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THESIS

A GAME THEORY ANALYSIS OF PROPOSED STRATEGIES OF AN AERIAL SEARCH FOR A MOBILE SURFACE TARGET

by

Charles James Smith, Jr.

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Charles James Smith, Jr.
Lieutenant, United States Naval Reserve
B.S., Boston College, 1959

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Signature of Author: Charles J Smith Jr
Operations Research/Systems Analysis Curriculum
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Certified by: Samuel H. Parry
Thesis Advisor

Accepted by: [signature] for J.R. Borsting
Chairman, Department of Operations Analysis

Approved by: R. F. Rinehart
Academic Dean

ABSTRACT

This thesis investigates the problem of aerial search for a mobile surface target. The initial position of the target is known, and a certain amount of time elapses before the search begins. The target has a variety of choices as to speed and direction, and the search aircraft has a choice of search patterns to be flown. The method proposed to solve the problem of maximizing or minimizing the probability of detection is the game theoretic analysis of various strategies available to the searcher and the evader. Four strategies for each are compared and evaluated, with optimal strategies, within the set of strategies investigated, being derived.

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I. Introduction.

A frequently encountered problem of attack aircraft is the detection and destruction of mobile enemy installations at some time after initial detection. For example, a flight may be fired upon by a mobile surface-to-air missile site. Upon return to the air base or carrier the location of the attack is reported and a strike against it is readied and sent out. In the meantime, the missile site has often had time to depart the position in which it was originally situated. There are many other ways in which a mobile target may be detected, such as aerial reconaissance, airborne electronic detection, or visual sighting. Some of these may give a probabilistic rather than a deterministic position.

The mobile target, if it knows it has been discovered, has the options of remaining in its present position or moving. If it moves, it has some variety in its choices of direction and speed. Often these choices of movement are to some degree limited. The search aircraft are then confronted with the problem of finding and attacking the moving strike objective. This can be hampered by such variables as time to sunset, terrain, weather, and other uncontrollable elements. The target may be found by methods other than visual if it is emitting electronic signals, i.e., if it has not moved or if it has moved and is set up and operating in another position. The most difficult problem arises when the method of detection must be strictly visual, as when the objective is moving or if it has set up in another position and is not emitting. The degree of difficulty also depends on how much freedom of motion the target has. If the direction of travel is limited rather strictly, as when there are only a small number of roads along which it may travel and off-road travel is not possible, the searcher's task is somewhat simplified. If, however, it is free to travel in any or almost any direction, the problem becomes much more difficult. This thesis examines some of the problems confronting a searcher faced with the problem of discovering a target with unrestricted direction of movement, and with varying maximum speeds.

This problem may be attacked in two fashions. One method is to consider the movements of the target as being determined probabilistically, with such variables as direction, velocity, and time of movement all being described by some density function, so that an estimate of its most likely position, or most likely set of positions, can be found using the usual rules of probability. In fact, all these variables have some probabilistic limitations, but within these limitations there is another influence: the judgment of the commander of the site. It would be somewhat naive to think that the commander would be guided solely by the constraints imposed by the state of nature. Rather, it is more reasonable to assume that he would try to vary his behavior within the limits of his physical constraints so as to attain maximum liklihood of evasion. If this did not occur immediately, then certainly experience would lead him to adopt more sophisticated evasive tactics. The intelligent searcher would then alter his search pattern to counter the evasive tactics to as great a degree as possible. This development brings the problem into the area of game theory, and it is with this aspect that this paper concerns itself.

To simplify the problem a number of probabilistic variables are treated deterministically. The effect of weather, time of day, terrain and variance in terrain over the search area, and other external influences are ignored. These and other assumptions are discussed in the next section. A small number of strategies for both searcher and evader are discussed. While an attempt has been made to consider the most advantageous strategies, there are probably many of merit which have not been included. It is felt that an approach has been made to the solution of the problem, however, and some possibly beneficial tactics derived.

The general approach is to select several tactics, both for the searcher and the target. These tactics are then evaluated in pairwise combinations, using game theory criteria. This allows conclusions to be drawn concerning the relative merits of the strategies.

The next section presents the basic assumptions of the model and attempts to justify them. The ensuing sections develop the general theoretical framework and the specific mathematical formulations of the proposed strategies. After comparing these strategies, some conclusions as to the value of each are inferred.

II. Theoretical Basis for the Model.

2.1 Assumptions.

It would be an almost impossible task to obtain a universal solution to this search problem, in which all relevant factors were included. On the other hand, elimination of a critical variable would leave the conclusions worthless. The assumptions made in the development of the model are set forth here in an attempt to explain its limitations and clarify its objectives.

The space in which the game is played is a two-dimensional Euclidean geographical space. The terrain problem is ignored in that the area is considered to be perfectly flat, and in no way affects either target mobility or the searcher's visibility. The target is free to move in any direction and at any velocity up to its maximum for any duration of time. The searcher has a fixed range sighting probability; within this range the probability of detection is 1.0, outside this range the probability of detection is 0. The searcher is capable of flying any pattern exactly. The initial position of the target is known exactly. The target knows he has been detected, and takes action accordingly. No variables such as weather, time of day, or other external phenomena are considered. The time at which the aircraft will arrive to begin his search is known to the target as soon as the initial detection occurs.

There are other minor assumptions in the model. These cited are considered to be the most relevant. To be able to claim that the model has any relevance to the real world some of these assumptions must be justified.

First, consider the playing area. Unless the terrain is extremely rough, the aircraft acts essentially as if it were flying above a two-dimensional area. The behavior of the target is more affected by this assumption. Since the target-chosen factors are speed and direction, the effect of terrain on these is relevant. In a real-world situation, the target might indeed be somewhat terrain limited. However, most missile and radar sites are near cities, which in turn are usually situated in flat terrain. Also, there is usually a heavy density of roads near any city, which gives the target a great degree of flexibility in its movements, both as to direction and speed. If the area around the initial position is not travelable in almost any direction, the problem is simplified to some extent, but is susceptible to the same type analysis as is proposed here.

The assumption that the target can move at any speed up to its maximum for the duration of the search is reasonable if the search aircraft arrive within a reasonable length of time, on the order of four hours after initial detection, and stay on station a moderate length of time. Since most attack circraft can fly at low altitudes for only about two hours, the total time the target must be able to keep moving is about six or seven hours. This is well within the capability of most mobile sites.

The method of first detection of the target is often visual or photo graphic, in which cases the position is known exactly. If the position is not known exactly, the point of highest probability is usually taken as the true position and treated deterministically in any actual search. The effect of large errors in the position of the original site is not examined in this discussion.

The implications of a fixed range of detection are well known and no further implications have arisen in this analysis. [3] However, the idea that an aircraft can fly any desired pattern exactly will strike any experienced aviator as a flagrant misconception. In fact, this is probably the most critical assumption of the model. Recent years have seen the development of Doppler navigational equipment which enables an aircraft to fly a precise track, and airborne radar in a control aircraft can guide other aircraft over a desired pattern with good accuracy. There is no reason to believe that such techniques could not be applied to search aircraft if the situation warranted.

Another critical assumption is that the target knows, as soon as detected, at what time the search aircraft will arrive. Some examination of the actual procedures reveals this assumption to be less restrictive than it initially appears. If the target moves, the probability of detection diminishes with time in any of the strategies considered. The search aircraft will therefore want to begin the search as rapidly as possible. The target often knows the position of the base or carrier from which the attack aircraft will be launched, and can therefore predict fairly accurately their arrival time. An estimate might also be obtained from observing past arrival times.

While these assumptions restrict the application of the model to a limited situation, they allow the formulation of a general mathematical model. Refinements of the general approach may be made to deal with specific deviations.

2.2 Outline of Game Theory Approach.

The searcher, denoted player A, and target, denoted player B, have available to them a large and possibly infinite set of strategies. Each attempts to maximize his payoff by choosing judiciously among these strategies. B may choose any direction and any speed up to his maximum, and may stop his movement at any time. A may choose any search pattern over the playing area which does not exceed his range. The payoff function for A is the probability of target detection. The payoff to B is either the negative of the probability of detection, -P(D), or the probability that A does not detect B, 1-P(D). If the former is defined as B's payoff the game is zero-sum; if the latter the game is constant-sum and the sum is one.

The set of strategies for player A is denoted Γ and each element of this strategy set is denoted α. The set of strategies for player B is denoted Σ and the elements of the set are denoted β. The pairs of strategies (α, β) are elements of the Cartesian product of Γ and Σ, denoted $\Gamma \times \Sigma$. The payoff to A for a given pair of strategies, α_o and β_o, is $M(\alpha_o, \beta_o) = P(A \text{ detects } B / \alpha = \alpha_o \cap \beta = \beta_o)$. For an optimal strategy to exist, there must exist strategies α^* and β^* such that

$$\min_{\beta} \max_{\alpha} M(\alpha, \beta) = \max_{\alpha} \min_{\beta} M(\alpha, \beta) = M(\alpha^*, \beta^*) \quad [4]$$

Whether in fact such strategies exist has not been determined. To determine the existance of such strategies, either all possible strategies must be examined, or it must be proven topologically that such strategies exist. The first method is difficult due to the large number of possible strategies. There exist some existence theorems which pertain to a pursuit-and-evasion game, but none were found which were applicable to the game under discussion. An existence theorem developed by Ryll-Nardzewski [1] which was the most general as to existence criteria, indicated that a solution involving pure strategies did not exist.

It is also possible that an optimal solution which is a convex combination of various strategies exists. To find this, it is necessary to examine the payoff matrix of the various strategies, and to apply the usual methods of game theory to find a solution. This requires developing the probability of detection for various strategies. Therefore it is worthwhile to develop payoff functions for individually considered strategies of A and B.

The general procedure then is to choose some strategy α_0 and some strategy β_0. β_0 specifies a rule for choosing direction, speed and stopping point for the target. α_0 specifies a pattern to be flown over the playing area. The payoff is P(A detects b/$\alpha = \alpha_0 \cap \beta = \beta_0$). These strategies may be probabilistic and specify a probability function for choosing speed or initial aircraft search point, or may be deterministic, such as always start the search at a radius of five miles, or always remain at the initial position. If the set of attractive strategies is compared pairwise in this manner, the matrix of strategy payoffs is generated.

III. Development of Strategies.

3.1 Strategies Considered.

As has been noted, there are a great many possible strategies for both searcher and target. After considering a large number of strategies, the following were selected for investigation on the basis of their intuitive attractiveness or because they are frequently used in practice. A glossary of terms is given in table one. The strategies as proposed by the author are:

α_1:

Player A searches over v_t in the most efficient manner which also searches over all directions of possible target travel, $(0, 2\pi)$. This is taken to be a spiral search, and the choice of the target velocities to be searched is determined in a game theoretic manner.

Table I. Glossary of Terms

v_t target speed

v_a search aircraft speed

$v_{t_{max}}$ maximum target speed

θ direction of target movement or polar angular coordinate of aircraft position

R aircraft range

T_{min} minimum time after sighting at which target may move

T_{arr} time from initial detection of aircraft arrival

t_0 $T_{arr} - T_{min}$ or time the target has been in motion at aircraft arrival time

r radius from initial target position

r_0 radius at which the aircraft starts its search

S_i circumference of the i^{th} circuit searched by the aircraft

W fixed sighting range

A diagram of a spiral search is given in Figure 1.

α_2:

This is essentially the same as α_1 with the additional condition that the original target position is always investigated.

α_3:

The aircraft search randomly in the circle of radius ($v_{t_{max}} \cdot t$), or the circle within which the target must be.

α_4:

The aircraft search uniformly over v_t and $(0, 2\pi)$

The evasive strategies are

β_1:

The target chooses velocity and direction to counter α_1 according to a game theory criteria.

β_2:

The target chooses velocity and direction to counter α_2 according to game theory criteria.

β_3:

The target chooses speed uniformly on $(0, v_{t_{max}})$ and direction uniformly on $(0, 2\pi)$.

β_4:

The target chooses direction uniformly on $(0, 2\pi)$. The target proceeds radially outward so as to distribute himself uniformly over the geographic area bounded by $r = v_{t_{max}} \cdot t$.

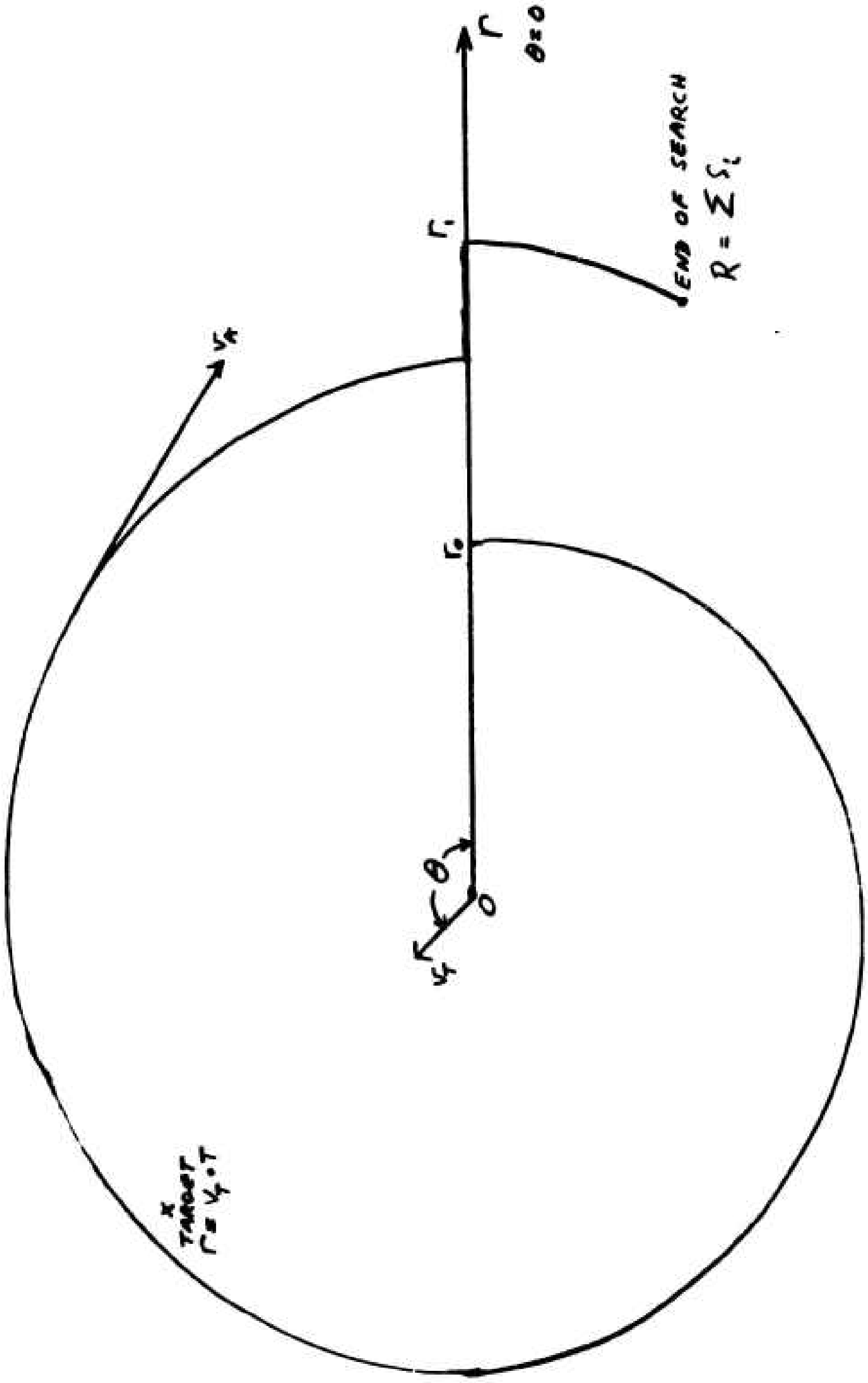

Figure 1

Diagram of Spiral Search

3.3 Spiral Search.

It is possible to demonstrate that a spiral search for a target moving radially outward from a fixed point is the most efficient way of searching over various target velocities, if the target direction is uniformly distributed on $(0, 2\pi)$. This may be intuitively justified as follows: Suppose a searcher has a given fixed range. He desires a search pattern which will minimize his path length while not searching over any velocity range more than once. For a single circuit, i.e., a search of θ from 0 to 2π, the target velocities depend upon search width W and the initial radius of the search, r_0. If the velocity range is to remain constant throughout the circuit, the search radius must move outward with the velocity which would have put the target at r_0, which is r_0/t_0. If the searcher moves outward in such a manner, and increases the radial component of his velocity with each subsequent circuit such that $r_n = r_{n-1} - (r_{n-1} / t_0) \cdot t$, he will not re-search any previously searched velocity. This pattern may be flown in several ways. The outward radial component of velocity may be chosen to remain uniform throughout the search. Another method is to vary the radial component after the completion of each circuit, thereby flying a series of spirals joined by short cross-legs. While this is slightly less efficient than a continuous spiral, it has the advantage that it searches over a continuous target velocity range. This makes it more compatible with an analysis of the optimum velocity choices of the target, and for this reason this method is used.

The equation for a single circuit is derived as follows:

The search aircraft flies an increment of the circuit of length ΔS in time Δt. During this time the target moves radially outward with velocity v_t.

$$r = v_t \cdot \Delta t = v_t \frac{S}{v_a} \quad \text{since} \quad t = \frac{S}{v_a}$$

or for a complete circuit, $r = \frac{v_t}{v_a} S.$

To derive S, take θ to be in radians.

$$(1) \qquad dS = r d\theta$$

then since $r = \frac{v_t}{v_a} S$,

$$dS = \frac{v_t}{v_a} S d\theta$$

or

$$\frac{dS}{S} = \frac{v_t}{v_a} d\theta$$

Integrating,

$$\int \frac{dS}{S} = \int \frac{v_t}{v_a} d\theta \Rightarrow \ln S + c_1 = \frac{v_t}{v_a} \theta + c_2$$

or equivalently,

$$(2) \qquad S = c_1 e^{\frac{v_t}{v_a} \theta} + c_2$$

Since (1) $r = \frac{dS}{d\theta}$,

$$(3) \qquad r = \frac{v_t}{v_a} c_1 e^{\frac{v_t}{v_a} \theta}$$

Imposing boundary conditions,

$$\theta = 0 \qquad \begin{matrix} r = r_o \\ S = 0 \end{matrix}$$

$$c_1 = \frac{v_a}{v_t} r_o \quad \text{from (3)}$$

$$c_2 = -\frac{v_a}{v_t} r_o \text{ from (2).}$$

The final forms of the expressions are

$$S = \frac{v_a}{v_t} r_o \left(e^{\frac{v_t}{v_a}\theta} - 1 \right)$$

$$r = r_o e^{\frac{v_t}{v_a}\theta}$$

To get these expressions in a more manageable form, express $e^{\frac{v_t}{v_a}\theta}$ as an infinite series.

$$e^{\frac{v_t}{v_a}\theta} = 1 + \frac{v_t}{v_a}\theta + \frac{\frac{v_t^2}{v_a^2}\theta^2}{2!} + \ldots .$$

For $v_t >> v_a$,

$$e^{\frac{v_t}{v_a}\theta} \approx 1 + \frac{v_t}{v_a}\theta$$

Since v_a is usually over 300 and v_t is usually less than 30, this is a valid approximation.

Then

$$S = \frac{v_a}{v_t} r_o \left(1 + \frac{v_t}{v_a}\theta - 1 \right) = \theta r_o$$

and

$$r = r_o \left(1 + \frac{v_t}{v_a}\theta \right)$$

At the end of each circuit, the searcher must position himself to search over a new velocity increment. If the detection range either side of the flight path is W, the minimum velocity unsearched is $(r_0 + W)/ t_0$, which is that velocity which would put the target at $r_0 + W$ at the start of the search. Since the search width is 2W, a radius of $r_0 + 2W$ will search velocities from $(r_0 + W) / t_0$ to $(r_0 + 3W) / t_0$, if the velocity is assumed to be constant across the search width. This is not strictly true, since $\frac{r_0 + W}{t_0} < \frac{r_0 + 2w}{t_0} < \frac{r_0 + 3W}{t_0}$; however since this distance is small it will be ignored. For example, if $W = 1$, $t_0 = 2$, then W/t_0 is .5, which is the maximum deviation from the velocity at the center of the search strip. As the time increases, this deviation becomes smaller.

If the target were at $r_0 + 2W$ at t_0, it would move radially outward during the time the first circuit was being flown. The time required to search the first velocity increment is

$$\frac{S_0}{v_a} = \frac{2\ r_0}{v_a}$$

Then the initial radius for the following circuit is

$$r_1 = r_0 + 2W + \frac{r_0 + 2w}{t_0} \left(\frac{S_0}{v_a}\right)$$

The distance from the position at the end of the first circuit to the beginning point of the next circuit is

$$r = r_1 - r_0 \left(1 + \frac{2\ r_0}{v_a t_0} \right) = 2W + \frac{4W\ r_0}{t_0 v_a}$$

If $r_0 = 100$, $W = 1$, $v_a = 300$, and $t_0 = 2$,

$$r = 2 + \frac{8 \cdot 100}{2 \cdot 300} = 12.16$$

Since these cross-legs are small compared to total aircraft range, they will be ignored in further computations.

Then to compute any r, it is necessary to take its position at t_0 and to add to it the distance traveled while the aircraft was searching previous circuits.

$$r_n = r_0 + 2nW + \left(\frac{r_0 + 2nW}{t_0}\right) \frac{\sum_{i=0}^{n-1} S_i}{v_a}$$

since

$$S_n = 2\pi r_n,$$

$$S_n = 2\pi \left[r_0 + 2nW + \left(\frac{r_0 + 2nW}{t_0}\right) \frac{\sum_{i=0}^{n-1} S_i}{v_a} \right]$$

Because of the term $\sum_{i=0}^{n-1} S_i$, a recursive relation exists.

A solution for $N = n_{max}$ is possible if the total path flown is restricted by the aircraft range, or

$$R = \sum_{i=0}^{N} S_i$$

However, because of the complexity of the expression an analytic solution is very difficult and an iterative method of solution for the maximum velocity searched as a function of r_0 is laborious. For these reasons a computer program was used to solve the expression. Thus a method exists for solving for the range of target velocities searched for any r_0, with input parameters W, R, v_a, $v_{t_{max}}$ and t_0. This solution is necessarily discrete, since the r_0's are discrete; however, any degree of accuracy desired is obtainable.

3.4 Game Theory Evaluations of the Proposed Strategies.

3.4.1 Discussion of Strategy α_1.

Using the previously derived expression for the range of velocities searched over for any r_0, the minimum and maximum target velocities may be obtained. The probability of detection for any given initial search radius, say r_o^*, is

$$P\left[v_{min}(r_o^*) < v_t < v_{max}(r_o^*) \;/\; r_o = r_o^*\right]$$

Then

$$M(\alpha_1, \beta_1) = P(D)$$

$$= P\left(v_{min}(r_o^*) < v_t < v_{max}(r_o^*) \middle/ r_o = r_o^*\right) P\left(r_o = r_o^*\right)$$

The lower the value of r_0^* the greater is the range of target velocities searched. For the last circuit, generally not all values of θ are searched since the circuit must terminate when maximum range is reached. The target direction of travel is distributed uniformaly on (0, 2), which means the probability of detection on the last circuit is the percentage of the circumference searched. This probability is denoted θ_N. The density function for $P(D/r_0 = r_0^*)$ is

$$f(r_0^*) = \begin{cases} 0 & v_t < \dfrac{r_0 - W}{t_0} = v_{min}(r_0^*) \\ 1 & \dfrac{r_0 - W}{t_0} \leq v_t < \dfrac{r_0 + 2NW - W}{t_0} \\ \theta_N & \dfrac{r_0 + 2NW - W}{t_0} \leq v_t \leq \dfrac{r_0 + 2NW + W}{t_0} = v_{max}(r_0^*) \\ 0 & v_{max}(r_0^*) \leq v_t \end{cases}$$

The graph of this density function is depicted in Figure 2.

Denote the range of velocities searched by v_r. The payoff function is $M(v_r^*, v_t^*) = P(D / v_r = v_r^* \text{ and } v_t = v_t^*)$.

The distribution functions for v_r and v_t are $F(v_r)$ and $G(v_t)$. These are not fixed distributions, but are any possible distributions. The expected value of the game for two distributions F and G is denoted E(F,G).

$$E(F,G) = \int_0^{v_{t_{max}}} \int_0^{v_{t_{max}}} M(v_r, v_t)\, dF(v_r)\, dG(v_t)$$

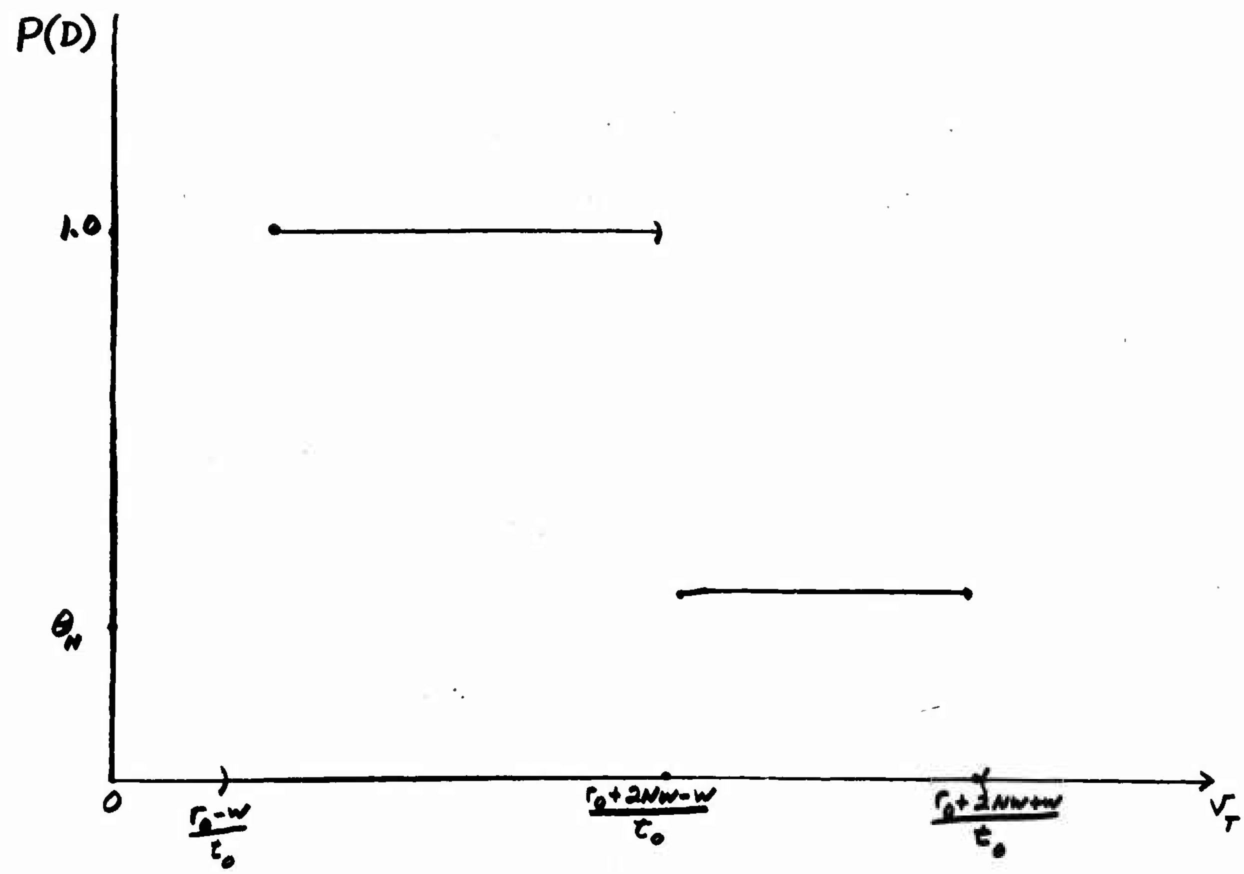

Figure 2

Conditional Density Function of the Probability of Detection

If $M(v_r, v_t)$ is continuous then a solution is guaranteed. However, in the case of spiral search M is not continuous, and therefore no solution is guaranteed. However, an examination of the proof of this theorem in 4 indicates that the requirement of a continuous payoff function may be too restrictive. The assumption of continuity is necessary in the proof to guarantee the existance of the Stieltjes integral of

$$\int M(x, y)\, dF(x)$$

and

$$\int M(x, y)\, dG(y)$$

and to guarantee the existance of certain minima and maxima over the domain of the payoff function. It has not been determined if the existence of a solution to the problem under discussion exists.

The existence of a game theory solution has been presumed and density functions for the searcher and target derived. Due to the complexity of the analytic expression giving the range of target velocities searched for a given r_0, an approximation to the solution using discrete values of r_0 and v_t has been used.

If r_0 and v_t are allowed to take on discrete values and the probability of detection derived for each combination of these discrete values, a matrix approximating the continuous-domained payoff function may be obtained. The degree of accuracy is dependent on the fineness of the increments of v_t and r_0; for purposes of the present investigation increments of one unit of speed, i.e., one knot or one mile per hour were used. The resultant matrix is m x m+1, where $m = v_{t_{max}}$. There are several methods to obtain the game theory solution of this matrix, but the precise methods of solution have the detriment of being extremely complex for a matrix of any size. For this reason an algorithm which gives an approximate solution was used. The algorithm and the computer program used to perform the calculations are discussed in Appendix I. An extensive discussion with proof of convergence is given in Karlin [2]. The relevant properties of the method are:

(1) An approximate strategy for A and B may be found.

(2) Upper and lower bounds on the value of the game are generated. The accuracy of the solution depends on the number of iterations performed. Two thousand iterations are considered to give a usable approximation for the matrix under discussion. As an accuracy check one program

was run which performed twenty thousand iterations. The upper and lower values of the game, denoted $\overline{V}$ and $\underline{V}$, were .1254 and .0845 for 2000 iterations and .1153 and .1017 for 20,000 iterations. The total computer time time for the more accurate method is four minutes and seven seconds. With 2,000 iterations ten sets of input parameters may be run in three minutes and fifty seconds. Using this program optimal strategies for A and B were found, with A restricted to using only spiral search. The strategies for searcher and target for varying $v_{t_{max}}$ and t_0 are tabulated in Appendix II.

3.4.2. Discussion of Strategy α_2.

All of the theory and general discussion relating to α_1 is applicable to α_2. The same computer program with appropriate modifications was used to solve the payoff matrix. The range of the search aircraft was diminished by the radius of initial search to compensate for searching $v_t = 0$ on each search pattern, which had little effect on the outcome. This strategy forced B to always move from the position of initial detection, which to some extent alleviates the assumption that the time the target begins its movement is a deterministic rather than a probabilistic quantity. The reason for this assumption is that there is a lower bound on the length of time the site takes to disassemble components and makes ready to move. If B is forced to move, the longer he is in motion the more difficult is the problem of detection. If B knows the initial position will always be searched, he is more likely to make an effort to become highly efficient in preparing to move, and therefore the range of times at which he moves cluster more closely around the minimum move time. The solution generated by the computer for this strategy indicates that B will move at a very low speed in about the same proportion that he did not move when A plays strategy α_1.

3.4.3. Discussion of Strategy α_3.

This strategy requires A to search randomly in a circle of radius $(t_0 \cdot v_{t_{max}})$. It has the advantage of not requiring any expenditure of effort on navigation, and in fact is often used in practice. It is an inefficient search method and is included largely as a basis of comparison for more effective strategies. The area in which it searches is not constant if the target is moving radially outward, and the probability of detection is less than is given in the following formula. However, since the probability of detection is low for this method, no accuracy was lost in the general conclusions.

The probability of detection is given by

$$P(D) = 1 - e^{\frac{-2WR}{A}} \qquad [3]$$

The derivation of this formula is given by Koopman. [3]

3.4.4. Discussion of Strategy α_4.

In this strategy A searches all velocities of B with equal likelihood, or v_r is distributed uniformaly on $(0, v_{t_{max}})$. There is a problem in this strategy arising from the fact that searching some velocities requires less distance traveled than others. The strategy is therefore formulated as the sequential choice of velocities to be searched with each choice being uniformly distributed on the target velocity range. The expected probability of detection based on the expected range of velocities searched is used in the payoff matrix of the various strategies.

3.4.5. Discussion of Strategies β_1 and β_2.

These strategies are defined in the course of deriving strategies

α_1 and α_2. As with α_1 and α_2, discrete values are used to approximate the continuous strategy.

3.4.6. Discussion of Strategy β_3.

B moves at time t_0 with a velocity chosen uniformly on $(0, v_{t_{max}})$ and continues to move with this velocity throughout the time period. The direction of travel is chosen uniformly on $(0, 2\pi)$. This distribution is also obtained if the target moves with velocity $v_{t_{max}}$ and stops at times uniformly distributed on $(0, t_0)$ where 0 denotes the minimum time at which the target may move.

3.4.7. Discussion of Strategy β_4.

In this strategy player B distributes himself uniformly over the geographical area of radius $v_{t_{max}} \cdot t_0$. The probability of detection is

$$P(D) = \frac{a}{A}$$

where

$$a = \pi r^2$$

The distribution function in terms of radius, r, is

$$F(r) = P(D) = \frac{r^2}{A} = \frac{r^2}{(v_{t_{max}} t_0)^2} = \frac{r^2}{(v_{t_{max}} t_0)^2}$$

The density function is

$$f(r) = dF(r) = \frac{2r}{(v_{t_{max}} t_0)^2} dr$$

Since $r = t_0 v$,

$$dr = t_0 dv$$

The density function in terms of v is

$$f(v) = \frac{2t_0 v t_0 dv}{v_{t_m}^2 t_0^2} = \frac{2v}{v_{t_m}^2} dv$$

and the distribution is

$$P(v_t < v_r) = \int_0^{v_r} f(v) = \frac{v_r^2}{v_{t_m}^2}$$

Since strategies α_1 and α_2 have been approximated using discrete velocity increments this strategy may be defined in velocity increments using the distribution function, which gives an expression compatible with α_1 and α_2.

IV. Evaluation of Strategies.

4.1. Method of Evaluation of Strategies

The pairwise comparison of all combinations of strategies considered may be made and the results tabulated in a matrix of payoffs. Examination of this matrix then reveals the optimum or combination of optimum strategies. The elements of the matrix are

$$a_{ij} = P\left[D \mid \alpha = \alpha_i \cap \beta = \beta_j\right]$$

The general expression for the probability of detection for strategies α_1 and α_2 is

$$P(D) = \sum_{i=1}^{v_{t_{max}}} P(r_0 = r_i)\, P(D / r_0 = r_i) = \sum_{i=1}^{v_{t_{max}}} P(r_0 = r_i)\Big(\Big(P(v_{min}(r_i) \quad v_t \quad v_{max}(r_i))$$

The comparison of α_1 and β_1, and of α_2 and β_2 have been made in the solution of the game. The solution for elements a_{12} and a_{21} may be made using the general formula directly. To solve for elements a_{13}, a_{14}, a_{23}, a_{24} the following modification of the basic formula is used:

$$P(D) = \sum_{i=1}^{v_{t_{max}}} P(r_0 = r_i) \int_{v_{min}(r_i)}^{v_{max}(r_i)} f(v_t) \qquad \text{for } \beta_3 \text{ and } \beta_4$$

where

$$\int_{v_{min}}^{v_{max}} f(v_t) = \frac{v_{max}^2 - v_{min}^2}{v_{t_{max}}^2} \qquad \text{for } \beta_4$$

and

$$\int_{v_{min}}^{v_{max}} f(v_t) = \frac{v_{max} - v_{min}}{v_{t_{max}}} \qquad \text{for } \beta_3$$

$$\frac{2\sqrt{2}\ \frac{W}{t_0}}{v_{t_{max}}}$$

for a single circuit. Therefore a_{44} was taken to be

$$\frac{2\sqrt{2}\ \frac{W}{t_0}}{v_{t_{max}}}\ (E(C))$$

The payoff matrix for parameters R = 500, t_0 = 2, W = 1, $v_{t_{max}}$ = 30 and v_2 = 300 is

	1	2	3	4
1	.108	.103	.180	.071
2	.103	.101	.174	.074
3	.041	.041	.041	.041
4	.068	.064	.077	.054

This matrix has a saddle point at a_{24}. Therefore pure strategies exist for players A and B. It appears that searching by strategies α_1 and α_2 is highly desirable for player A, since the values of a_{1j} and a_{2j} are very close and a deviation by B from strategy β_4 greatly increases the probability of detection. If B distributes himself uniformly over his range of velocities then A increases the probability of detection by more than a factor of two if strategies α_1 and α_2 are used. To determine whether the existence of a saddle point in this case was unique, values of the elements of the first two rows of the matrix were calculated for six other sets of input parameters and in each case a saddle point existed at either a_{14} or a_{24}.

For α_3 the formula for the probability of detection of a uniformly distributed target is used although this is not true for strategies β_1, β_2 and β_3. However, the low probability of detection for random search indicates that more precise computation is not necessary.

For β_4 there is a difficulty in that path length for searching different velocities is not constant. This has been handled by using the expected value of the path length for a single circuit. The expected value of v_t is calculated for each strategy of B and this is used to get the expected number of circuits. This is then used to calculate the probability of detection.

The expected target velocity is

$$E(v_t) = \sum_{0}^{v_{t_{max}}} v_{t_i} \; P(v_t = v_{t_i})$$

Then

$$E\left[\text{path length}\right] = E[L] = 2\pi t_0 \; E(v_t)$$

Thus

$$E\left[\text{number of circuits}\right] = E[C] = \frac{R}{E[L]}$$

While it would be possible to solve for the exact probability of detection of α_4 for β_1 and β_2 this was not done since rough calculations indicated that such a value would be very low. The expected number of circuits may be used to solve for a_{43} directly by

$$\frac{E(C) \times \frac{2W}{t_0}}{v_{t_{max}}}$$

since the target speed is distributed uniformaly on $(0, v_{t_{max}})$. The expected probability of detection for a_{44} has been calculated to be

V. Conclusions.

5.1. Evaluations of Strategies

The low value of strategies $_3$ and $_4$: the random and uniform on v_t searches, indicate that their use in an actual situation would not be optimal. However, the use of a simple strategy that corrects some of the deficiencies of these strategies may be used. If an area search is used which does not re-search any area, the probability of detection will be approximately

$$\frac{2WR}{v_{t_{max}} \cdot t_0}$$

This is very slightly less than the probability of detection using strategies α_1 and α_2, the spiral searches, and is very much simpler to apply.

It must be concluded that a uniform geographic distribution of the target is optimal in the set of strategies discussed. Since no counter to this strategy seems to exist, this may well be an optimal solution over all strategies available to player B. If the equation for probability of detection used for non-overlapping flight path over a circle of radius $v_{t_{max}} \cdot t_0$ is used, a very close approximation to the probability of detection given by the saddle point is obtained. This indicates that strategies α_1 and α_2 are optimal in the set of strategies studied by virtue of their characteristic of non-overlapping flight paths rather than because they derive an advantage because of some other characteristic. If this is true and if B plays strategy β_4, uniform geographical distribution, the effort needed to apply strategies α_1 and α_2 seems unjustified. However, the distinction between uniform geographical distribution and uniform velocity distribution is not obvious, and conceivably a naive player B could choose to play strategy β_3 and distribute uniformly on $(0, t_{t_{max}})$. Since using strategies α_1 or α_2 in this case would considerably increase the probability of detecting the target, the anticipated benefits might outweigh the increased effort. This might be augmented by an attempt to estimate the type of strategy applied by the evader by goodness-of-fit tests, to determine the probability that B is using some strategy other than β_4 and to use a counter-strategy which takes maximum advantage of B's mistakes.

BIBLIOGRAPHY

1. Ryll-Nardszewski, C. A theory of pursuit and evasion, Advances in Game Theory, Princeton University Press 1964

2. Karlin, S. Mathematical Methods and Theory in Games, Programming and Mathematical Economics, Vol. I, Addison-Wesley Publishing Co., 1959

3. Koopman, B.O. Search and Screening (OEG Report No. 56) Office of the Chief of Naval Operations, 1946

4. McKinsey, J.C.C. Introduction to the Theory of Games, McGraw-Hill Book Company, Inc. 1952

APPENDIX I

Computer Program for Solving for Strategies $_1$ and $_2$

This program has two basic parts. The first develops the range of target velocities searched for various values of r_0 using a spiral search, and generates a payoff matrix using increments of one unit of speed between successive values of r_0 and v_t. The second part of the program solves the matrix using a method for approximating the solution of matrix games.

The inputs are:

RO	initial radius searched
VA	search aircraft speed
TO	length of time the target has been in motion when search begins
RNG	search aircraft range
DELR	amount the initial radius is incremented for each successive search
W	search width
TMAX	maximum target speed

The outputs are:

VMIN	minimum target speed searched for a given RO
VMAX	maximum target speed searched for a given RO
THETA	percent of circle searched on last circuit
RO	radius corresponding to VMIN, VMAX and THETA on same line of printout
A	matrix of detection probabilities
X(i)	approximation of optimum percent of time the search should start at RO equal to $i \cdot t_0 + W$
Y(i)	optimum percent of time the target should choose $v_t = i - 1$

V1 upper value of game

V2 lower value of game

Internal symbols are:

SUM total distance flown

S circumference of present circuit

SLAST percent of circumference searched on last circuit

VT(j) target velocity equal to (j − 1)

SUMA (IA) $\sum_{i=1}^{v_{t_{max}}} a_{i,j}$ for j $\ni$ SUMB(j) is maximum over j

SUMB(IB) ${}_{j=1}^{v_{t_{max}}+1} \; a_{i,j}$ for i $\ni$ SUMA(i) is minimum over i

NA minimum SUMB (j)

NB maximum SUMA (i)

YB (IE) number of times the target has picked velocity YB (IE - 1)

XA (IX) number of times the search aircraft has picked initial velocity searched XA= IX up to present iteration.

This program is written in Fortran 63 for use on the CDC 1604 computer. The first part of the program uses the equation developed in Section 3.3 to generate minimum and maximum target velocities searched for various values of RO. The initial RO is an input to the program, as is the increment of RO, designated DELR. For purposes of this investigation increments of one unit of speed were desired, and therefore the statement DELR = TO has been inserted. RO normally begins at W, which assures search of the initial target position. Each successive RO defines a row of the payoff matrix. After VMIN and VMAX are found, the spectrum of target velocities are subjected to IF tests to find if they fall within or without the detected velocities. They are assigned probabilities of detection of 0 or 1. Then the element of the row corresponding to VMAX is assigned a probability of detection THETA. This process is continued until RO = VTMAX · TO. At this point the matrix is complete.

Part II of the program obtains an approximate game theory solution for t is matrix. Obtaining an exact solution for a large matrix is difficult unless a saddle point exists, which requires that

$$\max_{i} \min_{j} a_{ij} = \min_{j} \max_{i} a_{ij}$$

But for each j in the matrix there exists an element a_{ij} such that $a_{ij} = 0$. For each i there exists an a_{ij} such that a_{ij} 0. Therefore no saddle point exists.

The exact methods of computation involve examination of all adjoints of the submatrices of the matrix. The generation of the adjoints requires evaluation of the inverse and the determinant of the submatrix. The number of submatrices of a 30 x 30 matrix is $\sum_{n=1}^{30} \binom{30}{n}\binom{30}{n}$. Therefore a method approximating the actual solution has been used. This uses an iterative process in which player A chooses his strategy by looking at the column vector which is the sum of all columns chosen by B, and playing the row corresponding to the maximum element in this vector. B looks at a row vector composed of the sum of all rows chosen by A and chooses the minimum element for his next play. The number of times each strategy is chosen divided by the total number is an approximation of the optimal strategies. The minimum element in the row vector and the maximum element in the column vector divided by the number of iterations give lower and upper bounds on the value of the game.

The row vector of sums is SUMB in the program, and the column vector of sums is SUMA. The number of times each row and column has been chosen are XA and YB. Statements 70 through 80 choose the maximum over SUMA. Statements 74 through 77 choose the minimum over SUMB.

The program as presented generates the solution for strategy α_2. To obtain the matrix and solution for strategy α_1, statement 97 and the preceding statement may be deleted, along with the fourth statement after statement 17.

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