 + i\delta\phi_o \right], \tag{9}$$

where $\delta\phi_o$ is the incremental phase error in replacing S by $\bar{S}$ and S' by $\bar{S}'$. Note that $\langle \delta\phi_o \rangle \neq 0$. If we use Eqs. (5) and (9) in (4) we obtain

$$\langle E_s E_s^* \rangle = \frac{k_o^4}{(4\pi R)^2} \iiint_V d^3x \iiint_V d^3x'\, M(\underline{x}, \underline{x}') \Big\langle [\varepsilon(\underline{x})-1][\varepsilon(\underline{x}')-1]$$

$$\times [1 + i\delta\phi_o] \exp\left\{ \frac{ik_o}{2} \int_S \frac{\delta\varepsilon(\underline{x}'')}{\sqrt{\bar{\varepsilon}(\underline{x}'')}}\, ds'' - \frac{ik_o}{2} \int_{S'} \frac{\delta\varepsilon(\underline{x}'')}{\sqrt{\bar{\varepsilon}(\underline{x}'')}}\, ds'' \right\} \Big\rangle \tag{10}$$

where

$$M(\underline{x}, \underline{x}') \equiv e^{i\underline{k}_s \cdot (\underline{x}-\underline{x}')} \left[\bar{\varepsilon}(\underline{x}) \bar{\varepsilon}(\underline{x}') \right]^{-1/4} \left\{ \frac{d\sigma[\bar{\varepsilon}(\underline{x})]}{d\sigma_o} \frac{d\sigma[\bar{\varepsilon}(\underline{x}')]}{d\sigma_o} \right\}^{-1/2}$$

$$\times \exp\left\{ ik_o \int_{\bar{S}} \sqrt{\bar{\varepsilon}(\underline{x}'')}\, ds'' - ik_o \int_{\bar{S}'} \sqrt{\bar{\varepsilon}(\underline{x}'')}\, ds'' \right\}.$$

As our next step we desire to expand the exponential within the ensemble average in Eq. (10) in a Taylor series. We therefore require that the phase fluctuation

$$\delta\psi = \frac{k_o}{2} \int_S \frac{\delta\varepsilon(\underline{x}'')}{\sqrt{\bar{\varepsilon}(\underline{x}'')}}\, ds'' << 1 .$$

We can estimate

$$\langle \delta\psi^2 \rangle = \frac{k_o^2}{4} \int_{\bar{S}} ds'' \int_{\bar{S}} ds''' \frac{\langle \delta\varepsilon(\underline{x}'')\delta\varepsilon(\underline{x}''') \rangle}{[\bar{\varepsilon}(\underline{x}'')\bar{\varepsilon}(\underline{x}''')]^{1/2}}$$

$$\simeq \frac{k_o^2}{4} \int_{\bar{S}} \frac{ds''}{\bar{\varepsilon}(x'')} \int_{-\infty}^{\infty} d\xi\, B(\underline{x}'', \xi) , \tag{11}$$

where we have assumed that $\overline{\varepsilon}$ doesn't change radically over a correlation length and therefore have approximated $\overline{\varepsilon}(x''')$ by $\overline{\varepsilon}(x'')$, and assumed the average ray path $\overline{S}$ is locally a straight line, so that $S''' - S'' \simeq x''' - x'' = \xi$. Also, we have assumed the turbulence is locally stationary so that

$$\langle \delta\varepsilon(\underline{x}'')\,\delta\varepsilon(\underline{x}''')\rangle = B\left(\frac{\underline{x}''+\underline{x}'''}{2},\ \underline{x}'' - \underline{x}'''\right)$$

$$\simeq B\,(\underline{x}'',\ \underline{x}'' - \underline{x}''')\ .$$

Upon estimating the order of magnitude of Eq. (11) we have

$$\langle \delta\psi^2\rangle \simeq \frac{k_o^2\, z\, L_o}{4\overline{\varepsilon}}\,\langle \delta\varepsilon^2\rangle\ .$$

Therefore, if $\langle \delta\psi^2\rangle$ is to be small compared with unity we have

$$\frac{k_o^2}{4}\int_S \frac{ds''}{\overline{\varepsilon}(\underline{x}'')}\int_{-\infty}^{\infty} B(\underline{x}'', \xi)\, d\xi \sim \frac{k_o^2\, z\, L_o}{4\overline{\varepsilon}}\,\langle \delta\varepsilon^2\rangle \ll 1. \qquad (12)$$

When Eq. (12) holds we can expand the last term in Eq. (10) in a Taylor series.* We then get

$$\langle E_s E_s^*\rangle = \frac{k_o^4}{(4\pi R)^2}\iiint_V d^3x \iiint_V d^3x'\, M(\underline{x}, \underline{x}')\Big\langle [\,\overline{\varepsilon}(\underline{x}) - 1 + \delta\varepsilon(\underline{x})\,]$$

$$\times\left[\overline{\varepsilon}(\underline{x}') - 1 + \delta\varepsilon(\underline{x}')\right]\ \left[1 + i\delta\phi_o\right]\left[1 + i\delta\phi_1 - \frac{\delta\phi_1^2}{2}\right]\ \Big\rangle, \qquad (13)$$

---

*It can also be shown that multiple scatter is unimportant whenever the inequality expressed by Eq. (12) holds. That is, it was shown previously[7] that the total field inside the plasma can be approximated by the incident field if $\int_S ds/\ell_t \ll 1$ where ℓ_t is the mean free-path for photon scatter by the turbulence. It can be shown that since $\ell_t^{-1} \simeq k_o^2 L_o \langle \delta\varepsilon^2\rangle/4$ this is equivalent to the condition in Eq. (12). Thus, the total field in the plasma $E = E_i + E_M$ can be approximated by E_i, as we did in Eq. (2), if the condition expressed in Eq. (12) holds.

7. Fante, R. (1973) Propagation of electromagnetic waves through turbulent plasma using transport theory, IEEE Trans. on Antennas and Propagation **AP-21**:750-755.

where

$$\delta\phi_1 = \frac{ik_o}{2}\int_S \frac{\delta\varepsilon(\underline{x}'')}{\sqrt{\bar{\varepsilon}(\underline{x}'')}}\,ds'' - \frac{ik_o}{2}\int_{S'} \frac{\delta\varepsilon(\underline{x}'')}{\sqrt{\bar{\varepsilon}(\underline{x}'')}}\,ds'' .$$

If we expand the terms within the ensemble average, using the fact that $\langle\delta\phi_1\rangle = 0$ and $\langle\delta\varepsilon\rangle = 0$, we get, correct to second order in $\langle\delta\varepsilon^2\rangle$,

$$\begin{aligned}\langle E_s E_s^*\rangle = \frac{k_o^4}{(4\pi R)^2} & \iiint_V d^3x \iiint_V d^3x'\, M(\underline{x}, \underline{x}') \\ & \times \Big\{ \left[\bar{\varepsilon}(\underline{x})-1\right]\left[\bar{\varepsilon}(\underline{x}')-1\right] + \langle\delta\varepsilon(\underline{x})\delta\varepsilon(\underline{x}')\rangle \\ & + i\left[\bar{\varepsilon}(\underline{x})-1\right]\left[\varepsilon(\underline{x}')-1\right]\langle\delta\phi_o\rangle + i\left[\bar{\varepsilon}(\underline{x}')-1\right]\langle\delta\varepsilon(\underline{x})\delta\phi_1\rangle \\ & + i\left[\bar{\varepsilon}(\underline{x})-1\right]\langle\delta\varepsilon(\underline{x}')\delta\phi_1\rangle - \frac{1}{2}\left[\bar{\varepsilon}(\underline{x})-1\right]\left[\bar{\varepsilon}(\underline{x}')-1\right]\langle\delta\phi_1^2\rangle \Big\} . \end{aligned} \tag{14}$$

The first, third, and sixth terms in Eq. (14) are sharply peaked in the forward direction and in the backscatter direction are at most of order $C_1(\bar{\varepsilon}-1)^2 V^{2/3} k_o^{-4}$, where C_1 is a constant common to all the terms in Eq. (14) [for example, $k_o^4/(4\pi R)^2 \ldots$], whereas the second term in Eq. (14) is of order $C_1 \langle\delta\varepsilon^2\rangle L_o^{-1/3} k_o^{-4} V$. Therefore, the ratio of the first, third, or sixth term in Eq. (14) to the second is at most

$$\frac{\langle\delta\varepsilon^2\rangle}{(\bar{\varepsilon}-1)^2}\left(\frac{L_o^3}{V}\right)^{1/3} .$$

If

$$\frac{\langle\delta\varepsilon^2\rangle}{(\bar{\varepsilon}-1)^2}\,\frac{L_o}{V^{1/3}} << 1 ,$$

we can therefore neglect the first, third, and sixth terms in Eq. (14) to obtain for the backscattered intensity

$$\langle E_s E_{s'}^* \rangle \simeq \frac{k_o^4}{(4\pi R)^2} \iiint_V d^3x \iiint_V d^3x' \, M(\underline{x}, \underline{x}') \Big\{ \langle \delta\varepsilon(\underline{x}) \delta\varepsilon(\underline{x}') \rangle$$

$$+ i \left[\bar{\varepsilon}(\underline{x}') - 1 \right] \langle \delta\varepsilon(\underline{x}) \delta\phi_1 \rangle + i \left[\bar{\varepsilon}(\underline{x}) - 1 \right] \langle \delta\varepsilon(\underline{x}') \delta\phi_1 \rangle \Big\} . \qquad (15)$$

If we could neglect the last two terms on the right-hand side of Eq. (15), we will have justified the use of the distorted-wave Born approximation. An estimate is obtained by substituting $\delta\phi_1$ into Eq. (15). If we approximate $\sqrt{\bar{\varepsilon}}$ by unity, and for purposes of an order of magnitude estimate only, replace

$$\exp \left\{ ik_o \int_S \sqrt{\varepsilon} \, ds'' - ik_o \int_{S'} \sqrt{\varepsilon} \, ds'' \right\}$$

by $\exp \{ i\underline{k}_i \cdot (\underline{x} - \underline{x}') \}$ and define $\underline{k} = \underline{k}_i - \underline{k}_s$ we obtain for the last term in Eq. (15):

$$\text{Last term in (15)} \simeq k_o C_1 \iiint_V d^3x \iiint_V d^3x' [\bar{\varepsilon}(\underline{x}) - 1] \, e^{i\underline{k} \cdot (\underline{x} - \underline{x}')}$$

$$\times \int_S \langle \delta\varepsilon(x') \delta\varepsilon(x'') \rangle \, ds'' \simeq k_o C_1 \iiint_V d^3x \, e^{i\underline{k} \cdot \underline{x}} [\bar{\varepsilon}(\underline{x}) - 1]$$

$$\times \int_S ds'' \iiint d^3x' \, B(\underline{x}' - \underline{x}'') \, e^{-i\underline{k} \cdot \underline{x}'} \quad ,$$

where $B(\underline{\xi}) = \langle \delta\varepsilon(\underline{x}) \delta\varepsilon(\underline{x} + \underline{\xi}) \rangle$ is the correlation function. Upon recalling that the wavenumber spectrum $\Phi(\underline{k}) = \iiint d^3\xi \, B(\underline{\xi}) \exp \{ -i\underline{k} \cdot \underline{\xi} \}$, we see that the above can be rewritten as

$$\text{Last term in (15)} \simeq ik_o C_1 \Phi(\underline{k}) \iiint_V d^3x \, e^{i\underline{k} \cdot \underline{x}} [\bar{\varepsilon}(\underline{x}) - 1] \int_S ds'' \, e^{-i\underline{k} \cdot \underline{x}''} .$$

If we approximate S by a straight line terminating at $\underline{x}$, and write $x'' = x - \Delta S$, where ΔS is the distance measured from x along the ray, we have $\underline{k} \cdot \underline{x}'' = \underline{k} \cdot \underline{x} - \underline{k} \cdot \underline{\Delta S} \simeq \underline{k} \cdot \underline{x} - k\xi$. Then

$$\text{Last term in (15)} \simeq ik_o C_1 \Phi(k) \iiint_V d^3x [\bar{\varepsilon}(\underline{x}) - 1] \int_x^o d\xi \, e^{ik\xi} \sim ik_o C_1 \Phi(\underline{k}) V [\bar{\varepsilon} - 1] .$$

In the backscatter direction $k = |\underline{k}_i - \underline{k}_s| = 2k_o$. Also $\Phi(k) \sim \langle \delta\varepsilon^2 \rangle L_o^{-1/3} k_o^{-4}$; consequently the ratio of the last term in Eq. (15) to the first is of order

$$\frac{\text{Last term in (15)}}{\text{First term in (15)}} \sim \frac{[\bar{\varepsilon}-1]\, V C_1 L_o^{-1/3} k_o^{-4} \langle \delta\varepsilon^2 \rangle}{2C_1 \langle \delta\varepsilon^2 \rangle L_o^{1/3} k_o^{-4} V} = \frac{[\bar{\varepsilon}-1]}{2} . \tag{16}$$

Therefore, in order to be able to neglect the last term in (15) in comparison with the first we must require that

$$\bar{\varepsilon} - 1 \ll 1 . \tag{17}$$

The same conclusion can be shown to hold for the second term on the right-hand side of Eq. (15). Therefore when Eq. (17) holds we can approximate Eq. (15) by

$$\langle E_s E_s^* \rangle = \frac{k_o^4}{(4\pi R)^2} \iiint_V d^3x \iiint_V d^3x' \, M(\underline{x}, \underline{x}') \langle \delta\varepsilon(\underline{x}) \delta\varepsilon(\underline{x}') \rangle , \tag{18}$$

which is the distorted-wave Born approximation.* However, as we showed in the course of our derivation the following conditions must hold if Eq. (18) is to be a valid representation of the backscattered intensity:

(1) $$\frac{\langle \delta\varepsilon^2 \rangle}{\bar{\varepsilon}^2} \ll 1 , \tag{19}$$

(2) $$\frac{1}{2}\left(\frac{k_o}{L_o}\right) \int_S \sqrt{\bar{\varepsilon}} \left[\int^{\xi} d\eta \, \langle \delta\varepsilon^2(\eta) \rangle \right] d\xi \ll 1 , \tag{20}$$

(3) $$\frac{k_o^2}{4} \int_S \frac{ds''}{\bar{\varepsilon}(\underline{x}'')} \int_{-\infty}^{\infty} B(\underline{x}'', \xi) d\xi \ll 1 , \tag{21}$$

(4) $$\frac{\langle \delta\varepsilon^2 \rangle}{(\bar{\varepsilon}-1)^2} \left(\frac{L_o^3}{V} \right)^{1/3} \ll 1 , \tag{22}$$

(5) $$\bar{\varepsilon} - 1 \ll 1 . \tag{23}$$

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* Eq. (18) has been evaluated by Finson; the result for the radar cross section of a cylindrical wake is $\sigma = (5k_o^4/4\pi)\sin^{-4}\alpha \, \Phi(k) \pi r_1^2 \tau$, where τ is the pulse length and $\Phi(k) = 15.6 \, L_o^3 \cdot (1+4k_o^2 L_o^2)^{-11/6}$.

In addition, in writing Eq. (1) we inherently assumed that

$$L_o/\lambda >> 1 \quad \text{and} \quad V/\lambda^3 >> 1 \; .$$

In Section 4 we will apply Eqs. (19) to (23) to a decoy wake to determine when the distorted-wave Born approximation is applicable.

3. COMMENTS ON SOLUTION NEAR TURNING POINT

As we commented earlier, Eq. (1) is incorrect near the turning point. Near the turning point we can rewrite the incident field as

$$E_i(\underline{x}) = A(\underline{x}) \exp \left\{ ik_o \int_S \sqrt{\varepsilon}\, ds \right\} .$$

We note that the phase function is still given by $\int_S \sqrt{\varepsilon}\, ds$ except that the phase jumps by $\pi/2$ at the turning point (caustic). It is extremely difficult to write down the amplitude function $A(\underline{x})$. For a one dimensionally stratified random medium $A(\underline{x})$ can be expressed in terms of the Airy functions. It can then be shown that if $n_e/n_c = \exp(-\beta z)$ near the turning point, where β is a random variable, that as long as $\delta\beta/\langle\beta\rangle << 1$ the amplitude fluctuations in $A(x)$ will not be significant. This in turn implies the requirement $-\delta n_e/\overline{n}_e\, [\ln(\sin^2\alpha)]^{-1} << 1$, where α is the aspect angle defined in the next section. Since $\delta n_e/\overline{n}_e << 1$, then as long as α is small this condition will clearly hold. Therefore it is mainly the phase fluctuations which will concern us, and these have been treated carefully in the last section. For small aspect angles (see next section) the assumption that $\delta\varepsilon << \overline{\varepsilon}$ made in the last section will still hold and the treatment near the turning point for the phase parallels that at points distant from the turning point.

4. APPLICATION OF EQS. (19) TO (23) TO TURBULENT WAKE BACKSCATTER

For a turbulent wake we have

$$\overline{\varepsilon} = 1 - \frac{n_e}{n_c} ,$$

$$\delta\varepsilon = - \frac{\delta n_e}{n_c} ,$$

where n_e is the mean electron density (electrons/cm^3) and $n_c = \omega^2/3.18\times 10^9$ is the critical electron density. In terms of n_e, the condition of Eq. (23) becomes

$$\frac{n_e}{n_c} \ll 1 , \tag{24}$$

whereas Eq. (19) becomes [assuming Eq. (24) is satisfied]

$$\frac{\langle \delta n_e^2 \rangle}{n_c^2} \ll 1 . \tag{25}$$

Since $\langle \delta n_e^2 \rangle \sim 0.3\,\bar{n}_e^2$ in a turbulent wake, it is clear that if Eq. (24) is satisfied, Eq. (25) will also be satisfied.

Next consider the condition expressed by Eq. (22). This is

$$\frac{\langle \delta\epsilon^2 \rangle}{(\epsilon-1)^2}\left(\frac{L_o^3}{V}\right)^{1/3} = \frac{\langle \delta n_e^2 \rangle}{\bar{n}_e^2}\left(\frac{L_o^3}{V}\right)^{1/3} \simeq 0.3\,\frac{L_o}{V^{1/3}} \ll 1 . \tag{26}$$

From Eq. (26) we see that the wake volume must be much greater than the outer scale length of the turbulence. Generally, this condition is satisfied, since L_o is generally of order of 1/8 to 1/16 of the wake radius.

Before going on to the conditions expressed by Eqs. (20) and (21) let us first examine in more detail when Eq. (24) will be satisfied. From ray optics it is clear that if a ray is incident on a wake at aspect angle α (see Figure 2), the turning point will occur when $n_e = n_c \sin^2\alpha$. This point is proven in Appendix A. Therefore

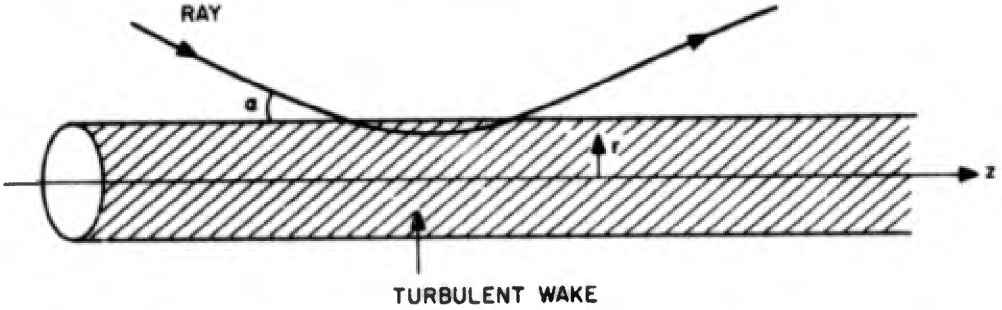

Figure 2. Path of a Ray Incident on a Turbulent Plasma Wake

$$\frac{n_e}{n_c} \sim \sin^2\alpha << 1 \ , \tag{27}$$

so that as long as the aspect angle α is small Eq. (24), and consequently Eq. (25), will hold. In view of the fact that $n_e/n_c << 1$ we can therefore approximate $\bar{\epsilon}$ by unity. With this approximation the conditions of Eqs. (20) and (21) become

$$\frac{1}{2}\frac{k_o}{L_o}\int_{\bar{S}} d\xi \int^{\xi} d\eta \langle \delta\epsilon^2 \rangle << 1 \ , \tag{20a}$$

$$\frac{k_o^2}{4}\int_{\bar{S}} ds'' \int_{-\infty}^{\infty} d\xi \ B(\underline{x}'', \xi) << 1 \ . \tag{21a}$$

Using $\langle \delta\epsilon^2 \rangle = \langle \delta n_e^2 \rangle / n_c^2 \simeq 0.3 \ \bar{n}_e^{\,2}/n_c^2$, we can rewrite Eq. (20a) as

$$0.15 \frac{k_o}{L_o n_c^2} \int_{\bar{S}} d\xi \int^{\xi} d\eta \ n_e^2(\eta) << 1 \ . \tag{28}$$

Also writing $B(\underline{x}, \xi) = \dfrac{\langle \delta n_e^2(\underline{x}) \rangle}{n_c^2} \exp\left(-\dfrac{|\xi|}{L_o}\right) \simeq 0.3 \dfrac{\bar{n}_e^2(x)}{n_c^2} \exp\left(-\dfrac{|\xi|}{L_o}\right)$

gives in Eq. (21a) the condition

$$\frac{0.15 \ k_o^2 L_o}{n_c^2} \int_{\bar{S}} \bar{n}_e^2(\xi) d\xi << 1 \ . \tag{29}$$

To proceed further let us approximate $\bar{S}$ by two straight line segments, as shown on Figure 3, and assume that $\bar{n}_e$ is radially stratified according to $\bar{n}_e = n_o \exp(-r^2/2r_1^2)$. Then since $ds = dr/\sin\alpha$ and $\xi = r/\sin\alpha$ we have from Eqs. (28) and (29) the requirements

$$\frac{0.30 \ k_o n_o^2}{L_o n_c^2 \sin^2\alpha} \int_{\infty}^{r_c} dr \int_{\infty}^{r} \exp\left(-\frac{r'^2}{r_1^2}\right) dr' = -\frac{0.15\sqrt{\pi} \ k_o r_1 n_o^2}{2L_o n_c^2 \sin^2\alpha} \int_{\infty}^{r_c} dr \ \mathrm{erfc}\left(\frac{r}{r_1}\right) << 1 \tag{30}$$

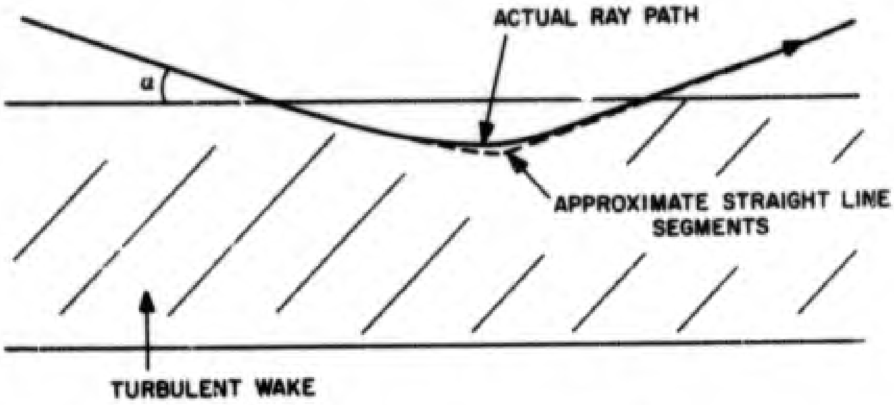

Figure 3. Approximate Path Used in Deriving Eqs. (30) and (31)

and

$$\frac{0.30\,k_o^2L_on_o^2}{n_c^2\sin\alpha}\int_\infty^{r_c} dr\,\exp\left(-\frac{r^2}{r_1^2}\right) = \frac{0.15\sqrt{\pi}\,k_o^2L_on_o^2r_1}{2n_c^2\sin\alpha}\,\mathrm{erfc}\left(\frac{r_c}{r_1}\right) << 1\,. \tag{31}$$

To solve for r_c we set $n_e = n_c\sin^2\alpha = n_o\exp\left(-r_c^2/2r_1^2\right)$, or

$$r_c = r_1\sqrt{-2\ln\left(\frac{n_c}{n_o}\sin^2\alpha\right)}\,. \tag{32}$$

If we perform the integration in Eq. (30) we get the condition

$$-\frac{0.15\sqrt{\pi}\,k_or_1^2n_o^2}{L_on_c^2\sin^2\alpha}\left[\frac{r_c}{r_1}\,\mathrm{erfc}\left(\frac{r_c}{r_1}\right) - \frac{1}{\sqrt{\pi}}\exp\left(-\frac{r_c^2}{r_1^2}\right)\right] << 1\,, \tag{33}$$

where erfc is the complementary error function. For $\alpha << 1$ it is clear that $r_c/r_1 >> 1$. Upon using the asymptotic expansion for erfc (. . .) in Eq. (33) we get the condition

$$\frac{0.133\,k_or_1^2n_o^2}{L_on_c^2\sin^2\alpha}\cdot\frac{\exp\left(-\frac{r_c^2}{r_1^2}\right)}{\sqrt{\pi}\left(\frac{r_c}{r_1}\right)^2} << 1\,. \tag{34}$$

Upon substituting for r_c/r_1 from Eq. (32) we obtain

$$- \frac{0.0752\, k_o r_1^2 \sin^2\alpha}{L_o \ln\left(\frac{n_c^2}{n_o^2} \sin^4\alpha\right)} << 1 \ . \tag{35}$$

Similarly, if we use Eq. (32) in (31) we get the other condition

$$- \frac{0.266\, k_o^2 L_o r_1 n_o^2}{n_c^2 \sin\alpha} \operatorname{erfc} \left\{ \sqrt{-2 \ln\left(\frac{n_c}{n_o} \sin^2\alpha\right)} \right\} \ . \tag{36}$$

For $\alpha << 1$ we can approximate Eq. (36) further by

$$\frac{0.150\, k_o^2 L_o r_1 \sin^3\alpha}{\sqrt{-\ln\left(\frac{n_c^2}{n_o^2} \sin^4\alpha\right)}} << 1 \ . \tag{37}$$

Therefore, if the conditions expressed by Eqs. (35) and (37) are to hold, it is clear that α must be quite small, since $k_o L_o >> 1$ and $k_o r_1 >> 1$.

5. SUMMARY OF SUFFICIENT CONDITIONS

In the last section we derived sufficient conditions for application of the distorted-wave Born approximation to a wake in which the turbulent eddy size is large compared with a signal wavelength. With the assumption that the mean electron density has the distribution $\bar{n}_e = n_o \exp(-r^2/2r_1^2)$ and that $\langle \delta n_e^2 \rangle \sim 0.3\, \bar{n}_e^2$ we derived the following sufficient conditions

$$\sin^2\alpha < 1 \ , \tag{38}$$

$$-0.0752\, (k_o r_1) \left(\frac{r_1}{L_o}\right) \frac{\sin^2\alpha}{\ln\left(\frac{n_c^2}{n_o^2} \sin^4\alpha\right)} << 1 \ , \tag{39}$$

$$\frac{0.150\,(k_oL_o)(k_or_1)\sin^3\alpha}{\sqrt{-\ln\left(\frac{n_c^2}{n_o^2}\sin^4\alpha\right)}} << 1\ . \qquad (40)$$

When these conditions hold we can be certain that the distorted-wave Born approximation (see Eq. (18) gives a good estimate of the wake radar cross section.

As an example, suppose we consider a C-band radar signal of 5-GHz frequency, incident at an aspect angle α of 10° on a wake in which $n_o = 3\,n_c$, and $r_1 = 32$ cm. If $L_o = 4$ cm, we then have $k_oL_o = 4.2$ and $k_or_1 = 33.5$. Consequently we get

$$-\,0.0752\,(k_or_1)\left(\frac{r_1}{L_o}\right)\frac{\sin^2\alpha}{\ln\left(\frac{n_c^2}{n_o^2}\sin^4\alpha\right)} = 0.066$$

and

$$\frac{0.150\,(k_oL_o)(k_or_1)\sin^3\alpha}{\sqrt{-\ln\left(\frac{n_c^2}{n_o^2}\sin^4\alpha\right)}} = 0.0356.$$

Therefore, for these wake conditions the inequalities expressed by Eqs. (39) and (40) are clearly satisfied. Likewise Eq. (38) holds for $\alpha = 10^\circ$, since $\sin^2(10^\circ) = 0.0303$.

In view of the sufficiency conditions we have derived it is expected that the distorted-wave Born approximation will give reasonable results for a large variety of turbulent wakes, provided that the aspect angle is small. This explains why various researchers have met with some success in employing this method for small aspect-angle wake backscatter. We have used the ray-optics approximation for the incident field in the plasma; a more accurate representation is possible using the full wave solutions, as done by Bisbing.[8] We have not found it necessary to use that method here, since we were only interested in deriving the conditions sufficient for application of the distorted-wave Born approximation.

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[8] Bisbing, P. (1974) Private communication.

References

1. Keller, J. (1964) Stochastic equations and wave propagation in random media, Proc. of Symposia in Applied Mathematics, Vol XVI, (edited by R. Bellman) American Mathematical Society, Providence, R.I.
2. Finson, M., Monsler, M., and Kaplan, L. (1970) Aspect Dependence Wake Scattering—Final Report, AVCO Everett Research Laboratory Document No. 70-945.
3. Finson, M. (1971) Reentry Experiments Analysis and Predictions—Final Report, AVCO Everett Research Laboratory Document No. 71-520.
4. Finson, M. (1972) Interpretation of Recent Delco Ballistic Range Observations, AVCO Everett Research Laboratory, Document No. 72-385.
5. Guthart, H. and Graf, K. (1974) Radar backscatter by turbulent wakes, Radio Science (to be published).
6. Chernov, L. (1960) Wave Propagation in a Random Medium, McGraw Hill, New York.
7. Fante, R. (1973) Propagation of electromagnetic waves through turbulent plasma using transport theory, IEEE Trans. on Antennas and Propagation AP-21:750-755.
8. Bisbing, P. (1974) Private communication.

Appendix A

Consider a ray in a radially stratified plasma cylinder having an index of refraction $\mu(r)$; then by virtue of Fermat's principle

$$\delta \int \mu(r) ds$$

$$= \delta \int \mu(r) \sqrt{1 + r^2 \dot{\theta}^2 + \dot{z}^2} \; dr = 0 \; . \qquad \text{(A1)}$$

we have that equations of motion of the ray are

$$\frac{\mu(r) r^2 \dot{\theta}}{\sqrt{1 + r^2 \dot{\theta}^2 + \dot{z}^2}} = c_1 \; , \qquad \text{(A2)}$$

$$\frac{\mu(r) \dot{z}}{\sqrt{1 + r^2 \dot{\theta}^2 + \dot{z}^2}} = c_2 \; , \qquad \text{(A3)}$$

where $\dot{\theta} = d\theta/dr$, $\dot{z} = dz/dr$ and r, θ, z are the coordinates in a cylindrical system. The constants c_1 and c_2 are determined by considering the case when $\mu = 1$. We then find that $c_1 = b \sin\alpha$ and $c_2 = \cos\alpha$ where α is the aspect angle and b is the distance of closest approach of the ray to the z axis when $\mu = 1$. Therefore, Eqs. (A2) and A3) can be written

$$\frac{dr}{d\theta} = \frac{r}{b \sin\alpha} \sqrt{(\mu^2 - \cos^2\alpha) r^2 - b^2 \sin^2\alpha} \; , \qquad \text{(A4)}$$

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$$\frac{dr}{dz} = \frac{1}{r\cos\alpha} \sqrt{(\mu^2 - \cos^2\alpha)r^2 - b^2\sin^2\alpha} \quad . \tag{A5}$$

From Eq. (A5) we see that the maximum penetration of the ray into the plasma occurs when $dr/dz = 0$. That is, the ray penetrates to a radius r_o given by

$$\mu^2(r_o) = \cos^2\alpha + \frac{b^2}{r_o^2}\sin^2\alpha \quad . \tag{A6}$$

Upon recalling that $\mu^2 = 1 - n_e/n_c$ we can rewrite Eq. (A6) as

$$\frac{n_e(r_o)}{n_c} = \left(1 - \frac{b^2}{r_o^2}\right)\sin^2\alpha \quad . \tag{A7}$$

For a given aspect angle α, the ray penetrates to the highest value of electron density when $b = 0$; that is, it then penetrates to $n_e(r_o) = n_c\sin^2\alpha$. Therefore, to be pessimistic in our results we have used in Eq. (27),

$$\frac{n_e}{n_c} = \sin^2\alpha \quad .$$

For nonzero impact parameters the turning point occurs at smaller values of n_e, as given by Eq. (A7).