COLUMBIA UNIVERSITY

IN THE CITY OF NEW YORK

ELECTRONICS RESEARCH LABORATORIES

632 WEST 125th STREET

NEW YORK, NEW YORK 10027

CU-48-66-AF-1478-ERL

July 15, 1966

DEFENSE WITH DAMAGE ASSESSMENT

TECHNICAL REPORT T-67322

By

L. R. Abramson
M. Shapiro

Prepared for

Director
Advanced Research Projects Agency
Washington, D.C. 20301

and

Electronics Division
Directorate of Engineering Sciences
Air Force Office of Scientific Research
Office of Aerospace Research
U.S. Air Force
Arlington, Virginia 22209

Contract No. AF 49(638)-1478
ARPA Order No. 279

TABLE OF CONTENTS

AUTHORIZATION

The research described in this report was performed at the Electronics Research Laboratories of Columbia University. This report was prepared by L. R. Abramson and M. Shapiro.

This project is directed by the Advanced Research Projects Agency of the Department of Defense and is administered by the Air Force Office of Scientific Research under Contract AF 49(638)-1478.

Submitted by:

H. Dym
Assistant Director

Approved by:

For L. H. O'Neill
Professor of Electrical Engineering
Director

DEFENSE WITH DAMAGE ASSESSMENT

L. R. Abramson and M. Shapiro

ABSTRACT

A farm of missile sites is sequentially attacked by m rounds, with each round a simultaneous attack of one attacker at each of the sites. A salvo of area interceptors is used to defend each of the live sites at each round. The defense is evaluated by the expected number of surviving sites.

This paper treats the offense enforceable case where m and the kill probability of an unintercepted attacker are both known to the defense. The problem is to find an optimal damage assessment firing doctrine.

Recursion formulas in m are found for the maximum expected number of survivors and the underlying optimal firing doctrine. These results are derived by both the exact approach and an approximate approach. The exact approach takes full account of the random nature of each engagement while the approximate approach replaces the outcome of each engagement by its expected value.

The approximate formulas are explicitly solved for the case of perfect interceptors and the results are compared to the optimal solutions when there is no damage assessment.

1. Introduction and summary.

The target system is a farm of n identical missile sites defended by a total of nk identical area interceptors (each interceptor can be used to defend any of the sites). We assume that the radar and interceptor stockpile are not attacked and that the state of the farm is known to the control center at all times.

The attack comprises m identical sequential attackers directed at each site with each round arriving simultaneously at all of the sites. For each round of the attack, a salvo of interceptors (which may vary from site to site) is used to defend each of the live sites. The defense is evaluated by the expected number of sites which survive the attack.

This paper treats the offense enforceable case where m is known to the defense. We also assume that the defense knows the kill probability of an unintercepted attacker (denoted by p) as well as the single shot kill probability of an interceptor (denoted by Q). The problem is to find an optimal defensive firing doctrine using damage assessment, i.e., a firing doctrine which maximizes the expected number of survivors when dead sites are never defended.

An approximate solution to the problem is found by assuming that the outcome of each engagement is its expected outcome and that salvo sizes and numbers of surviving sites may vary continuously. Under these assumptions, let $E_m(n,k)$ be the expected number of survivors of n sites attacked by m sequential attackers each if k interceptors per site are available and an optimal damage assessment firing doctrine is used. In Sec. 2, we show that the interceptors used on each round should be allocated as uniformly as possible among the live sites. In Sec. 3, we find a recursive expression in m for $E_m(n,k)$ and show how to construct an optimal firing doctrine. In Sec. 4, we specialize these results to the perfect interceptor case $(Q = 1)$ and find explicit expressions for $E_m(n,k)$ and the optimal firing doctrine. For perfect interceptors, every optimal firing doctrine has the following form: For some $r = r(m,k)$, none of the sites are defended for the first $r - 1$ rounds, some of the surviving sites are defended for the $r^{\underline{th}}$ round, and all of the surviving sites are defended for the last $m - r$ rounds.

Without damage assessment, a sequential attack of m attackers with known m is equivalent to a simultaneous attack of m attackers. If the interceptors are perfect, let $F_m(n,k)$ be the expected number of survivors of n sites attacked simultaneously by m attackers each if k interceptors

per site are available and an optimal firing doctrine is used. In Sec. 5, we evaluate the effect of damage assessment by comparing $F_m(n,k)$ with $E_m(n,k)$ for $Q = 1$. With perfect interceptors and with no damage assessment, every optimal firing doctrine defends some of the sites throughout the attack, i.e., defends some of the sites all of the time. This is in contrast to the optimal firing doctrine <u>with</u> damage assessment which defends some of the sites some of the time. Also, the damage assessment firing doctrine is a function of Q while the no damage assessment firing doctrine is not.

In Sec. 6, an exact damage assessment recursion formula for the maximum expected number of survivors is derived in which full account is taken of the distribution of the number of surviving sites at each stage of the attack.

2. Optimal single round firing doctrine

We begin by solving the single round allocation problem: How should the interceptors used on any round be allocated among the live sites?

Let $P(x)$ be the probability that a site survives one attacker if defended by a salvo of x interceptors. Assuming that the interceptors operate independently, we have

$$P(x) = 1 - p(1-Q)^x, \qquad x = 0,1,2,\cdots \quad . \tag{1}$$

To avoid uninteresting special cases, we assume that $p > 0$ and $Q > 0$, i.e., both the attackers and the interceptors have some capability.

Suppose that K interceptors are available to defend N sites for one round. A uniform defense is a firing doctrine such that salvo sizes at different sites differ by at most one interceptor. Let $s = [\frac{K}{N}]$, where $[u]$ is the largest integer $\leq u$. Then a uniform defense uses a salvo of s interceptors at each of $(N - K + Ns)$ sites and a salvo of $s + 1$ interceptors at each of $(K - Ns)$ sites. Moreover, since the sites are identical, it does not matter which sites are defended by which salvos and hence the uniform defense is unique.

Let $E(N,K)$ be the expected number of survivors of a uniform defense of N sites with K interceptors. Writing $t = \frac{K}{N} - [\frac{K}{N}]$, we have $E(N,K) = N(1-t)P(s) + NtP(s+1)$. Substituting for $P(s)$ and $P(s+1)$ from (1) we get

$$\frac{1}{N} E(N,K) = 1 - p(1-Q)^s(1-tQ) . \qquad (2)$$

This result suggests that we extend the definition of $P(x)$ to all x by

$$P(x) = 1 - p(1-Q)^{[x]}(1 - \langle x \rangle Q), \qquad x \geq 0, \qquad (3)$$

where $\langle x \rangle = x - [x]$. (Note that (3) reduces to (1) for integral x.) Then (2) can be written as

$$\frac{1}{N} E(N,K) = P(\frac{K}{N}) . \qquad (4)$$

This result implies that $P(x)$ can be thought of as a site survival probability even if x is not an integer.

<u>Theorem 1</u>. The uniform defense maximizes the expected number of survivors for each round of the attack.

<u>Proof</u>. Suppose that N sites are to be defended by K interceptors for some round of the attack. Consider the firing doctrine which defends each of n_i sites with a salvo of k_i interceptors $(i = 1,2,\cdots,M)$, where

$$\sum_{i=1}^{M} n_i = N \quad \text{and} \quad \sum_{i=1}^{M} n_i k_i = K . \tag{5}$$

The expected number of survivors is

$$E(\underline{n},\underline{k}) = \sum_{i=1}^{M} n_i P(k_i) , \tag{6}$$

where

$$\underline{n} = (n_1 , n_2 , \cdots , n_M) \quad \text{and} \quad \underline{k} = (k_1 , k_2 , \cdots , k_M) .$$

It is easy to see that $P(x)$ as defined by (3) is a concave function of x. The graph of $P(x)$ is composed of straight line segments connecting the points at integral values of x defined by (1). But since (1) is a concave function of x, it follows that (3) is also. Hence

$$\sum_{i=1}^{M} a_i P(x_i) \leq P\left(\sum_{i=1}^{M} a_i x_i \right), \tag{7}$$

for all x_i and a_i such that $a_1 + a_2 + \ldots + a_M = 1$ and $a_i \geq 0$, $i = 1, 2, \ldots , M$.

Applying (7) and (5) to (6), we have

$$\begin{aligned} E(\underline{n},\underline{k}) &= N \sum_{i=1}^{M} \frac{n_i}{N} P(k_i) \\ &\leq NP\left(\sum_{i=1}^{M} \frac{n_i}{N} k_i \right) \\ &= NP\left(\frac{K}{N}\right) . \end{aligned}$$

But, by (4), this means that $E(\underline{n},\underline{k}) \leq E(N,k)$. Since $\underline{n}$ and $\underline{k}$ are arbitrary, the theorem is proven.

3. Expected value approximation.

Unless the interceptors and attackers are perfect ($Q = p = 1$), the number of sites surviving each round is a random variable. We can get an approximate solution to the problem of finding an optimal firing doctrine by assuming that the outcome of each engagement is its expected outcome. (The exact solution will be discussed later.) We will also ignore the fact that salvo sizes and numbers of surviving sites must be integers and allow them to vary continuously. However, we shall use (3) as the site survival probability for a salvo of size x. In view of (4) and the definition of $E(N,K)$, this means that we are essentially using a uniform defense against each round of the attack. Since the exact optimal firing doctrine is composed of uniform defenses (Theorem 1), this implies that our approximate solution should be close to the exact solution if the expected value approximation is a good one.

Under the above assumptions, let $E_m(n,k)$ be the expected number of survivors of n sites attacked by m sequential attackers each if k interceptors per site are available and an optimal damage assessment firing doctrine is used. In other words, $E_m(n,k)$ is the maximum number of survivors taken over all firing doctrines which never defend dead sites.

Theorem 2. For all integral $m \geq 1$,

$$E_m(n,k) = \max_{0 \leq x \leq k} E_{m-1}\left(nP(x), \frac{k-x}{P(x)}\right) \tag{8}$$

and

$$E_0(n,k) = n .$$

Proof. The formula for $m=0$ is obvious. For $m \geq 1$, let nx be the total number of interceptors used on the first round. Then $n_1 = nP(x)$ sites survive the first round and there are $k_1 = (k-x)/P(x)$ interceptors per site left to defend against the next $m-1$ rounds. Given x, the maximum number of survivors is obviously $E_{m-1}(n_1, k_1)$. Clearly, an optimal firing doctrine uses an x which maximizes this, i.e., (8) holds.

Corollary. Let $D_m(k) = \frac{1}{n} E_m(n,k)$. Then $D_m(k)$ is independent of n and satisfies

$$D_m(k) = \max_{0 \leq x \leq k} P(x) D_{m-1}\left(\frac{k-x}{P(x)}\right) \tag{9}$$

for all integral $m \geq 1$ and $D_0(k) = 1$.

Proof. We use induction on m. Setting $m=1$ in (8), we have

$$E_1(n,k) = \max_{0 \leq x \leq k} nP(x) = nP(k) .$$

Hence $D_1(k) = P(k)$ is independent of n and satisfies (9).

Now suppose that the corollary holds for $m = M$. Using (6) with $m = M + 1$, we have

$$D_{M+1}(k) = \frac{1}{n} E_{M+1}(n,k)$$

$$= \frac{1}{n} \max_{0 \le x \le k} E_M\left(nP(x), \frac{k-x}{P(x)}\right)$$

$$= \frac{1}{n} \max_{0 \le x \le k} nP(x) D_M\left(\frac{k-x}{P(x)}\right)$$

$$= \max_{0 \le x \le k} P(x) D_M\left(\frac{k-x}{P(x)}\right).$$

But this is (9) with $m = M + 1$ and, since $D_M\left(\frac{k-x}{P(x)}\right)$ is independent of n, so is $D_{M+1}(k)$. Hence the corollary holds for $m = M + 1$ and the induction is complete.

It is clear from its definition that $D_m(k)$ is the probability that a randomly chosen site will survive the attack. The fact that it is also independent of n is a consequence of the expected value approximation.

Clearly, any x which maximizes (8) also maximizes (9) and is an optimal first round salvo size against m attackers if k interceptors per site are available. Denote the smallest such x by $\bar{x}_m(k)$. Then an optimal second round salvo size is $\bar{x}_{m-1}\left(\frac{k - \bar{x}_m(k)}{P(\bar{x}_m(k))}\right)$, etc. Hence an optimal firing doctrine can be constructed from the $\bar{x}_m(k)$'s.

4. Perfect interceptors.

An important and instructive case occurs when the interceptors are perfect, i.e., $Q = 1$. By (3),

$$P(x) = \begin{cases} 1 - p + px , & 0 \leq x \leq 1 , \\ 1 , & x > 1 . \end{cases} \tag{10}$$

(Obviously, nothing can be gained by having more than one perfect interceptor in a salvo.) Setting $m = 1$ in (9), we have

$$D_1(k) = \begin{cases} 1 - p + pk , & 0 \leq k \leq 1 , \\ 1 , & k > 1 . \end{cases} \tag{11}$$

Next, set $m = 2$ in (9) to get

$$\begin{aligned} D_2(k) &= \max_{0 \leq x \leq k} P(x) D_1\left(\frac{k-x}{P(x)}\right) \\ &= \max_{0 \leq x \leq \min\{k,1\}} P(x) D_1\left(\frac{k-x}{P(x)}\right) . \end{aligned} \tag{12}$$

(To see that $0 \leq x \leq k$ may be replaced by $0 \leq x \leq \min\{k,1\}$, recall that x is the first round salvo size and, since the interceptors are perfect, should therefore never exceed one.) Substituting (10) into (12), we have

$$D_2(k) = \max_{0 \le x \le \min\{k,1\}} (1 - p + px)D_1(h(x)) , \qquad (13)$$

where

$$h(x) = \frac{k - x}{1 - p + px} .$$

Since $h(x)$ may be less than or greater than one, there are several cases to consider.

<u>Case 1</u>. $0 \le k \le 1 - p$.

Then

$$h(x) \le \frac{1 - p - x}{1 - p + px} \le 1$$

and hence

$$D_2(k) = \max_{0 \le x \le k} (1 - p + px)(1 - p + ph(x))$$

$$= \max_{0 \le x \le k} ((1 - p)^2 + pk - p^2x)$$

$$= (1 - p)^2 + pk . \qquad (14)$$

<u>Case 2</u>. $1 - p < k \le 2$.

By definition, $h(x) \le 1$ if and only if $x \ge a = \frac{k - 1 + p}{1 + p}$. (Note that $0 < a \le 1$.) Since $D_1(h(x)) = 1$ for all $x < a$, it follows from (13) and (11) that

$$D_2(k) = \max_{a \le x \le \min\{k,1\}} (1 - p + px)(1 - p + ph(x))$$

$$= \max_{a \le x \le \min\{k,1\}} ((1 - p)^2 + pk - p^2x)$$

$$= (1 - p)^2 + pk - p^2 a$$

$$= \frac{1 - p + pk}{1 + p} . \tag{15}$$

Case 3. $k > 2$.

Since each site can be defended throughout the attack, we have

$$D_2(k) = 1 . \tag{16}$$

Now let x_i be the $i^{\underline{th}}$ round salvo size (average number of interceptors used per surviving site) required to maximize the expected number of survivors ($1 = 1,2, \ldots , m$). Then $(x_1, x_2, \ldots , x_m)$ is an optimal firing doctrine.

Inspection of (13) shows that x_1 is the maximizing x and $x_2 = h(x_1)$. Hence the results for $m = 2$ can be summarized by the following table.

Case	x_1	x_2	$D_2(k)$
$0 \leq k \leq 1 - p$	0	$\frac{k}{1 - p}$	$(1 - p)^2 + pk$
$1 - p < k \leq 2$	$\frac{k - 1 + p}{1 + p}$	1	$\frac{1 - p + pk}{1 + p}$
$k > 2$	1	1	1

(17)

Note that there is only one optimal firing doctrine when $k \leq 2$. Uniqueness fails only when more perfect interceptors are available than are needed. (If $k > 2$, then all firing doctrines with $x_1 \geq 1$ and $x_2 \geq 1$ are optimal).

We could continue in this way and find $D_m(k)$ and its associated optimal firing doctrine for $m = 3, 4, \ldots$. However, this will not be necessary. For as we shall see, the general solution can be deduced directly from the solution for $m = 2$.

We begin by relating the general optimal firing doctrine to the optimal firing doctrine for $m = 2$. Let $\underline{x} = (x_1, x_2, \ldots, x_m)$ be an optimal firing doctrine for some fixed m and k. Then, by the corollary to Theorem 2, $\underline{x}$ is optimal for any n. For some fixed n, let n_i be the number of sites surviving the first $(i - 1)$ rounds $(i = 1, 2, \ldots, m + 1)$. Let $K_i = n_i x_i$ be the total number of interceptors used by $\underline{x}$ on the $i^{\underline{th}}$ round $(i = 1, 2, \ldots, m)$. Thus, $K_i + K_{i+1}$ interceptors are used to defend n_i sites against two rounds of the attack. But since $\underline{x}$ is an optimal firing doctrine, it must make optimal use of the $K_i + K_{i+1}$ interceptors for rounds i and $i + 1$. In other words, given n_i and $K_i + K_{i+1}$, n_{i+2} must be a maximum. Thus

$$E_2\left(n_i, \frac{K_i + K_{i+1}}{n_i}\right) = n_{i+2}. \tag{18}$$

and the optimal firing doctrine for these two rounds of the attack must be (x_i, x_{i+1}).

As it stands, this observation is of little use. Since $K_i + K_{i+1}$ is a function of x_i and x_{i+1}, we cannot use (18) to calculate (x_i, x_{i+1}) directly. However, we can use it to deduce the <u>structure</u> of $\underline{x}$. Examination of the firing doctrine from (17) shows that $x_{i+1} = 1$ whenever $x_i > 0$. In other words, as soon as any sites are defended, <u>all</u> surviving sites must be defended on the next round. Starting with $i = 1$ and applying this principle round by round, we conclude that every optimal firing doctrine has the following structure:

$$x_1 = \cdots = x_{r-1} = 0 \ , \ x_r > 0 \ , \ x_{r+1} = \cdots = x_m = 1 \ , \qquad (19)$$

for some $r = r(m, k)$, $1 \leq r \leq m$.

In words, as soon as any sites are defended, all surviving sites must be defended for the remainder of the attack.

<u>Theorem 3</u>. If $Q = 1$ and $0 < k \leq m$,

$$D_m(k) = \frac{(1 - p)^r + pk}{1 + p(m - r)} \ , \qquad (20)$$

where $r = r(m,k)$ satisfies

$$(1 - p)^r(m - r) < k \leq (1 - p)^{r-1}(m - r + 1) \ . \qquad (21)$$

The optimal firing doctrine is given by (19) with

$$x_r = \frac{k - (1 - p)^r(m - r)}{(1 - p)^{r-1}(1 + pm - pr)} . \tag{22}$$

<u>Proof</u>. It is sufficient to check that the firing doctrine given by (19) and (22) yields (20) and uses k interceptors per site. Starting with a farm of n sites, we have from (19) that $n(1 - p)^{r-1}$ sites survive the first $(r - 1)$ rounds of the attack. Hence

$$n(1 - p)^{r-1}x_r = n \cdot \frac{k - (1 - p)^r(m - r)}{1 + p(m - r)} \tag{23}$$

sites are defended and

$$n(1 - p)^{r-1}(1 - x_r) = n \cdot \frac{(1 - p)^{r-1}(m - r + 1) - k}{1 + p(m - r)} \tag{24}$$

sites are left undefended on the $r^{\underline{th}}$ round. (Note that (21) implies $0 < x_r \leq 1$.) Since the probability that an undefended site survives the $r^{\underline{th}}$ round is $(1 - p)$, the number of sites surviving the first r rounds is

$$n(1 - p)^{r-1}x_r + n(1 - p)^r(1 - x_r) = n \cdot \frac{(1 - p)^r + pk}{1 + p(m - r)} . \tag{25}$$

But all sites surviving the first r rounds survive the attack. Hence (25) is $E_m(n, k)$ and (20) is proven.

Finally, by (19), (23) and (25), the total number of interceptors used is $n(1-p)^{l-1}x_r + (m-r)\,\varepsilon_m(n,k) = nk$ and the proof is complete.

<u>Corollary</u>. If $Q = 1$ and

$$k = (1-p)^s(m-s) \tag{26}$$

for some $s = 0, 1, \cdots, m-1$, then

$$D_m(k) = (1-p)^s = \frac{k}{m-s}\,. \tag{27}$$

<u>Proof</u>. By (26), $r = s+1$ in (21) and (20) becomes (27).

The corollary presents the solution for the special cases where the sites are left undefended for the first s rounds and all surviving sites are defended for the last $m-s$ rounds $(s = 0, 1, \cdots, m-1)$.

For completeness, we note that $D_m(0) = (1-p)^m$ and $D_m(k) = 1$ for $k \geq m$. The result for $k = 0$ could have incorporated in Theorem 3 by allowing equality in the left-hand-side of (21). This, however, would have led to non-unique $r(m,k)$ for the cases of the corollary.

5. No damage assessment.

We can evaluate the effect of damage assessment by comparing the expected number of surviving sites with damage assessment to the expected number of surviving sites without damage assessment. We will restrict ourselves to the case of perfect interceptors.

Without damage assessment, a sequential attack of m attackers with known m is equivalent to a simultaneous attack of m attackers. Under the same assumptions made in Section 3, let $F_m(n, k)$ be the expected number of survivors of n sites attacked simultaneously by m attackers each if k interceptors per site are available and an optimal firing doctrine is used. We can evaluate the effect of damage assessment by comparing $F_m(n, k)$ with $E_m(n, k)$.

Let y_i be the number of perfect interceptors used to defend the $i^{\underline{th}}$ site $(i = 1, 2, \cdots, n)$. Then the expected number of survivors of a simultaneous attack of m attackers is

$$F(y_1, y_2, \cdots, y_n) = \sum_{i=1}^{n} (1 - p)^{m-y_i} \quad , \tag{28}$$

Then

$$F_m(n,k) = \max_S F(y_1, y_2, \cdots, y_n) \; , \tag{29}$$

where S is the set of all integers $y_1, y_2, \cdots, y_n$ such that $0 \le y_i \le m$ and $y_1 + y_2 + \cdots + y_n = nk$.

For the special case of perfect attackers ($p = 1$) it is easy to see that $P_m(n, k) = [\frac{nk}{m}]$ and the optimal firing doctrine is to defend $[\frac{nk}{m}]$ sites with m interceptors each. Since the disposition of the remaining $nk - m[\frac{nk}{m}]$ interceptors does not matter, we may suppose that they are all assigned to one of the sites. As we shall see, this firing doctrine is optimal in general.

<u>Theorem 4</u>. If $0 = 1$ and $k \le m$,

$$P_m(n,k) = s + (1 - p)^{m-t} + (n - s - 1)(1 - p)^m , \qquad (30)$$

where $s = [\frac{nk}{m}]$ and $t = nk - ms$.

An optimal firing doctrine is to defend s sites with m interceptors each and 1 site with t interceptors (the remaining $n - s - 1$ sites are left undefended). If $0 < p < 1$, this firing doctrine is unique.

<u>Proof</u>. If $p = 1$, no site can survive unless it is defended against all of its attackers. The largest number of sites which can be so defended is s; hence the theorem.

For $0 < p < 1$, we can write

$$F(y_1, y_2, \cdots, y_n) = (1 - p)^m \sum_{i=1}^{n} a^{y_i} , \qquad (31)$$

where $a = 1/(1-p)$. Hence, by (29), we must maximize

$$G(y_1, y_2, \cdots, y_n) = \sum_{i=1}^{n} a^{y_i} \tag{32}$$

over S for some $a > 1$. Introducing Lagrange multipliers, we are led to the solution $y_1 = y_2 = \cdots = y_n = k$. This, however, minimizes G and so we must use a different approach.

Let $\underline{y} = (y_1, y_2, \cdots, y_n)$ be any firing doctrine in S with $0 < y_1 \leq y_2 < m$. Set $y_1' = y_1 - 1$ and $y_2' = y_2 + 1$. Then $\underline{y}' = (y_1', y_2', y_3, \cdots, y_n)$ is also in S. But

$$\begin{aligned} G(\underline{y}') - G(\underline{y}) &= a^{y_1'} + a^{y_2'} - a^{y_1} - a^{y_2} \\ &= (a - 1)(a^{y_2} - a^{y_1 - 1}) \\ &> 0 \end{aligned}$$

since $a > 1$ and $y_2 > y_1 - 1$. Thus $G(\underline{y}') > G(y)$. By symmetry, we conclude that $G(y_1, y_2, \cdots, y_n)$ can always be increased if any two of the y_i's lie between 0 and m. It follows that $G(y_1, y_2, \cdots, y_n)$ is maximized when at most one of the y_i's is neither 0 nor m. In other words, the optimal firing doctrine defends the largest possible number of sites with m interceptors each and defends one site with the remaining interceptors. The theorem now follows immediately.

The structure of the optimal firing doctrine without damage assessment is quite different from that of the opti-

mal firing doctrine with damage assessment. Without damage assessment, we defend some of the sites all of the time. With damage assessment, we defend all of the sites surviving to some round for all of the remaining rounds, i.e., we defend some of the sites some of the time. Another characterization of their difference is that the damage assessment firing doctrine is a function of p while the no damage assessment firing doctrine is not.

To facilitate the comparison between $F_m(n, k)$ and $E_m(n, k)$, let us suppose that $(1 - p)^m \approx 0$. Then by Theorem 4,

$$F_m(n, k) \approx \frac{nk}{m} . \tag{33}$$

(This shows that even the no damage assessment outcome is approximately independent of p.) In view of the corollary to Theorem 3,

$$E_m(n,k) \approx \frac{nk}{m - u} , \tag{34}$$

where $u = u(m, k)$ satisfies $(1 - p)^u(m - u) = k$. Hence the relative loss due to lack of damage assessment is

$$\frac{E_m(n,k) - F_m(n,k)}{E_m(n,k)} \approx \frac{u}{m} . \tag{35}$$

The relative loss is a minimum when $k = m$ (damage assessment is unnecessary when there are enough interceptors

to defend all of the sites throughout the attack) and increases all k decreases (the fewer interceptors there are the more important it is to use them most efficiently).

6. Exact recursion formula.

We will now drop the assumption that the outcome of each engagement is its expected outcome and derive an exact recursion formula for the problem as stated. Let $E_m^*(n,k)$ be the expected number of surviving sites with an optimal damage assessment firing doctrine when full account is taken of the distribution of the number of surviving sites at each stage of the attack.

Theorem 5. For all integral $m \geq 1$,

$$E_m^*(n,k) = \max_{x=\frac{0}{n},\frac{1}{n},\cdots,\frac{nk}{n}} \sum_{j=0}^{n} p_j(n,x) E_{m-1}^*\left(j, \frac{nk - nx}{j}\right), \tag{36}$$

where

$$p_j(n,x) = \sum_{i=0}^{j} \binom{n-t}{i}\binom{t}{j-i}\{P(s)\}^i\{1-P(s)\}^{n-t-i}\{P(s+1)\}^{j-i}\{1-P(s+1)\}^{t-j+i} \tag{37}$$

$(j = 0,1,\cdots,n)$,

$s = [x]$, $t = nx - ns$, and $E_0^*(n,k) = n$.

Proof. The proof is patterned after that of Theorem 2. For $m \geq 1$, let $nx = ns + t$ be the total number of interceptors used on the first round. By Theorem 1, the optimal first round firing doctrine is the uniform defense. Thus, $n - t$ sites are defended with s interceptors each and t

sites are defended with $s + 1$ interceptors each. Let $p_j(n, x)$ be the probability that exactly j sites survive the first round. Then

$$p_j(n,x) = \sum_{i=0}^{j} \Pr\left\{\begin{array}{l} i \text{ survivors in the group defended by} \\ s \text{ interceptors and } j - i \text{ survivors} \\ \text{in the group defended by } s+1 \text{ intercep-} \\ \text{tors} \end{array}\right\} . \tag{38}$$

Since the outcomes are independent from site to site, each term in (38) is a product of two binomial probabilities and (37) follows immediately. For every j, $E^*_{m-1}\left(j, \frac{nk - nx}{j}\right)$ is the expected number of survivors if an optimal firing doctrine is used on the last $m - 1$ rounds. Hence the summation in (36) is the expected number of survivors if nx interceptors are used on the first round and an optimal firing doctrine is used thereafter. Since nx must be an integer, x is restricted to the values $0, \frac{1}{n}, \cdots, \frac{nk}{n}$ and (36) follows.*

---

* This research was sponsored by the Advanced Research Projects Agency and technically monitored by the Air Force Office of Scientific Research, Contract No. AF 49(638)-1478 as part of Project DEFENDER Studies.